ON (m,∞)-ISOMETRIES. EXAMPLES.

TERESA BERMÚDEZ AND HAJER ZAWAY

Abstract. An operator T on a Banach space X is said to be an (m,∞)-isometry, if

$$\max_{\substack{0\le k\le m\\ k \text{ even}}} \|T^k x\| = \max_{\substack{0\le k\le m\\ k \text{ odd}}} \|T^k x\| \ ,$$

for all $x \in X$. In this paper, we study unilateral weighted shift operators which are (m,∞)-isometries for some integers m. In particular, we show that any power of an (m,∞)-isometry is not necessarily an (m,∞)-isometry. We also study strict $(3,\infty)$-isometries on $\mathbb{R}^2$ and give an example of a strict $(2n-1,\infty)$-isometry on $\mathbb{C}^2$, for any odd integer n.

1. Introduction

Let H be a complex Hilbert space and $L(H)$ be the C^*-algebra of all bounded linear operators on H. Let m be a positive integer. A bounded linear operator T defined on a Hilbert space H is said to be an *m-isometry* if it satisfies

$$\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} T^{*k} T^k = 0 \ ,$$

where T^* denotes the adjoint operator of T. It is easy to prove that T is an m-isometry if and only if

$$\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \|T^k x\|^2 = 0 \ , \tag{1.1}$$

for all $x \in H$. This notion of m-isometry was introduced by Agler [2] and it was later studied by many other authors. See [1, 4, 5, 6, 9, 11, 12].

Date: January 19, 2024.

2010 *Mathematics Subject Classification.* 47A05.

Key words and phrases. (m,p)-isometry, (m,∞)-isometry, weighted shift operator.

Similarly, for Banach spaces, in [4] Bayart and in [12] Hoffmann, Mackey and Ó Searcóid gave the following definition of (m, p)-isometry on a Banach space: Given a positive integer m and a positive real number p, a bounded linear operator T on a Banach space X is called an (m, p)-*isometry* if

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\|T^k x\|^p = 0\ , \tag{1.2}$$

for any $x \in X$.

So, if T is an (m, p)-isometry on X, then

$$\left(\sum_{\substack{0\le k\le m \\ k \text{ even}}}\binom{m}{k}\|T^k x\|^p\right)^{1/p} = \left(\sum_{\substack{0\le k\le m \\ k \text{ odd}}}\binom{m}{k}\|T^k x\|^p\right)^{1/p} . \tag{1.3}$$

Hoffmann, Mackey and Ó Searcóid in [12] have introduced the following definition, taking limits as p tends to infinity in equality (1.3). See also [13, 14].

An operator T defined on a Banach space X is an (m, ∞)-*isometry* if it satisfies

$$\max_{\substack{0\le k\le m \\ k \text{ even}}} \|T^k x\| = \max_{\substack{0\le k\le m \\ k \text{ odd}}} \|T^k x\|\ , \tag{1.4}$$

for all $x \in X$. It is said that T is a strict (m, ∞)-isometry if T is an (m, ∞)-isometry and is not an $(m-1, \infty)$-isometry.

In the following proposition, we summarize some basic properties of (m, ∞)-isometries that hold also valid for (m, p)-isometric operators.

Proposition 1.1. [12, Propositions 6.2, 6.3, 6.4 & 6.5] *Let $T \in L(X)$. If T is an (m, ∞)-isometry, then the following assertions hold:*

(1) T is bounded below.

(2) If $m = 2$, then $\|Tx\| \geq \|x\|$, for all $x \in X$.

(3) T is an $(m + 1, \infty)$-isometry.

(4) If T is invertible, then T^{-1} is an (m, ∞)-isometry. Moreover, if m is even, then T is an $(m - 1, \infty)$-isometry.

(5) The spectrum of T, $\sigma(T)$, is the closed unit disc or a closed subset of the unit circle.

A first natural problem is to study which are the (m,∞)-isometries that are (n,p)-isometries simultaneously where m,n are positive integers and $p>0$. In [12, Proposition 6.1], it is proved that the intersection of (m,∞)-isometries and (n,p)-isometries are the isometric operators.

The main purpose of this paper is to present that some properties of (m,p)-isometries are not enjoyed by (m,∞)-isometries.

The paper is organized as follows. In Section 2, we study powers of unilateral weighted shifts which are $(2,\infty)$-isometric operators on $\ell^2(\mathbb{N})$ and we give a complete characterization of the weights that are strict $(3,\infty)$-isometries on the canonical basis. Moreover, we construct an example of unilateral weighted shift which is a strict $(5,\infty)$-isometry on $\ell^2(\mathbb{N})$. In Section 3, we prove that any power of a $(2,\infty)$-isometry is also a $(2,\infty)$-isometry but this result isn't valid for $(3,\infty)$-isometry. In particular, we obtain an example of a strict $(3,\infty)$-isometry on $\ell^\infty(\mathbb{N})$ such that any power is not a $(3,\infty)$-isometry. In the final section, we prove some partial results about $(3,\infty)$-isometries on $\mathbb{R}^2$. We also prove that on $\mathbb{C}^2$ it is possible to define a strict $(2n-1,\infty)$-isometry, for any odd n.

2. Unilateral weighted shift

Let S_λ be a unilateral weighted shift operator with weight sequence $\lambda := (\lambda_k)_{k\in\mathbb{N}} \subseteq \mathbb{C}$ defined by

$$(S_\lambda x)_k \;:=\; \begin{cases} 0, & \text{if } k=1 \\ \lambda_{k-1}x_{k-1} & \text{if } k\geq 2\,, \end{cases}$$

for all $x=(x_1,x_2,...)\in \ell^2(\mathbb{N})$ or equivalently, $S_\lambda e_k := \lambda_k e_{k+1}$, where $e_k := (0,\ldots,0,1,0,\ldots)$.

Several authors have studied the unilateral weighted shift operators which are (m,p)-isometries on $\ell^p(\mathbb{N})$. See [1, 5, 8, 9, 11].

2.1. **On $(2,\infty)$-isometries.**

In the following proposition some properties of $(2,\infty)$-isometries are given.

Proposition 2.1. [12, Proposition 5.8] *Assume that $T \in L(X)$. Then the following conditions are equivalent*

(1) T is a $(2,\infty)$-isometry.

(2) $\|T^2x\| = \|Tx\|$ and $\|Tx\| \geq \|x\|$ for all $x \in X$.

(3) T is an isometry on $R(T)$ and satisfies $\|Tx\| \geq \|x\|$ for all $x \in X$.

Notice that by part (1) of Proposition 1.1, all (m,∞)-isometries are injective. As a consequence, if S_λ is an (m,∞)-isometry with weight sequence $\lambda = (\lambda_k)_{k\in\mathbb{N}}$, then $\lambda_k \neq 0$ for all $k \geq 1$.

In the next theorem, we study $(2,\infty)$-isometries with the unilateral weighted shift operators.

Theorem 2.2. *Let S_λ be a unilateral weighted shift on $\ell^2(\mathbb{N})$ with weight sequence $\lambda = (\lambda_k)_{k\in\mathbb{N}}$ and n be a positive integer. Then S_λ^n is a $(2,\infty)$-isometry if and only if*

$$|\lambda_k \cdots \lambda_{n+k-1}| \geq 1, \quad \textit{for } k = 1,2,...,n \quad \textit{and} \quad |\lambda_k \cdots \lambda_{n+k-1}| = 1, \quad \textit{for all} \quad k \geq n+1\ . \tag{2.5}$$

Proof. Assume that S_λ^n is a $(2,\infty)$-isometry on $\ell^2(\mathbb{N})$. By part (2) of Proposition 2.1, $\|S_\lambda^{2n}x\| = \|S_\lambda^n x\|$ and $\|S_\lambda^n x\| \geq \|x\|$ for all $x \in \ell^2(\mathbb{N})$. In particular taking $x := e_k$, then $|\lambda_k \cdots \lambda_{2n+k-1}| = |\lambda_k \cdots \lambda_{n+k-1}|$ and $|\lambda_k \cdots \lambda_{n+k-1}| \geq 1$, for all $k \in \mathbb{N}$. Hence

$$|\lambda_k \cdots \lambda_{n+k-1}| \begin{cases} \geq 1 & \text{if } k = 1,2,\ldots,n \\ = 1 & \text{if } k \geq n+1\ . \end{cases}$$

Conversely, if the weight sequence $(\lambda_k)_{k\in\mathbb{N}}$ satisfies (2.5), then it is easy to prove part (2) of Proposition 2.1. □

Corollary 2.3. *Let S_λ be a unilateral weighted shift on $\ell^2(\mathbb{N})$ with weight sequence $\lambda = (\lambda_k)_{k\in\mathbb{N}}$. Then*

(1) S_λ *is a* $(2,\infty)$*-isometry if and only if* $|\lambda_1| \geq 1$ *and* $|\lambda_k| = 1$*, for all* $k \geq 2$*.*

(2) If S_λ *is a* $(2,\infty)$*-isometry, then any power* S_λ^n *is a* $(2,\infty)$*-isometry.*

In general, the converse of part (2) of Corollary 2.3 is not true.

