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Optimization of an inventory system with partial backlogging from a financial investment perspective

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Abstract

In some real inventory systems, item sales can be adjusted to a stable demand rate along the inventory cycle. For instance, home appliances, electrical products, lounge furniture, home water supply, etc., are items with a stable demand rate. In this work, we analyze an inventory system for an item of this type, where the demand rate is constant. Shortages are allowed, and it is assumed that a fraction of demand during the stock-out period is backlogged. It is supposed that the shortage costs (backorder cost and lost sales cost) have an affine structure: a fixed cost plus a linear cost that depends on the period of time where shortages exist. In this paper, instead of the maximization of the profit per unit time, or the minimization of the average inventory cost per unit time, the objective is the maximization of the return on inventory investment, which is a quotient defined as the average profit divided by the average inventory cost. The optimal inventory policy is obtained in a closed form under this new perspective. Moreover, it is shown that the optimal policy that maximizes the return on inventory investment is, in general, different from the one that maximizes the profit per unit time. In addition, the new optimal perspective offers some advantages. The optimal inventory policy that maximizes the return on investment does not depend on the unit selling price. Therefore, the inventory manager does not need to change his/her inventory policy if this price changes. These advantages are not usually present when the objective is the maximization of the profit per unit time. Numerical examples are provided to illustrate the theoretical results developed in this work. A sensitivity analysis of the optimal policy with respect to the system input parameters is also developed.

Keywords: inventory; return on inventory investment; partial backlogging; shortage cost

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1. Introduction

Most existing inventory models in the literature consider as their objective to determine the optimal inventory policy that maximizes the profit per unit time or minimizes the cost per unit time. Some recent papers in this line are Zhao et al. (2022) and Xu et al. (2022), which consider the profit maximization problem, and Meneses et al. (2021), which uses cost minimization. However, one of the main objectives for a company in managing assets (an inventory is considered as such) is to maximize the profitability of their investments to generate profits. In general, inventories may be considered a short-term investment. Thus, they can be treated as a current asset on the balance sheet. A classical measurement of profitability is the return on investment (ROI), defined as the ratio between the obtained profit in each inventory cycle and the sum of all the costs involved in managing the inventory (purchasing cost, ordering cost, holding cost, backordering cost, etc.). Although it could exist doubts about what should be considered an investment, it seems better to us to include all the costs of the inventory system, to evaluate profitability.

Eilon (1957) studied the consequences of selecting a lot-size assuming, among others, the criteria of maximizing the return (ratio of profit to investment) or maximizing the rate of return (taking into account the time in which this yield is effected). Eilon (1959) presented a method for determining the economic order quantity when the goal is to maximize the rate of return per unit time for a multiproduct schedule. Tate et al. (1964) analyzed several ways to maximize the return and concluded that the lot size that minimizes the total cost is equal to or greater than the lot size that maximizes any of the alternative returns. Schroeder and Krishnan (1976) discussed the suitability of the ROI as an appropriate criterion for many types of inventories and derived optimal decision rules for some common assumptions. Trietsch (1995) showed that the conclusion of Tate et al. (1964) was not always appropriate and concluded that the adoption of the ROI policy instead of the economic order quantity (EOQ) by some firms can reduce the volatility of business cycles. Otake et al. (1999) developed an ROI model and determined the global optimal policy when there exists an option to invest in setup operations. Chen (2001) presented an inventory model under ROI maximization for intermediate firms and determined the optimal quantity, the best price, and the optimal quality level. Otake and Min (2001) formulated and analyzed an inventory problem with variable quality, employing the criterion of ROI maximization, and they determined the unique global optimal solution. More recently, Marchioni and Magni (2018) studied the coherence of the average ROI and the net present value (NPV), concluding that the average ROI can be reliably associated with NPV, providing consistent pieces of information.

In some real inventory systems, demand for items is approximately stable over time. Thus, the assumption that the demand rate is constant along the inventory cycle can be useful to represent the behavior of those items. For example, home appliances, electrical products, lounge furniture, home water supply, etc., are items with an approximately stable demand rate. For this reason, there are many works in the literature on inventory control that have considered this hypothesis in the development of their models. Another common feature in some inventory models is to allow shortages. In that situation, some clients are willing to wait to the next replenishment of products, while others are not agreeable to waiting and they decide to buy items from other vendors. As it is well known, in the area of marketing and logistic business, stock-out might generate different effects, depending on the type of good and the relevance of its utility for the customer. In such a situation, either demand can be backordered until a new order arrives to the inventory system, or demand during the stock-out period may be lost. Thus, in this last situation, customers decide to cancel their orders and generate lost sales. Inventory management looks at the holding cost, ordering cost, and shortage cost. The return on inventory investment could be maximized if planned shortages are assumed. If the sum of the ordering and holding costs is significantly greater than the shortage cost, then permitting shortages may be a good business practice, because the ROI could be incremented without losing business. This has motivated several researchers to consider, in the model, that a fraction of demand is lost along the stockout period, while the remaining fraction is backordered.

The first authors who accepted this partial backlogging situation in their inventory models were Montgomery et al. (1973), Rosenberg (1979), and Park (1982). Their models were later extended in several ways bySan-José et al. (2009a, 2009b). Also, Pentico and Drake (2009) presented an alternative approach to modeling the EOQ inventory problem with partial backordering, determining the optimal values for when and how much to order. Drake and Pentico (2010) extended their Pentico and Drake (2009) model for the deterministic EOQ with fixed partial backordering, allowing the possibility of offering a discounted price to customers who order items during the stockout period, in order to keep some of those who would otherwise be lost sales to backorder. Toews et al. (2011) also extended the Pentico and Drake (2009) model for the EOQ with fixed partial backordering, allowing the fraction of backordered demand to increase linearly as the time until delivery decreases. Taleizadeh et al. (2012) developed an EOQ model with partial backordering for three scenarios with a special sale price. Taleizadeh et al. (2013a) considered an EOQ model with partial backordering in which a fraction of purchasing cost must be paid at the beginning of the inventory cycle, and the remaining amount can be paid later. Taleizadeh and Pentico (2013) proposed EOQ inventory models with partial backordering for two scenarios with an announced price increase. Taleizadeh et al. (2013b) developed an EOQ model with multiple partial prepayments and partial backordering. Taleizadeh (2014) studied an EOQ model with partial backordering and partial consecutive prepayments for a deteriorating product. Pentico et al. (2014) presented two heuristics for the basic economic order quantity and economic production quantity with partial backordering by using the time between orders and the percentage of demand filled from stock as the decision variables. Taleizadeh and Pentico (2014) developed a solution procedure for the EOQ model with all-unit discounts and fixed partial backordering. Taleizadeh et al. (2015) determined the optimal inventory policy for an EOQ model in which the supplier offers incremental quantity discounts. Pentico et al. (2015) studied the accuracy of approximating the EOQ with a backordering rate, which is either an exponential or a rational function of the time remaining until the backorder can be filled, by the EOQ with either a constant or linear backordering rate. Sharifi et al. (2015) developed an EOQ model for imperfect quality items with partial backordering and screening errors. Wang et al. (2015) extended the EOQ inventory model with fixed partial backlogging to imperfect quality items, using the time interval as a decision variable. Taleizadeh et al. (2016) studied an EOQ inventory model with partial backlogging and imperfect products, which are valuable and repairable. Taleizadeh (2017a) studied the lot-sizing problem under partial backordering and prepayment when the buyer may face disruption. Diabat et al. (2017) presented an EOQ model in supply chains with partial downstream delayed payment and partial upstream advance payment for perishable products. Taleizadeh (2017b) developed the vendor-managed inventory policy with partial backordering for evaporating products. Lin (2018) included the sustainability concept in the EOQ inventory model with partial backlogging, integrating environmental and economic perspectives. Lashgary et al. (2018) considered an economic order quantity model with a hybrid payment policy offered by the supplier. Godichaud and Amodeo (2019) developed three EOQ models for disassembly systems with two-level bills of material. Other recent research articles based on the EOQ inventory model with partial backlogging are, for example, Krommyda et al. (2019), Thinakaran et al. (2019), and Taleizadeh et al. (2020). A common characteristic of all the papers cited above is that the demand rate is a known constant.

An interesting reasoning about the affine structure of the shortage cost can be seen in Hadley and Whitin (1963, p. 18). Montgomery et al. (1973) were the first to include a fixed cost per backordered unit in the inventory models with partial backordering. That fixed cost caused the nonconvexity of the cost function, so they could not guarantee that their proposed solution was optimal. Perhaps that may be why not many research papers use this shortage cost structure. Among other authors who have also used an affine structure in the backorder cost are Rosenberg (1979), Drake and Pentico (2010), San-José and García-Laguna (2009); San-José et al. (2014, 2015, 2017), and Sicilia et al. (2009, 2012). Table 1 summarizes some characteristics of the previously cited papers.

