**Universidad de La Laguna**

# Entanglement, decoherence and the transition from quantum to classical

Educational review of "Entanglement of quantum clocks through gravity" (Ruiz et al., 2017)

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# Abstract

In this work we present the behaviour of open quantum systems, as they get entangled with their environment and the decoherence process that it leads. This is, in fact, the transition to quantum to classical world, the lost of information in this process and how long it takes are the principal topics we study in this lines.

We will make a educational review of an article in modern research in quantum physics "Entanglement of quantum clocks through gravity" (Ruiz et al., 2017), where quantum clocks located in nearby worldlines get entangled through the gravitational interaction, leading to a loss of coherence of a single clock and an uncertainty in the time measuring process.

# Resumen

En este trabajo presentamos el comportamiento de sistemas cuánticos abiertos, a medida que se entrelazan con su entorno y el proceso de decoherencia que esto provoca. Esta es, de hecho, la transición del mundo cuántico al clásico, la pérdida de información en este proceso y cuánto dura son los principales temas que estudiamos en estas líneas.

Haremos una revisión educativa de un artículo de investigación moderna en física cuántica "Entrelazamiento de relojes cuánticos a través de la gravedad"(Ruiz et al., 2017), donde los relojes cuánticos ubicados en lineas universo cercanas se entrelazan a través de la interacción gravitatoria, llevando a una pérdida de coherencia de uno de los relojes y a una incertidumbre en el proceso de medida del tiempo.

# Contents

# 1 Introduction

Physics predicts the evolution of systems, the evolution of the whole universe over time, or even in passed times, from the most tiny of the particles to the most humongous galaxy we could imagine.The determinism of physics itself makes us think that the world must be understood completely, and nothing could get away from our understanding. Quantum mechanics explains most of the science we know nowadays, and its advance along the last century has created new branches of knowledge.

The evolution in quantum mechanics is deterministic, given by the Schrödinger equation

$$i\hbar\frac{d}{dt}|\psi\rangle = H|\psi\rangle \tag{1.1}$$

According to initial and boundary conditions.

This deterministic evolution equation, enters in dispute with the probabilistic results when a measurement is done. The evolution of the state that this Eq. (1.1) gives to us, sometimes shows many alternatives that are never seen to co-exist in our world (Zurek, 2002). This deterministic equation could give to us a superposition that we will never be able to measure and we will never see.

Assume we want to measure, a physical magnitude. Our system, just before the measurement, is in a state represented by $|\psi\rangle$. From the postulates of quantum mechanics, we know that the only possible result of the measurement is one of the eigenvalues of the corresponding observable we are measuring, our physical magnitude. We know the probabilities of obtaining the different results, and only one of the possibilities is achieved. Now, after the measurement, we posses additional information, and the state of the system after the measurement, will be different, due to this extra info we own now. (Cohen-Tannoudji et al., 1986) .

In quantum systems the only reality which we have access is the results of measurements.

Before a measurement, our information about the system is non-existent and the only quantum state we could suppose is a superposition. When a measurement is done the wave function collapse and it is, in fact, an update of the information of the system due to this measurement.

And with it, decoherence plays a key role in the quantum measurement process, which is one of the most important points when we try to understand the transition from quantum to classical world.

Any quantum system is open, it is never perfectly isolated from the environment. Open quantum systems are conditioned by their surroundings. When we study a quantum system, the environment conduct indirect measurements on the system, changing the information we have about it. After a certain time, the coherences, the superposition of states are eliminated, by this interaction. We have a lost of information due to this contact of the system-environment, where the coherences decay until being destroyed. It is then the superposition of states is not present, and a preferred basis thrives on the system. This lost of information occurs almost instantaneously, in short periods of time, the populations of the reduced density matrix barely evolve, while the coherences decay on extremely short times.

We will see that a system in contact with the environment, can be described by an interaction Hamiltonian, which distinguishes a specific set of basis states, that are not influenced by the environmental interaction. This interaction Hamiltonian is taken to be $H_I = \sum_n A_n \otimes B_n$, and with it, the evolution of the total system leads to a basis states not affected by the dynamics, not affected during the time evolution.

# 2 An exactly solvable model

To illustrate our discussion, we will see an specific system-reservoir model, that allow an exact analytic solution, following the referenced book (Breuer et al., 2002).

In this study, we consider a system in contact with a reservoir of harmonic oscillators to describe how is the time evolution of the total system and how the preferred basis emerges. Then, considering a reservoir in thermal equilibrium at certain temperature $T$, we will show how is the evolution with time of the coherences and how their decay is described by the so called, decoherence function.

## 2.1 Time evolution of the total system

Considering a two-state system coupled to a reservoir of harmonic oscillators. The total Hamiltonian in the Schrödinger's picture (operators not depend on time) written as:

$$H = H_S + H_B + H_I = H_0 + H_I = \frac{\omega_0}{2}\sigma_3 + \sum_k \omega_k b_k^\dagger b_k + \sum_k \sigma_3 (g_k b_k^\dagger + g_k^* b_k) \quad (2.1)$$

where $H_0$ is the Hamiltonian of the system and the reservoir and $H_I$ is the interaction Hamiltonian. $\omega_0$ is the level spacing of the two-state system and $k$ labels the reservoir modes with frequencies $\omega_k$. The terms $b_k^\dagger$ and $b_k$ are the bosonic creation an annihilation operators. They satisfy the following commutation relation

$$[b_k, b_{k'}^\dagger] = \delta_{kk'}, \quad (2.2)$$

and $g_k$ are coupling constant which describe the coupling of the two-state system to the reservoir modes $b_k$ through the Pauli matrix $\sigma_3$.

Let us introduce a basis of states vectors of the qubit through $\sigma_3$ :
$\sigma_3 |0\rangle = -|0\rangle$
$\sigma_3 |1\rangle = +|1\rangle$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, |0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (2.3)$$

The Hamiltonian is of the form of $H = H_S + H_B + H_I = H_0 + H_I$ and $H_I = \sum_n |n\rangle \langle n| \otimes B_n = A_n \otimes B_n$.

In this case, the Pauli matrix $\sigma_3$ is a conserved quantity, because it commutes with the Hamiltonian:

$$[H, \sigma_3] = 0. \text{ As } H = H_0 + H_I \rightarrow [\sigma_3 \frac{\omega_0}{2}, \sigma_3] = 0, [\sigma_3, \sum_k \omega_k b_k^\dagger \omega_k] = 0 \quad (2.4)$$

Just because they belong to different subspaces. And as well:

$$[\sigma_3, \sigma_3(g_k b_k^\dagger + g_k^* b_k)] = [\sigma_3, \sigma_3 g_k b_k^\dagger] + [\sigma_3, \sigma_3 g_k^* b_k]. \tag{2.5}$$

Writing the density matrix of the total system $\rho(t)$, since $\sigma_3$ is a conserved quantity, the populations are constant in time and are given by:

$$\rho_{11} = tr_{S+B}(|1\rangle\langle 1|\rho(t)) = \langle 1|\rho_S(t)|1\rangle\,,\ \rho_{00} = tr_{S+B}(|0\rangle\langle 0|\rho(t)) = \langle 0|\rho_S(t)|0\rangle\,. \tag{2.6}$$

Our target is how decoherence appears, and for that, we must study the decoherence function. In order to do that, we observe that in the interaction picture, its Hamiltonian is described as follows

$$H_I(t) = e^{iH_0 t} H_I e^{-iH_0 t} = \sum_k \sigma_3(g_k b_k^\dagger e^{i\omega_k t} + g_k^* b_k e^{-i\omega_k t}), \tag{2.7}$$

and the unitary time-evolution operator in the interaction picture can be written as:

$$U(t) = T_{\leftarrow} exp\left[-i\int_0^t ds H_I(s)\right]. \tag{2.8}$$

With $T_{\leftarrow}$ the time order operator. Since the commutator of the interaction Hamiltonian at two different times is a c-number function:

$$[H_I(t), H_I(t')] = -2i\sum_k |g_k|^2 \sin\omega_k(t-t') \equiv -2i\varphi(t-t'), \tag{2.9}$$

this last equation can be written as next

$$[H_I(t), H_I(t')] = H_I(t)H_I(t') - H_I(t')H_I(t) = \tag{2.10}$$

$$\sum_k \sigma_3(g_k b_k^\dagger e^{i\omega_k t} + g_k^* b_k e^{-i\omega_k t})\cdot\sum_k \sigma_3(g_k b_k^\dagger e^{i\omega_k t'} + g_k^* b_k e^{-i\omega_k t'}) -$$

$$-\sum_k \sigma_3(g_k b_k^\dagger e^{i\omega_k t'} + g_k^* b_k e^{-i\omega_k t'})\cdot\sum_k \sigma_3(g_k b_k^\dagger e^{i\omega_k t} + g_k^* b_k e^{-i\omega_k t}) =$$

$$\sum_k \sigma_3^2|g_k|^2[b_k^\dagger b_k 2i\sin\omega_k(t-t') - b_k b_k^\dagger 2i\sin\omega_k(t-t')] = -2i\sum_k \sigma_3^2|g_k|^2\sin\omega_k(t-t').$$

Using in this last step $[b_k, b_k^\dagger] = 1$, equation (2.9) follows.