Example 2.4. Let S_λ be a unilateral weighted shift operator on $\ell^2(\mathbb{N})$ with weight sequence $\lambda = (\lambda_k)_{k\in\mathbb{N}}$ given by

$$\lambda_k := \begin{cases} 2 & \text{if } k = 2 \\ 1 & \text{if } k \neq 2 \,. \end{cases}$$

That is, $S_\lambda(x_1, x_2, \ldots) = (0, x_1, 2x_2, \ldots)$. Then S_λ^2 is a $(2,\infty)$-isometry but S_λ is not.

Recall different characterizations of (m,p)-isometries.

Theorem 2.5. *Let* S_λ *be a weighted shift operator on* $\ell^p(\mathbb{N})$ *with weight sequence* $\lambda = (\lambda_k)_{k\in\mathbb{N}}$*. The following assertions are equivalent:*

(1) S_λ *is an* (m,p)*-isometry.*

(2) [5, Theorem 3.4] *For* $n \geq 1$*,*

$$|\lambda_n|^p = \frac{\displaystyle\sum_{k=0}^{m-1} (-1)^{m-1-k} \frac{n \cdots \overbrace{(n-k)} \cdots (n-m+1)}{k!(m-1-k)!} \Lambda_k}{\displaystyle\sum_{k=0}^{m-1} (-1)^{m-1-k} \frac{(n-1) \cdots \overbrace{(n-1-k)} \cdots (n-m)}{k!(m-1-k)!} \Lambda_k} > 0 \,, \tag{2.6}$$

where $\Lambda_k := |\lambda_0 \lambda_1 \cdots \lambda_k|^p$*, with* $\lambda_0 := 1$ *and* $\overbrace{(n-k)}$ *denotes that the factor* $(n-k)$ *is omitted.*

(3) [1, Theorem 1] *&* [11, Corollary 4.6] *There exists a polynomial* q *of degree less than or equal to* $m-1$ *with real coefficients such that for all integers* $n \geq 1$*,* $q(n) > 0$ *and*

$$|\lambda_n|^p = \frac{q(n)}{q(n-1)} \,.$$

Next, we prove a "similar" result of part (3) of Theorem 2.5 for powers of $(2,\infty)$-isometries.

Theorem 2.6. *Let S_λ be a unilateral weighted shift on $\ell^2(\mathbb{N})$ with weight sequence $\lambda = (\lambda_k)_{k\in\mathbb{N}}$. If S_λ^n is a $(2,\infty)$-isometry, then there exists a function f defined by*

$$f(k) := \sum_{\ell=0}^{n-1} a_\ell e^{\frac{2i\ell k\pi}{n}}, \quad \textit{for all} \quad k \geq n+1,$$

with $(a_\ell)_{\ell=0}^{n-1} \subset \mathbb{C}$ where $f(k)$ is nonzero for all $k \geq n+1$ and such that for all integers $k \geq n+1$, we have that

$$|\lambda_k|^2 = \frac{f(k+1)}{f(k)} \ .$$

Proof. Define the sequence $(f(k))_{k\geq n+1}$ as follows $f(k) := \prod_{j=1}^{k-1} |\lambda_j|^2$ for $k \geq n+1$. Then $f(k)$ is nonzero for all $k \geq n+1$.

Since S_λ^n is a $(2,\infty)$-isometry on $\ell^2(\mathbb{N})$, so we have $|\lambda_k \cdots \lambda_{n-1+k}| = 1$ for all $k \geq n+1$. Then

$$\frac{f(k+n)}{f(k)} = \prod_{j=k}^{n+k-1} |\lambda_j|^2 = 1 \ \text{ for all } k \geq n+1 \ .$$

That is,

$$f(k+n) - f(k) = 0, \quad \text{for all } k \geq n+1. \tag{2.7}$$

The characteristic equation of (2.7) is giving by $r^n - 1 = 0$ and the characteristic roots, $e^{\frac{2i\ell\pi}{n}}$, are distinct values with $\ell = 0, ..., n-1$.

Thus, by [10, Section 2.3]

$$f(k) = \sum_{\ell=0}^{n-1} a_\ell e^{\frac{2i\ell k\pi}{n}},$$

for all $k \geq n+1$ where $(a_\ell)_{\ell=0}^{n-1}$ are complex numbers. Hence the proof is achieved. □

Notice that by Theorem 2.5, the operator S_λ on $\ell^2(\mathbb{N})$ is a $(2,2)$-isometry if and only if the weight sequence $\lambda = (\lambda_k)_{k\in\mathbb{N}}$ satisfies

$$|\lambda_n|^2 = \frac{q(n)}{q(n-1)} \ ,$$

where $q(n) := a_1 n + a_0$, with $a_0, a_1 \in \mathbb{R}$ and $q(n) > 0$ for all n. However, by Theorem 2.6, if S_λ on $\ell^2(\mathbb{N})$ is a $(2,\infty)$-isometry, then the weight sequence satisfies $|\lambda_n|^2 = 1$ for all $n \geq 2$.

2.2. **On $(3,\infty)$-isometries.**

Several authors have focussed on characterizations of unilateral weighted shifts which are (m,p)-isometries, [1, 5, 8, 11]. In general, the study on a Hilbert space is easier than on a general Banach space. For example, Abdullah and Le proved that for every nonzero complex number λ_1, it is possible to define a weight sequence $\lambda = (\lambda_k)_{k\in\mathbb{N}}$ such that the weighted shift operator S_λ on $\ell^2(\mathbb{N})$ is a strict 3-isometry where λ_1 is the first weight [1, Theorem 1]. And also by [5], [8] and [11] the weight sequence $\lambda = (\lambda_k)_{k\in\mathbb{N}}$ of a $(3,p)$-isometry is given in terms of the first two terms, that is λ_1 and λ_2. What can we say about $(2,\infty)$ and $(3,\infty)$-isometries?

Definition 2.7. Let $T \in L(\ell^p(\mathbb{N}))$ with $p \geq 1$. It is said that T is an (m,∞)-*isometry on the canonical basis* $\{e_n \, : \, n \in \mathbb{N}\}$ if

$$\max_{\substack{0\leq k\leq m \\ k \text{ even}}} \|T^k e_n\| = \max_{\substack{0\leq k\leq m \\ k \text{ odd}}} \|T^k e_n\| \,, \tag{2.8}$$

for all $n \in \mathbb{N}$.

By Corollary 2.3, a unilateral weighted shift operator S_λ on $\ell^2(\mathbb{N})$ is a $(2,\infty)$-isometry if and only if S_λ is a $(2,\infty)$-isometry on the canonical basis. This is not true for $m > 2$. The following example satisfies that S_λ is a $(3,\infty)$-isometry on the canonical basis but is not a $(3,\infty)$-isometry.

Example 2.8. Let S_λ be a unilateral weighted shift on $\ell^2(\mathbb{N})$ with weight sequence $(\lambda_k)_{k\in\mathbb{N}}$ given by

$$\lambda_k := \begin{cases} 3 & \text{if } k = 2 \\ 1 & \text{if } k = 3j \\ \dfrac{1}{2} & \text{if } k = 3j+1 \\ 2 & \text{if } k = 1 \text{ and } k = 3j+2 \,, \end{cases}$$

with $j \geq 1$. Then it is not difficult to check that S_λ is a $(3,\infty)$-isometry on the canonical basis. Moreover, S_λ is not a $(3,\infty)$-isometry on $\ell^2(\mathbb{N})$ since

$$\max\{\sqrt{2}, \|S_\lambda^2(e_1+e_2)\|\} \neq \max\{\|S_\lambda(e_1+e_2)\|, \|S_\lambda^3(e_1+e_2)\|\} .$$

The upcoming theorem allows us to derive admissible weights of a $(3,\infty)$-isometry on the canonical basis.

Theorem 2.9. *Let S_λ be a unilateral weighted shift on $\ell^2(\mathbb{N})$, with weight sequence $\lambda = (\lambda_k)_{k\in\mathbb{N}}$. Then S_λ is a $(3,\infty)$-isometry on the canonical basis if and only if, any block of three consecutive weights has the following behavior:*

(1) If $|\lambda_k| \geq 1$ and $|\lambda_{k+1}| = 1$, then $|\lambda_{k+2}| \leq 1$.
(2) If $|\lambda_k| = 1$ and $|\lambda_{k+1}| < 1$, then $|\lambda_{k+2}| \leq \dfrac{1}{|\lambda_{k+1}|}$.
(3) If $|\lambda_k| < 1$ and $|\lambda_{k+1}| < \frac{1}{|\lambda_k|}$, then $|\lambda_{k+2}| = \dfrac{1}{|\lambda_k\lambda_{k+1}|}$.
(4) If $|\lambda_k| < 1$ and $|\lambda_{k+1}| = \dfrac{1}{|\lambda_k|}$, then $|\lambda_{k+2}| = 1$.
(5) If $|\lambda_k| > 1$ and $|\lambda_{k+1}| > 1$, then $|\lambda_{k+2}| = 1$.