In this paper, an inventory model with partial backlogging, considering maximizing the ROI as the goal, is developed. To the best of our knowledge, this problem has not been investigated in the literature and it is interesting because inventory managers can wish to know the inventory policy that leads to obtain the maximum return on their investments. In addition, by allowing a partial backlogging in the inventory model, customer behavior is better represented and makes the model more realistic.

The main contribution of this work is to determine the optimal inventory policy that maximizes the return on inventory investment, assuming that only a fixed fraction of the demand during the stock-out period is satisfied with the arrival of the next replenishment. This last assumption makes the model more realistic, because considering the proportion of demand which is backordered, a variety of real practical situations can be modeled. As we will see later, the inventory policy that maximizes ROI is, in general, different from the other one that maximizes the profit per unit time.

The rest of this article is organized as follows. Section 2 presents the problem description, the basic assumptions, and notation. Section 3 provides the mathematical formulation of the model. Section 4 develops the theoretical results which determine the optimal inventory policy. Section 5 studies the behavior of the policy that maximizes the return on inventory investment with respect to the policy that maximizes the profit per unit time. Also, in this section, a numerical sensitivity analysis of the optimal policy and of the maximum return on inventory investment with respect to the system input parameters are developed. Finally, Section 6 gives some conclusions and suggestions for future research lines.

2. Problem description

Consider a situation in which a firm with limited resources wants to invest in different projects. Faced with this circumstance, the objective of the manager is to select those projects that provide a greater ROI. Assume that one of the projects consists of commercializing (purchasing, holding, replenishing, and sale) a particular item, whose demand is approximately stable over time. We suppose, as in Axsäter (2015, p. 50), that the unit holding cost per unit time has two components. The first term is a fixed cost independent of the purchasing cost (warehouse rental, taxes, insurances, etc.), and the second component is a variable cost depending on the unit purchasing cost.

Table 1
Summary of literature

Paper	ROI model	Constant demand rate	No shortage	Partial backlogging	Backorder cost		Goodwill lost sale cost
					Linear	Affine	
Chen (2001)	✓		✓				
Diabat et al. (2017)				✓	✓		Constant
Drake and Pentico (2010)		✓		✓		✓	Constant
Eilon (1957, 1959)	✓	✓	✓				
Godichaud and Amodeo (2019)		✓					Constant
Krommyda et al. (2019)		✓		✓	✓		Constant
Lashgary et al. (2018)		✓		✓	✓		Constant
Lin (2018)		✓		✓	✓		Constant
Marchioni and Magni (2018)	✓		✓				
Montgomery et al. (1973)		✓		✓		✓	
Otake and Min (2001)	✓	✓	✓				
Otake et al. (1999)	✓	✓	✓				
Park (1982)		✓		✓	✓		Constant
Pentico and Drake (2009)		✓		✓	✓		Constant
Pentico et al. (2014, 2015)		✓		✓	✓		Constant
Rosenberg (1979)		✓		✓		✓	
San-José and García-Laguna (2009)		✓				✓	
San-José et al. (2009a)		✓		✓	✓		Constant
San-José et al. (2009b, 2014)		✓		✓		✓	Affine
San-José et al. (2015)		✓		✓		✓	Constant
San-José et al. (2017)				✓		✓	Affine
Schroeder and Krishnan (1976)	✓	✓	✓				
Sharifi et al. (2015)		✓		✓	✓		Linear
Sicilia et al. (2009, 2012)		✓		✓		✓	Constant
Taleizadeh (2014, 2017a, 2017b)		✓		✓	✓		Constant
Taleizadeh and Pentico (2013, 2014)		✓		✓	✓		Constant
Taleizadeh et al. (2012, 2013a, 2013b, 2015, 2016)		✓		✓	✓		Constant
Taleizadeh et al. (2020)				✓	✓		Constant
Tate et al. (1964)	✓	✓	✓				
Thinakaran et al. (2019)		✓		✓		✓	Constant
Toews et al. (2011)		✓		✓	✓		Constant
Trietsch (1995)	✓	✓	✓				
Wang et al. (2015)		✓		✓	✓		Constant
This paper	✓	✓		✓		✓	Affine

When shortages of this item occur, some customers are willing to wait for the arrival of the next replenishment, while other customers are not willing to wait and go to buy items from other sellers. As usual in practice, customers make the decision to wait until the next order or not, according to the possible compensation they would receive from the firm, if they decide to wait. It seems reasonable that this compensation depends on the time that customers would have to wait to satisfy their demand. Therefore, both in the case of backorders and in the case of lost sales, the commercial prestige rests on the time elapsed until the arrival of the next order. As a result, we can suppose that both, backorder unit cost and lost sale unit cost, include a fixed cost and a variable cost which is proportional to the length of the shortage time. In general, the unit shortage cost for a patient customer willing to wait is different from another impatient customer unwilling to wait. Thus, four parameters must be considered to determine the shortage costs. Also, we assume a known ordering cost, and that the unit holding cost per unit time has a fixed term and a variable component, which depends on the unit purchasing cost. Then, the inventory model proposed in this paper is based on the following assumptions:

(1) The item is a single product with a constant demand rate.
(2) The replenishment is instantaneous, and the inventory system is continuously reviewed.
(3) Shortages are allowed, and they are partially backordered.
(4) Ordering cost is fixed regardless of the lot size.
(5) The holding cost is a function based on average inventory.
(6) The backorder cost per unit has a constant cost and a cost which is proportional to the length of time for which backorder exists. Similarly, the unit goodwill cost also includes a fixed cost and a cost which is proportional to the length of time for which lost sales exist.

The notation to be used is summarized in Table 2.

Table 2
Notation

λ	Demand rate per unit time (> 0).
K	Ordering cost (> 0).
c	Unit purchasing cost (> 0).
s	Unit selling price ($s \geq c$).
h_0	Fixed unit holding cost per unit time.
i	Fraction of the variable holding cost per unit time (> 0).
h	Unit holding cost per unit time, that is, $h = h_0 + ic$ (> 0).
ω_0	Constant cost per backordered unit (≥ 0).
ω	Shortage cost per backordered unit and per unit time (≥ 0).
η_0	Constant goodwill cost per lost unit (≥ 0).
η	Unit goodwill cost per unit time (≥ 0).
ρ	Fraction of demand which is backordered ($0 \leq \rho \leq 1$).
β_0	Fixed average shortage cost per unit, that is, $\beta_0 = \omega_0\rho + \eta_0(1 - \rho)$.
β_1	Time-dependent average shortage cost, that is, $\beta_1 = \omega\rho + \eta(1 - \rho)$.
T	Length of the inventory cycle where the net stock is positive (≥ 0 , *decision variable*).
Ψ	Length of the inventory cycle when net stock is less than or equal to zero (≥ 0, *decision variable*).
α	Inventory cycle, that is, $\alpha = T + \Psi$ (> 0).
$I(t)$	Net inventory level at time t, with $0 \leq t \leq \alpha$.
S	Maximum level of the stock, that is, $S = \lambda T$ (≥ 0).
b	Demanded quantity during the stock-out period, that is, $b = \lambda\Psi$ (≥ 0).
Q	Lot size per cycle, that is, $Q = S + \rho b = \lambda(T + \rho\Psi)$ (≥ 0).

3. The model

At the beginning of the inventory cycle, there are S units in stock. Next, the inventory level decreases due to customer demand. At time $t = T$, the inventory level is zero. Later, the inventory drops into shortage and begins the stock-out period of length Ψ. At the end of that period, the net inventory level is $-\rho b$, where $b = \lambda\Psi$ is the total quantity demanded during the stock-out period and ρ represents the fraction of backordered demand. Next, to meet the pending demand and replenish the inventory, $Q = S + \rho b$ units are ordered.

From the previous hypotheses, it can easily be seen that the net inventory level $I(t)$ is described by

$$I(t) = \begin{cases} S - \lambda t = \lambda(T - t) & \text{if } 0 \le t < T \\ \lambda\rho(T - t) & \text{if } T \le t < \alpha. \end{cases}$$

At $I(T) = 0$, the maximum inventory level is $S = \lambda T$ and the lot size is $Q = \lambda(T + \rho\Psi)$.