Consequently, we obtain the time-evolution operator:

$$U(t) = \exp\left[-\frac{1}{2}\int_0^t ds\int_0^t ds'[H_I(s), H_I(s')]\Theta(s-s')\right]\exp\left[-i\int_0^t ds H_I(s)\right]. \tag{2.11}$$

To derive this last equation we proceed as follows:

First of all, we rewrite the time-evolution operator using the Magnus expansion (Blanes et al., 2010).

$$U(t,t_0) = e^A \tag{2.12}$$

$$A = -\frac{i}{\hbar}\int_{t_0}^{t} ds_1 H(S_1) + \frac{1}{2}\left(-\frac{i}{\hbar}\right)^2 \int_{t_0}^{t} ds_2 \int_{t_0}^{s_2} ds_1 [H(t_2), H(t_1)] + \tag{2.13}$$
$$+\frac{1}{4}\left(-\frac{i}{\hbar}\right)^3 \int_{t_0}^{t} ds_3 \int_{t_0}^{s_3} ds_2 \int_{t_0}^{s_2} ds_1 [H(s_3),[H(s_2),H(s_1)]] + ...$$

If $[H(t_2), H(t_1)] = f(t_2, t_1)$ is a c-number then

$$A = -\frac{i}{\hbar}\int_{t_0}^{t} ds_1 H(S_1) + \frac{1}{2}\left(-\frac{i}{\hbar}\right)^2 \int_{0}^{t} ds_1 \int_{0}^{s_1} ds_2 f(s_1, s_2), \tag{2.14}$$

and therefore

$$U(t,0) = e^A = \exp\left\{-\frac{1}{2}\frac{1}{\hbar^2}\int_0^t ds_1 \int_0^{s_1} ds_2 [H(s_1), H(s_2)]\right\} \times \tag{2.15}$$

$$\times \exp\left(-\frac{i}{\hbar}\int_0^t ds_1 H(s_1)\right) \tag{2.16}$$

In this solution, the right part (2.15) is a c-number and (2.16) is an operator.

Using the Heaviside function defined by

$$\Theta(x) = \begin{cases} 1 & ;\ \mathrm{x} > 0 \\ 0 & ;\ \mathrm{x} < 0 \end{cases} \tag{2.17}$$

we will change the upper limits of the integrals. The time-evolution operator will be described with only one common limit t.

$$U(t,0) = \exp\left\{-\frac{1}{2}\frac{1}{\hbar^2}\int_0^t ds_1 \int_0^{t} ds_2 [H(s_1), H(s_2)]\Theta(s_1 - s_2)\right\} \times \tag{2.18}$$

$$\times \exp\left(-\frac{i}{\hbar}\int_0^t ds_1 H(s_1)\right) \tag{2.19}$$

We will rename (2.19) as $V(t)$. Notice that $[H(s_1), H(s_2)] = -2i\varphi(s_1 - s_2)$, then we can conclude that the time evolution operator has the form:

$$U(t,0) = \exp\left\{\frac{1}{\hbar^2}\int_0^t ds_1 \int_0^{t} ds_2\ \varphi(s_1 - s_2)\Theta(s_1 - s_2)\right\} \cdot V(t) \tag{2.20}$$

Showing (2.11) is correct.

If we focus on the unitary operator $V(t)$, from equation (2.7) it follows that

$$V(t) = \exp\left[-i\sum_k \sigma_3 g_k b_k^\dagger \int_0^t e^{i\omega_k s} ds + \sum_k \sigma_3 g_k b_k^* \int_0^t e^{-i\omega_k s} ds\right] = \tag{2.21}$$

$$= \exp\left[\sum_k \sigma_3 g_k \left[b_k^\dagger \left(\frac{1-e^{i\omega_k t}}{\omega_k}\right) - b_k^* \left(\frac{1-e^{-i\omega_k t}}{\omega_k}\right)\right]\right] =$$

$$= \exp\left[\frac{1}{2}\sigma_3 \sum_k (\alpha_k b_k^\dagger - \alpha_k^* b_k^*)\right] = \exp\left[\frac{1}{2}\sigma_3 \sum_k (\alpha_k b_k^\dagger - \alpha_k^* b_k)\right],$$

with the $\alpha_k$ the amplitudes

$$\alpha_k = 2g_k \frac{1-e^{i\omega_k t}}{\omega_k}. \tag{2.22}$$

We see the evolution of the total system is determined by $V(t)$, ignoring a global phase factor. If we choose an arbitrary reservoir state $|\phi\rangle$ and the coherent state generator $D(\alpha_k)$ as:

$$D(\alpha_k) = \exp\left[\alpha_k b_k^\dagger - \alpha_k^* b_k\right] \tag{2.23}$$

$$V(t)\ (|0\rangle \otimes |\phi\rangle) = |0\rangle \otimes \prod_k D(-\alpha_k/2)\,|\phi\rangle \equiv |0\rangle \otimes |\phi_0(t)\rangle \tag{2.24}$$

$$V(t)\ (|1\rangle \otimes |\phi\rangle) = |1\rangle \otimes \prod_k D(+\alpha_k/2)\,|\phi\rangle \equiv |1\rangle \otimes |\phi_1(t)\rangle\,. \tag{2.25}$$

The interaction of the system with its environment creates correlations between the system states $|0\rangle$ and $|1\rangle$ and a certain reservoir states $|\phi_0(t)\rangle$ and $|\phi_1(t)\rangle$, respectively. Since now the states are entangled states system-reservoir according by a superposition of states $|n\rangle \otimes |\phi_n(t)\rangle$. Due to this interaction, the reservoir carries information on the system state.

If we consider this "coherent state generator", we can rewrite:

$$D(\alpha_k) = e^{[\alpha_k b_k^\dagger - \alpha_k^* b_k]} = e^{(\alpha_k b_k^\dagger)} \cdot e^{(-\alpha_k^* b_k)} \cdot e^{-[\alpha_k b_k, -\alpha_k^* b_k]/2} = \tag{2.26}$$

$$= e^{\alpha_k b_k^\dagger} e^{-\alpha_k^* b_k} e^{-|\alpha_k|^2/2}.$$

Notice that in (2.26) we have used the property:

$$e^{A+B} = e^A e^B e^{-[A,B]/2}, \tag{2.27}$$

and $[b_k^\dagger, b_k] = -1$.

So, taking into account that $b_k |0\rangle = 0$ and $D(\alpha_k)|0\rangle = e^{\alpha_k b_k^\dagger} e^{-|\alpha_k|^2/2} |0\rangle$, everything conduces to: $D(\alpha_k)|0\rangle = e^{-|\alpha_k|^2/2} e^{\alpha_k b_k^\dagger} |0\rangle$.

Let us consider a coherent state (Martín Fierro et al., 2004) defined by

$$|\alpha_k\rangle \equiv \sum_n e^{-\frac{|\alpha_k|^2}{2}} \frac{\alpha_k^n}{\sqrt{n!}} |n\rangle . \tag{2.28}$$

Therefore, this coherent state generator acting to a state $|0\rangle$ leads to

$$D(\alpha_k)|0\rangle = e^{-|\alpha_k|^2/2} e^{\alpha_k b_k^\dagger} |0\rangle = e^{-|\alpha_k|^2/2} \sum_{n=0}^{\infty} \frac{1}{n!} (\alpha_k)^n (b_k^\dagger)^n |0\rangle , \tag{2.29}$$

but $(b_k^\dagger)^n |0\rangle = \sqrt{n!}\,|n\rangle$, therefore

$$D(\alpha_k)|0\rangle = e^{\frac{|\alpha_k|^2}{2}} \sum_{n=0}^{\infty} \frac{1}{n!} (\alpha_k)^n \sqrt{n!}\,|n\rangle = e^{\frac{|\alpha_k|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha_k^n}{\sqrt{n!}} |n\rangle \equiv |\alpha_k\rangle . \tag{2.30}$$

We have show that the "coherent state generator" acting on $|0\rangle$ will lead to a coherent state. If we choose another state, the operator will give another, different coherent state.

So the reservoir states, are also coherent states with amplitudes $\pm\alpha_k/2$

$$|\phi_0(t)\rangle = \prod_k |-\alpha_k/2\rangle , \; |\phi_1(t)\rangle = \prod_k |+\alpha_k/2\rangle . \tag{2.31}$$

## 2.2 Decoherence function

Let us consider a reservoir in a thermal equilibrium state at temperature T. Considering an initial state of the total system as

$$\rho(0) = \rho_S(0) \otimes \rho_B \text{ with } \rho_B = \frac{1}{Z_B} e^{-\beta H_B}, \tag{2.32}$$

with $\beta = 1/k_B T$ and $Z_B$ is the reservoir partition function.

The decoherence function describes the decay of the off-diagonal terms of the reduced density matrix, the coherences.

To obtain it, first of all we have to know the matrix elements of the system's density matrix, that are related with the decoherence function through the evolution with time of the coherences. The evolution of the matrix elements of the density matrix are given, in the same way we mentioned in (2.6) by

$$\rho_{ij}(t) = \langle i| \rho_S(t) |j\rangle = \langle i| tr_B\{V(t)\rho(0)V^{-1}(t)\} |j\rangle, \tag{2.33}$$

the populations does not change with time, as they are the diagonal terms and the time dependence disappears when the product is done. On the other hand coherences, evolve with time. The evolution is given by

$$\rho_{10}(t) = \rho_{10}(0)e^{\Gamma(t)}, \tag{2.34}$$

where we have introduce the decoherence function $\Gamma(t)$.