Remark 2.10. (1) If $|\lambda_k| > 1$ and $|\lambda_{k+1}| < 1$, then it is impossible to find λ_{k+2} such that S_λ is a $(3,\infty)$-isometry.

(2) The admissible weights of a $(3,\infty)$-isometry is different from a $(3,p)$-isometry on $\ell^p(\mathbb{N})$. The election of the first two weights of a $(3,p)$-isometry gives all the other weights. However this is not true for a $(3,\infty)$-isometry.

In general, we are interested in $|\lambda_k|$ instead of λ_k. For that reason, we suppose without lost of generality that $(\lambda_k)_{k\in\mathbb{N}}$ is a sequence of positive numbers.

In the following picture, we can see the admissible weight λ_3 for a $(3,\infty)$-isometry on the canonical basis for positive weights. Indeed, this representation works for every three consecutive weights.

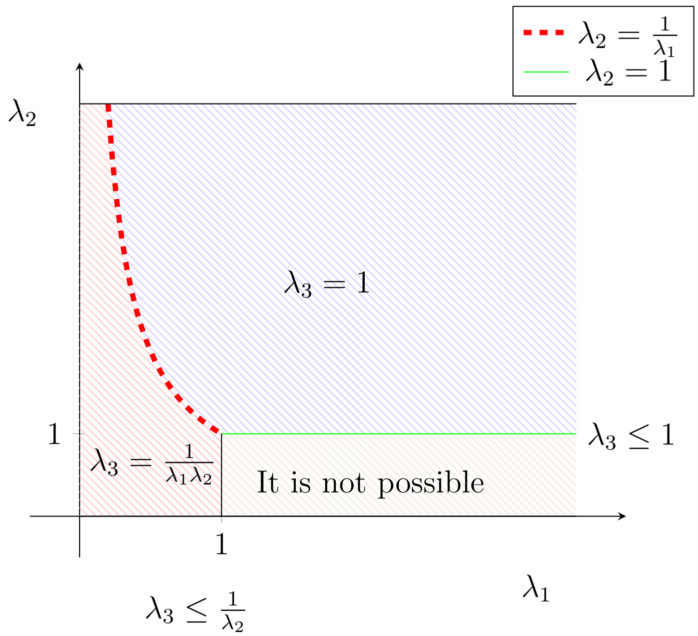


FIGURE 1. Graphical representation of the first two terms of a $(3,\infty)$-isometry on the canonical basis for positive weights.

Remark 2.11. Let S_λ be a unilateral weighted shift on $\ell^2(\mathbb{N})$ with weight sequence $(\lambda_k)_{k\in\mathbb{N}}$ such that S_λ is a $(3,\infty)$-isometry.

(1) If $|\lambda_k| = 1$ and $|\lambda_{k+1}| < 1$, then $|\lambda_{k+2}| \le |\lambda_{k+1}|^{-1}$.

(2) If $|\lambda_k| = 1$, $|\lambda_{k+1}| < 1$ and $|\lambda_{k+2}| = |\lambda_{k+1}|^{-1} > 1$, then $|\lambda_{k+3}| = 1$ and $|\lambda_{k+4}| \le 1$.

(3) If $|\lambda_k| = 1$, $|\lambda_{k+1}| < 1$ and $|\lambda_{k+2}| < |\lambda_{k+1}|^{-1}$, then $|\lambda_{k+3}| = |\lambda_{k+2}\lambda_{k+1}|^{-1} > 1$ and $|\lambda_{k+4}| = 1$.

2.3. **On $(5,\infty)$-isometries.**

The next result gives an example of strict $(5,\infty)$-isometry.

Theorem 2.12. *Let $\lambda = (\lambda_n)_{n\in\mathbb{N}}$ be the weight sequence given by*

$$\lambda_n \; := \begin{cases} a & \text{if } n = 3\ell+1 \\ b & \text{if } n = 3\ell+2 \\ c & \text{if } n = 3\ell+3, \end{cases}$$

where $\ell \in \mathbb{N} \cup \{0\}$ *and* a, b, c *are different positive real numbers, different from* 1*, such that* $a.b.c = 1$*. Then* S_λ *is a strict* $(5, \infty)$*-isometry on* $\ell^2(\mathbb{N})$*.*

Proof. Without lost of generality we suppose that $a < b < c$. So, $c > 1$ and $a < 1$.
For $x := e_2 + e_5$, we obtain that

$$\max\{\|x\|^2, \|S_\lambda^2 x\|^2, \|S_\lambda^4 x\|^2\} = 2b^2c^2 \neq \max\{2, 2b^2\} = \max\{\|S_\lambda x\|^2, \|S_\lambda^3 x\|^2\} \ ,$$

since $2b^2c^2 > 2b^2$ and $2b^2c^2 = 2\dfrac{1}{a^2} > 2$. Therefore S_λ is not a $(4, \infty)$-isometry.
The condition of the weights gives that S_λ^3 is an isometry and hence S_λ is a $(5, \infty)$-isometry by [12, Proposition 5.9]. □

3. Powers of (m, ∞)-isometries

In [6], it was proved that any power of an (m, p)-isometry is also an (m, p)-isometry. See also [15, Theorem 2.3]. The converse, in general, is not true. Indeed, sufficient conditions for the converse were given in [6, Theorem 3.6 & Corollary 3.7].

We summarize some results in the following proposition.

Proposition 3.1. [6, Theorems 3.1, 3.6 & Corollary 3.7] *Let* $T \in L(X)$*,* m *be a positive integer and* $p \geq 1$ *be a real number.*

(1) If T *is an* (m, p)*-isometry, then any power* T^r *is also an* (m, p)*-isometry.*
(2) Let r, s, m, ℓ *be positive integers and* $p \geq 1$*. If* T^r *is an* (m, p)*-isometry and* T^s *is an* (ℓ, p)*-isometry, then* T^t *is an* (h, p)*-isometry, where* t *is the greatest common divisor of* r *and* s*, and* h *is the minimum of* m *and* ℓ*.*
(3) If T^r *and* T^{r+1} *are* (m, p)*-isometries, then* T *is an* (m, p)*-isometry.*

Our aim is now to study similar properties for the class of (m, ∞)-isometric operators.

The next result improves part (2) of Corollary 2.3 for the class of $(2, \infty)$-isometries, that is, any power of a $(2, \infty)$-isometry is also a $(2, \infty)$-isometry.

Theorem 3.2. *Assume that $T \in L(X)$. If T is a $(2,\infty)$-isometry, then any power T^n is also a $(2,\infty)$-isometry.*

Proof. We will prove the following equality

$$\max\{\|x\|, \|T^{2n}x\|\} = \|T^n x\|,$$

for all $x \in X$. By Proposition 2.1, we have that $\|Tx\| \geq \|x\|$ and $\|T^2x\| = \|Tx\|$ for all $x \in X$, since T is a $(2,\infty)$-isometry. Then $\|T^{2n}x\| \geq \|x\|$, for all $x \in X$ and $n \in \mathbb{N}$. Then $\max\{\|x\|, \|T^{2n}x\|\} = \|T^{2n}x\| = \|T^n x\|$ for all $x \in X$. □

Part (3) of Proposition 3.1 does not work for $(2,\infty)$-isometries as proves the following theorem.

Theorem 3.3. *Let S_λ be a unilateral weighted shift on $\ell^2(\mathbb{N})$ with weight sequence $(\lambda_k)_{k\in\mathbb{N}}$. Assume that S_λ^n and S_λ^{n+1} are $(2,\infty)$-isometries. Then S_λ is a $(2,\infty)$-isometry if and only if $|\lambda_k| = 1$ for $k = 2, 3, \dots, n$.*

Proof. Assume that S_λ is a $(2,\infty)$-isometry. Then by part (1) of Corollary 2.3, we obtain that $|\lambda_k| = 1$ for $k = 2, \dots, n$.

Now, suppose that $|\lambda_k| = 1$ for $k \in \{2, \dots, n\}$. Let us prove that $|\lambda_k| = 1$ for all $k > n$ and $|\lambda_1| \geq 1$. By Theorem 2.2,

$$|\lambda_k \cdots \lambda_{n+k-1}| = 1, \text{ for all } k \geq n+1 \tag{3.9}$$

and

$$|\lambda_k \cdots \lambda_{n+k}| = 1, \text{ for all } k \geq n+2\ . \tag{3.10}$$

Then $|\lambda_{n+k}| = |\lambda_k| = 1$, for all $k \geq n+1$. Moreover, $|\lambda_1 \cdots \lambda_n| \geq 1$. Hence, by hypothesis we obtain that $|\lambda_1| \geq 1$. □

Corollary 3.4. *Let S_λ be a unilateral weighted shift on $\ell^2(\mathbb{N})$ with weight sequence $(\lambda_k)_{k\in\mathbb{N}}$. Assume that S_λ^2 and S_λ^3 are $(2,\infty)$-isometries. Then S_λ is a $(2,\infty)$-isometry if and only if $|\lambda_2| = 1$.*

Example 3.5. Let $(\lambda_k)_{k\in\mathbb{N}}$ be a sequence of weights given by

$$\lambda_k := \begin{cases} 2 & \text{if } k = 1, 2 \\ 1 & \text{if } k \geq 3\ . \end{cases}$$

By Theorem 2.2, we have that S_λ^2 and S_λ^3 are $(2,\infty)$-isometries on $\ell^2(\mathbb{N})$ and S_λ is not.