The total profit per cycle $PC(T, \Psi)$ is the difference between the revenue per cycle $sQ = s\lambda(T + \rho\Psi)$ and the sum of the ordering cost K, the purchasing cost $cQ = c\lambda(T + \rho\Psi)$, the holding cost, the backordering cost, and the lost sale cost per cycle. The holding cost per cycle is given by $HC(T) = \int_0^T hI(t)\mathrm{d}t = \int_0^T (h_0 + ic)I(t)\mathrm{d}t = (h_0 + ic)\lambda T^2/2$. The backorder cost is $BC(\Psi) = \int_T^{T+\Psi} (\omega_0\rho\lambda - \omega I(t))\mathrm{d}t = \omega_0\lambda\rho\Psi + \omega\lambda\rho\Psi^2/2$, and the goodwill lost sale cost is given by $LC(\Psi) = \int_T^{T+\Psi} (\eta_0(1 - \rho)\lambda + \eta \int_T^t (1 - \rho)\lambda \mathrm{d}u)\mathrm{d}t = \eta_0\lambda(1 - \rho)\Psi + \eta\lambda(1 - \rho)\Psi^2/2$. Thus, the total profit in a cycle is

$$PC(T, \Psi) = (s - c)\lambda(T + \rho\Psi) - (K + HC(T) + BC(\Psi) + LC(\Psi)).$$

Therefore, the return on inventory investment (ROI) is given by

$$ROI(T, \Psi) = \frac{PC(T, \Psi)}{CC(T, \Psi)}, \tag{1}$$

where $CC(T, \Psi)$ is the total cost during an inventory cycle. This cost is given by $CC(T, \Psi) = c\lambda(T + \rho\Psi) + K + HC(T) + BC(\Psi) + LC(\Psi)$.

4. The optimal policy

To find the optimal inventory policy that maximizes the return on inventory investment, we consider two possible situations, depending on the value of the fraction ρ of demand which is backordered. Thus, we have to study the following alternative cases: (i) $\rho > 0$ and (ii) $\rho = 0$ (full lost sales case). Next, we analyze the case when $\rho > 0$.

4.1. Backlogging case ($\rho > 0$)

Since $\rho > 0$ and $T + \Psi > 0$, it follows that $Q = \lambda(T + \rho\Psi) > 0$ and, after a few algebraic manipulations, the ROI defined by Equation (1) can be expressed in the form

$$ROI(T, \Psi) = \frac{s}{c + AI(T, \Psi)} - 1, \tag{2}$$

where $AI(T, \Psi)$ represents the average inventory cost per unit of item, without considering the purchasing cost. That is,

$$\begin{aligned} AI(T, \Psi) &= \frac{1}{Q}(K + HC(T) + BC(\Psi) + LC(\Psi)) \\ &= \frac{1}{\lambda(T + \rho\Psi)}\left(K + (h_0 + ic)\lambda\frac{T^2}{2} + \beta_0\lambda\Psi + \beta_1\lambda\frac{\Psi^2}{2}\right), \end{aligned}$$

where $\beta_0 = \omega_0\rho + \eta_0(1 - \rho)$ and $\beta_1 = \omega\rho + \eta(1 - \rho)$.

From (2), it is clear that the optimal inventory policy that minimizes $AI(T, \Psi)$ is the same as the optimal policy that maximizes the return on inventory investment $ROI(T, \Psi)$. Thus, our goal is to solve the nonlinear problem

$$\min_{(T,\Psi)\in\Pi} AI(T, \Psi), \tag{3}$$

where $\Pi = \{(T, \Psi) \mid T \geq 0, \Psi \geq 0 \text{ and } T + \Psi > 0\}$.

Note that, in this inventory system, the average inventory cost per unit of item, $AI(T, \Psi)$ is different from the minimum inventory cost per unit time, $CC(T, \Psi)/\alpha$. Thus, as we show in Section 5, the optimal inventory policy that maximizes the return on inventory investment is different from the one for the minimization of the inventory cost per unit time.

To solve the problem (3), we first assume $\Psi \in [0, \infty)$ to be fixed and $T \geq 0$ to be variable. Thus, we consider the function $AI_\Psi(T) = AI(T, \Psi)$. It is immediate that the function $AI_\Psi(T)$ is strictly convex and attains its minimum at the point

$$T^*_\Psi = \sqrt{\frac{2K + 2\beta_0\lambda\Psi + \lambda\big(\beta_1 + (h_0 + ic)h\rho^2\big)\Psi^2}{\lambda(h_0 + ic)}} - \rho\Psi, \tag{4}$$

with the value

$$\begin{aligned} Z(\Psi) &= AI_\Psi(T^*_\Psi) \\ &= \sqrt{\frac{2K(h_0 + ic)}{\lambda} + 2\beta_0(h_0 + ic)\Psi + (h_0 + ic)\big(\beta_1 + (h_0 + ic)\rho^2\big)\Psi^2} - (h_0 + ic)\rho\Psi \\ &= (h_0 + ic)T^*_\Psi. \end{aligned} \tag{5}$$

The first derivative of $Z(\Psi)$ is given by

$$Z'(\Psi) = \frac{\beta_0(h_0+ic) + (h_0+ic)\big(\beta_1 + (h_0+ic)\rho^2\big)\Psi}{\sqrt{\frac{2K(h_0+ic)}{\lambda} + 2\beta_0(h_0+ic)\Psi + (h_0+ic)\big(\beta_1+(h_0+ic)\rho^2\big)\Psi^2}} - (h_0+ic)\rho.$$

After some algebraic manipulations, this derivative can be rewritten as

$$Z'(\Psi) = \frac{L(\Psi)}{\lambda M(\Psi)N(\Psi)},$$

where

$$L(\Psi) = \lambda\beta_1\big(\beta_1 + (h_0+ic)\rho^2\big)\Psi^2 + 2\lambda\beta_0\beta_1\Psi + \lambda\beta_0^2 - 2K(h_0+ic)\rho^2, \tag{6}$$

$M(\Psi) = T^*_\Psi + \rho\Psi$ and $N(\Psi) = \beta_0 + (\beta_1 + (h_0+ic)\rho^2)\Psi + (h_0+ic)\rho M(\Psi)$.

As $M(\Psi) > 0$ and $N(\Psi) > 0$ for all $\Psi \geq 0$, we get *sign* $(Z'(\Psi)) =$ *sign* $(L(\Psi))$. Next, we provide a result to calculate the optimal value of the decision variable Ψ.

Theorem 1. *Let* $\beta_0 = \omega_0\rho + \eta_0(1-\rho)$, $\beta_1 = \omega\rho + \eta(1-\rho)$, *and* $\Gamma = \lambda\beta_0^2 - 2K(h_0+ic)\rho^2$. *The optimum stock-out period* Ψ^* *is determined as follows:*

A. *If* $\Gamma > 0$, *then* $\Psi^* = 0$.
B. *If* $\Gamma = 0$ *and* $\beta_1 > 0$, *then* $\Psi^* = 0$.
C. *If* $\Gamma = 0$ *and* $\beta_1 = 0$, *then the function* $Z(\Psi)$ *is reduced to* $Z(\Psi) = \sqrt{2K(h_0+ic)/\lambda}$ *for all* $\Psi \geq 0$, *and, consequently, the optimum stock-out period is any point on the interval* $[0, \infty)$.
D. *If* $\Gamma < 0$ *and* $\beta_1 > 0$, *then the optimum stock-out period* Ψ^* *is attained at the point*

$$\Psi_1 = \frac{1}{\beta_1 + (h_0+ic)\rho^2}\left(\sqrt{\frac{2K(h_0+ic)\rho^2\big(\beta_1 + (h_0+ic)\rho^2\big) - \lambda\beta_0^2(h_0+ic)\rho^2}{\lambda\beta_1}} - \beta_0\right).$$

E. *If* $\Gamma < 0$ *and* $\beta_1 = 0$, *then* $\inf_{\Psi\geq 0} Z(\Psi) = \lim_{\Psi\to\infty} Z(\Psi) = \beta_0/\rho$. *In this case, the best inventory policy is* $\Psi^* = \infty$.

Proof. See Appendix A. ■

Remark 1. Note that β_1 represents the average shortage cost per unit and per unit time. What does Γ mean? Since $\lambda\Gamma = (\beta_0\lambda)^2 - (\rho\sqrt{2K(h_0+ic)\lambda})^2$, it follows that $\lambda\Gamma$ is the difference between the square of the fixed average cost per unit of time (excluding the loss of profit) if the demand is all short, $\beta_0\lambda$, and the square of the fraction of backordered shortage of the optimum cost if there is no shortage, $\rho\sqrt{2K(h_0+ic)\lambda}$. Then, Γ represents the above difference per demanded unit.