That can be written in general :

$$\Gamma_{nm}(t) = \ln \left|\langle V_m^{-1}(t)V_n(t)\rangle\right|. \tag{2.35}$$

Where the angular brackets are the expectation value taken over the initial density of the reservoir $\rho_B$, it can be as well written as

$$\Gamma(t) = \ln\ tr_B \left\{ \exp\left[\sum_k (\alpha_k b_k^\dagger - \alpha_k^* b_k)\right] \rho_B \right\} = \sum_k \ln \left\langle \exp\left[\alpha_k b_k^\dagger - \alpha_k^* b_k\right]\right\rangle. \tag{2.36}$$

If we define the expectation value by mean of the Wigner characteristic function of the bath mode $k$ (Ferraro et al., 2005).

$$\chi(\alpha_k, \alpha_k^*) \equiv \left\langle \exp\left[\alpha_k b_k^\dagger - \alpha_k^* b_k\right]\right\rangle. \tag{2.37}$$

We find after some arrangements,

$$\chi(\alpha_k, \alpha_k^*) \equiv \left\langle \exp\left[\alpha_k b_k^\dagger - \alpha_k^* b_k\right]\right\rangle = tr\rho_B e^{\alpha_k b_k^\dagger - \alpha_k^* b_k}, \tag{2.38}$$

where $\rho_B$ can be written, using (2.32) and (2.1):

$$\rho_B = \frac{e^{-\beta\hbar\omega_k b_k^\dagger b_k}}{Tr_B e^{-\beta\hbar\omega_k b_k^\dagger b_k}}, \tag{2.39}$$

Let us introduce a coherent state basis, with $|n\rangle$ Fock states and $|\alpha\rangle$ a coherent state defined in (2.28)

$$\hbar\omega_k b_k^\dagger b_k |n\rangle = \hbar\omega_k n |n\rangle \text{ and } |\alpha\rangle \equiv e^{-\frac{|\alpha|^2}{2}} \sum_n \frac{\alpha_n}{\sqrt{n!}} |n\rangle \tag{2.40}$$

The basis is an overcomplete one, as the coherent states are not orthogonal, the resolution of the identity is not unique. One of them could be $1 = \int \frac{d^2\alpha}{\pi} |\alpha\rangle \langle\alpha|$ with $d^2\alpha = d(Re\,\alpha)d(Im\,\alpha)$.

We have to compute (2.38), the Wigner characteristic function,

$$\chi(\alpha_k, \alpha_k^*) = tr\rho_B e^{-\alpha_k^* b_k + \alpha_k b_k^\dagger} = tr\rho_B e^{-\alpha_k^* b_k} e^{\alpha_k b_k^\dagger} e^{-\frac{1}{2}[-\alpha_k^* b_k, \alpha_k b_k^\dagger]} \tag{2.41}$$

Where we have used the same property we used in (2.27). Introducing the identity

$$\chi(\alpha_k, \alpha_k^*) = Tr \int \frac{d^2\alpha}{\pi} \rho_B e^{-\alpha_k^* b_k} |\alpha\rangle \langle\alpha| e^{\alpha_k b_k^\dagger} e^{|\alpha_k|^2/2} = \int \frac{d^2\alpha}{\pi} e^{|\alpha_k|^2/2} e^{\alpha_k\alpha^* - \alpha_k^*\alpha} \langle\alpha| \rho_B |\alpha\rangle . \tag{2.42}$$

In this last step we have introduced $Tr\{\rho_B |\alpha\rangle \langle\alpha|\} = \langle\alpha| \rho_B |\alpha\rangle$.

Let us calculate this last term $\langle\alpha| \rho_B |\alpha\rangle$, using (2.39) and (2.40)

$$\langle\alpha| \rho_B |\alpha\rangle = Z^{-1} \sum_n e^{-\beta\hbar\omega n} \langle\alpha|n\rangle \langle n|\alpha\rangle , \tag{2.43}$$

with $\langle n|\alpha\rangle = e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}}$ and $Z^{-1} = \sum_n e^{-\beta\hbar\omega n} = \frac{1}{1-e^{-\beta\hbar\omega}}$. Then

$$\langle\alpha| \rho_B |\alpha\rangle = Z^{-1} \sum_n e^{-\beta\hbar\omega n} e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!} = (1 - e^{-\beta\hbar\omega}) \exp(-|\alpha|^2(1 - e^{-\beta\hbar\omega})) \equiv \\ \equiv A \cdot \exp(-|\alpha|^2 A) \tag{2.44}$$

With this result, we can go back to (2.42) and write

$$\chi(\alpha_k, \alpha_k^*) = e^{|\alpha_k|^2/2} \int \frac{d^2\alpha}{\pi} e^{\alpha_k \alpha^* - \alpha_k^* \alpha} A \cdot e^{-|\alpha|^2 A} = \tag{2.45}$$
$$= e^{|\alpha_k|^2/2} A \int_{-\infty}^{\infty} \frac{dxdy}{\pi} e^{\alpha_k(x-iy) - \alpha_k^*(x+iy)} e^{-(x^2+y^2)A}.$$

Solving the integral we conclude that

$$\chi(\alpha_k, \alpha_k^*) = e^{|\alpha_k|^2/2(1-\frac{2}{A})} = \exp\left(-\frac{|\alpha_k|^2}{2} \coth\left(\frac{\beta\hbar\omega}{2}\right)\right). \tag{2.46}$$

To conclude with the proper form of the Wigner characteristic function of the bath mode $\chi(\alpha_k, \alpha_k^*)$, notice that

$$\bar{n} = \langle b_k^\dagger b_k \rangle = (e^{\beta\hbar\omega} - 1)^{-1} \tag{2.47}$$

And also:

$$\langle \{b_k^\dagger, b_k\} \rangle = \langle b_k^\dagger b_k + b_k b_k^\dagger \rangle = \langle 2 b_k^\dagger b_k + 1 \rangle = 2\bar{n} + 1 = \frac{2}{e^{\beta\hbar\omega} - 1} + 1 = coth\left(\frac{\beta\hbar\omega}{2}\right) \tag{2.48}$$

Finally, the Wigner function is written as

$$\chi(\alpha_k, \alpha_k^*) = \exp\left\{\frac{-|\alpha_k|^2}{2} \langle \{b_k^\dagger, b_k\} \rangle\right\}. \tag{2.49}$$

With this result, the decoherence function (2.36) will be

$$\Gamma(t) = -\sum_k \frac{1}{2} |\alpha_k|^2 \langle \{b_k, b_k^\dagger\} \rangle = -\sum_k \frac{4|g_k|^2}{{\omega_k}^2} \coth\left(\frac{\beta\hbar\omega_k}{2}\right) (1 - \cos(\omega_k t)) \tag{2.50}$$

With $\beta = \frac{1}{k_B T}$.

This is the result for the decoherence function for a system in contact with a reservoir in thermal equilibrium at temperature T. The coherences that evolve in time are described by this decoherence function, which depends on the reservoir state. In this case, it depends on $|g_k|^2$, that are the coupling constants that describes the coupling of the two-state system to the reservoir modes $k$. It depend as well on the oscillation modes of the reservoir $\omega_k$, and of the temperature of the bath.

## 2.3 Continuum limit

It is of interest to consider the continuum limit of the bath modes. If we define the spectral density $J(\omega)$, through the density of the modes of frequency $f(\omega)$

$$J(\omega) = 4f(\omega)|g(\omega)|^2. \tag{2.51}$$

Using it, we can write the decoherence function and obtain the explicit expression for the function of this model.

$$\Gamma(t) = -\int_0^\infty d\omega J(\omega) \coth\left(\frac{\omega}{2k_BT}\right)\left(\frac{1-\cos\omega t}{\omega^2}\right) \tag{2.52}$$

As an example, let us take a spectral density with exponential form, as it is obtained in the quantum optical regime (Breuer et al., 2002), with $g(\omega) \approx \sqrt{\omega}$, and with a mode density $f(\omega) =$ constant. That represent a one - dimensional field of bath modes.

$$J(\omega) = A\omega e^{-\omega/\Omega}. \tag{2.53}$$

We take this form because we assume a linear increase of $J(\omega)$ for small frequencies and an exponential frequency cutoff at $\Omega$.

To determine the decoherence function, we split it in two parts, a vacuum part and a thermal, $\Gamma(t) = \Gamma_{vac}(t) + \Gamma_{th}(t)$.

The vacuum part describes how the fluctuations of the field vacuum affect to the coherence of the open system. Depends only on the cutoff frequency $\Omega$, it does not depend of the temperature:

$$\Gamma_{vac}(t) \equiv -\int_0^\infty d\omega e^{-\omega/\Omega}\frac{1-cos\omega t}{\omega} = -\frac{1}{2}\ln\left(1+\Omega^2t^2\right). \tag{2.54}$$

The thermal contribution is given by

$$\Gamma_{th}(t) \equiv -\int_0^\infty d\omega e^{-\omega/\Omega}\left[\coth\left(\frac{\omega}{2k_BT}\right)-1\right]\frac{1-\cos\omega t}{\omega} = \tag{2.55}$$
$$= \frac{-1}{\beta}\int_0^t ds \int_0^\infty dx e^{-k_BTx/\Omega}[\coth(x/2)-1]\sin(sx/\beta).$$

If we now assume that $k_BT << \Omega$, $e^{-k_BTx} \to 1$. The thermal contribution it is approximated by

$$\Gamma_{th}(t) \approx \frac{-1}{\beta}\int_0^t ds \int_0^\infty dx[\coth(x/2)-1]\sin(sx/\beta) = -\ln\left[\frac{\sinh(t\tau_B)}{t/\tau_B}\right]. \tag{2.56}$$

Where $\tau_B$ is the thermal correlation time

$$\tau_B = \frac{\beta}{\pi} = \frac{1}{\pi k_B T} \tag{2.57}$$

We can now write the total decoherence function:

$$\Gamma(t) = -\frac{1}{2}\ln\big(1+\Omega^2 t^2\big) - \ln\left[\frac{\sinh(t/\tau_B)}{t/\tau_B}\right] \tag{2.58}$$

We can distinguish three time regimes: a short time regime, with $t << \Omega^{-1}$, a vacuum regime, that appears when $\Omega^{-1} << t << \tau_B$ and the thermal, when $\tau_B << t$.

- **Short time regime ($t << \Omega^{-1}$):**

  The thermal term does not affect to the decoherence function in this regime. It only involves the vacuum term, that can be written as

  $$\Gamma(t) \approx -\frac{1}{2}\Omega^2 t^2. \tag{2.59}$$

  In this short- time regime, the decoherence effects are fully determined by the vacuum contribution.

