Ergodic and dynamical properties of m-isometries

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Abstract

An example of a weakly ergodic 3-isometry is provided in [3], we give new examples of weakly ergodic 3-isometries and study numerically hypercyclic m-isometries on finite and infinite dimensional Hilbert spaces. In particular, all weakly ergodic strict 3-isometries on a Hilbert space are weakly numerically hypercyclic. Adjoints of unilateral forward weighted shifts which are strict m-isometries on $\ell^2(\mathbb{N})$ are shown to be hypercyclic.

1 Introduction

Throughout this article X stands for a Banach space, the symbol $B(X)$ denotes the space of bounded linear operators defined on X.

Given $T \in B(X)$, we denote the *Cesàro mean* by

$$M_n(T)x := \frac{1}{n+1}\sum_{k=0}^{n} T^k x \quad \text{for all } x \in X.$$

We need to recall some definitions concerning the behaviour of the sequence of Cesàro means $(M_n(T))_{n\in\mathbb{N}}$.

Definition 1.1. A linear operator T on a Banach space X is called

1. *Uniformly ergodic* if $M_n(T)$ converges uniformly.
2. *Mean ergodic* if $M_n(T)$ converges in the strong operator topology of X.
3. *Weakly ergodic* if $M_n(T)$ converges in the weak operator topology of X.
4. *Absolutely Cesàro bounded* if there exists a constant $C > 0$ such that

$$\sup_{N\in\mathbb{N}} \frac{1}{N}\sum_{j=1}^{N} \|T^j x\| \le C\|x\| \quad \text{for all } x \in X.$$

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5. *Cesàro bounded* if the sequence $(M_n(T))_{n\in\mathbb{N}}$ is bounded.

6. *Uniformly Kreiss bounded* if there is a $C > 0$ such that

$$\|M_n(\lambda T)\| \leq C \quad \text{for } |\lambda| = 1 \text{ and } n = 0, 1, 2, \cdots . \tag{1}$$

An operator T is said *power bounded* if there is a $C > 0$ such that $\|T^n\| < C$ for all n.

Remark 1.1. 1. On finite-dimensional Hilbert spaces, the classes of uniformly Kreiss bounded and power bounded operators are equal, [21, page 762].

2. Any absolutely Cesàro bounded operator is uniformly Kreiss bounded.

The class of absolutely Cesàro bounded operators was introduced by Hou and Luo in [18].

The following implications for operators on Hilbert spaces among various concepts in ergodic theory are a direct consequence of the corresponding definitions and results in [6]:

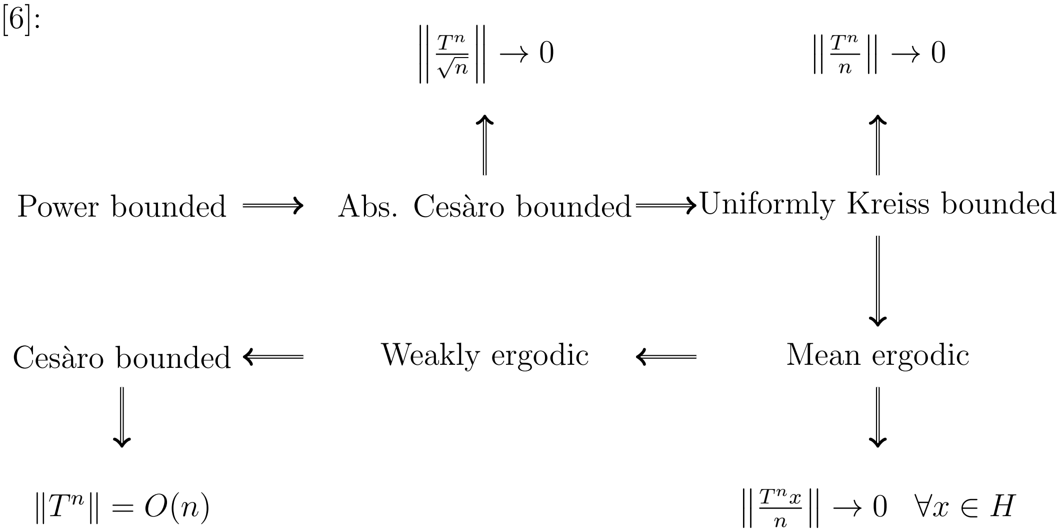

Figure 1: Relations between different definitions in ergodic theory in Hilbert spaces.