In the following theorem, we prove that Theorem 3.2 does not work for $(3,\infty)$-isometries.

Theorem 3.6. *Fixed a positive integer $n > 1$, there exist a sequence of weights $(\lambda_n(k))_{k\in\mathbb{N}}$ and a positive integer a_n such that:*

(a) *$S_{\lambda_n}^\ell$ is a strict $(3,\infty)$-isometry on $\ell^\infty(\mathbb{N})$ for $\ell \in \{1\} \cup A_n$,*

(b) *$S_{\lambda_n}^\ell$ is not a $(3,\infty)$-isometry on $\ell^\infty(\mathbb{N})$ for $\ell \in B_n$,*

(c) *$S_{\lambda_n}^\ell$ is a strict $(2,\infty)$-isometry on $\ell^\infty(\mathbb{N})$ for $\ell > 2a_n + 1$,*

where $A_n := \{a_n + 1, ..., 2a_n + 1\}$ and $B_n := \{2, ..., a_n\}$.

Proof. Assume that

$$\mathbb{N} \cup \{0\} = \bigcup_{i=0}^{2} E_i\ ,$$

where

$$E_i := \{3j + i \ \ : \ \ j \in \mathbb{N} \cup \{0\}\}\ .$$

Fixed $n \in E_{i_0} \backslash \{0, 1\}$, there exists $j_0 \in \mathbb{N} \cup \{0\}$ such that $n = 3j_0 + i_0$. We define the sequence of weight $(\lambda_n(k))_{k\in\mathbb{N}}$ as follows:

$$\lambda_n(k) \ := \ \begin{cases} 2 & \text{if } k = 1 \text{ or } k = 3(h+1) \\ 3 & \text{if } k = 2 \\ 1 & \text{if } k = 3,\ k = 3h + 1 \text{ or } k \geq 2a_n + 3 \\ \dfrac{1}{2} & \text{if } k = 3h + 2\ , \end{cases}$$

where $h \in \{1, ..., 2j_0+1\}$ and $a_n := 3j_0+2$. That is,

$$(\overbrace{2, \quad 3, \quad 1}^{\text{1st block of 3 weights}}, \quad \underbrace{1, \quad \tfrac{1}{2}, \quad 2}_{\text{2nd block of 3 weights}}, \quad \ldots, \quad \overbrace{1, \quad \tfrac{1}{2}, \quad 2}^{(2j_0+2)\text{th block of 3 weights}}, \quad 1, \quad 1, \quad 1, \quad \ldots)$$

(a) Let us prove that $S_{\lambda_n}^{\ell}$ is a strict $(3,\infty)$-isometry on $\ell^\infty(\mathbb{N})$, for $\ell \in \{1\} \cup A_n$.

Let $x \in \ell^\infty(\mathbb{N})$. Assume $\ell = 1$. We consider two cases, $\|x\|_\infty = |x_k|$, for some $k \in \mathbb{N}$ or $\|x\|_\infty \neq |x_k|$, for any $k \in \mathbb{N}$.

Case 1. $\|x\|_\infty = |x_k|$, for some $k \in \mathbb{N}$.

Let's assume, without lost of generality, that $\|x\|_\infty = |x_4|$, since other cases are similar. Then, we get that

$$\|S_{\lambda_n}x\|_\infty = \max\left\{2|x_1|,\ 3|x_2|,\ |x_4|,\ 2|x_k|,\ k = 3h\ ,\ h \in \{2, ..., 2j_0+2\}\right\}.$$

Case 1.1. If $\|S_{\lambda_n}x\|_\infty = 2|x_1|$, then $\|S_{\lambda_n}^2 x\|_\infty = \|S_{\lambda_n}^3 x\|_\infty = 6|x_1|$, which implies that

$$\max\{\|x\|_\infty, \|S_{\lambda_n}^2 x\|_\infty\} = \max\{\|S_{\lambda_n}x\|_\infty, \|S_{\lambda_n}^3 x\|_\infty\}.$$

Case 1.2. If $\|S_{\lambda_n}x\|_\infty = 3|x_2|$, then $\|S_{\lambda_n}^2 x\|_\infty = \|S_{\lambda_n}^3 x\|_\infty = \max\{6|x_1|,\ 3|x_2|\}$, which implies that

$$\max\{\|x\|_\infty, \|S_{\lambda_n}^2 x\|_\infty\} = \max\{\|S_{\lambda_n}x\|_\infty, \|S_{\lambda_n}^3 x\|_\infty\} = 3|x_1|.$$

Case 1.3. If $\|S_{\lambda_n}x\|_\infty = 2|x_k|$, with $k = \dot{3}$ such that $3 < k \leq 6j_0+6$, where $\dot{3}$ denotes a multiple of 3, then $\|S_{\lambda_n}^2 x\|_\infty = \max\{6|x_1|,\ 2|x_k|\}$, which implies that

$$\max\{\|x\|_\infty, \|S_{\lambda_n}^2 x\|_\infty\} = \max\{\|S_{\lambda_n}x\|_\infty, \|S_{\lambda_n}^3 x\|_\infty\} = \max\{6|x_1|, |x_4|, 2|x_k|\}.$$

Case 1.4. If $\|S_{\lambda_n}x\|_\infty = |x_4|$, then $\|S_{\lambda_n}^3 x\|_\infty = \max\{6|x_1|,\ |x_4|\}$, which implies that

$$\max\{\|x\|_\infty, \|S_{\lambda_n}^2 x\|_\infty\} = \max\{\|S_{\lambda_n}x\|_\infty, \|S_{\lambda_n}^3 x\|_\infty\} = \max\{|x_4|, 6|x_1|\}.$$

Case 2. $\|x\|_\infty \neq |x_k|$, for any $k \in \mathbb{N}$.

Assume that $\beta := \|x\|_\infty$. Hence, we obtain that

$$\begin{aligned}
\|S_{\lambda_n} x\|_\infty &= \max\{2|x_1|,\ 3|x_2|,\ \beta,\ 2|x_k|,\ \text{with } k = 3h\ ,\ h \in \{2, ..., 2(j_0+1)\}\} \\
\|S^2_{\lambda_n} x\|_\infty &= \max\{6|x_1|,\ 3|x_2|,\ \beta,\ 2|x_k|,\ \text{with } k = 3h\ ,\ h \in \{2, ..., 2(j_0+1)\}\} \\
\|S^3_{\lambda_n} x\|_\infty &= \max\left\{6|x_1|,\ 3|x_2|,\ \beta,\ 2|x_{6(j_0+1)}|\right\}\ .
\end{aligned}$$

Then, we conclude that

$$\max\{\|x\|_\infty, \|S^2_{\lambda_n} x\|_\infty\} = \max\{\|S_{\lambda_n} x\|_\infty, \|S^3_{\lambda_n} x\|_\infty\} = \|S^2_{\lambda_n} x\|_\infty,$$

for any $x \in \ell^\infty(\mathbb{N})$.

On the other hand, we have that S_{λ_n} is not a $(2, \infty)$-isometry, since

$$\max\{1, \|S^2_{\lambda_n} e_1\|_\infty\} \neq \|S_{\lambda_n} e_1\|_\infty.$$

Hence, S_{λ_n} is a strict $(3, \infty)$-isometry on $\ell^\infty(\mathbb{N})$.

Let $\ell \in A_n$, where $A_n := \{a_n + 1, ..., 2a_n + 1\}$. The case $n = 2$ is easy since $\|S^k_{\lambda_n} x\|_\infty = \|S^6_{\lambda_n} x\|_\infty$ for any $k \geq 6$ and $\|S^i_{\lambda_n} x\|_\infty \leq \|S^6_{\lambda_n} x\|_\infty$ for $i \in \{3, 4, 5\}$, for all $x \in \ell^\infty(\mathbb{N})$. Henceforth $S^\ell_{\lambda_n}$ is a strict $(3, \infty)$-isometry for $\ell \in A_n$. Assume that $n > 2$, then $j_0 \in \mathbb{N}$. So $\ell = 3(j_0 + J) + i$, with $i \in \{0, 1, 2\}$ and $J \in \{1, ..., j_0 + 1\}$.