4.2. Full lost sales case ($\rho = 0$)

Now, we assume that all shortages are lost sales, that is, $\rho = 0$. In this case, if $T = 0$, then $Q = 0$ and, from (1), we obtain that the ROI is $ROI(0, \Psi) = -1$ for all $\Psi > 0$. On the other hand,

$ROI(T, \Psi) > -1$ for all $T > 0$. Thus, we can assume that $T > 0$ and, as in the previous subsection, our aim is to minimize the average inventory cost per unit of item $AI(T, \Psi)$. In this case, as $\rho = 0$, we have $\beta_0 = \eta_0$ and $\beta_1 = \eta$. Thus, the function $Z(\Psi)$ is simplified to

$$Z_0(\Psi) = \sqrt{\frac{2K(h_0 + ic)}{\lambda} + 2\eta_0(h_0 + ic)\Psi + (h_0 + ic)\eta\Psi^2}.$$

Next, we show how to determine the optimal policy in this situation of full lost sales.

Theorem 2. *Let $T_0 = \sqrt{2K/(\lambda(h_0 + ic))}$. The inventory policy (T^*, Ψ^*) that maximizes the return on inventory investment $ROI(T, \Psi)$ can be determined as follows:*

1. *If $\eta_0 + \eta > 0$, then $T^* = T_0$ and $\Psi^* = 0$.*
2. *Otherwise (i.e., $\eta_0 = \eta = 0$), each point of the ray $R = \{(T_0, \Psi), \Psi \geq 0\}$ is an optimal policy.*

Proof. See Appendix A. ■

Remark 2. In the case that all shortages are lost sales and there exists a positive goodwill cost, the inventory policy that maximizes the return on inventory investment coincides with the optimal policy of the classic lot size system.

4.3. Some results related to the optimal policy

Next, we present various results which show the optimal values of other variables in the inventory system.

Corollary 1. *In the cases (i) $\Gamma > 0$ and (ii) $\Gamma = 0$ and $\beta_1 > 0$, we have*

1. *The optimal stock-in period is $T_0 = \sqrt{\frac{2K}{\lambda(h_0+ic)}}$.*
2. *The optimal inventory cycle is $\alpha_0 = T_0$.*
   The optimal lot size is $Q_0 = \sqrt{\frac{2K\lambda}{h_0+ic}}$.
3. *The minimum average inventory cost per unit of item is $AI_0 = \sqrt{\frac{2K(h_0+ic)}{\lambda}}$.*
4. *The maximum return on inventory investment is $ROI_0 = \frac{(s-c)\lambda - \sqrt{2K(h_0+ic)\lambda}}{c\lambda + \sqrt{2K(h_0+ic)\lambda}}$.*

Proof. See Appendix A. ■

Remark 3. The cases considered in Corollary 1 assume that the fixed average shortage cost per unit must be greater or equal than the fraction of backlogged demand times the optimal cost of the system without shortage. In these cases, the optimal inventory policy consists of not allowing shortages and the optimal lot size is the economic order quantity.

Corollary 2. *If $\Gamma = 0$ and $\beta_1 = 0$, then*

1. *The optimal stock-in period is $T_0 = \sqrt{\frac{2K}{\lambda(h_0+ic)}}$.*

2. *The optimal inventory cycle is* $\alpha = T_0 + \Psi$, *for all* $\Psi \geq 0$.
   The optimal lot size is $Q = \sqrt{\frac{2K\lambda}{h_0+ic}} + \lambda\rho\Psi$, *for all* $\Psi \geq 0$.
3. *The minimum average inventory cost per unit of item is* $AI_0 = \sqrt{\frac{2K(h_0+ic)}{\lambda}}$.
4. *The maximum return on inventory investment is* $ROI_0 = \frac{(s-c)\lambda - \sqrt{2K(h_0+ic)\lambda}}{c\lambda + \sqrt{2K(h_0+ic)\lambda}}$.

Proof. See Appendix A. ■

Remark 4. The case considered in Corollary 2 is unlikely to occur in a real-world situation, because the shortage cost does not depend on the time that customer should wait until the next order arrives, and, furthermore, the shortage unit cost has to match the fraction of backlogged demand times the optimal cost of the system without shortage.

Corollary 3. *If* $\Gamma < 0$ *and* $\beta_1 > 0$, *then*

1. *The optimal stock-in period is*

$$T_1 = \sqrt{\frac{2K}{\lambda(h_0+ic)} - \frac{\beta_0^2}{(h_0+ic)\big(\beta_1+(h_0+ic)\rho^2\big)}}\sqrt{\frac{\beta_1}{\beta_1+(h_0+ic)\rho^2}} + \frac{\beta_0\rho}{\beta_1+(h_0+ic)\rho^2}.$$

2. *The optimal inventory cycle is*

$$\alpha_1 = \sqrt{\frac{2K}{\lambda(h_0+ic)} - \frac{\beta_0^2}{(h_0+ic)\big(\beta_1+(h_0+ic)\rho^2\big)}}\frac{\beta_1+(h_0+ic)\rho}{\sqrt{\beta_1\big(\beta_1+(h_0+ic)\rho^2\big)}} - \frac{\beta_0(1-\rho)}{\beta_1+(h_0+ic)\rho^2}.$$

3. *The optimal lot size is*

$$Q_1 = \sqrt{\frac{2K\lambda}{(h_0+ic)} - \frac{(\lambda\beta_0)^2}{(h_0+ic)\big(\beta_1+(h_0+ic)\rho^2\big)}}\sqrt{\frac{\beta_1+(h_0+ic)\rho^2}{\beta_1}}.$$

4. *The minimum average inventory cost per unit of item is*

$$AI_1 = \sqrt{\frac{2K(h_0+ic)}{\lambda} - \frac{\beta_0^2(h_0+ic)}{\beta_1+(h_0+ic)\rho^2}}\sqrt{\frac{\beta_1}{\beta_1+(h_0+ic)\rho^2}} + \frac{\beta_0(h_0+ic)\rho}{\beta_1+(h_0+ic)\rho^2}.$$

5. *The maximum return on inventory investment is*

$$ROI_1 = \frac{\sqrt{\lambda}\big((s-c)\big(\beta_1+(h_0+ic)\rho^2\big)-\beta_0(h_0+ic)\rho\big)-\sqrt{\big(2K\big(\beta_1+(h_0+ic)\rho^2\big)-\lambda\beta_0^2\big)(h_0+ic)\beta_1}}{\sqrt{\lambda}\big(c\big(\beta_1+(h_0+ic)\rho^2\big)-\beta_0(h_0+ic)\rho\big)+\sqrt{\big(2K\big(\beta_1+(h_0+ic)\rho^2\big)-\lambda\beta_0^2\big)(h_0+ic)\beta_1}}.$$

Proof. See Appendix A. ■

Remark 5. If $\Gamma < 0$ and $\beta_1 > 0$, it is obvious that $ROI_1 > ROI_0$ (see Lemma 1 in Appendix A for more details). That is, the optimal return on inventory investment is greater than the corresponding return on inventory investment associated with the EOQ model without shortage.

Note that if $\Gamma < 0$ and $\beta_1 = 0$, then Theorem 1 shows that the best inventory policy is $\Psi^* = \infty$. Obviously, this solution should be interpreted as the absence of an inventory system (see, for instance, Sicilia et al., 2009). For this reason, we have not considered this situation in the previous corollaries.

5. Comparison of the best ROI policy with the maximum profit policy

In this section, we use several numerical examples to show that, in general, the optimal inventory policy for maximizing the return on inventory investment *ROI* is different from the optimal policy for maximizing the profit per unit time.

San-José et al. (2009b) determined the optimal policy $(T^{\#}, \Psi^{\#})$ that maximizes the average profit per unit of time (i.e., the function $B(T, \Psi) = PC(T, \Psi)/\alpha$) for the inventory system with the same assumptions as the one studied here.

Note that if $\rho = 1$ (full backordering case), then the optimal inventory policy that maximizes the return on inventory investment $ROI(T, \Psi)$ is the same as the optimal policy that maximizes the average profit per unit time and also minimizes the total cost per unit time. However, this is not generally true when there is partial backlogging ($0 < \rho < 1$) in the inventory system.

For example, let us consider the following parameters for the inventory problem: $\lambda = 1000$, $K = 1000$, $c = 4$, $s = 8$, $h_0 = 0.8$, $i = 0.3$, $\omega_0 = 0.25$, $\omega = 2$, $\eta_0 = 0$, $\eta = 0$, and $\rho = 0.6$. Then $h = 2$, $\beta_0 = 0.15$, $\beta_1 = 1.2$, and $\Gamma = -1417.5$. Applying Corollary 3 proposed in this paper, we obtain that the inventory policy that maximizes the ROI is $T^* = T_1 = 0.835125$ and $\Psi^* = \Psi_1 = 0.710125$. However, using Theorem 1 of San-José et al. (2009b), we obtain the inventory policy that maximizes the profit per unit time, which is $T^{\#} = 0.989093$ and $\Psi^{\#} = 0.190155$.