In general, the converse implications of the above figure are not true. See [6] and references therein.

Let H be a Hilbert space. For a positive integer m, an operator $T \in B(H)$ is called an *m-isometry* if for any $x \in H$,

$$\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \|T^k x\|^2 = 0 .$$

We say that T is a *strict m-isometry* if T is an m-isometry but it is not an $(m-1)$-isometry.

Remark 1.2. 1. For $m \geq 2$, the strict m-isometries are not power bounded. Moreover, $\|T^n\| = O(n)$ for strict 3-isometries and $\|T^n\| = O(n^{\frac{1}{2}})$ for strict 2-isometries, [8, Theorem 2.1].

2. There are no strict m-isometries on finite dimensional spaces for m even. See [2, Proposition 1.23].

3. An example of a weakly ergodic 3-isometry is provided in [3].

We recall the following definition that allow us to study some properties of orbits of the m-isometries or adjoint of m-isometries.

A point $x \in X$ is called a *periodic point* of $T \in B(X)$ if there is some $n \geq 1$ such that $T^n x = x$.

An operator $T \in B(X)$ is said to be *hypercyclic* if there exists a point $x \in X$ such that for every nonempty open subset U of X, the set $\{n \in \mathbb{N} : T^n x \in U\}$ is nonempty, T is *mixing* if for every nonempty open sets $U, V \subset X$, there exists $n_0 \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$ for all $n \geq n_0$ and T is *Devaney chaotic* if it is hypercyclic and has a dense set of periodic points.

Examples of absolutely Cesàro bounded mixing operators on $\ell^p(\mathbb{N})$ are given in [6] (see also [18], [10], [11]).

Definition 1.2. Let H be a Hilbert space. $T \in B(H)$ is called *numerically hypercyclic* if there exists a unit vector $x \in H$ such that the set $\{\langle T^n x, x\rangle : n \in \mathbb{N}\}$ is dense in $\mathbb{C}$.

Clearly numerical hypercyclicity is preserved by unitary equivalence but in general not by similarity, [23, Theorem 1.13 & Remark 1.17]. This leads to the following definition:

Definition 1.3. Let $T \in B(X)$. It is said that T is *weakly numerically hypercyclic* if T is similar to a numerically hypercyclic operator.

In [23, Proposition 1.5], Shkarin proved that $T \in B(H)$ is weakly numerically hypercyclic if and only if there exist $x, y \in H$ such that the set $\{\langle T^n x, y\rangle : n \in \mathbb{N}\}$ is dense in $\mathbb{C}$.

The paper is organized as follows: Section 2 studies ergodic properties of m-isometries on finite or infinite dimensional Hilbert spaces. For example, strict m-isometries with $m > 3$ are not Cesàro bounded, and we give new examples of weakly ergodic 3-isometries. In Section 3, we analyze numerical hypercyclicity of m-isometries. In particular, we obtain that the adjoint of any strict m-isometry unilateral forward weighted shift on $\ell^2(\mathbb{N})$ is hypercyclic. Moreover, we prove that weakly ergodic 3-isometries are weakly numerically hypercyclic.

2 Ergodic properties for m-isometries in Hilbert spaces

The purpose of this section is to study ergodic properties of m-isometries. It is clear that isometries (1-isometries) are power bounded. What can we say about strict m-isometries and the definitions of Figure 1 on finite or infinite Hilbert spaces?

The following example is due to Assani. See [15, page 10] and [3, Theorem 5.4] for more details.

Example 2.1. Let H be $\mathbb{R}^2$ or $\mathbb{C}^2$ and $T = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}$. It is clear that

$$T^n = \begin{pmatrix} (-1)^n & (-1)^{n-1}2n \\ 0 & (-1)^n \end{pmatrix}$$

and $\sup_{n\in\mathbb{N}} \|M_n(T)\| < \infty$. Then T is Cesàro bounded and $\frac{\|T^n x\|}{n}$ does not converge to 0 for some $x \in H$. Hence T is not mean ergodic. Note that T is a strict 3-isometry.

The above example shows that on a 2-dimensional Hilbert space there exists a 3-isometry which is Cesàro bounded, not mean ergodic, and so not weakly ergodic. This example could be generalized to any Hilbert space of dimension greater or equal to 2.