First, we will prove that $S^\ell_{\lambda_n}$ is a $(3, \infty)$-isometry. By [12, Proposition 5.8], it is sufficient to prove that

$$\|S^{3\ell}_{\lambda_n} x\|_\infty = \|S^{2\ell}_{\lambda_n} x\|_\infty,\ \|S^{3\ell}_{\lambda_n} x\|_\infty \geq \|S^{\ell}_{\lambda_n} x\|_\infty \text{ and } \|S^{3\ell}_{\lambda_n} x\|_\infty \geq \|x\|_\infty, \text{ for any } x \in \ell^\infty(\mathbb{N})\ .$$

It is easy to check that $\|S_{\lambda_n}^{2\ell}x\|_\infty = \|S_{\lambda_n}^{3\ell}x\|_\infty$, for any $x \in \ell^\infty(\mathbb{N})$, since $S_{\lambda_n}^k x = S^k y$, for $k > 2a_n + 1$, where $y := (y(k))_{k\in\mathbb{N}}$ is given by

$$y(k) := \begin{cases} 6x_1 & \text{if } k = 1 \\ 3x_2 & \text{if } k = 2 \\ 2x_k & \text{if } k = 3h \\ x_k & \text{if } k \neq 1,\ 2,\ 3h\ , \end{cases}$$

with $h \in \{2, ..., 2(j_0+1)\}$ and $S(y_1, y_2, ...) := (0, y_1, y_2, ...)$ is the unweighed shift operator. To show that $\|S_{\lambda_n}^{3\ell}x\|_\infty \geq \|S_{\lambda_n}^{\ell}x\|_\infty$, we study three cases, depending on ℓ, that is, when ℓ is given by $3(j_0+J)$, $3(j_0+J)+1$ or $3(j_0+J)+2$, with $J \in \{1, ..., j_0+1\}$. Notice that $\|S_{\lambda_n}^{3\ell}x\|_\infty = \|S^{3\ell}y\|_\infty = \|y\|_\infty$.

Case 1. If $\ell = 3(j_0+J)$, then $S_{\lambda_n}^{3(j_0+J)}x = S^{3(j_0+J)}y_0$, where $y_0 := (y_0(k))_{k\in\mathbb{N}}$ is given by

$$y_0(k) \ := \begin{cases} 6x_1 & \text{if } k = 1 \\ 3x_2 & \text{if } k = 2 \\ \frac{1}{2}x_3 & \text{if } k = 3 \\ 2x_k & \text{if } k = 3h \\ x_k & \text{if } k \neq 1,\ 2,\ 3h\ , \end{cases}$$

with $h \in \{j_0 - J + 3, ..., 2(j_0+1)\}$. So $\|S_{\lambda_n}^{3\ell}x\|_\infty = \|S^{3\ell}y\|_\infty = \|y\|_\infty \geq \|y_0\|_\infty = \|S^{\ell}y_0\|_\infty = \|S_{\lambda_n}^{\ell}x\|_\infty$.

Case 2. If $\ell = 3(j_0+J)+1$, with $J \in \{1, ..., j_0\}$, then $S_{\lambda_n}^{3(j_0+J)+1}x = S^{3(j_0+J)+1}y_1$, where $y_1 := (y_1(k))_{k\in\mathbb{N}}$ is given by

$$y_1(k) \ := \begin{cases} 6x_1 & \text{if } k = 1 \\ \frac{3}{2}x_2 & \text{if } k = 2 \\ 2x_k & \text{if } k = 3h,\ h \in \{2, ..., 2(j_0+1)\} \\ \frac{1}{2}x_k & \text{if } k = 3h+2,\ h \in \{1, ..., j_0 - J + 1\} \\ x_k & \text{if } k \neq 1,\ 2,\ 3h,\ 3h+2 \end{cases}$$

and $S_{\lambda_n}^{3(2j_0+1)+1}x = S^{3(2j_0+1)+1}y_1'$, where $y_1' := (y_1'(k))_{k\in\mathbb{N}}$ is given by

$$y_1'(k) \quad := \quad \begin{cases} 6x_1 & \text{if } k=1 \\ \frac{3}{2}x_2 & \text{if } k=2 \\ 2x_k & \text{if } k=3h \\ x_k & \text{if } k\neq 1,\ 2,\ 3h\ , \end{cases}$$

with $h \in \{2, ..., 2(j_0+1)\}$.

Case 3. If $\ell = 3(j_0+J)+2$, with $J \in \{1, ..., j_0\}$, then $S_{\lambda_n}^{3(j_0+J)+2}x = S^{3(j_0+J)+2}y_2$, where $y_2 := (y_2(k))_{k\in\mathbb{N}}$ is given by

$$y_2(k) \quad := \quad \begin{cases} 3x_k & \text{if } k=1 \text{ or } k=2 \\ 2x_k & \text{if } k=3h,\ h\in\{2,...,2(j_0+1)\} \\ \frac{1}{2}x_k & \text{if } k=3h+1,\ h\in\{1,...,j_0-J+1\} \\ x_k & \text{if } k\neq 1,\ 2,\ 3h,\ 3h+1\ , \end{cases}$$

and $S_{\lambda_n}^{3(2j_0+1)+2}x = S^{3(2j_0+1)+2}y_2'$, where $y_2' := (y_2'(k))_{k\in\mathbb{N}}$ is given by

$$y_2'(k) \quad = \quad \begin{cases} 3x_k & \text{if } k=1 \text{ or } k=2 \\ 2x_k & \text{if } k=3h \\ x_k & \text{if } k\neq 1,\ 2,\ 3h\ , \end{cases}$$

with $h \in \{2, ..., 2(j_0+1)\}$. Hence, we obtain that $\|S_{\lambda_n}^{3\ell}x\|_\infty \geq \|S_{\lambda_n}^{\ell}x\|_\infty$, for any $x \in \ell^\infty(\mathbb{N})$. Moreover, $S_{\lambda_n}^{\ell}$ is not a $(2,\infty)$-isometry, since

$$\max\{1, \|S_{\lambda_n}^{2(3(j_0+J)+i)}e_{3-i}\|_\infty\} \neq \|S_{\lambda_n}^{3(j_0+J)+i}e_{3-i}\|_\infty,$$

for $i \in \{0,1,2\}$ and $J \in \{1, ..., j_0+1\}$. Hence, we get the result.

(b) Now, we prove that $S_{\lambda_n}^{\ell}$ is not a $(3,\infty)$-isometry, for any $\ell \in B_n$, where $B_n := \{2, ..., a_n\} = \{2, ..., 3j_0+2\}$. That is, there exists $x_\ell \in \ell^\infty(\mathbb{N})$ such that

$$\max\{\|x_\ell\|_\infty, \|S_{\lambda_n}^{2\ell}x_\ell\|_\infty\} \neq \max\{\|S_{\lambda_n}^{\ell}x_\ell\|_\infty, \|S_{\lambda_n}^{3\ell}x_\ell\|_\infty\}.$$

For $j_0 := 0$, we have that

$$\lambda_2(k) := \begin{cases} 2 & \text{if } k=1 \text{ or } k=6 \\ 3 & \text{if } k=2 \\ \frac{1}{2} & \text{if } k=5 \\ 1 & \text{if } k\neq 1,\ 2,\ 5,\ 6\ . \end{cases}$$

Then

$$\max\{1, \|S^4_{\lambda_2} e_2\|_\infty\} = \frac{3}{2} \neq 3 = \max\{\|S^2_{\lambda_2} e_2\|_\infty, \|S^6_{\lambda_2} e_2\|_\infty\}.$$

For $j_0 := 1$. That means $n=3$ or $n=4$ or $n=5$, we have that

$$\lambda_n(k) := \begin{cases} 2 & \text{if } k=1 \text{ or } k=6 \text{ or } k=9 \\ 3 & \text{if } k=2 \\ \frac{1}{2} & \text{if } k=5 \text{ or } k=8 \\ 1 & \text{if } k\neq 1,\ 2,\ 5,\ 6,\ 8,\ 9\ . \end{cases}$$

Then

$$\begin{cases} \max\{1, \|S^{2\ell}_{\lambda_n} e_2\|_\infty\} = \frac{3}{2} \neq 3 \max\{\|S^{\ell}_{\lambda_n} e_2\|_\infty, \|S^{3\ell}_{\lambda_n} e_2\|_\infty\} & \text{if } \ell \in \{2,5\} \\ \max\{1, \|S^{8}_{\lambda_n} e_1\|_\infty\} = 3 \neq 6 = \max\{\|S^{4}_{\lambda_n} e_1\|_\infty, \|S^{12}_{\lambda_n} e_1\|_\infty\} \\ \max\{1, \|S^{6}_{\lambda_n} e_6\|_\infty\} = 1 \neq 2 = \max\{\|S^{3}_{\lambda_n} e_6\|_\infty, \|S^{9}_{\lambda_n} e_6\|_\infty\} & . \end{cases}$$