- **Vacuum regime, ($\Omega^{-1} << t << \tau_B$):**

  In this case, $t\Omega >> 1$ and $\frac{t}{\tau_B} << 1$, so the decoherence function can be approximated by

  $$\Gamma(t) \approx -\frac{1}{2}\ln\big(\Omega^2 t^2\big) - \ln\left(\frac{\frac{t}{\tau_B}}{\frac{t}{\tau_B}}\right) = -\ln(\Omega t). \tag{2.60}$$

- **Thermal regime. Long time regime ($\tau_B << t$):**

  In this case, $\frac{t}{\tau_B} >> 1$ and we can approximate $\Gamma(t)$ by

  $$\Gamma(t) \approx \frac{-1}{2}\ln\big(1+\Omega^2 t^2\big) - \ln\left(\frac{e^{t/\tau_B}}{t/\tau_B}\right) \tag{2.61}$$

  $$\Gamma(t) \approx -\frac{1}{2}\ln\big(1+\Omega^2 t^2\big) - \frac{t}{\tau_B} + \ln\left(\frac{t}{\tau_B}\right) \approx -\frac{t}{\tau_B} \tag{2.62}$$

  Where in this last step, the lineal term with $t$ is dominant over the $ln$ term for $t >> \tau_B$.

In Figure 1 it is represented in a semilogarithmic plot the Decoherence Function $\Gamma(t)$ (2.58). Together with the three approximations previously announced (2.59), (2.60),(2.62).

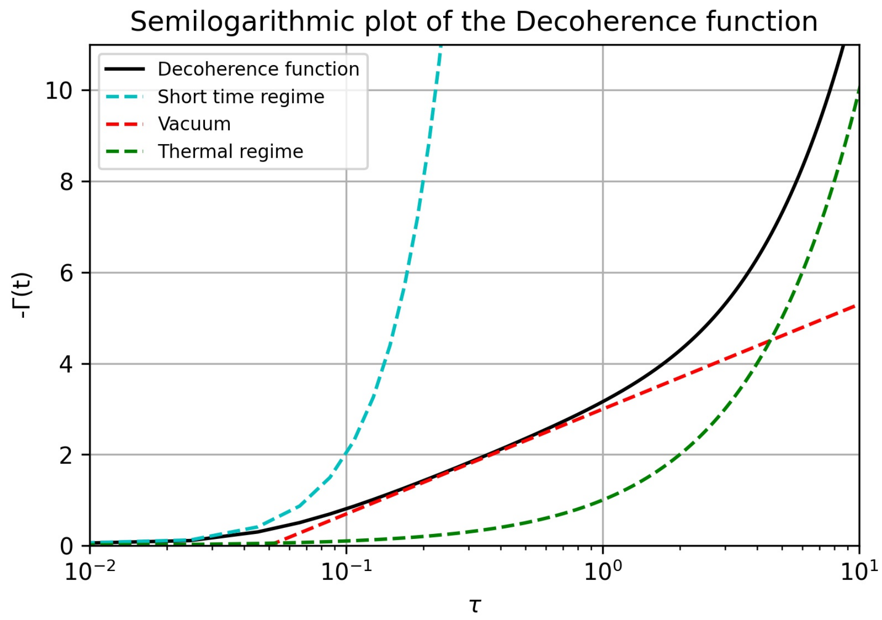


Figure 1: Semilogarithmic plot of the decoherence function $\Gamma(t)$ as a function of $\tau = t/\tau_B$.

We have used that the cutoff($\Omega$) frequency have a value where $\Omega\tau_B = 20$.

It is showed the difference between the three regimes, and how the approximations performed the decoherence function in each time regime.

For longer times involved in the interaction of reservoir - system, the decoherence process is faster, as the decoherence function increases rapidly. For very short times, coherences are still present in the reduced state and the superposition is still possible. But, for larger times, the decoherence function get a value that makes the coherences to decay fast to zero.

# 3 Entanglement of quantum clocks through gravity

In this section, our motivation is to see an application of this decoherence phenomena, and how entanglement and decoherence appears in a particular context.

We are going to review and follow the article "Entanglement of quantum clocks through gravity" written by Esteban Castro Ruiz et al (Ruiz et al., 2017) where they show that different clocks get entangled through time dilation effect, which eventually leads to a loss of coherence of a single clock.

After the solvable model we have just studied we will follow the article in which, we model the clocks, defining an operational concept of clock, as quantum systems in a superposition of energy eigenstates. It will leads to an entanglement of nearby clocks. And, due to this, we will show the limitations when time is measured.

We will see how the solvable model that we studied in the past section provides the basis to understand this article. And it helps us to understand how the entanglement of nearby quantum clocks appears as well as the decoherence time.

To show that, we consider the simplest system as a reference clock, one in superposition of energy eigenstates like a two level system to which one defines time evolution. Considering that the observer does not need to be located next to the clock, in fact, the observer could perform measurements sending a probe quantum system to interact with the clock, and then measuring the probe at his/her location.

First of all, we describe the internal Hamiltonian of the particle we are studying as

$$H_{int} = E_0 \left|0\right\rangle \left\langle 0\right| + E_1 \left|1\right\rangle \left\langle 1\right| \tag{3.1}$$

And we will consider that the energy $E_0 = 0$ and call $\Delta E = E_1 - E_0 = E_1$. It has the same structure that the one we studied before in (2.3).

Let us consider the initial state of the clock like the simplest state in a superposition of eigenstates. It could be described by:

$$\left|\psi_{in}\right\rangle = \frac{1}{\sqrt{2}}(\left|0\right\rangle + \left|1\right\rangle). \tag{3.2}$$

If we consider this state, it leads us to define an "orthogonalization time $t_\perp$" of the clock, as the passage of a unit time, defined by the time needed for the initial state to become orthogonal to itself. With it, the clock model has a defined precision. As this orthogonalization time quantifies his precision, the uncertainty of time will be determined by it. For the system we are considering, the two level system, from the initial state (3.2) and the internal Hamiltonian, we can easily deduce the value of $t_\perp$ to be $t_\perp = \frac{\pi\hbar}{\Delta E}$. It gives to us the time uncertainty of the clock.

## 3.1 Gravitational interaction between two clocks

We now consider two clocks interacting gravitationally. The gravitational interaction would be described with the simplest approximation of the Einstein equations, with the Newtonian gravitational energy: $U(x) = \frac{-Gm_A m_B}{x}$. Here "x" represents the coordinate distance between the two clocks according to the far away observer, which is fixed. And the labels A and B names the two clocks and therefore $m_A, m_B$ are their masses. This masses represent the two terms of the mass-energy equivalence of the gravitational field: the static rest mass and the dynamical mass, where this last one correspond to the internal degrees of freedom $H_{int}$. And the only one we are going to considerate.

As we are considering the gravitational interaction and concerning to time measuring and time dilation effects, it is relevant for us the time dilation effect due to the mass - energy equivalence. With two clocks, the first one concerning to measure time, have an uncertainty given by $t_\perp$. The other clock, that is located at a distance x of the first one, suffers a time dilation through gravity effects, and, the time for it will run different, according to $t + \Delta t = t[1 + G\Delta E/(c^4 x)]$ .

Therefore, the time uncertainty will be given by the value of the precision of the clock $t_\perp$, also should take into account the time dilation effect due to the different gravitational fields. This two effects can be described by

$$t_\perp \Delta t = \frac{\pi\hbar G t}{c^4 x}. \tag{3.3}$$

To describe the interaction between clocks, we could use the masses and the mass - energy equivalence in the interaction Hamiltonian, resulting that the interaction is written in terms of $m + H_{int}/c^2$. To simplify it, we assume that the static mass is negligible compared with the dynamical one.

Therefore, for the two clock system, the Hamiltonian leads

$$\hat{H} = \hat{H}_A + \hat{H}_B - \frac{G}{c^4 x}\hat{H}_A \hat{H}_B. \tag{3.4}$$

We assume that $H_A$ and $H_B$ have equal energies:

$$\hat{H}_A = E_0 \left|0\right\rangle \left\langle 0\right| + E_1 \left|1\right\rangle \left\langle 1\right| \quad \text{and} \quad \hat{H}_B = E_0 \left|0\right\rangle \left\langle 0\right| + E_1 \left|1\right\rangle \left\langle 1\right|, \tag{3.5}$$

where we can take $E_0 = 0$ and $E_1 = \Delta E$.

In matrix form

$$H_A = \begin{pmatrix} E_0 & 0 \\ 0 & E_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \Delta E \end{pmatrix}, \quad H_B = \begin{pmatrix} E_0 & 0 \\ 0 & E_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \Delta E \end{pmatrix}. \tag{3.6}$$

The energy spectra of this Hamiltonian will be:

$$\hat{H} \left|0,0\right\rangle = 0 \left|0,0\right\rangle \tag{3.7}$$

$$\hat{H}\,|0,1\rangle = \Delta E\,|0,1\rangle$$
$$\hat{H}\,|1,0\rangle = \Delta E\,|1,0\rangle$$
$$\hat{H}\,|1,1\rangle = (2\Delta E - \frac{G}{c^4 x}\Delta E^2)\,|1,1\rangle$$

So the time evolution operator of the total system have the form, as we showed in the previous section:

$$U = diag\left\{1, e^{-i\Delta Et/\hbar}, e^{-i\Delta Et/\hbar}, e^{-i\Delta Et\,(2-\frac{G}{c^4 x}\Delta E)/\hbar}\right\}. \tag{3.8}$$

We now consider the initial state of both clocks uncorrelated $|\psi(t=0)\rangle = [\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)]^{\otimes 2}$.

The evolution of the initial state will be performed by the time evolution operator

$$U\,|\psi(t=0)\rangle = \frac{1}{2}\left[|00\rangle + e^{-i\Delta Et/\hbar}\,|10\rangle + e^{-i\Delta Et/\hbar}\,|01\rangle + e^{-i\Delta Et(2-\frac{G}{c^4 x}\Delta E)/\hbar}\,|11\rangle\right]. \tag{3.9}$$

So the state at time $t$ according to the far-away observer is:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle\,|\varphi_0\rangle + e^{\frac{-it}{\hbar}\Delta E}\,|1\rangle\,|\varphi_1\rangle\right) \tag{3.10}$$

Where $|\varphi_0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{\frac{-it}{\hbar}\Delta E}\,|1\rangle\right)$, and $|\varphi_1\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{\frac{-it}{\hbar}\Delta E(1-\frac{G\Delta E}{c^4 x})}\,|1\rangle\right)$

With this, we see that the clocks get entangled through the gravitational interaction and the time running of one clock is correlated with the energy of the other clock. We realise that the interacting term of the Hamiltonian (3.4) is the only one which creates the entanglement between the clocks A and B. The gravitational interaction is the cause of the entanglement.