Let H be a Hilbert space and $T \in B(H)$. Tomilov and Zemánek in [24] considered the Hilbert space $\mathcal{H} = H \oplus H$ with the norm

$$\|x_1 \oplus x_2\|_{H\oplus H} = \sqrt{\|x_1\|^2 + \|x_2\|^2}\ ,$$

and the bounded linear operator $\widehat{T}$ on $\mathcal{H}$ given by the matrix

$$\widehat{T} := \begin{pmatrix} T & T-I \\ 0 & T \end{pmatrix} .$$

Lemma 2.1. *[24, Proof of Lemma 2.1] Let $T \in B(H)$ and n a positive integer number. Then*

$$\widehat{T}^n = \begin{pmatrix} T^n & nT^{n-1}(T-I) \\ 0 & T^n \end{pmatrix} .$$

In [24], the authors obtained the following relations of ergodic properties between the operators $\widehat{T}$ and T.

Lemma 2.2. *[24, Lemmma 2.1] Let $T \in B(H)$. Then*

1. *$\widehat{T}$ is Cesàro bounded if and only if T is power bounded.*
2. *$\widehat{T}$ is mean ergodic if and only if T^n converges in the strong topology of H.*
3. *$\widehat{T}$ is weakly ergodic if and only if T^n converges in the weak topology of H.*

Recall some properties of m-isometries.

Lemma 2.3. *Let $T \in B(H)$ and $m \in \mathbb{N}$. Then*

1. *[8, Theorem 2.1] T is a strict m-isometry if and only if $\|T^n x\|^2$ is the value at n of a polynomial of degree less than or equal to $m-1$ for all $x \in H$, and there exists $x_m \in H$ such that $\|T^n x_m\|^2$ is a polynomial of degree exactly $m-1$.*

2. *[9, Theorem 2.7] If H is a finite dimensional Hilbert space, then T is a strict m-isometry with odd m if and only if there exist a unitary $U \in B(H)$ and a nilpotent operator $Q \in B(H)$ of order $\frac{m+1}{2}$ such that $UQ = QU$ with $T = U+Q$.*

3. *[9, Theorem 2.2] If $A \in B(H)$ is an isometry and $Q \in B(H)$ is a nilpotent operator of order n which commutes with A, then $A+Q$ is a strict $(2n-1)$-isometry.*

Example 2.2. Let H be a Hilbert space and $T \in B(H)$ such that $T = I + Q$ where $Q^n = 0$ for some $n \geq 2$ and $Q^{n-1} \neq 0$. Define the Hilbert space $\mathcal{H}$ and the bounded linear operator $\widehat{T}$ on $\mathcal{H}$ as above. By construction $\widehat{T} = A + \mathcal{Q}$ where

$$A := \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \qquad \mathcal{Q} := \begin{pmatrix} Q & Q \\ 0 & Q \end{pmatrix}$$

where $\mathcal{Q}^n = 0$ and $\mathcal{Q}^{n-1} \neq 0$. By part (3) of Lemma 2.3, T is a strict $(2n-1)$-isometry and hence not power bounded. Thus, by Lemma 2.2 we have that $\widehat{T}$ is not Cesàro bounded. Newly by part (3) of Lemma 2.3, $\widehat{T}$ is a strict $(2n-1)$-isometry.

Example 2.3. Let λ be a unimodular complex number different from 1. Then

$$\widehat{I_\lambda} := \begin{pmatrix} \lambda & \lambda - 1 \\ 0 & \lambda \end{pmatrix}$$

is a Cesàro bounded operator on $\mathbb{C}^2$ by Lemma 2.2 (since $\sup_n |\lambda^n| < \infty$), it is not mean ergodic (since $\lambda^n x$ does not converge) and by Lemma 2.3, $\widehat{I_\lambda}$ is a 3-isometry on $\mathbb{C}^2$.

Now we give some ergodic properties of m-isometries.

Example 2.1 is a Cesàro bounded 3-isometry. However, an uniformly Kreiss bounded operator on a Hilbert space satisfies that $\lim_{n\to\infty} n^{-1}\|T^n\| = 0$, [6, Theorem 2.2] and by Lemma 2.3, we obtain the following.