Assume that $j_0 > 1$. We obtain

$$\begin{cases} \max\{1, \|S^{2\ell}_{\lambda_n} e_2\|_\infty\} = \frac{3}{2} \neq 3 \max\{\|S^{\ell}_{\lambda_n} e_2\|_\infty, \|S^{3\ell}_{\lambda_n} e_2\|_\infty\} & \text{if } \ell = 3J+2,\ J \in \{0,...,j_0\} \\ \max\{1, \|S^{2\ell}_{\lambda_n} e_1\|_\infty\} = 3 \neq 6 = \max\{\|S^{\ell}_{\lambda_n} e_1\|_\infty, \|S^{3\ell}_{\lambda_n} e_1\|_\infty\} & \text{if } \ell = 3J+1,\ J \in \{1,...,j_0\} \\ \max\{1, \|S^{2\ell}_{\lambda_n} e_3\|_\infty\} = 1 \neq \frac{1}{2} = \max\{\|S^{\ell}_{\lambda_n} e_3\|_\infty, \|S^{3\ell}_{\lambda_n} e_3\|_\infty\} & \text{if } \ell = \dot{3} \in \{3,...,2j_0+1\} \\ \max\{1, \|S^{2\ell}_{\lambda_n} e_6\|_\infty\} = 1 \neq 2 = \max\{\|S^{\ell}_{\lambda_n} e_6\|_\infty, \|S^{3\ell}_{\lambda_n} e_6\|_\infty\} & \text{if } \ell = \dot{3} \in \{2(j_0+1),...,3j_0\}\ . \end{cases}$$

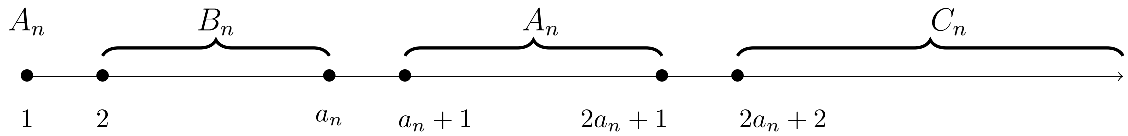
Hence $S^\ell_{\lambda_n}$ is not a $(3,\infty)$-isometry, for any $\ell \in B_n$.

(c) Finally, we need to show that $S^\ell_{\lambda_n}$ is a strict $(2,\infty)$-isometry, for $\ell > 2a_n + 1 = 6j_0 + 5$.

For $\ell > 2a_n+1$, we have $\|S^{2\ell}_{\lambda_n} x\|_\infty = \|S^{\ell}_{\lambda_n} x\|_\infty$ and $\|S^{\ell}_{\lambda_n} x\|_\infty \geq \|x\|_\infty$, for any $x \in \ell^\infty(\mathbb{N})$.

So $S^\ell_{\lambda_n}$ is a $(2,\infty)$-isometry and $\|S^\ell_{\lambda_n} e_1\|_\infty \neq 1$. The proof is now completed. □

Remark 3.7. Fixed $n \in E_{i_0} \setminus \{0,1\}$, there exists $j_0 \in \mathbb{N} \cup \{0\}$ such that $n = 3j_0 + i_0$, where $E_i := \{3j+i \quad : \quad j \in \mathbb{N} \cup \{0\}\}$, for $i \in \{0,1,2\}$. Theorem 3.6 proves that there exist a sequence of weights $(\lambda_n(k))_{k\in\mathbb{N}}$ and a positive integer $a_n := 3j_0 + 2$ such that we obtain the following diagram:



where

$$
\begin{aligned}
A_n &:= \{k \in \mathbb{N} : \ S_{\lambda_n}^k \text{is a strict } (3,\infty)\text{-isometry on } \ell^\infty(\mathbb{N})\} \\
B_n &:= \{k \in \mathbb{N} : \ S_{\lambda_n}^k \text{is not a } (3,\infty)\text{-isometry on } \ell^\infty(\mathbb{N})\} \\
C_n &:= \{k \in \mathbb{N} : \ S_{\lambda_n}^k \text{is a strict } (2,\infty)\text{-isometry on } \ell^\infty(\mathbb{N})\}.
\end{aligned}
$$

In the following example, we have that part (1) of Proposition 3.1 does not valid for the class of $(3,\infty)$-isometries. The main idea is to define an operator as in the proof of Theorem 3.6 using the first block and repeating the second one continuously. Notice that the blocks of ones, in the proof of Theorem 3.6, are to obtain that some powers of S_λ are strict $(2,\infty)$-isometries.

Example 3.8. Let S_λ be a unilateral weighted shift on $\ell^\infty(\mathbb{N})$ with weight sequence $(\lambda(k))_{k\in\mathbb{N}}$ given by

$$
\lambda_k := \begin{cases} 2 & \text{if } k = 1 \text{ or } k = 3(h+1) \\ 3 & \text{if } k = 2 \\ 1 & \text{if } k = 3 \text{ or } k = 3h+1 \\ \frac{1}{2} & \text{if } k = 3h+2 \ , \end{cases}
$$

with $h \in \mathbb{N}$, that is,

$$
(\overbrace{2, \ 3, \ 1}^{\text{1st block of 3 weights}}, \ \underbrace{1, \ \tfrac{1}{2}, \ 2}_{\text{2nd block of 3 weights}}, \ \overbrace{1, \ \tfrac{1}{2}, \ 2}^{\text{3rd block of 3 weights}}, \ \ldots\ ) \ .
$$

Then S_λ is a strict $(3,\infty)$-isometry and S_λ^n is not a $(3,\infty)$-isometry, for any integer $n \geq 2$.

Part (2) of Proposition 3.1 is not valid for the class of $(3,\infty)$-isometries as proves the following example.

Example 3.9. Consider a unilateral weighted shift S_λ on $\ell^\infty(\mathbb{N})$ with weight sequence $\lambda = (\lambda_k)_{k\in\mathbb{N}}$ defined by

$$\lambda_k := \begin{cases} 2 & \text{if } k=1 \text{ or } k=3(h+1) \\ 3 & \text{if } k=2 \\ 1 & \text{if } k=3,\ k=3h+1 \text{ or } k\geq 13 \\ \dfrac{1}{2} & \text{if } k=3h+2\ , \end{cases}$$

where $h \in \{1,2,3\}$. By Theorem 3.6, with $n=5$, we obtain that

(*a*) S_λ^ℓ is a strict $(3,\infty)$-isometry for $\ell \in \{1,6,7,...,11\}$,

(*b*) S_λ^ℓ is not a $(3,\infty)$-isometry for $\ell \in \{2,3,4,5\}$,

(*c*) S_λ^ℓ is a strict $(2,\infty)$-isometry for $\ell \geq 12$.

In the next results, we prove that with additional conditions related to [12, Propositions 5.8 & 5.9] any power of an (m,∞)-isometry is also an (m,∞)-isometry.

Proposition 3.10. *Let $T \in L(X)$ and $m \in \mathbb{N}$, with $m \geq 2$ such that*

$$\|T^m x\| = \|T^{m-1}x\| \ \ \textit{and} \ \ \|T^m x\| \geq \|T^\ell x\|,$$

for any $\ell \in \{0,1,...,m-2\}$ and $x \in X$. Then T^k is an (m,∞)-isometry for any $k \in \mathbb{N}$.

Proof. The case $k=1$ was proved in [12, Proposition 5.8].
Notice that $\|T^{m+i}x\| = \|T^{m-1}x\|$, for every $i \geq 0$ and $x \in X$. Fixed $k \in \mathbb{N}$, we have that $\|T^{km}x\| = \|T^{k(m-1)}x\|$. If $k\ell < m$, then $\|T^{km}x\| = \|T^m x\| = \|T^{m-1}x\| \leq \|T^{k\ell}x\|$. If $k\ell \geq m$, then $\|T^{k\ell}x\| = \|T^m x\| = \|T^{m-1}x\|$. The result is a consequence of [12, Proposition 5.8] for the operator T^k. □

Proposition 3.11. *Let $T \in L(X)$.*

(1) *If T^{2n} is an isometry, then T is a $(2n+1, \infty)$-isometry if and only if T is a $(2n-1, \infty)$-isometry.*

(2) *If T^2 is an isometry, then T is an (m, ∞)-isometry if and only if T is an isometry.*

Proof. It is clear by the definition of (m, ∞)-isometry. □

Theorem 3.12. *Let $T \in L(X)$ such that T^n is an isometry for an odd number n. Then T^k is a $(2n-1, \infty)$-isometry for any positive integer number k.*

Proof. For $k = 1$, it was proved in [12, Proposition 5.9].
For the general case, take into account that if T^n is an isometry, then $(T^k)^n$ is an isometry. Then the result is an immediate consequence of [12, Proposition 5.9]. □

4. (m, ∞)-isometries on finite dimensional space

Notice that if T is an (m, ∞)-isometry on a finite dimensional Banach space, then the spectrum of T is equal to the eigenvalues of T, $\sigma_p(T)$, and it is a finite subset of the unit circle, [12, Proposition 6.5].