In this same situation of partial backlogging, the differences between the two optimal policies can be even more pronounced. For instance, assume the same parameters as in the previous example, but modifying the values of ω and ρ to $\omega = 0$ and $\rho = 0.5$, respectively. Now, $\beta_0 = 0.125$, $\beta_1 = 0$, and $\Gamma = -984.375$. From Theorem 1 given above, we obtain that the best policy that maximizes the ROI is $\Psi^* = \infty$ (shortage period is very large) and, from (4), $T^* = 0.125$. However, applying Theorem 1 of San-José et al. (2009b), we deduce that the inventory policy that maximizes the profit per unit time is $T^{\#} = 1$ and $\Psi^{\#} = 0$ (shortage period is null).

These differences between the two optimal policies can also occur in the case of full lost sales ($\rho = 0$). It is enough for this to now consider the following values for the parameters: $\lambda = 1000$, $K = 2000$, $c = 4$, $s = 8$, $h_0 = 6$, $i = 0.5$, $\eta_0 = 0.05$, $\eta = 0$, and $\rho = 0$. Then $h = 8$, $\beta_0 = \eta_0 = 0.05$, and $\beta_1 = \eta = 0$. Applying Theorem 2 proposed in this paper, we get $T^* = 0.707107$ and $\Psi^* = 0$. However, from Theorem 1 given by San-José et al. (2009b), it is concluded that the inventory policy that maximizes the average profit is $T^{\#} = 0.50625$ and $\Psi^{\#} = \infty$.

In summary, the two approaches generally lead to different inventory policies. This is due, among other reasons, to the fact that the inventory policy that maximizes the ROI does not depend on the unit purchasing cost or the unit selling price, unlike what happens with the optimal inventory policy of the maximum profit problem. Obviously, using one or the other approach will rest on what is the ultimate objective of the inventory manager.

In Appendix B, it is also compared the optimal inventory policy that maximizes the ROI with the one that maximizes the NPV. Thus, in that appendix, we have solved a numerical example which shows that the optimal solution for maximizing the ROI is different from the optimal policy obtained using the NPV approach.

5.1. Numerical sensitivity analysis

Let us consider a firm, which is dedicated to the storage and commercialization of chemical products. It supplies liquid nitrogen in bulk (among other formats) to small plants in different industrial sectors (food, health, cosmetics, etc.). The inventory manager wishes to maximize the ROI from storing this product. It can be accepted that the inventory system of that product satisfies the assumptions made in this paper. Thus, its demand is roughly stable over time. The inventory system allows shortages, but not all the customers have the same behavior towards them. There are some customers who cannot wait for the next order, while other customers who are willing to wait to meet their demand in the system. It can be assumed that, at any time, the fraction of backordered demand ρ is fixed.

To illustrate the behavior of the optimal inventory policy that maximizes the ROI for this firm, we include the following numerical example.

Example 1. We consider a unit purchasing cost $c = 10$, and a unit selling price $s = 20$ currency units. The demand rate is $\lambda = 1000$ items per unit time. The replenishment cost for the inventory, including shipping and handling costs, is $K = 500$ currency units. In addition, the storage of a unit has a fixed cost $h_0 = 1.5$ currency units and a carrying charge $i = 30\%$ of the purchasing cost. Thus, the holding cost per unit and per unit time is $h = 4.5$ currency units. Also, we suppose that the backorder cost parameters are $\omega_0 = 0.1$ and $\omega = 5$, and there is no goodwill cost, that is, $\eta_0 = \eta = 0$.

If $\rho = 0$, applying Theorem 2, we obtain that the maximum return on inventory investment is attained at any point of the ray $R = \{(0.471405, \Psi), \Psi \geq 0\}$. In this case, the policy that maximizes the total inventory profit per unit time is unique and is the vertex of the ray, that is, $T^{\#} = 0.471405$ and $\Psi^{\#} = 0$. In this case, the maximum inventory profit per unit time is $B^{\#} = B(T^{\#}, \Psi^{\#}) = 7878.68$.

However, if $\rho > 0$, then $\beta_1 = 5\rho > 0$, $\Gamma = -4490\rho^2 < 0$ and, now applying Theorem 1, we obtain $\Psi^*(\rho) = \Psi_1(\rho)$. The optimal policy $(T^{\#}, \Psi^{\#})$ that maximizes the average profit per unit time $B(T, \Psi)$ and the optimal inventory policy (T^*, Ψ^*) that maximizes the return on inventory investment $ROI(T, \Psi)$ are shown in Table 3 for different values of the fraction of backordered demand ρ. The value $\Delta ROI(\%)$ is defined as $\Delta ROI = 100[ROI(T^*, \Psi^*)/ROI(T^{\#}, \Psi^{\#}) - 1]$.

From these results, we can make the following comments: if $\rho > 0$ increases, then (i) the optimal stock-in period T^* and the optimal stock-out period Ψ^* are strictly decreasing; (ii) the maximum return on inventory investment ROI^* is strictly increasing; (iii) there is a value, say $\widetilde{\rho}$, such that ΔROI is strictly increasing when $\rho < \widetilde{\rho}$, while ΔROI is strictly decreasing if $\rho > \widetilde{\rho}$; and (iv), as expected, the profit per unit time is less for the solution with a maximum ROI than for the one with maximum profit. This implies that the solution with maximum ROI has, in general, a total inventory cost per unit of time (i.e., $CC(T, \Psi)/\alpha$) less than the solution for the problem of maximum profit per unit time.

Example 2. Next, we analyze the fluctuations of the optimal inventory policies when some variations or changes in the parameters of the inventory system are allowed. Thus, first we consider the

Table 3
Numerical results associated with Example 1

ρ	T^*	Ψ^*	$B(T^*, \Psi^*)$	$ROI^*(\%)$	$T^\#$	$\Psi^\#$	$B(T^\#, \Psi^\#)$	$ROI^\#(\%)$	$\Delta ROI(\%)$
0	0.471405	$\Psi \geq 0$	$B_0(\Psi)$[a]	64.9985	0.471405	0.00000	7878.68	64.9985	0.00000
0.1	0.453317	0.387985	4656.20	66.1140	0.471405	0.00000	7878.68	64.9985	1.71610
0.3	0.422929	0.360637	5488.22	68.0223	0.471405	0.00000	7878.68	64.9985	4.65205
0.7	0.377663	0.319897	7158.55	70.9477	0.471405	0.00000	7878.68	64.9985	9.15282
0.8	0.368578	0.311720	7576.98	71.5472	0.471306	0.010219	7879.12	65.4630	9.29402
0.85	0.364292	0.307863	7786.28	71.8315	0.458536	0.112567	7936.59	69.1817	3.83015
0.9	0.360163	0.304146	7995.63	72.1062	0.431664	0.189442	8057.51	71.1011	1.41365
0.95	0.356181	0.300563	8205.03	72.3719	0.395605	0.249520	8219.78	72.1448	0.31481
1	0.352339	0.297105	8414.47	72.6292	0.352339	0.297105	8414.47	72.6292	0.00000

[a] $B_0(\Psi) = 1000(\frac{10\sqrt{2}-3}{\sqrt{2}+3\Psi})$.