It is important to mention that the state at a time $t_{mix} = \frac{\pi\hbar c^4 x}{\Delta E^2 G}$ or $\tau_{mix} = \frac{\pi\xi}{\varepsilon^2}$ gets maximally entangled (the modules of the Schmidt coefficients of the Schmidt decomposition are all equal (Dür et al., 2000)). Where in this last equation, for reasons of simplicity, we introduced the dimensionless variables Planck units, defined as: $\tau = \frac{t}{t_P}$ with $t_P = \frac{l_P}{c}$ and $l_P = \sqrt{\frac{\hbar G}{c^3}}$, $\varepsilon = \frac{\Delta E}{E_P}$ with $E_P = \frac{\hbar}{t_P}$ and $\xi = \frac{x}{l_P}$.

Let us go deep in this entanglement studying the reduced state of the first clock. As we see in the section 2.2, the matrix elements of the system's density matrix are related with the decoherence function through the evolution with time of the coherences.

First of all, we must calculate the density matrix of the complete two clocks system. With our state at time t (3.10), the density matrix of our two clock system must be:

$$\rho = |\psi\rangle\langle\psi| = \frac{1}{4}\begin{pmatrix} 1 & e^{i\tau\xi\varepsilon}e^{-i\frac{\tau\varepsilon^2}{\xi}} & e^{i\tau\xi\varepsilon}e^{-i\frac{\tau\varepsilon^2}{\xi}} & e^{2i\tau\xi\varepsilon}e^{-i\frac{\tau\varepsilon^2}{\xi}} \\ e^{-i\tau\xi\varepsilon}e^{i\frac{\tau\varepsilon^2}{\xi}} & 1 & 1 & e^{i\tau\xi\varepsilon} \\ e^{-i\tau\xi\varepsilon}e^{i\frac{\tau\varepsilon^2}{\xi}} & 1 & 1 & e^{i\tau\xi\varepsilon} \\ e^{-2i\tau\xi\varepsilon}e^{i\frac{\tau\varepsilon^2}{\xi}} & e^{-i\tau\xi\varepsilon} & e^{-i\tau\xi\varepsilon} & 1 \end{pmatrix}. \tag{3.11}$$

With this density matrix of the complete system, we can glimpse the entanglement between the clocks, as the superposition is present, and moreover, if we focus on how is the effect between both, the reduced state of the zeroth clock would be given by the trace over the first clock, as it performs like a system doing indirect measurements to the other one.

To know how is the state of "the zeroth clock", due to this interaction between both, we suppose the first clock is doing indirect measurements to clock zero, as they are in nearly world lines. Then, the reduced state of the zeroth clock will be given by $\rho_{0(2)} = Tr_1\rho$. Where the (2) in $\rho_{0(2)}$ labels that we are considering a 2 clock system, while the 1 in $Tr_1\rho$ labels that we are tracing over the first system.

$$\rho_{0(2)} = Tr_1\rho = \frac{1}{2}\begin{pmatrix} 1 & \frac{1}{2}\left[e^{i\tau\varepsilon\xi}\left(1+e^{-i\frac{\tau\varepsilon^2}{\xi}}\right)\right] \\ \frac{1}{2}\left[e^{-i\tau\varepsilon\xi}\left(1+e^{i\frac{\tau\varepsilon^2}{\xi}}\right)\right] & 1 \end{pmatrix} \tag{3.12}$$

If we focus in the coherences, we notice that they evolve with time, while populations do not. This is the same behaviour we mentioned in the previous chapter; equation (2.34). The coherences evolve in time ($\tau$) and, the diagonal terms of the reduced state lose their time dependence.

In fact, the two clocks, located near each other interact due to the gravitational effect, and eventually get entangled, which leads to a loss of coherence of one of the clocks.

To describe how this effect arises, in the next section we will focus our attention in how this coherences evolve with time, according to the decoherence function and how long it takes, through the decoherence time.

This will be illustrated with a generalization of this two clocks case, with $N+1$ clocks.

## 3.2 Gravitational interaction between N+1 clocks

For this section, we will consider three clocks, and after the case of $N+1$ clocks in a region of the space characterized by the coordinate distance $x$. So we could generalize the Hamiltonian of two interaction clocks (3.4) as:

$$\hat{H} = \sum_{a=0}^{N} \hat{H}_a - \frac{G}{c^4 x} \sum_{a<b} \hat{H}_a \hat{H}_b. \tag{3.13}$$

Let us analyze the evolution in the interaction picture of a three clock initial state with an equal superposition of energies, which has the form: $|\psi_{in}\rangle = [\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)]^{\otimes 3}$.

Following (Ruiz et al., 2017), we will not use the complete Hamiltonian (3.13). For simplicity, we will focus in the interacting part, which is the only responsible of the entanglement between the systems. According to (3.13), for three clocks, the interacting Hamiltonian has the form:

$$\hat{H}_{int} = -\frac{G}{c^4 x} \left[ \hat{H}_1 \hat{H}_2 + \hat{H}_2 \hat{H}_3 + \hat{H}_1 \hat{H}_3 \right] \tag{3.14}$$

Following the same procedure of the section 3.1, we conclude that

$$|\psi(t)\rangle = U\,|\psi(t=0)\rangle = \frac{1}{(\sqrt{2})^3}[|000\rangle + |001\rangle + |010\rangle + e^{-i\frac{\tau\varepsilon^2}{\xi}} + |011\rangle + \\ + |100\rangle + e^{-i\frac{\tau\varepsilon^2}{\xi}}\,|110\rangle + e^{-i\frac{\tau\varepsilon^2}{\xi}}\,|101\rangle + e^{-3i\frac{\tau\varepsilon^2}{\xi}}\,|111\rangle]. \tag{3.15}$$

To achieve our target of study the evolution of the coherences, we are going to focus on the reduce state of the zeroth clock due to the interaction of the clocks on their surrounding. It can be written as $\rho_{0(3)} = Tr_1 Tr_2\,\rho$:

$$\rho_{0(3)} = Tr_1 Tr_2\,\rho = \frac{1}{2}\begin{pmatrix} 1 & \frac{1}{4}\left(1+e^{-i\frac{\tau\varepsilon^2}{\xi}}\right)^2 \\ \frac{1}{4}\left(1+e^{i\frac{\tau\varepsilon^2}{\xi}}\right)^2 & 1 \end{pmatrix}. \tag{3.16}$$

Generalizing for a system of N+1 clocks, where the initial state is $|\psi_{in}\rangle = [\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)]^{\otimes N+1}$. The resultant reduce state for the zeroth clock $\rho_{0(N+1)}$ read

$$\rho_{0(N+1)} = \frac{1}{2}\begin{pmatrix} 1 & \left[\frac{1}{2}\left(1+e^{-i\frac{\tau\varepsilon^2}{\xi}}\right)\right]^N \\ \left[\frac{1}{2}\left(1+e^{i\frac{\tau\varepsilon^2}{\xi}}\right)\right]^N & 1 \end{pmatrix}. \tag{3.17}$$

As mentioned before, the coherences evolve in time, in this case, as well as in the two clocks, the state gets maximally entangled at the time $\tau_{mix}$. To study how this coherences evolve with time and know at what time the coherences get reduced, it is defined the Visibility as: $V = |(\rho_0)_{12}|^2$ where $(\rho_0)_{12}$ are the off-diagonal terms of the reduced state:

$$V = 2|(\rho_0)_{12}| = \left[\frac{1}{2}\left(1+\cos\frac{\tau\varepsilon^2}{\xi}\right)\right]^N \approx 1-\left(\frac{\sqrt{N}\tau\varepsilon^2}{2\xi}\right)^2 \approx e^{-\left(\frac{\sqrt{N}\tau\varepsilon^2}{2\xi}\right)^2}. \tag{3.18}$$

In the first approximation we have considered

$$\frac{1}{2}\left(1+cos\frac{\tau\varepsilon^2}{\xi}\right)^N \approx \left[\frac{1}{2}\left(2-\frac{\left(\frac{\tau\varepsilon^2}{\xi}\right)^2}{2}\right)\right]^N = \left(1-\left(\frac{\tau\varepsilon^2}{2\xi}\right)^2\right)^N, \tag{3.19}$$

and then, according to the binomial series $(1+x)^k \approx 1+kx+...$, it is concluded that: $\frac{1}{2}(1+cos\frac{\tau\varepsilon^2}{\xi})^N \approx \left(1-\frac{\sqrt{N}\tau\varepsilon^2}{2\xi}\right)^2 \approx e^{-\left(\frac{\sqrt{N}\tau\varepsilon^2}{2\xi}\right)^2}$ (3.18). Everything , assuming that $\tau << \frac{2\xi}{\sqrt{N}\varepsilon^2}$.

From this result, we can identify the time at which the coherences disappears and with them the superposition. According to the definition we have made of quantum clocks at the beginning of this section as "quantum system in a superposition of energy eigenstates", they do not perform as clocks anymore, their ability to measure has been lost. This decoherence time in the Planck Units is given by

$$\tau_D = \frac{2\xi}{\sqrt{N}\tau\varepsilon^2}. \tag{3.20}$$

In the initial units it takes the form:

$$t_D = \frac{2\hbar c^4 x}{\sqrt{N}G(\Delta E)^2}. \tag{3.21}$$

To get deeper on the understanding of this clock decoherence time $t_D$ we perform a logarithmic plot of the decoherence time as a function of the difference of energies involved in the process ($\Delta E$) and the separation between the clocks $x$. Taking $N = 10^{23}$ and $G$ the Gravitational constant see Figure 2.