In the following proposition, we prove that some types of operators can not be a strict $(3, \infty)$-isometry on $\mathbb{R}^2$.

Proposition 4.1. *If $T := \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ or $T := \begin{pmatrix} a & 0 \\ b & d \end{pmatrix}$, where $a, b, d \in \mathbb{R}$, is a $(3, \infty)$-isometry on $\mathbb{R}^2$, then T is an isometry.*

Proof. Assume that $T = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$. The other case is similar. It is clear that $\sigma_p(T) = \{a, d\}$ and by [12, Proposition 6.5] we obtain that $a = \pm 1$ and $d = \pm 1$.

Assume that $a = d = 1$. That is, $T = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$. It is clear that T is a 3-isometry. By [12, Proposition 6.1] the m-isometries that are (m, ∞)-isometries are the isometries.

If $a = 1$ and $d = -1$, then T^2 is an isometry. Therefore, the result is a consequence of part (2) of Proposition 3.11. The other cases are similar. $\square$

Theorem 4.2. *Let $T \in L(X)$ such that $T^2 = T$. Then T is a $(3,\infty)$-isometry if and only if $T = I$.*

Proof. Assume that T is a $(3,\infty)$-isometry and $T^2 = T$. Then $T(Tx) = Tx$ implies that $Tx = x$ for all $x \in X$. So $T = I$. $\square$

By [3, 7] the strict m-isometries on $\mathbb{C}^n$ are of odd m and less than or equal to $2n-1$. Denote

$$I_m(\mathbb{C}^n) := \{T \in L(\mathbb{C}^n) : \ \ T \text{ is an } m\text{-isometry}\}.$$

Then

$$I_1(\mathbb{C}^n) = I_2(\mathbb{C}^n) \subsetneq I_3(\mathbb{C}^n) = I_4(\mathbb{C}^n) \subsetneq ... \subsetneq I_{2n-2}(\mathbb{C}^n) \subsetneq I_{2n-1}(\mathbb{C}^n) = I_m(\mathbb{C}^n)\ , \tag{4.11}$$

for all $m \geq 2n-1$.

The next theorem proves that on $\mathbb{C}^2$ it is possible to define a strict $(2n-1,\infty)$-isometry for every odd number n. So it is not possible to translate (4.11) to the class of (m,∞)-isometries.

Theorem 4.3. *Let n be an odd number. There exists $T \in L(\mathbb{C}^2)$ such that T is a strict $(2n-1,\infty)$-isometry.*

Proof. Assume that n is an odd number such that $n > 1$. Rewrite $n = 2j+1$, for some $j \in \mathbb{N}$ and define on $\mathbb{C}^2$

$$T := \begin{pmatrix} a & 1 \\ 0 & a^2 \end{pmatrix},$$

where $a := e^{i\theta}$ such that $\theta = \frac{2\pi}{n}$. By the assumptions on n and θ, we have that a and a^2 are different complex numbers on $\partial\mathbb{D}\backslash\{1\}$.

First, we will prove that T is a $(2n-1,\infty)$-isometry. By [12, Proposition 5.9], it is sufficient to prove that $T^n = I$. It is straight forward that

$$T^k = \begin{pmatrix} a^k & a^{k-1}\sum_{\ell=0}^{k-1} a^\ell \\ & \\ 0 & a^{2k} \end{pmatrix} . \tag{4.12}$$

Hence $T^n = I$, since $a^n = 1$.

To conclude the proof we need to prove that T is a strict $(2n-1,\infty)$-isometry or equivalently, since T is invertible, that T is not a $(2n-3,\infty)$-isometry by parts (4) and (5) of Proposition 1.1. That is, there exists $x_0 \in \mathbb{C}^2$ such that

$$\max_{\substack{0\le k\le 2n-3 \\ k \text{ even}}} \|T^k x_0\| \neq \max_{\substack{0\le k\le 2n-3 \\ k \text{ odd}}} \|T^k x_0\| . \tag{4.13}$$

It is easy to see that (4.13) is satisfied for $n = 3$ and $n = 5$ with $x_0 := (1,1)$. Assume that $n \geq 7$, that is, $j \geq 3$ where $n = 2j+1$. In particular, we will prove that

$$\max_{\substack{0\le k\le 2n-3 \\ k \text{ even}}} \|T^k T^{j+1} x_0\| \neq \max_{\substack{0\le k\le 2n-3 \\ k \text{ odd}}} \|T^k T^{j+1} x_0\| , \tag{4.14}$$

where $x_0 := (1,1)$.

Since $T^n = I$, (4.14) is equivalent to

$$\max_{\substack{0\le k\le 2n-3 \\ k \text{ even}}} \|T^k T^{j+1} x_0\| = \max_{\substack{0\le k\le n-1 \\ k\ \neq n-2}} \|T^k T^{j+1} x_0\| \neq \max_{0\le k\le n-2} \|T^k T^{j+1} x_0\| = \max_{\substack{0\le k\le 2n-3 \\ k \text{ odd}}} \|T^k T^{j+1} x_0\|. \tag{4.15}$$

We need three claims.

Claim 1. *Let $x_0 := (1,1)$. Then*

$$\max_{0\le k\le n-1} \|T^k T^{j+1} x_0\|^2 = \max_{-j\le k\le j} \left\{ 1 + \frac{\left|a^k + a^2 - a - 1\right|^2}{|a-1|^2} \right\} .$$

Proof. Using (4.12),

$$\|T^k T^{j+1} x_0\|^2 = 1 + \left| a + \sum_{\ell=0}^{k+j} a^\ell \right|^2 .$$

Then

$$\begin{aligned}
\max_{0\leq k\leq n-1}\|T^kT^{j+1}x_0\|^2 &= \max_{j\leq k\leq 3j}\left\{1+\left|a+\sum_{\ell=0}^{k}a^\ell\right|^2\right\} = \max_{j\leq k\leq 3j}\left\{1+\left|a+\frac{a^{k+1}-1}{a-1}\right|^2\right\} \\
&= \max_{j+1\leq k\leq 3j+1}\left\{1+\frac{\left|a^k+a^2-a-1\right|^2}{|a-1|^2}\right\} \\
&= \max_{n-j\leq k\leq n+j}\left\{1+\frac{\left|a^k+a^2-a-1\right|^2}{|a-1|^2}\right\} \\
&= \max_{-j\leq k\leq j}\left\{1+\frac{\left|a^k+a^2-a-1\right|^2}{|a-1|^2}\right\} ,
\end{aligned}$$

since $n=2j+1$. Hence, we get the result. □

Denote $U_k := \left|a^k+a^2-a-1\right|^2$. It is straightforward that

$$U_k = 2\left(2+\cos\left(\frac{2k-4}{n}\pi\right)-\cos\left(\frac{2k-2}{n}\pi\right)-\cos\left(\frac{2k\pi}{n}\right)-\cos\left(\frac{4\pi}{n}\right)\right) ,$$

since $a=e^{\frac{2\pi i}{n}}$.

Claim 2. *The sequence* $(U_k)_{-j}^{j}$ *satisfies that* $U_k-U_{-k}>0$, *for any* $k\in\{1,...,j\}$.

Proof. We have

$$\begin{aligned}
U_k-U_{-k} &= 2\left(2\cos\left(\frac{2k-4}{n}\pi\right)-\cos\left(\frac{2k+4}{n}\pi\right)-\cos\left(\frac{2k-2}{n}\pi\right)-\cos\left(\frac{2k+2}{n}\pi\right)\right) \\
&= 4\sin\left(\frac{2k\pi}{n}\right)\left(\sin\left(\frac{4\pi}{n}\right)-\sin\left(\frac{2\pi}{n}\right)\right) \\
&= 4\sin\left(\frac{2k\pi}{n}\right)\sin\left(\frac{2\pi}{n}\right)\left(2\cos\left(\frac{2\pi}{n}\right)-1\right) .
\end{aligned}$$

Since $\frac{2k\pi}{n}\in(0,\pi)$ for $k\in\{1,...,j\}$ and $n\geq 7$, then $U_k-U_{-k}>0$. □

Claim 3. *The sequence* $(U_k)_{0\leq k\leq j}$ *is strictly increasing. That is,*

$$U_k-U_{k-1}>0, \tag{4.16}$$

for any $k\in\{1,...,j\}$.

Proof. We have

$$U_k - U_{k-1} = 2\left(2\cos\left(\frac{2k-4}{n}\pi\right) - \cos\left(\frac{2k-6}{n}\pi\right) - \cos\left(\frac{2k\pi}{n}\right)\right) \tag{4.17}$$

$$= 2\cos\left(\frac{2k-4}{n}\pi\right)\left(2-\cos\left(\frac{4\pi}{n}\right)-\cos\left(\frac{2\pi}{n}\right)\right)$$

$$+ \; 2\sin\left(\frac{2k-4}{n}\pi\right)\left(\sin\left(\frac{4\pi}{n}\right)-\sin\left(\frac{2\pi}{n}\right)\right) .$$

Let's assume that j is even. The other case is similar.