Corollary 2.1. *There is no uniformly Kreiss bounded strict 3-isometry.*

Theorem 2.1. *Assume that H is a finite n-dimensional Hilbert space. Then*

1. *If $n \geq 2$, then there exists a Cesáro bounded strict 3-isometry.*

2. *The isometries are the only mean ergodic strict m-isometries on H.*

Proof. *(1)* Let

$$\widehat{I_\lambda} := \begin{pmatrix} \lambda & \lambda - 1 \\ 0 & \lambda \end{pmatrix}$$

be the operator on $\mathbb{C}^2$ considered in Example 2.3. Write $H = \mathbb{C}^2 \oplus \mathbb{C}^{n-2}$ and let $\mathcal{B} := \widehat{I_\lambda} \oplus I_{\mathbb{C}^{n-2}}$. Then $\mathcal{B}$ is a strict 3-isometry which is Cesàro bounded (and not power bounded).

(2) Suppose that T is a strict m-isometry with $m > 1$ on a finite dimensional Hilbert space, then $m \geq 3$ by part (2) of Remark 1.2. Using part (1) of Lemma 2.3, it is easy to prove that $\frac{\|T^n x\|}{n}$ does not converges to 0 for some $x \in H$. So, T is not mean ergodic. □

In Hilbert space of infinite dimensional we can say more.

Theorem 2.2. *Let H be an infinite-dimensional Hilbert space and $T \in B(H)$ be a strict m-isometry. Then*

1. *If $m > 3$, then T is not Cesàro bounded.*
2. *If $m \geq 3$, then T is not mean ergodic.*

Proof. By part (1) of Lemma 2.3, there exists $x \in H$ such that $\|T^n x\|^2$ is a polynomial at n of order $m - 1$ exactly. Since

$$\frac{T^n}{n+1} = M_n(T) - \frac{n}{n+1} M_{n-1}(T) \ , \tag{2}$$

the proof is complete.

□

Since any weakly ergodic operator is Cesàro bounded, in particular there is no weakly ergodic strict m-isometry for $m > 3$.

Theorem 2.3. *There exists a Cesàro bounded and weakly ergodic strict 3-isometry.*

Proof. Let U be the bilateral shift. Define

$$\widehat{U} := \begin{pmatrix} U & U - I \\ 0 & U \end{pmatrix} .$$

First observe that $\widehat{U}$ is Cesàro bounded, by part (1) of Lemma 2.2. Since $U^n \to 0$ in the weak operator topology, $\widehat{U}$ is weakly ergodic by part (3) of Lemma 2.2. Therefore, $\widehat{U} = \mathcal{A} + \mathcal{Q}$, where

$$\mathcal{A} := \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} \text{ and } \mathcal{Q} := \begin{pmatrix} 0 & U - I \\ 0 & 0 \end{pmatrix} .$$

The conclusion is derived by part (3) of Lemma 2.3. □

In [3, Section 5.2], an example of a Cesàro bounded strict 3-isometry T on a Hilbert space H for which the sequence $\left(\frac{T^n}{n}\right)_{n\in\mathbb{N}}$ is bounded below for all $x \in H\setminus\{0\}$ is given. In particular, $(M_n(T)x)_{n\in\mathbb{N}}$ diverges for each $x \in H \setminus \{0\}$, and T is weakly ergodic.

We give a characterization of this property.

Given an m-isometry T, the *covariance operator* of T is defined by

$$\Delta_T := \frac{1}{(m-1)!}\sum_{j=0}^{m-1}(-1)^{m-1-j}\binom{m-1}{j}T^{*j}T^j\ .$$

Theorem 2.4. *Let T be a strict 3-isometry on a Hilbert space H. Then the sequence $\left(\frac{T^n x}{n}\right)_{n\in\mathbb{N}}$ is bounded below for all $x \in H\setminus\{0\}$ if and only if the covariance operator Δ_T is injective.*

Proof. If T is a strict 3-isometry and Δ_T is injective, then $\inf_n \frac{\|T^n x\|}{n} > 0$ for all $x \in H \setminus \{0\}$ (see the proof of [7, Theorem 3.4]).

If Δ_T is not injective, then there exists x such that $\langle \Delta_T x, x\rangle = 0$. By [7, Proposition 2.3], we have that $\inf_n \frac{\|T^n x\|}{n} \to \langle \Delta_T x, x\rangle = 0$, and thus the sequence $\frac{T^n x}{n}$ is not bounded below. □

There exist weakly ergodic strict 3-isometries with the covariance operator Δ_T injective by [3, Section 5.2] and not injective, see the proof of Theorem 2.3.