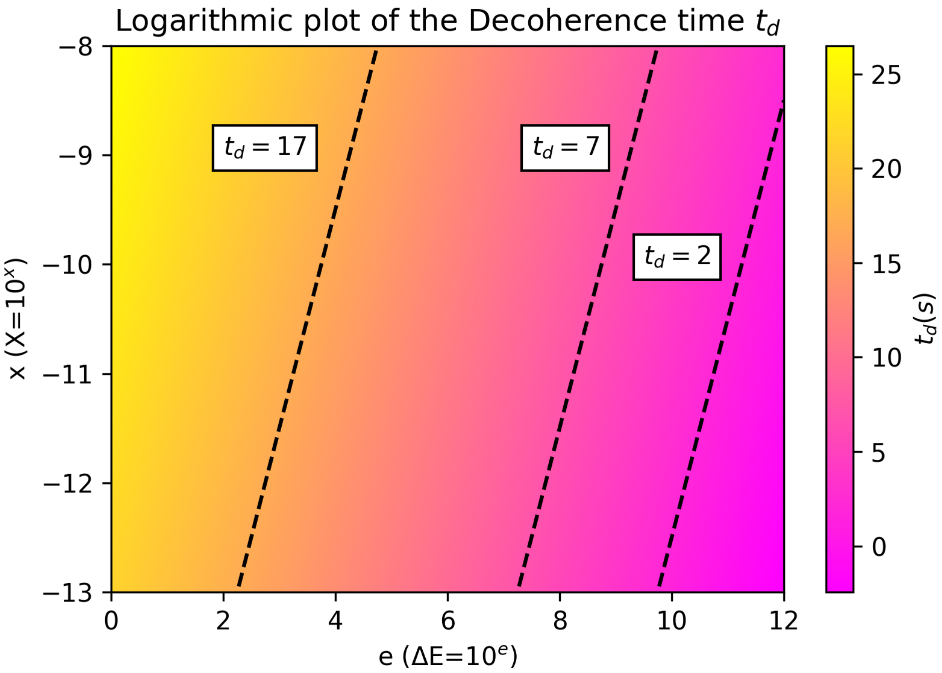


Figure 2: Logarithmic density plot of the decoherence time $t_D$ as a function $\Delta E$ and x.

Notice that for a macroscopic number of particles, and an energy gap between the clocks of $\Delta E = 10^2\,eV$, the decoherence time is the order of the age of the universe ($t_D = 10^{17}\,s$), something not operational for our capabilities at the moment. For a fixed value of x, the larger energy gap between the clocks (a more macroscopic total system), the decoherence time becomes shorter. For a shorter decoherence time, the quantum characteristic of the system disappears long before than for systems with low energy gap between them. That is , in fact, the entanglement between the clocks lasts further more in microscopic systems than in macroscopic ones, as the decoherence phenomena appears rapidly to show only the classical behaviour of the system.

# 4 Classical limit in quantum clocks interacting gravitationally

Our target in this section will be confirm that the classical definition of a clock is achieved with our model. Showing that the entanglement in that limit is negligible and a characteristic time (acting like a decoherence time) defines the classical limit.

To understand the classical limit of our clock model, considered before as a system in a superposition of energy eigenstates, we will consider the clock states as spin coherent states.

## 4.1 Spin coherent states

As we previously discussed, a coherent state is a specific quantum state for the quantum harmonic oscillator, which best resembles the classic harmonic oscillator. In our case, we will consider spin coherent states which are specific quantum states of a spin system that most closely resembles the classical behaviour of a spin(Lee Loh and Kim, 2015). An that is the closest quantum state to the classical state of clocks considered in this article.

The general theory of angular momentum shows that in a space with an arbitrary angular momentum $\hat{J}$, is always possible construct a basis set $|j,m\rangle$ as common eigenvectors of the operators of angular momentum $J^2$ and $J_z$, as they perform in the basis $|j,m\rangle$: (for simplicity we take $\hbar = 1$)

$$J^2 |j,m\rangle = j(j+1) |j,m\rangle \tag{4.1}$$

$$J_z |j,m\rangle = m |j,m\rangle$$

With $m \in (-j, -j+1, ..., j-1, j)$

A spin coherent state is defined taking the maximal $z$ - angular momentum state, $|j, m = j\rangle$, and rotating it an angle $\theta$ along the $y$ axis first, and then performing another rotation an angle $\varphi$ along the $z$ axis.

From classical mechanics, we know that angular momentum is the generator of rotation, like the momentum is generator of translation or the Hamiltonian of time evolution.(Sakurai and Commins, 1995)

In particular, for a two level system, the rotation in quantum space along the $\hat{n}$ axis an angle $\phi$, of a ket $|\alpha\rangle$ is described, in the formalism of Pauli Matrix (convenient for our case) by: $|\alpha\rangle_R = \exp\left(\frac{-i\bar{\sigma}\hat{n}\phi}{2}\right)|\alpha\rangle$. The operator that perform the rotation is called rotation operator and can be written

$$D(\hat{n},\phi) = \exp\left(\frac{-i\bar{\sigma}\hat{n}\phi}{2}\right) = \mathbb{1}\cos\left(\frac{\phi}{2}\right) - i\bar{\sigma}\hat{n}\ \sin\left(\frac{\phi}{2}\right). \tag{4.2}$$

Which is the rotation operator according to a rotation along the $\hat{n}$- axis a $\phi$ angle. So, for our spin coherent state, we must perform two rotations, first one an angle $\theta$ along the y- axis and another an angle $\varphi$ along the z axis. To show this, we will do it for the spin 1/2 case and it will agree with the generalization for the $|j,m\rangle$, described in (Sakurai and Commins, 1995).

So, according to the definition of spin coherent state mention before, the coherent state for a $|j,j\rangle$ state will be, using the rotation operators

$$|j,j\rangle = D(z,\varphi)D(y,\theta)\,|j,j\rangle = \exp\left(\frac{-i\sigma_z\varphi}{2}\right)\,\exp\left(\frac{-i\sigma_y\theta}{2}\right)|j,j\rangle = \tag{4.3}$$
$$= \left[\mathbb{1}\cos\left(\frac{\varphi}{2}\right) - i\sigma_z\,\sin\left(\frac{\varphi}{2}\right)\right]\left[\mathbb{1}\cos\left(\frac{\theta}{2}\right) - i\sigma_y\,\sin\left(\frac{\theta}{2}\right)\right]|j,j\rangle\,.$$

In particular, for the spin 1/2, using the matrix representation for the state $\left|\frac{1}{2},\frac{1}{2}\right\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, according to equation (4.3), the coherent state will be

$$\left|\frac{1}{2},\frac{1}{2}\right\rangle = e^{-i\varphi/2}\cos\left(\frac{\theta}{2}\right)\left|\frac{1}{2},\frac{1}{2}\right\rangle + e^{i\varphi/2}\sin\left(\frac{\theta}{2}\right)\left|\frac{1}{2},-\frac{1}{2}\right\rangle. \tag{4.4}$$

In general, for a coherent state written as linear combination of $|j,m\rangle$, using the Wigner $D$ matrix (Sakurai and Commins, 1995):

$$|j,\theta,\varphi\rangle = \sum_{m=-j}^{m=j}\left(\frac{2j!}{j+m!j-m!}\right)^{1/2}\left(\cos\frac{\theta}{2}\right)^{j+m}\left(\sin\frac{\theta}{2}\right)^{j-m} e^{-im\varphi}\,|j,m\rangle \tag{4.5}$$

That in general, could be also written as

$$|j,\theta,\phi\rangle = (\cos\theta/2\,|0\rangle + e^{i\varphi}\sin\theta/2\,|1\rangle)^{\otimes 2j}. \tag{4.6}$$

Is important to emphasize our desire of choosing a quantum states that characterize the pointer of our clock in a precise way. To that, we have chosen the spin angular momentum and, in particular, the spin coherent states, as we want to study the classical limit of our model.

Now, understanding why we have chosen those states, we can follow with the Hamiltonian and how our two systems (two clocks) evolve. We will conclude that the gravitational interaction affects our systems characterized by the spin angular momentum $j$.

## 4.2 POVM's "Positive Operator-Valued Measurements"

To proceed further we shall introduced, the so called, POVM . The common kind of measurement study in basics courses of quantum mechanics are the ones where the system is projected in one of the eigenstates of the observable we are measuring. Those type of measurement are refer as Von Neumann measurements (or PVM "Projective-Valued Measurement"), and are only one of the possible kind of measurements that can be performed to quantum systems. (Jacobs and Steck, 2006)

This kind of measurements (the POVM's) can be used sending a probe system to interact unitarily with our subject of study, and then performing a PVM to this probe system.

They are described by a set of non-negative operators $\{M_k\}$ which satisfy

$$\sum_k M_k = I. \tag{4.7}$$

When the POVM is done to a quantum state described by $\rho$, the probability of obtaining the outcome $k$ is given by

$$p_k = Tr(M_k \rho). \tag{4.8}$$

We have interest in describe a POVM to measure our system, defining an operator which is going to discern between the set of orthogonal states and return exactly the position of the pointer clock.

With this in mind, we define for our system a POVM defined by the set of operators $\{M_k\}_{k=1}^{2\pi/R}$ with

$$M_k = \frac{2j+1}{4\pi} \int_0^{\pi} d\theta \, sin\theta \int_{(k-1)R}^{kR} d\varphi \, |\theta, \varphi\rangle \langle\theta, \varphi| \,. \tag{4.9}$$

With this definition for the operators, when a measurement is done to a state $|\theta, \varphi, j\rangle$, the result gives us exactly the angle $\varphi$ in which the state is located, with a resolution of $R$. The result of the measurement tell us if the angle $\varphi$ is inside the bin of dimension $R$. If the state is not in that section of the circumference, the result of the measurement will be 0. We use the angle $\varphi$ as it is the one that takes the values from $[0, 2\pi]$ as the pointer, the spin indicates the fragmentation of the angle, and that is, in fact, a way of time measurement.