We divide the proof into four steps depending on k.

Step 1*:* Let $k \in \{1, ..., \frac{j}{2}\}$.

It is clear that equation (4.16) is satisfied for $k = 1$ and $k = 2$.

Assume that $k \in \{3, ..., \frac{j}{2}\}$. Since $\dfrac{(2k-4)\,\pi}{n} \in (0, \dfrac{\pi}{2})$, then $\cos\left(\frac{2k-4}{n}\pi\right) > 0$ and $\sin\left(\frac{2k-4}{n}\pi\right) > 0$. Moreover, $\sin\left(\frac{4\pi}{n}\right) - \sin\left(\frac{2\pi}{n}\right) > 0$, for all $n \geq 7$. Thus, Equation (4.16) is satisfied for any $k \in \{1, ..., \frac{j}{2}\}$.

Step 2*:* Let $k = \frac{j}{2} + 1$.

We have

$$U_{\frac{j}{2}+1} - U_{\frac{j}{2}} = 2\cos\left(\frac{j-2}{n}\pi\right)\left(2-\cos\left(\frac{4\pi}{n}\right)-\cos\left(\frac{2\pi}{n}\right)\right)$$

$$+ \; 2\sin\left(\frac{j-2}{n}\pi\right)\left(\sin\left(\frac{4\pi}{n}\right)-\sin\left(\frac{2\pi}{n}\right)\right) > 0 ,$$

since $\dfrac{(j-2)\,\pi}{n} \in (0, \dfrac{\pi}{2})$.

Step 3*:* $k \in \{\frac{j}{2} + 2, ..., j-1\}$.

Let $k=\dfrac{j}{2}+i$ with $i\in\{2,...,\dfrac{j}{2}-1\}$. By (4.17), we obtain

$$\begin{aligned} U_k-U_{k-1} &= U_{\frac{j}{2}+i}-U_{\frac{j}{2}+i-1} \\ &= 2\cos\left(\frac{j\pi}{n}\right)\left(2\cos\left(\frac{2i-4}{n}\pi\right)-\cos\left(\frac{2i-6}{n}\pi\right)-\cos\left(\frac{2i\pi}{n}\right)\right) \\ &+ 2\sin\left(\frac{j\pi}{n}\right)\left(-2\sin\left(\frac{2i-4}{n}\pi\right)+\sin\left(\frac{2i-6}{n}\pi\right)+\sin\left(\frac{2i\pi}{n}\right)\right) . \end{aligned}$$

By Step 1, we proved that $U_k-U_{k-1}=2\left(2\cos\left(\frac{2k-4}{n}\pi\right)-\cos\left(\frac{2k-6}{n}\pi\right)-\cos\left(\frac{2k\pi}{n}\right)\right)>0$, for $k\in\left\{1,...,\dfrac{j}{2}\right\}$, so for $i\in\{2,...,\dfrac{j}{2}-1\}$ we have

$$2\cos\left(\frac{2i-4}{n}\pi\right)-\cos\left(\frac{2i-6}{n}\pi\right)-\cos\left(\frac{2i\pi}{n}\right)>0\ .$$

Moreover, $\cos\left(\dfrac{j\pi}{n}\right)>0$ and $\sin\left(\dfrac{j\pi}{n}\right)>0$, since $\dfrac{j\pi}{n}\in(0,\dfrac{\pi}{2})$.
Finally, we need to prove that

$$-2\sin\left(\frac{2i-4}{n}\pi\right)+\sin\left(\frac{2i-6}{n}\pi\right)+\sin\left(\frac{2i\pi}{n}\right)>0\ ,$$

where $i\in\{2,...,\dfrac{j}{2}-1\}$.
Define f on $[2,\dfrac{j}{2}-1]$ as:

$$f(x):=-2\sin\left(\frac{2x-4}{n}\pi\right)+\sin\left(\frac{2x-6}{n}\pi\right)+\sin\left(\frac{2x\pi}{n}\right).$$

It is easy to see that the derivative of f is negative and $f\left(\dfrac{j}{2}-1\right)>0$. So, $U_k-U_{k-1}>0$, for $k\in\{\dfrac{j}{2}+2,...,j-1\}$.

Step 4. Let $k=j$.

We need to show that $U_j-U_{j-1}>0$.

By (4.17), we have

$$\begin{aligned} U_j - U_{j-1} &= 2\left(2\cos\left(\frac{2j-4}{n}\pi\right) - \cos\left(\frac{2j-6}{n}\pi\right) - \cos\left(\frac{2j\pi}{n}\right)\right) \\ &= 2\left(-2\cos\left(\frac{5\pi}{n}\right) + \cos\left(\frac{7\pi}{n}\right) + \cos\left(\frac{\pi}{n}\right)\right). \end{aligned}$$

Define

$$h(x) := -2\cos\left(\frac{5\pi}{x}\right) + \cos\left(\frac{7\pi}{x}\right) + \cos\left(\frac{\pi}{x}\right),$$

for $x \in [7, +\infty)$. Using the same ideas as before, we obtain $h(x) > 0$, since its derivative is negative and $\lim_{x\to\infty} h(x) = 0$. Hence $U_j - U_{j-1} > 0$, for any $n \geq 7$. □

By the three claims, we get that

$$\max_{\substack{0\leq k\leq n-1 \\ k\neq n-2}} \|T^{k+j+1}x_0\|^2 = \|T^{n+j}x_0\|^2 > \|T^{n+j-1}x_0\|^2 = \max_{0\leq k\leq n-2} \|T^{k+j+1}x_0\|^2 .$$

Hence the proof is completed. □

At the present time, we do not know the answer of the following question.

Question 1. *Is it possible to define a strict $(3, \infty)$-isometry on $\mathbb{C}^2$ with the euclidean norm?*

Acknowledgments: The first author is partially supported by grant of Ministerio de Ciencia e Innovación, Spain, project no. MTM2013-47093-P. The second author is supported by a grant of University of Gabes, UNG 933989527 and by a grant of Department of Mathematical Analysis of University of La Laguna.

The authors wish to thank the referee for many helpful comments.

References

[1] B. Abdullah, T. Le, The structure of m-isometric weighted shift operators, Oper. Matrices, **10** (2) (2016), 319-334.

[2] J. Agler, A disconjugacy theorem for Toeplitz operators, *Amer. J. Math.*, **112** (1) (1990), 1-14.

[3] J. Agler, W. Helton, M. Stankus, Classification of hereditary matrices, *Linear Algebra Appl.*, **274** (1998), 125-160.

[4] F. Bayart, m-isometries on Banach spaces, *Math. Nachr.*, **284** (17-18) (2011), 2141-2147.

[5] T. Bermúdez, A. Martinón, E. Negrín, Weighted shift operators which are m-isometries, *Integral Equations and Operator Theory,* **68** (3) (2010), 301-312.

[6] T. Bermúdez, C. Díaz Mendoza, A. Martinón, Powers of m-isometries, Studia Math., **208** (3) (2012), 249-255.

[7] T. Bermúdez, A. Martinón, J. A. Noda, An isometry plus a nilpotent operator is an m-isometry. Applications, *J. Math. Anal. Appl.*, **407** (2) (2013), 505-512.

[8] T. Bermúdez, A. Martinón, J. A. Noda, Weighted shift and composition operators on ℓ_p which are (m,q)-isometries, Linear Algebra Appl., **505** (2016), 152-173.

[9] F. Botelho, On the existence of n-isometries on ℓ_p spaces, *Acta Sci. Math. (Szeged),* **76** (2010), 183-192.

[10] S. Elaydi, An introduction to difference equations, third ed., Undergraduate Texts in Mathematics, Springer, New York, 2005. MR 2128146 (2005j:39001)

[11] C. Gu, The (m,q)-isometric weighted shifts on ℓ_p spaces, Integral Equations Operator Theory, **82** (2) (2015), 157-187.

[12] P. Hoffmann, M. Mackey, M. Ó Searcóid, On the second parameter of an (m,p)-isometry, *Integral Equations Operator Theory*, **71** (3) (2011), 389-405.

[13] P. Hoffmann, M. Mackey, (m,p)-isometric and (m,∞)-isometric operator tuples on normed spaces, Asian-Eur. J. Math, **8** (2) (2015), (32 pages).

[14] P. Hoffmann, A note on operator tuples which are (m,p)-isometric as well as (μ,∞)-isometric, Oper. Matrices, **11** (3) (2017), 623-633.

[15] Z. J. Jabłonski, Complete hyperexpansivity, subnormality and inverted boundedness conditions, Integral Equations Operator Theory, **44** (3) (2002), 316-336.

E-mail address: tbermude@ull.es

E-mail address: hajer_zaway@live.fr

Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de La Laguna,, 38271 La Laguna (Tenerife), Spain

Department of Mathematics, Faculty of Sciences, University of Gabès, 6072 Gabès, Tunisia