The Uniform ergodic theorem of Lin [20, Theorem] asserts that if $\frac{\|T^n\|}{n} \to 0$, then T is uniformly ergodic if and only if the range of $I - T$ is closed. On the other hand, T is uniformly ergodic if and only if $\frac{\|T^n\|}{n} \to 0$ and 1 is a pole of the resolvent operator, [14, Theorem 3.16].

Corollary 2.2. *For $m > 1$, there is no uniform ergodic strict m-isometry on a Hilbert space.*

Proof. By part (2) of Theorem 2.2, there is no mean ergodic strict m-isometry for $m \geq 3$. For $m = 2$ the result follows from the fact that the spectrum of any strict 2-isometry is $\sigma(T) = \overline{\mathbb{D}}$ and, thus, 1 is not an isolated point of $\sigma(T)$. □

There exists a strict 3-isometry T which is weakly ergodic (thus Cesàro bounded), but it is not mean ergodic. For 2-isometries something else can be established.

Proposition 2.1. *Let H be an infinite dimensional Hilbert space and let T be a strict 2-isometry. Then the following assertions are equivalent:*

1. *T is mean ergodic.*
2. *T is weakly ergodic.*
3. *T is Cesàro bounded.*

Proof. It is a consequence of part (1) of Lemma 2.3, since $\frac{T^n x}{n}$ converges to zero for all $x \in H$. $\square$

The following example provides a 2-isometry that is not Cesàro bounded.

Example 2.4. On $\ell^2(\mathbb{N})$ we consider the operator T given by $T(x_1, x_2, \dots) := (x_1, x_1, x_2, x_3, \dots)$. Then, it is clear that

$$\|T^2 x\|^2 - 2\|Tx\|^2 + \|x\|^2 = 0 \ ,$$

for any $x \in \ell^2(\mathbb{N})$. Notice that $T^k e_1 = e_1 + e_2 + \cdots + e_k$, so

$$\|M_n(T) e_1\| = \frac{1}{n+1} \sqrt{\frac{(n+1)(n+2)(2n+3)}{6}}$$

is not bounded. Then T is a 2-isometry which is not Cesàro bounded.

Proposition 2.2. *There is no Cesàro bounded weighted forward shift on $\ell^2(\mathbb{N})$, which is a strict 2-isometry.*

Proof. Assume that T is a weighted forward shift with weights $(w_n)_{n\in\mathbb{N}}$. By [1, Theorem 1] (see also [8, Remark 3.9]), if T is a strict 2-isometry, then

$$|w_n|^2 = \frac{p(n+1)}{p(n)} \ ,$$

where p is a polynomial of degree 1, that is, $p(n) := an + b$.

First, suppose that $b = 0$. Then $w_n = \sqrt{\frac{n}{n-1}}$, since $a \neq 0$. Hence $T^* e_n := \sqrt{\frac{n}{n-1}} e_{n-1}$. By [6, Proposition 2.1], T^* is not Cesàro bounded. Since Cesàro boundedness is preserved by taking adjoints, T is not Cesàro bounded.

Now, assume that $b \neq 0$, then $w_n(c) := \sqrt{\frac{cn+1}{c(n-1)+1}}$ with $c \neq 0$. Denote $T_c e_n := w_n(c) e_{n+1}$ and the diagonal operator $V e_n := \alpha_n e_n$, where $\alpha_n := \sqrt{\frac{c(n-1)+1}{n}}$. Then V is invertible and satisfies that $VT_1 = VT_c$. Moreover, T_1 is not Cesàro bounded, by following an argument as in [6, Proposition 2.1]. Using that Cesàro boundedness is preserved by similarities, we obtain that T_c is not Cesàro bounded. $\square$

Corollary 2.3. *There is no absolutely Cesàro bounded strict 2-isometry on a Hilbert space.*

Proof. It is immediate by [6, Theorem 2.5] and part (1) of Lemma 2.3. $\square$

Question 2.1. Is it possible to construct a Cesàro bounded strict 2-isometry on an infinite dimensional Hilbert space?

3 Numerically hypercyclic properties of m-isometries

In this section we study numerically hypercyclic m-isometries. For simplicity we discuss only operators on Hilbert spaces.

Recall that an operator $T \in B(X)$ is called weakly hypercyclic (weakly supercyclic) if there is a vector on X with a weakly dense orbit (weakly projective dense orbit).