According to the probability of the result of a measurement (4.8) with this definition of the $\{M_k\}_{k=1}^{2\pi/R}$:

$$p_k = Tr(M_k \rho) = \frac{2j+1}{4\pi} \int_0^{\pi} \int_{(k-1)R}^{kR} d\theta d\varphi \, \sin\theta \;\; \langle\theta, \varphi| \, \rho \, |\theta, \varphi\rangle \,. \tag{4.10}$$

Where the product $\langle\theta,\varphi|\,\rho\,|\theta,\varphi\rangle$ (the trace of the density matrix over the basis coherent states $\{|\theta,\varphi\rangle\}$) can be renamed as the Husimi function $Q_p(\theta,\varphi)$ of the density matrix $\rho$. It is one of the simplest distributions of quasiprobability in phase space. Therefore, it will quantify the probability of finding the clock pointer in certain place. This Husimi function, has a characteristic width proportional to $j^{-1/2}$, so, due to it, the experimental resolution $R$ should be $R >> j^{-1/2}$ to distinguish the classical behaviour, with no quantum fluctuations.

In the next section, we will introduce this coarse- grained measurement to our two-clocks system and the evolution of the system as coherent states, to discuss how the classical time dilation of relativistic effects appears in the limit of a coarse grained measurement and clocks in coherent states.

## 4.3 Two-clock model in the classical limit

Let us consider that our initial state of the clock is in a spin coherent state characterized by $\theta = \pi/2$ and $\varphi = 0$, according to (4.5) and (4.6).

$$|\psi_{in}\rangle = |\theta = \pi/2, \varphi = 0, j\rangle = \sum_{m=-j}^{m=j} \left(\frac{2j!}{j+m!j-m!}\right)^{1/2} \frac{1}{(\sqrt{2})^{2j}} |j,m\rangle \tag{4.11}$$

Which it is, in fact, the extension to angular momentum of the initial state (3.2) defined in section 3

$$|\psi_{in}\rangle = |\theta = \pi/2, \varphi = 0, j\rangle = \left[\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right]^{\otimes 2j}.$$

The generalization of the Hamiltonian (3.1), for the angular momentum $j$ and the angular momentum in the $z$ direction $\hat{J}_z$, for the two level system ($j = 1/2$) is written as $H_{int} = \Delta E(\frac{1}{2}\mathbb{1} - \hat{J}_z)$, and for a system with spin $j$:

$$H_{free} = \Delta E(j\mathbb{1} - \hat{J}_z) \tag{4.12}$$

Let us now consider two clocks, labeled as A and B. The clocks interact gravitationally, in the same the way we have described in the two clocks section 3.1:

$$\hat{H} = \hat{H}_A + \hat{H}_B - \frac{G}{c^4 x}\hat{H}_A\hat{H}_B, \tag{4.13}$$

where, each one ($H_A$ and $H_B$) are described by a $j$ spin system Hamiltonian according to (4.12), $H_A = \Delta E(j_A\mathbb{1} - \hat{J_{z_A}})$ and $H_B = \Delta E(j_B\mathbb{1} - \hat{J_{z_B}})$.

Choosing the initial state of our two clocks system, as $|\psi_{in}\rangle$ in (4.11)

$$|\psi_{in}\rangle = |\theta = \pi/2, \varphi = 0, j_A\rangle \otimes |\theta = \pi/2, \varphi = 0, j_B\rangle = \tag{4.14}$$

$$=\sum_{m=-j_A}^{m=j_A}\sum_{m=-j_B}^{m=j_B}\left(\frac{2j_A!}{j_A+m_A!j_A-m_A!}\right)^{1/2}\left(\frac{2j_B!}{j_B+m_B!j_B-m_B!}\right)^{1/2}\frac{1}{(\sqrt{2})^{2j_A}}\frac{1}{(\sqrt{2})^{2j_B}}\left|j_A,m_A\right\rangle\left|j_B,m_B\right\rangle.$$

To study the entanglement between the two clocks, let us study the time evolution of the system.

Considering, the time evolution of the initial state, as well as we made in previous sections, with the unitary time evolution operator $U=\exp\left(-\frac{i}{\hbar}\hat{H}t\right)$

Conforming, the evolution of our initial state given by (4.14):

$$\left|\psi\right\rangle=U\left|\psi_{in}\right\rangle=\sum_{m=-j_A}^{m=j_A}\sum_{m=-j_B}^{m=j_B}\left(\frac{2j_A!}{j_A+m_A!j_A-m_A!}\right)^{1/2}\left(\frac{2j_B!}{j_B+m_B!j_B-m_B!}\right)^{1/2}\times \tag{4.15}$$
$$\times\frac{1}{2^{j_A}}\frac{1}{2^{j_B}}e^{\frac{-it}{\hbar}\left[\Delta E(j_A-m_A)+\Delta E(j_B-m_B)-\frac{G\Delta E^2}{c^4x}(j_A-m_A)(j_B-m_B)\right]}\left|j_A,m_A\right\rangle\left|j_B,m_B\right\rangle.$$

We should know the reduced state for the B clock at time $t$, as it will give to us information about the entanglement between both clocks

$$\rho_B=Tr_A\left|\psi\right\rangle\left\langle\psi\right|=\sum_{m=-j_A}^{m=j_A}\left\langle j_A,m|\psi\right\rangle\left\langle\psi|j_A,m\right\rangle= \tag{4.16}$$

$$=\sum_m\sum_{m_Am_B}\sum_{m'_Am'_B}\left(\frac{2j_A!}{j_A+m_A!j_A-m_A!}\right)^{1/2}\left(\frac{2j_A!}{j_A+m'_A!j_A-m'_A!}\right)^{1/2}\times$$
$$\times\left(\frac{2j_B!}{j_B+m_B!j_B-m_B!}\right)^{1/2}\left(\frac{2j_B!}{j_B+m'_B!j_B-m'_B!}\right)^{1/2}\times$$

$$\times\frac{1}{2^{2j_A}}\frac{1}{2^{2j_B}}\exp\left(\frac{-it}{\hbar}\left[\Delta E(j_A-m_A)+\Delta E(j_B-m_B)-\frac{G\Delta E^2}{c^4x}(j_A-m_A)(j_B-m_B)\right]\right)\times$$
$$\times\ \exp\left(\frac{it}{\hbar}\left[\Delta E(j_A-m'_A)+\Delta E(j_B-m'_B)-\frac{G\Delta E^2}{c^4x}(j_A-m'_A)(j_B-m'_B)\right]\right)\times$$
$$\times\left\langle j_A,m|j_A,m_A\right\rangle\left|j_B,m_B\right\rangle\left\langle j_Bm'_B\right|\left\langle j_Am'_A|j_Am\right\rangle,$$

where the products $\langle j_A,m|j_A,m_A\rangle$ and $\langle j_Am'_A|j_Am\rangle$ due to orthogonalization are $\delta(m_A,m)=(0,1)$(0 if $m_A\neq m$ or 1 if $m_A=m$) and $\delta(m'_A,m)=(0,1)$ (0 if $m'_A\neq m$ or 1 if $m'_A=m$). So, rewriting it with this conditions, and simplifying, (4.16) :

$$\rho_B=\sum_m\sum_{m_Bm'_B}\left(\frac{2j_A!}{j+m!j-m!}\right)\left(\frac{2j_B!}{j_B+m_B!j_B-m_B!}\right)^{1/2}\left(\frac{2j_B!}{j_B+m'_B!j_B-m'_B!}\right)^{1/2}\times \tag{4.17}$$

$$\times \frac{1}{4^{j_A}} \frac{1}{4^{j_B}} e^{\frac{-it}{\hbar}\left[-\Delta E m_B - \frac{G\Delta E^2}{c^4 x}(j_A - m) m_B\right]} e^{\frac{it}{\hbar}\left[-\Delta E m'_B - \frac{G\Delta E^2}{c^4 x}(j_A - m) m'_B\right]} \left|j_B m_B\right\rangle \left\langle j_B m'_B\right|$$

Rewriting $e^{\frac{-it}{\hbar}\left[-\Delta E m'_B - \frac{G\Delta E^2}{c^4 x}(j_A - m) m'_B\right]} = e^{-im_B\varphi_k}$. With $\varphi_k = \frac{-t}{\hbar}\Delta E\left(1 - \frac{G\Delta E}{c^4 x}(j-m)\right)$ And with this result, we can change the variable $(j_A - m)$ by $k$ as:

$$\sum_{m=-j_A}^{j_A} f(j_A - m) = \sum_{k=0}^{2j_A} f(k), \tag{4.18}$$

So, introducing this new variables it follows

$$\rho_B = \frac{1}{4^{j_A}} \sum_{k=0}^{2j_A} \sum_{m_B m'_B} e^{-im_B\varphi_k} \binom{2j_A}{k} \left(\frac{2j_B!}{j_B + m_B! j_B - m_B!}\right)^{1/2} \times \tag{4.19}$$

$$\times \left(\frac{2j_B!}{j_B + m'_B! j_B - m'_B!}\right)^{1/2} e^{im'_B\varphi_k} \left|j_B m_B\right\rangle \left\langle j_B m'_B\right|.$$

We can identify the first part:

$$\left|\theta = \pi/2, \varphi = \varphi_k, j_B\right\rangle = e^{-im_B\varphi_k} \binom{2j_A}{k} \left(\frac{2j_B!}{j_B + m_B! j_B - m_B!}\right)^{1/2} \left|j_B m_B\right\rangle \tag{4.20}$$

And the last part:

$$\left\langle\theta = \pi/2, \varphi = \varphi_k, j_B\right| = \left(\frac{2j_B!}{j_B + m'_B! j_B - m'_B!}\right)^{1/2} e^{im'_B\varphi_k} \left\langle j_B m'_B\right| \tag{4.21}$$

Where the reduced state for the clock B at time $t$ finally is

$$\rho_B = \frac{1}{4^{j_A}} \sum_{k=0}^{2j_A} \binom{2j_A}{k} \left|\theta = \pi/2, \varphi_k, j_B\right\rangle \left\langle\theta = \pi/2, \varphi_k, j_B\right| \tag{4.22}$$

Where,

$$\varphi_k = \frac{-t}{\hbar}\Delta E\left(1 - \frac{G\Delta E}{c^4 x}k\right). \tag{4.23}$$

This result shows that the reduced state for the clock B shows is a sum of coherent states, which leads to a mixing of states, each one with different phase $\varphi_k$. As matter of fact this possible to properly define the classical limit. When the interaction of both clocks appears, the coherent states in $\rho_B$, that are mixed, evolve differently in time, with different time dilation factors, as they have different phases that are function of the different $k$.