Table 4
Effects of the parameters λ and ρ

	Δ	$\Delta T^*(\%)$	$\Delta\Psi^*(\%)$	$\Delta ROI^*(\%)$	$\Delta T^\#(\%)$	$\Delta\Psi^\#(\%)$	$\Delta B^\#(\%)$
λ	+20%	−8.50129	−9.04674	2.93532	−8.69383	−100.000	22.8082
	+10%	−4.54061	−4.83194	1.55889	−4.63382	−100.000	11.3720
	+5%	−2.35138	−2.50225	0.804757	−2.38960	−100.000	5.67539
	−5%	2.53460	2.69722	−0.861454	2.51122	130.293	−5.64217
	−10%	5.27752	5.61612	−1.78676	5.16274	267.864	−11.2507
	−20%	11.5156	12.2545	−3.86471	10.9849	569.943	−22.3655
ρ	+20%	−3.57472	−3.80408	1.22558	−17.7836	2443.61	4.78693
	+10%	−1.83994	−1.95799	0.629258	−5.85916	1475.92	1.57715
	+5%	−0.933776	−0.993687	0.318938	−1.87484	825.219	0.504664
	−5%	0.962823	1.02460	−0.327973	0.0208934	−100.000	−0.00562401
	−10%	1.95624	2.08175	−0.665428	0.0208934	−100.000	−0.00562401
	−20%	4.04163	4.30094	−1.37073	0.0208934	−100.000	−0.00562401

same parameters as in Example 1 together with the fraction of backordered demand $\rho = 0.8$. Then, we evaluate the percentage variations of the optimal policies assuming different values in each of the parameters while keeping all the others fixed. More specifically, for each input parameter, we have varied its value by $\pm 20\%$, $\pm 10\%$, and $\pm 5\%$. As a result, Table 4 shows the effects of the parameters related to the demand, that is, the demand rate λ and the fraction of backlogged demand ρ. From these computational results, we can establish the following managerial insights:

1. The optimal inventory policy that maximizes the return on inventory investment *ROI* is more sensitive to fluctuations in the demand rate λ than to changes in the fraction of backordered demand ρ.
2. The policy that gives a major *ROI* is more sensitive to negative than a positive variations in the parameter λ. This is also true for the parameter ρ.
3. The maximum *ROI* is not very sensitive to the changes considered. This does not usually happen regarding the maximum inventory profit per unit of time when changes in λ occur.

4. The maximum ROI decreases (increases) when the value of one of the two parameters decreases (increases). However, the stock-in and the stock-out periods decrease (increase) when the value of each parameter increases (decreases).
5. When the parameter λ varies, the behavior of the best stock-in period in the two policies is similar. However, it does not usually happen regarding the stock-out period.

Table 5 contains the percentage variations of the optimal policies with respect to the parameters K, h_0, i, ω, ω_0, c, and s. From these computational results, we can establish the following managerial insights:

1. As expected, the optimal policy obtained maximizing the ROI does not depend on the parameter s. However, this does not usually happen if the goal is the maximization of the profit per unit time.
2. Changes in the constant cost per backordered unit (w_0) have little effects on the policy that maximizes ROI.
3. The maximum ROI is very insensitive to changes in the parameters K , h_0, i, ω, or ω_0. However, it is very sensitive to changes in the purchasing cost or the selling price.
4. The maximum ROI increases (decreases) when the value of the selling price s increases (decreases), while it decreases (increases) when the value of any of the other parameters increases (decreases).

6. Conclusions

In this article, an inventory model with the constant demand rate and fixed partial backlogging is studied. We assume that the shortage costs (backorder cost and goodwill lost sales cost) have an affine structure: a fixed cost plus a linear cost that depends on the period of time when shortages exist. Instead of the maximization of the profit per unit time or the minimization of the average inventory cost per unit time, we consider the maximization of the return on inventory investment, defined as the ratio given by the average profit/average inventory cost, as our objective. In many real-world situations, a firm cannot have enough financial resources to invest in several projects, at least one of them being the commercialization of some product (purchase, holding, replenishing, and sale). Thus, the optimal inventory policy that maximizes the ROI is a better alternative to the one that maximizes the profit per unit time because, although this criterion reduces the profit, it requires a lower investment cost in inventory management. Therefore, it is more affordable. The optimal inventory policies are determined in a closed form for the different possible cases of the inventory system. Also, it is shown that the optimal policy that maximizes the return on inventory investment is, in general, different from the one that maximizes the profit per unit time. Moreover, the policy that maximizes the ROI does not depend on the unit selling price, and, hence, the inventory manager does not need to change his/her inventory policy if this price changes. However, this does not usually happen if the objective is the maximization of the profit per unit time. The numerical sensitivity analysis shows that the maximum return on inventory investment ROI^* is not very sensitive to changes in the parameters related to demand (the demand rate per unit time and

Table 5
Effects of the parameters K, h_0, i, ω, ω_0, c and s

	Δ	$\Delta T^*(\%)$	$\Delta\Psi^*(\%)$	$\Delta ROI^*(\%)$	$\Delta T^\#(\%)$	$\Delta\Psi^\#(\%)$	$\Delta B^\#(\%)$
K	+20%	9.31191	9.90936	−3.13480	8.95745	464.749	−2.41114
	+10%	4.76203	5.06756	−1.61341	4.66921	242.258	−1.25684
	+5%	2.40940	2.56398	−0.819046	2.38863	123.932	−0.642964
	−5%	−2.47048	−2.62899	0.845661	−2.51169	−100.000	0.676089
	−10%	−5.00694	−5.32818	1.72014	−5.11185	−100.000	1.37599
	−20%	−10.3010	−10.9619	3.56596	−10.5386	−100.000	2.83675
h_0	+20%	−4.45262	2.04021	−0.652186	−3.29573	163.498	−0.848235
	+10%	−2.28129	1.03862	−0.332455	−1.67455	83.1685	−0.431482
	+5%	−1.15494	0.524073	−0.167868	−0.844095	41.9485	−0.217630
	−5%	1.18471	−0.533886	0.171253	0.858048	−42.6964	0.221511
	−10%	2.40045	−1.07788	0.346001	1.73037	−86.1610	0.447007
	−20%	4.93038	−2.19747	0.706447	3.53146	−100.000	0.907297
i	+20%	−8.49750	3.94041	−1.25643	−6.38728	316.203	−1.64047
	+10%	−4.45262	2.04021	−0.652186	−3.29573	163.498	−0.848235
	+5%	−2.28129	1.03862	−0.332455	−1.67455	83.1685	−0.431482
	−5%	2.40045	−1.07788	0.346001	1.73037	−86.1610	0.447007
	−10%	4.93038	−2.19747	0.706447	3.53146	−100.000	0.907297
	+20%	10.4266	−4.57265	1.47472	7.43967	−100.000	1.85345
ω	+20%	3.33127	−13.7125	−1.13095	0.00342069	−16.5188	−0.000920763
	+10%	1.77327	−7.37542	−0.603346	0.00186284	−9.00304	−0.000501440
	+5%	0.916293	−3.83325	−0.312144	0.000974887	−4.71373	−0.000262416
	−5%	−0.982145	4.16299	0.335482	−0.00107522	5.20443	0.000289423
	−10%	−2.03770	8.70173	0.697088	−0.00226715	10.9804	0.000610270
	−20%	−4.40575	19.1395	1.51230	−0.00508633	24.6701	0.00136912
ω_0	+20%	0.484800	−0.767299	−0.165253	0.0129833	−38.4680	−0.00349479
	+10%	0.242854	−0.383166	−0.0828098	0.00725151	−19.1946	−0.00195194
	+5%	0.121541	−0.191462	−0.0414508	0.00381459	−9.58749	−0.00102679
	−5%	−0.121768	0.191220	0.0415426	−0.00418995	9.56801	0.00112784
	−10%	−0.243763	0.382199	0.0831771	−0.00875306	19.1167	0.00235612
	−20%	−0.488433	0.763432	0.166722	−0.0189900	38.1563	0.00511166
c	+20%	−8.49750	3.94041	−36.0248	−9.09245	1135.67	−26.1988
	+10%	−4.45262	2.04021	−19.4944	−4.16487	604.667	−13.2905
	+5%	−2.28129	1.03862	−10.1657	−1.95623	312.701	−6.69901
	−5%	2.40045	−1.07788	11.1214	1.73078	−100.000	6.79278
	−10%	4.93038	−2.19747	23.3407	3.53146	−100.000	13.5991
	−20%	10.4266	−4.57265	51.8000	7.43967	−100.000	27.2370
s	+20%	0.00000	0.00000	47.9536	0.0208934	−100.000	50.7614
	+10%	0.00000	0.00000	23.9768	0.0208934	−100.000	25.3779
	+5%	0.00000	0.00000	11.9884	0.0208934	−100.000	12.6861
	−5%	0.00000	0.00000	−11.9884	−0.628566	456.657	−12.5226
	−10%	0.00000	0.00000	−23.9768	−1.98704	875.444	−24.8487
	−20%	0.00000	0.00000	−47.9536	−6.40137	1624.95	−49.0440

the fraction of backordered demand). However, it is very sensitive to changes in the purchasing cost or in the selling price.

With respect to future research, the model can be extended in several ways. For example, (i) to consider a power demand pattern and obtain the optimal policy for this situation; (ii) to analyze the case of perishable or deteriorating items; (iii) to study the case of integer lot size; (iv) to assume a finite rate of replenishment; (v) to develop the inventory system with a price-dependent demand rate; and (vi) to consider stochastic demand in the system.

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Appendix A

In this appendix, we give the proofs of the main results.