Faghih and Hedayatian proved in [16] that m-isometries on a Hilbert space are not weakly hypercyclic. Moreover, m-isometries on a Banach space are not 1-weakly hypercyclic [5]. However, there are isometries that are weakly supercyclic [22] (in particular cyclic). Thus the first natural question is the following: are there numerically hypercyclic m-isometries?

Let H be a Hilbert space. Denote

$$I_m(H) := \{T \in B(H) \quad : \quad T \text{ is } m\text{-isometry}\} .$$

If H is an n dimensional Hilbert space, then by [9, Theorem 2.7] we have that

$$I_1(H) = I_2(H) \subsetneq I_3(H) = I_4(H) \subsetneq \cdots \subsetneq I_{2n-3}(H) = I_{2n-2}(H) \subsetneq I_{2n-1}(H) = I_m(H)$$

for all $m \geq 2n-1$.

Theorem 3.1. *There are no weakly numerically hypercyclic m-isometries in $B(\mathbb{C}^n)$ for $n \leq 3$.*

Proof. If $n = 1$, there are not weakly numerically hypercyclic operators. Let $n = 2$. By [23, Theorem 1.13], if $T \in B(\mathbb{C}^2)$ is a weakly numerically hypercyclic operator, then there exists $\lambda \in \sigma(T)$, with $|\lambda| > 1$ and thus T is not an m-isometry. For $n = 3$, it is the same by [23, Theorem 1.14]. □

We discuss the existence of weakly numerically hypercyclic m-isometries on n-dimensional spaces for $n \geq 4$.

We say that $\lambda_1, \lambda_2 \in \mathbb{T}$ are *rationally independent* if $\lambda_1^{m_1}\lambda_2^{m_2} \neq 1$ for every non-zero pair $m = (m_1, m_2) \in \mathbb{Z}^2$, or equivalently if $\lambda_j = e^{i\theta_j}$ with $\theta_j \in \mathbb{R}$ with π, θ_1, θ_2 are linearly independent over the field $\mathbb{Q}$ of rational numbers.

If $T \in B(X)$ and there are rationally independent $\lambda_1, \lambda_2 \in \mathbb{T}$ such that $ker(T - \lambda_j I)^2 \neq ker(T - \lambda_j I)$ for $j \in \{1, 2\}$, then T is weakly numerically hypercyclic [23, Theorem 1.9]. Moreover if X is a Hilbert space, then T is numerically hypercyclic [23, Proposition 1.12]. The following result gives an answer to the above question for some m-isometries.

Theorem 3.2. *There exists a numerically hypercyclic strict $(2m-1)$-isometry on $B(\mathbb{C}^n)$, with $n \geq 4$, for $2 \leq m \leq n-2$.*

Proof. Let $\ell \in \{2, 3, \ldots, n-2\}$. We will construct a numerically hypercyclic strict $(2\ell - 1)$-isometry. Define D the diagonal operator with diagonal

$$(\underbrace{\lambda_1, \cdots, \lambda_1}_{\ell}, \lambda_2, \lambda_2, \underbrace{1, \cdots, 1}_{n-\ell-2})$$

where λ_1 and λ_2 are rationally independent complex numbers with modulus 1 and Q by

$$\begin{aligned} Qe_i: &= e_{i-1} \text{ for } i \in \{2,3,\cdots,\ell\} \cup \{\ell+2\} \text{ and} \\ Qe_i: &= 0 \text{ for } i=1, i=\ell+1 \text{ and } i \geq \ell+3\,. \end{aligned}$$

It is clear that $Q^\ell = 0$ and $Q^{\ell-1}e_\ell = e_1 \neq 0$. Moreover,

$$\begin{aligned} QDe_i &= DQe_i = \lambda_1 e_{i-1} \text{ for } 2 \leq i \leq \ell \\ QDe_{\ell+2} &= DQe_{\ell+2} = \lambda_2 e_{\ell+1} \\ QDe_i &= DQe_i = 0 \text{ for } i = 1, \ell+1 \text{ and } \geq i \geq \ell+3\,. \end{aligned}$$

By part (3) of Lemma 2.3, $T := D + Q$ is a strict $(2\ell-1)$-isometry for any $\ell \in \{2,3,\cdots,n-2\}$.