The evolution of the clock, is characterized by two effects, one is the complete movement of the clock, that corresponds with the pointer with more probability for detection, the one with the phase $\varphi_{j_A} = \frac{-t}{\hbar}\Delta E\left(1 - \frac{G\Delta E}{c^4 x} j_A\right)$.Where in this phase the first term $\frac{-t}{\hbar}\Delta E$ represents the free movement of the clock, and the second term, $\frac{t}{\hbar}\frac{G\Delta E^2}{c^4 x} j_A$ the interaction term.

The second effect, is due to the separation of the coherent states with time, as they tend to spread from each other, resulting to a mixing of the reduced state. This contribution of the difference between coherent states we can define it through the separation angle between them: $\Delta\varphi = \varphi_{2j_A} - \varphi_0 = \frac{2Gj_A(\Delta E)^2 t}{c^4 x\hbar}$, which is proportional to $j_A$, and for that fact, the contribution to each state is different due to the binomial distribution, because, for larger $j_A$,we could approximate the probability (Husimi function) to a Gaussian function $p(k) \approx \sqrt{\frac{1}{\pi j_A}\exp\left(\frac{k-j_A}{\sqrt{j_A}}\right)^2}$ that has a characteristic width proportional to $\sqrt{j_A}$ and not to $j_A$.

In fact, this effective separation angle between coherent states, defines the characteristic time which delimit the quantum effects.

This result, gives a value of $\Delta\varphi_{eff} = \frac{G\sqrt{2j_A}(\Delta E)^2 t}{\hbar c^4 x}$.

From this, we could depict a characteristic time $t^*$

$$t^* = \frac{\hbar c^4 x}{G\sqrt{2j_A}(\Delta E)^2} \tag{4.24}$$

We can differentiate between two regimes

**For a time $t << t^*$.**

According to the POVM, all the coherent states in $\rho_B$ are inside one bin of dimension $R >> j^{-1/2}$ (the experimental resolution), and there are not quantum effects when time is measured. The only time dilation factor of the clock B is due to the average energy of the clock A. Here, entanglement is negligible, and this is, in fact, the classical limit of our system.

**For times $t > t^*$.**

The coherent states could be located at different bins, when the measurement is done, and it shows the effects of the quantum entanglement between the clocks.

If we compare with the previous result in section 3.1, we distinguish the dissipation of the coherences here as well, where the difference of phases between coherent states defines what operationally, was defined as a decoherence time in the past. Here it is showed as a characteristic time which distinct the rates at which entanglement became negligible and the classic behaviour come to light.

## 4.4 Analysis of the uncertainty when measuring time

In section 3, we made an analysis about the uncertainty when measuring time with our clock model and the result was given by equation (3.3). After the study of interacting clocks in the classical limit of the past section 4.3, we are going to achieve the same result for the case of two clocks.

We will consider now two interacting clocks, the A one, characterized by $j_A = 1/2$ and the B one, with $j_B >> 1$. This is the case the clock B will have coherent states that approximates to the classical limit, while clock A is purely quantum. For an analysis of the effects of time dilation between both clocks, we consider that the time dilation on B due to clock A is non-negligible, and the other case, the dilation effect on A due to cock B can be neglected. That is, $\frac{G\Delta E_A}{(c^4 x)} \gg 1$ and $\frac{G j_B \Delta E_B}{(c^4 x)} \ll 1$.

According to this, the reduced state of clock B will be given, specifying for the case of (4.22), with $\varphi_k = \frac{-t}{\hbar}\Delta E_B \left(1 - \frac{Gk\Delta E_A}{c^4 x}\right)$ $k = 0, 1$

$$\rho_B = \frac{1}{2}(\left|\theta = \pi/2, \varphi_0, j_B\right\rangle \left\langle\theta = \pi/2, \varphi_0, j_B\right| + \left|\theta = \pi/2, \varphi_1, j_B\right\rangle \left\langle\theta = \pi/2, \varphi_1, j_B\right|) \tag{4.25}$$

This reduced state of clock B simpler than in the past case, as $j_A$ only takes the value $j_A = 1/2$.

To pull out information of the system about time, it is needed to build an operator that give us the pointer position of clock B, and will return time. This is, an operator that will work as Mk (4.9) worked previously in the coarse- grained measurement, but, for this case, defining an operator which will give us information about the uncertainty when measure time with this operator.

This one, is given in the article we are following (Ruiz et al., 2017), to be

$$T^{j_B} = \frac{\hbar(2j_B + 1)}{4\pi\Delta E_B}\int_0^\pi d\theta\, sin\theta \int_0^{2\pi} d\phi\, \phi \left|\theta, \phi, j_B\right\rangle \left\langle\theta, \phi, j_B\right| \tag{4.26}$$

The variance of this operator just defined, is represented in the limit when $j_B \to \infty$, so, at this case, due to the way their are defined, the coherent states are orthonormal,and $\left|\theta = \frac{\pi}{2}, \varphi_k, j_B\right\rangle$ is eigenstate of the operator with the eigenvalue $\hbar\varphi_k/\Delta E_B$ for $\varphi_k \in (0, 2\pi)$, $k = 0, 1$. So, the variance of the operator $T^{j_B}$ can be written as

$$\Delta T^{j_B} = \frac{\hbar}{2\Delta E_B}(\varphi_1 - \varphi_0) = \frac{G\Delta E_A t}{2c^4 x} \tag{4.27}$$

If we consider now, mesuring time on clock A, as it has $j_A = 1/2$, the operator that will give us the probability of measuring one unit of time is $_A = \hbar/\Delta E_A \left|-\right\rangle \left\langle-\right|$. For the state of clock A, the reduced state is, according to (4.22) with $j_A = 1/2$, and $\varphi_k = \frac{-t}{\hbar}\Delta E_A \left(1 - \frac{G\Delta E_B}{c^4 x}k\right)$

$$\rho_A = \frac{1}{4^{j_B}} \sum_{k=0}^{2j_B} \binom{2j_B}{k} \left|\theta = \frac{\pi}{2}, \varphi_k, 1/2\right\rangle \left\langle\theta = \frac{\pi}{2}, \varphi_k, 1/2\right| \tag{4.28}$$

Therefore the $dT_A$ will be the time it takes for the average of $T_A$ to change significantly, being this a method to measure time uncertainty. $dT_A$ will be given by $dT_A = \hbar/\Delta E_A$, because we have consider that the dilation effect off clock B on clock A is neglible, so $\varphi_1$ do not contribute on $dT_A$.

Whit both considerations, the result is written

$$dT_A \Delta T^{j_B} = \frac{\hbar G t}{2c^4 x} \tag{4.29}$$

That is the equation we conclude before, in (3.3) considering only the uncertainty of measuring time with our definition of clock ($t_\perp$) and the gravitational effects ($\Delta t$) with a factor $\pi/2$.

To conclude, we confirm that between clocks located at nearby worldlines, an effect of time dilation and uncertainty in the time measuring with clocks located near from each other appears. And this agree with the uncertainty when considering a measure time at points of different gravitational fields.

# 5 Conclusions

In order to summarize everything we have achieve in this work, we started studying, how decoherence appears in open quantum systems, following the reference of (Breuer et al., 2002). Open quantum systems and the interaction with their surroundings creates correlations between the environmental states and the system. This interaction, embodied in the Hamiltonian through the interaction term is key for study the decoherence phenomena. As the evolution of the system presents entanglement between system-environment, when studying the reduced state density matrix, the phenomena comes to light. The coherences, change in time according to the decoherence function, the one that shows to us the rate at which the coherences get destroyed. In our study of the decoherence function in the section 2.2 we understood the case of a thermal reservoir interacting with our system, and how different time scales are relevant in a decoherence process.

With this more theoretical analysis and understanding how decoherence appears in quantum systems, we followed and reviewed the article (Ruiz et al., 2017) where, considering clocks as quantum systems (with an operational concept of clocks), they get entangled through gravitational interaction, and, as well as in the first sections 2.2, the reduced state of the system (3.17) shows decoherence with a characteristic time scale (3.21). This shows us how the clocks located at nearby worldlines, where the gravitational interaction is to be considered between them, leads to an uncertainty when we are using the clocks to measure time.

In the classical limit, we studied the case of two interacting clocks represented by spin coherent states, and through POVM the result was that the relativistic effects by the pass of time appeared. As well we could see that was a characteristic time which delimited when the quantum effects arises on the clock that was interacting with the other and the certain intervals where the quantum effect could not be noticed and the classical behaviour appeared.

To conclude, we show how all this treatment leads to an uncertainty in measuring time with clocks that are interacting gravitationally, and makes our precision conditioned by taking into account this effects. Emphasising that the clocks are not ideal objects that could rid of the quantum and relativistic effects. Therefore, the article reviewed makes us question about if the notion of time intervals when it is measured in nearby worldlines is well defined, because using clocks that are nearby located, makes that their measurement of time has a gravitatory effect due to the interaction between different clocks.

Is unbelievable how a simple reasoning, simplifying the clock as a simple two level system and studying it behaviour as if it were an open quantum system interacting with the other clocks, so deep results are achieved. The theory is simple, but its application and ways to bring to other questions are endless.

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