Proof of Theorem 1
First, from (6), we obtain $L(0) = \Gamma$. According to the definition of the function $L(\Psi)$, two cases are feasible: (i) $\beta_1 > 0$ and (ii) $\beta_1 = 0$.

(i) Suppose $\beta_1 > 0$. Since the first derivative of $L(\Psi)$ is $L'(\Psi) = 2\lambda\beta_1[\beta_0 + (\beta_1 + (h_0 + ic)\rho^2)\Psi] > 0$, the function $L(\Psi)$ is strictly increasing with $\lim_{\Psi\to\infty} L(\Psi) = \infty$.
Thus, if $\Gamma = L(0) \geq 0$ (cases A and B), then $L(\Psi) > 0$ for all $\Psi > 0$ and, therefore, the function $Z(\Psi)$ is strictly increasing on its domain. Consequently, it attains its minimum at the point $\Psi^* = 0$.
If $\Gamma < 0$ (case D), as $L(\Psi)$ is strictly increasing with $\lim_{\Psi\to\infty} L(\Psi) = \infty$, then $L(\Psi)$ has a unique root on the interval $(0, \infty)$. A trivial verification shows that this root is Ψ_1. Moreover, since $L(\Psi) < 0$ for $\Psi \in (0, \Psi_1)$ and $L(\Psi) > 0$ for $\Psi > \Psi_1$, it follows that $Z(\Psi)$ attains its minimum at Ψ_1.

(ii) Now, we consider that $\beta_1 = 0$. From (6), we obtain $L(\Psi) = \Gamma$ for all $\Psi \geq 0$.
Hence, if $\Gamma > 0$ (case A), then the function $Z(\Psi)$ is strictly increasing on $(0, \infty)$ and, therefore, attains its minimum at $\Psi^* = 0$.
If $\Gamma = 0$ (case C), then $Z(\Psi) = \sqrt{\beta_0^2/\rho^2 + 2(h_0 + ic)\beta_0\Psi + (h_0 + ic)^2\rho^2\Psi^2} - (h_0 + ic)\rho\Psi = \beta_0/\rho = \sqrt{2K(h_0 + ic)/\lambda}$ for all $\Psi \geq 0$. Therefore, the minimum is attained at any point in the interval $[0, \infty)$.

Finally, if $\Gamma < 0$ (case E), then the function $Z(\Psi)$ is strictly decreasing with $\lim_{\Psi\to\infty} Z(\Psi) = \beta_0/\rho$. ■

Proof of Theorem 2

The derivative of the function $Z_0(\Psi)$ is $Z_0'(\Psi) = \dfrac{(h_0+ic)(\eta_0+\eta\Psi)}{\sqrt{\frac{2K(h_0+ic)}{\lambda} + 2\eta_0(h_0+ic)\Psi + (h_0+ic)\eta\Psi^2}}$. Therefore

1. If $\eta_0 + \eta > 0$, then $Z_0'(\Psi) > 0$ for all $\Psi \in [0,\infty)$ and $Z_0(\Psi)$ is a strictly increasing function on the interval $[0,\infty)$. Hence, it attains its minimum at the point $\Psi^* = 0$ and, from (4), the length of the inventory cycle with positive net stock is $T^* = T_0^* = \sqrt{2K/(\lambda(h_0+ic))} = T_0$.
2. If $\eta_0 = 0$ and $\eta = 0$, then $Z_0(\Psi) = \sqrt{2K(h_0+ic)/\lambda}$ for all $\Psi \geq 0$. As $Z_0(\Psi)$ is a constant function on $[0,\infty)$, it reaches its minimum at any point $\Psi \geq 0$. Finally, from (4), $T^* = \sqrt{2K/(\lambda(h_0+ic))} = T_0$. ■

Proof of Corollary 1

If $\Gamma > 0$, we can distinguish two scenarios: (a) $\rho > 0$ and (b) $\rho = 0$. In the first one, we are in case (A) of Theorem 1 and, in the second, as necessarily $\eta_0 > 0$, we are in case (1) of Theorem 2. Thus, in both scenarios, we have that the optimal inventory policy is $\Psi^* = 0$.

If $\Gamma = 0$ and $\beta_1 > 0$, we fall into case (B) of Theorem 1 when $\rho > 0$, and in case (1) of Theorem 2 when $\rho = 0$. Therefore, in this situation, we also obtain $\Psi^* = 0$.

The rest of the proof follows from (4), (5), the relation $Q = \lambda(T + \rho\Psi)$, and (2). ■

Proof of Corollary 2

It follows immediately, taking into account that we are now in case (C) of Theorem 1 when $\rho > 0$, and in case (2) of Theorem 2 if $\rho = 0$. ■

Proof of Corollary 3

The optimal stock-in period follows from (4). The optimal inventory cycle α_1 is calculated as $\alpha_1 = T_1 + \Psi_1$. The economic lot size follows from $Q = \lambda(T + \rho\Psi)$, the minimum average inventory cost per unit of item follows from (5) and the maximum return on inventory investment follows immediately from (2). ■

Lemma 1. *If* $\Gamma < 0$ *and* $\beta_1 > 0$*, then* $ROI_1 > ROI_0$.

Proof. It is obvious that $(\rho\sqrt{2Kh/\lambda} - \beta_0)^2 > 0$. This last inequality is equivalent to

$$\beta_0^2 - 2\rho\beta_0\sqrt{\frac{2Kh}{\lambda}} > -\frac{2Kh}{\lambda}\rho^2 \Leftrightarrow \frac{h}{\beta_1 + h\rho^2}\left(\beta_0^2 - 2\rho\beta_0\sqrt{\frac{2Kh}{\lambda}}\right) > -\frac{2Kh^2\rho^2}{\lambda(\beta_1 + h\rho^2)}$$

$$\Leftrightarrow \frac{2Kh}{\lambda} + \frac{h}{\beta_1 + h\rho^2}\left(\beta_0^2 - 2\rho\beta_0\sqrt{\frac{2Kh}{\lambda}}\right) - \frac{\beta_0^2\beta_1 h}{(\beta_1 + h\rho^2)^2} > \frac{2Kh}{\lambda} - \frac{2Kh^2\rho^2}{\lambda(\beta_1 + h\rho^2)} - \frac{\beta_0^2\beta_1 h}{(\beta_1 + h\rho^2)^2}.$$

Thus, we have

$$\left(\sqrt{\frac{2Kh}{\lambda}}-\frac{\beta_0 h\rho}{\beta_1+h\rho^2}\right)^2>\left(\frac{2Kh}{\lambda}-\frac{\beta_0^2 h}{\beta_1+h\rho^2}\right)\frac{\beta_1}{\beta_1+h\rho^2}. \tag{A1}$$

Taking into account that $\Gamma<0$, it follows that $\rho>0$ and $\beta_0<\sqrt{2Kh/\lambda}\rho$, hence $\beta_0/\rho<\sqrt{2Kh/\lambda}$. Since $\beta_1>0$, we have $h\rho^2/(\beta_1+h\rho^2)<1$. Thus, we see that $\beta_0 h\rho/(\beta_1+h\rho^2)<\beta_0/\rho<\sqrt{2Kh/\lambda}$. Hence, Equation (A1) is equivalent to

$$\sqrt{\frac{2Kh}{\lambda}}-\frac{\beta_0 h\rho}{\beta_1+h\rho^2}>\sqrt{\frac{2Kh}{\lambda}-\frac{\beta_0^2 h}{\beta_1+h\rho^2}}\sqrt{\frac{\beta_1}{\beta_1+h\rho^2}}$$

and, taking into account that $h=h_0+ic$, we have $AI_0>AI_1$. Now, from (2), the lemma follows. ■

Appendix B

In this appendix, we develop a mathematical model to represent the inventory system analyzed in this paper, but now under the NPV approach. As we will see later, the objective functions proposed in the previous sections are very different from the objective function corresponding to the NPV criterion.

A NPV model

From the formulation of the NPV given by Dural-Selcuk and Cimen (2013) and the hypotheses given in Section 2, a *NPV* model for an inventory system with constant demand, under linear holding cost h_0, partial backlogging, and affine shortage costs can be developed as follows.

The revenue and the relevant inventory costs of the system for each cycle are obtained below.