Let us prove that T satisfies that $Ker(\lambda_i - T) \neq Ker(\lambda_i - T)^2$ for $i = 1,2$. By definition $e_2 \in Ker(\lambda_1 - T)^2 \setminus Ker(\lambda_1 - T)$ and $e_{\ell+1} \in Ker(\lambda_2 - T)^2 \setminus Ker(\lambda_2 - T)$. So by [23, Proposition 1.9], T is numerically hypercyclic. □

As a consequence of the proof of Theorem 3.2, we obtain

Corollary 3.1. *Let H be a complex Hilbert space with dimension at least 4. Then there exists a numerically hypercyclic strict 3-isometry on H.*

Proof. If H is an infinite dimensional Hilbert space, then $H = \mathbb{C}^4 \oplus H'$, where H' is an infinite dimensional Hilbert space. Then $S := T \oplus I$, where T is defined in Theorem 3.2 on $\mathbb{C}^4$ is a numerically hypercyclic strict 3-isometry on H.

□

Theorem 3.3. *An n-dimensional Hilbert space supports no weakly numerically hypercyclic strict k-isometry with $k = 2n-3$ or $k = 2n-1$.*

Proof. Let H be a finite-dimensional Hilbert space, $\dim H = n < \infty$. Suppose on the contrary that $T \in B(H)$ is a weakly numerically hypercyclic strict $(2n-1)$-isometry. Since $\|T^k x\|^2$ grows polynomially for each $x \in H$ and there exists $u \in H$ such that $\|T^k u\|^2$ is a polynomial of degree $2n-2$, the Jordan form of T has only one block corresponding to an eigenvalue λ with $|\lambda| = 1$. Since, if T has more than one Jordan block, then $\|T^n x\|^2$ is bounded by a polynomial of degree strictly less than $2n-2$, for each $x \in H$, which is a contradiction. Thus $T = \lambda I + Q$ where $Q^n = 0$. Thus

$$T^k = \sum_{j=0}^{k} \binom{k}{j} \lambda^{k-j} Q^j = \lambda^k \sum_{j=0}^{k} \binom{k}{j} \lambda^{-j} Q^j$$

for all $k \in \mathbb{N}$.

Let $x, y \in H$ and suppose that the set $\{\langle T^k x, y\rangle : k \in \mathbb{N}\}$ is dense in $\mathbb{C}$. We have $\langle T^k x, y\rangle = \lambda^k p(k)$ for some polynomial p of degree $\leq n-1$. If $\deg p \geq 1$ then $|\langle T^k x, y\rangle| \to \infty$ so the set $\{\langle T^k x, y\rangle : k \in \mathbb{N}\}$ is not dense in $\mathbb{C}$.

If $\deg p = 0$ then the set $\{\langle T^k x, y\rangle : k \in \mathbb{N}\}$ is bounded and again is not dense in $\mathbb{C}$. Hence T is not weakly numerically hypercyclic.

The case of $(2n-3)$-isometries can be treated similarly. If $T \in B(H)$ is a strict $(2n-3)$-isometry then the Jordan form of T has two blocks: one of dimension $n-1$ corresponding to an eigenvalue λ, $|\lambda| = 1$ and the second one-dimensional block corresponding to an eigenvalue $\mu, |\mu| = 1$. For $x, y \in H$ we have $\langle T^k x, y\rangle = \lambda^k p(k) + a\mu^k$ for some polynomial $p, \deg p \le n-2$ and a number $a \in \mathbb{C}$. Again one can show easily that the set $\{\langle T^k x, y\rangle : k \in \mathbb{N}\}$ cannot be dense in $\mathbb{C}$. Hence there are no weakly numerically hypercyclic $(2n-3)$-isometries on H. □

Theorem 3.4. *For $m \ge 2$, there exists a numerically hypercyclic strict m-isometry on $\ell^2(\mathbb{N})$.*

Proof. For $m \ge 2$, no strict m-isometry is power bounded [13, Theorem 2]. Also by [4, Proposition 8], there exist forward weighted shifts on $\ell^2(\mathbb{N})$ that are strict m-isometries for $m \ge 2$. Since a forward weighted shift on $\ell^p(\mathbb{N})$, $1 < p < \infty$ is numerically hypercyclic if and only if T is not power bounded ([19] & [23]), we obtain the result. □

Since both numerical hypercyclicity and m-isometry are properties preserved by unitary equivalence, we have that

Corollary 3.2. *Let H be an infinite dimensional separable complex Hilbert space and $m \ge 2$. Then there exists a numerically hypercyclic m-isometry on H.*

Theorem 3.5. *There exists a numerically hypercyclic Cesàro bounded strict 3-isometry on $\mathbb{C}^4$.*

Proof. Let T be the operator considered in the proof of Theorem 3.2

$$T := \begin{pmatrix} \lambda_1 & \lambda_1 - 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & \lambda_2 - 1 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix},$$

where $\lambda_1, \lambda_2 \in \mathbb{T}$ are rationally independent. By the proof of Theorem 3.2, it is clear that T is numerically hypercyclic.

Since both blocks

$$\begin{pmatrix} \lambda_1 & \lambda_1 - 1 \\ 0 & \lambda_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \lambda_2 & \lambda_2 - 1 \\ 0 & \lambda_2 \end{pmatrix}$$

are Cesàro bounded by Lemma 2.2, it is easy to see that T is Cesàro bounded. □

We know that there exist examples of numerically hypercyclic and weakly ergodic 3-isometries. The following result goes further in this direction.

Theorem 3.6. *Any weakly ergodic strict 3-isometry on a Hilbert space is weakly numerically hypercyclic.*

Proof. If T is a weakly ergodic strict 3-isometry, then there exists x such that $\frac{T^n x}{n}$ is weakly convergent but it is not norm convergent. Indeed for a strict 3-isometry T, there exists x such that $\frac{T^n x}{n}$ does not converge to zero in norm.

Then, since $x_n = \frac{T^n x}{n}$ is weakly convergent but it is not norm convergent, by [23, Lemma 6.1] there is $y \in H$ such that $\{n\langle x_n, y\rangle : n \in \mathbb{N}\}$ is dense on $\mathbb{C}$. Hence T is weakly numerically hypercyclic.

□

In particular, the example of a weakly ergodic 3-isometry defined in [3, Section 5.2] is weak numerically hypercyclic.

Question 3.1. Do there exist numerically hypercyclic weakly ergodic 3-isometries?

Let T be an m-isometry. By part (1) of Lemma 2.3, it is clear that T is not hypercyclic. What can we say about dynamical properties of T^*? In general we can not say anything. By [12, Theorem 1], it is obtained that for any m-isometry T on H satisfying that $\cap_{n=0}^{\infty} T^n(H) = \{0\}$ and $T^*T \geq I$, T^* is hypercyclic. Moreover, the following result gives a positive answer for forward weighted shift operators.

Theorem 3.7. *Let S_w be a forward weighted shift strict m-isometry on $\ell^2(\mathbb{N})$. Then*

1. *S_w^* is mixing if and only if $m \geq 2$.*
2. *S_w^* is chaotic if and only if $m \geq 3$.*

Proof. By [1, Theorem 1], a unilateral weighted forward shift on a Hilbert space is an m-isometry if and only if there exists a polynomial p of degree at most $m-1$ such that for any integer $n \geq 1$, we have that $p(n) > 0$ and $|w_n|^2 = \frac{p(n+1)}{p(n)}$. Thus for $m \geq 2$, S_w^* satisfies condition ii) of (b) from [17, Theorem 4.8] and S_w^* is mixing. For $m \geq 3$, S_w^* satisfies condition ii) of (c) from [17, Theorem 4.8] and S_w^* is chaotic.
□

Notice that, if S_w is a unilateral forward weighted shift and a strict m-isometry on $\ell^2(\mathbb{N})$ with $m \geq 2$, then S_w^* is hypercyclic operator.

By [1, Corollary 20], there exist bilateral forward weighted shifts which are strict m-isometries on $\ell^2(\mathbb{Z})$ for odd m, then we have

Theorem 3.8. *Let S_w be a bilateral forward weighted shift strict m-isometry on $\ell^2(\mathbb{Z})$ with $m > 1$. Then S_w^* is chaotic.*

Proof. By [1, Theorem 19 & Corollary 20], a bilateral weighted forward shift on a Hilbert space is a strict m-isometry if and only if there exists a polynomial p of degree at most $m-1$ such that for any integer n, we have $p(n) > 0$ and $|w_n|^2 = \frac{p(n+1)}{p(n)}$ and m is an odd integer. Hence, for $m \geq 3$, S_w^* satisfies condition ii) of (c) from [17, Theorem 4.13]. Thus S_w^* is chaotic.

□

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