- Revenue: $s(\int_0^T \lambda e^{-rt}dt+\lambda\rho\Psi e^{-r(T+\Psi)})=s\lambda(\frac{1-e^{-rT}}{r}+\rho\Psi e^{-r(T+\Psi)})$, where r denotes the discount rate.
- Ordering cost: K
- Purchasing cost: cQ
- Holding cost: $h_0\int_0^T \lambda(T-t)e^{-rt}dt=h_0\frac{\lambda}{r^2}(e^{-rT}+rT-1)$
- Backordering cost: $\omega_0\lambda\rho\Psi e^{-r(T+\Psi)}+\omega e^{-r(T+\Psi)}\int_T^{T+\Psi}\lambda\rho(t-T)e^{r(T+\Psi-t)}dt=\omega_0\lambda\rho\Psi e^{-r(T+\Psi)}+\frac{\omega\lambda\rho e^{-rT}}{r^2}(1-(1+r\Psi)e^{-r\Psi})$
- Lost sale cost: $\eta_0\lambda(1-\rho)\Psi e^{-r(T+\Psi)}+\frac{\eta\lambda(1-\rho)e^{-rT}}{r^2}(1-(1+r\Psi)e^{-r\Psi})$.

Thus, the NPV of the corresponding total profit of the inventory system is

$$s\lambda\left(\frac{1-\mathrm{e}^{-rT}}{r}+\rho\Psi\mathrm{e}^{-r(T+\Psi)}\right)-K-c\lambda(T+\rho\Psi)-h_0\frac{\lambda}{r^2}\left(\mathrm{e}^{-rT}+rT-1\right)$$
$$-\beta_0\lambda\Psi\mathrm{e}^{-r(T+\Psi)}-\frac{\beta_1\lambda\mathrm{e}^{-rT}}{r^2}\left(1-(1+r\Psi)\mathrm{e}^{-r\Psi}\right),$$

where, let us remember that, $\beta_0=\omega_0\rho+\eta_0(1-\rho)$ and $\beta_1=\omega\rho+\eta(1-\rho)$.

If we now consider a planning horizon of length H, the NPV for this planning horizon is (see Gurnani, 1983):

$$\left(\frac{1-\mathrm{e}^{-rH}}{1-\mathrm{e}^{-r(T+\Psi)}}\right)\left\{s\lambda\left(\frac{1-\mathrm{e}^{-rT}}{r}+\rho\Psi\mathrm{e}^{-r(T+\Psi)}\right)-K-c\lambda(T+\rho\Psi)\right.$$
$$\left.-h_0\frac{\lambda}{r^2}\left(\mathrm{e}^{-rT}+rT-1\right)-\beta_0\lambda\Psi\mathrm{e}^{-r(T+\Psi)}-\frac{\beta_1\lambda\mathrm{e}^{-rT}}{r^2}\left(1-(1+r\Psi)\mathrm{e}^{-r\Psi}\right)\right\}.$$

Therefore, the NPV for an infinite planning horizon is given by

$$\left(\frac{1}{1-\mathrm{e}^{-r(T+\Psi)}}\right)\left\{s\lambda\left(\frac{1-\mathrm{e}^{-rT}}{r}+\rho\Psi\mathrm{e}^{-r(T+\Psi)}\right)-K-c\lambda(T+\rho\Psi)\right.$$
$$\left.-h_0\frac{\lambda}{r^2}\left(\mathrm{e}^{-rT}+rT-1\right)-\beta_0\lambda\Psi\mathrm{e}^{-r(T+\Psi)}-\frac{\beta_1\lambda\mathrm{e}^{-rT}}{r^2}\left(1-(1+r\Psi)\mathrm{e}^{-r\Psi}\right)\right\}.$$

In order to compare the NPV approach to the profit per unit time and the return on inventory investment, we use the concept of annuity stream (see Van der Laan and Teunter, 2002). Thus, the total annuity stream for our inventory system is given by

$$AS(T,\Psi)=\left\{\frac{r}{1-\mathrm{e}^{-r(T+\Psi)}}\right\}\left\{s\lambda\left(\frac{1-\mathrm{e}^{-rT}}{r}+\rho\Psi\mathrm{e}^{-r(T+\Psi)}\right)-K-c\lambda(T+\rho\Psi)\right.$$
$$-h_0\frac{\lambda}{r^2}\left(\mathrm{e}^{-rT}+rT-1\right)-\beta_0\lambda\Psi\mathrm{e}^{-r(T+\Psi)}$$
$$\left.-\frac{\beta_1\lambda\mathrm{e}^{-rT}}{r^2}\left(1-(1+r\Psi)\mathrm{e}^{-r\Psi}\right)\right\}.$$

Therefore, the inventory problem would be to determine the policy (T,Ψ) such that maximizes the annuity stream $AS(T,\Psi)$.

Numerical example

Consider the same parameters as in Example 1 of Section 5 , but modifying the value of i to $i=0.15$. As usual in the literature, we suppose that the discount rate r is equal to the inventory opportunity cost rate i. Thus, we have $\lambda=1000$, $K=500$, $c=10$, $s=20$, $h_0=1.5$, $i=0.15$, $\omega_0=0.1$, $\omega=5$, $\eta_0=0$, $\eta=0$, and $r=0.15$.

Table B.1
Comparison of optimal policies under NPV, ROI and maximum profit criteria

ρ	T^*	Ψ^*	$B(T^*, \Psi^*)$	$ROI^*(\%)$	$T^\#$	$\Psi^\#$	$B(T^\#, \Psi^\#)$	$ROI^\#(\%)$	T^N	Ψ^N	$B(T^N, \Psi^N)$	$ROI^N(\%)$
0.00	0.577350	$\Psi \geq 0$	$B_0(\Psi)$[a]	70.4732	0.577350	0.000000	8267.95	70.4732	0.569136	0.000000	8267.77	70.4706
0.10	0.562606	0.317564	5613.06	71.1183	0.577350	0.000000	8267.95	70.4732	0.569136	0.000000	8267.77	70.4706
0.30	0.536444	0.301866	6275.70	72.2752	0.577350	0.000000	8267.95	70.4732	0.569136	0.000000	8267.77	70.4706
0.70	0.494122	0.276473	7600.85	74.1801	0.577350	0.000000	8267.95	70.4732	0.569136	0.000000	8267.77	70.4706
0.80	0.485133	0.271080	7932.00	74.5902	0.577350	0.000000	8267.95	70.4732	0.569136	0.000000	8267.77	70.4706
0.85	0.480835	0.268501	8097.55	74.7869	0.575962	0.033620	8272.11	71.5431	0.569136	0.000000	8267.77	70.4706
0.90	0.476658	0.265995	8263.07	74.9785	0.555598	0.128177	8333.21	73.7982	0.569136	0.000000	8267.77	70.4706
0.92	0.475021	0.265013	8329.28	75.0537	0.542377	0.159811	8372.87	74.3397	0.565005	0.008611	8279.86	70.7915
0.94	0.473402	0.264041	8395.48	75.1282	0.526864	0.188636	8419.41	74.7447	0.553078	0.029763	8315.87	71.5476
0.96	0.471800	0.263080	8461.67	75.2019	0.509301	0.214980	8472.10	75.0377	0.539315	0.049797	8358.64	72.2208
0.98	0.470217	0.262130	8527.86	75.2749	0.489858	0.239096	8530.43	75.2350	0.523843	0.068826	8407.65	72.8194
1.00	0.468650	0.261190	8594.05	75.3471	0.468650	0.261190	8594.05	75.3471	0.506748	0.086946	8462.53	73.3483

[a] $B_0(\Psi) = 1000(\frac{10-\sqrt{3}}{1+\sqrt{3}\Psi})$.

Table B.1 of this appendix shows, for different values of the fraction of the backordered demand, the following inventory strategies: the optimal policy (T^*, Ψ^*) that maximizes the return on inventory investment, the policy $(T^\#, \Psi^\#)$ that maximizes the average profit per unit time, and the policy (T^N, Ψ^N) that maximizes the annuity stream. As expected, the policy (T^N, Ψ^N) is different from the one that maximizes the return on inventory investment.

From this numerical example, we can make the following comments: (i) the policy (T^N, Ψ^N) provides lower ROI than the inventory policy $(T^\#, \Psi^\#)$ that maximizes the profit per unit time; (ii) the policy that maximizes the annuity stream is not very sensitive to fluctuations in the fraction of backordered demand ρ; (iii) in the strictly partial backlogging case $(0 < \rho < 1)$, the policy (T^N, Ψ^N) is closer to the policy $(T^\#, \Psi^\#)$ that maximizes the profit per unit time, than to the policy (T^*, Ψ^*) that maximizes the return on inventory investment; (iv) the optimal stock-out period with the NPV approach never is greater than those obtained with the other two criteria; and (v) in the backlogging case $(\rho > 0)$, the policy that maximizes the annuity stream provides greater profit per unit time than the one that maximizes the return on inventory investment, except for high values of the parameter ρ.