A semiclassical approach to quantum gravity

(An educated review of "Sourcing semiclassical gravity from spontaneously localized quantum matter" [AT16])

Daniel Panea Lichtig

June 7, 2017

Abstract

This work is an educated review of the article "Sourcing semiclassical gravity from spontaneously localized quantum matter" by Antoine Tilloy and Lajos Diósi [AT16]. A complete introduction to the mathematical formalism necessary to understand the original article is given. Tools commonly used in quantum measurement theory are developed and applied to the special case of the continuous measurement of mass density which we then use to construct a theory of semiclassical gravity. We then try to solve the resulting dynamics by giving an analytical approximation which leads to important conclusions about the dependence of the dynamics on the parameters of the theory. Afterwards we will visualize the complete dynamics of the equations by simulating the time evolution of the density matrix elements for two concrete systems: First for the simple case of a spin in a magnetic field with added decoherence and feedback and then for a localized particle in a spatial superposition. In this way we illustrate the gravitational interaction and decoherence predicted by the theory.

Abstract (Español)

Este trabajo es una revisión educada del artículo "Sourcing semiclassical gravity from spontaneously localized quantum matter" de Antoine Tilloy y Lajos Diósi [AT16]. Daremos una introducción completa al formalismo matemático necesario para entender el artículo original. Se desarrollarán herramientas comúnmente usadas en la teoría cuántica de la medida y se aplicarán al caso especial de la medida continua de la densidad de masa que luego usaremos para construir una teoría semiclásica de la gravedad. A continuación intentaremos resolver la dinámica resultante con una aproximación analítica que nos llevará a conclusiones importantes sobre la dependencia de la dinámica de los parámetros de la teoría. Luego visualizaremos la dinámica completa de las ecuaciones simulando la evolución temporal del operador densidad para dos sistemas concretos: Primero para el caso sencillo de un espín en un campo magnético añadiendo decoherencia y realimentación y luego para el caso de una partícula localizada en superposición espacial. De esta forma ilustraremos la interacción gravitatoria y la decoherencia predicha por la teoría.

Contents

1 Introduction

After nearly half a century of concentrated effort on marrying quantum mechanics and gravity in a unified theory there is still no consistent and accepted theory of quantum gravity. This problem has proven to be very hard to attack directly, which leads to considering a semiclassical approximation to quantum gravity before developing an eventually complete quantum-gravitational formalism. Such a semiclassical approach has proven to be very useful historically as in the development of ordinary quantum mechanics. The main objective of the article "Sourcing semiclassical gravity from spontaneously localized quantum matter" [AT16] is to provide such a semiclassical approximation to full quantum gravity. It is semiclassical because it considers the dynamics of quantum matter coupled to classical (non-quantized) Newtonian (non-relativistic) gravity.

In this article I will give an educated review of the work of Tilloy and Diósi on semiclassical gravity, explaining their procedures, elaborating on details that are discussed very briefly in their work and testing their proposed formalism on concrete examples.

The main problem in semiclassical gravity is how to couple quantum matter and classical spacetime. The standard approach is due to Møller [Mø62] and Rosenfeld [Ros63] : use the quantum mechanical average to get a classical quantity out of the energy-momentum tensor operator $\hat{T}_{\mu\nu}$ of quantized matter:

$$G_{\mu\nu} = 8\pi G \langle \Psi | \hat{T}_{\mu\nu} | \Psi \rangle \tag{1.1}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is the Einstein tensor which is used to source gravity in the framework of general relativity. This formalism however has several serious problems. First of all it completely ignores the quantum fluctuations of the energy-momentum tensor operator $\hat{T}_{\mu\nu}$. Secondly, since we are not only interested in how to source gravity *from* quantum matter, but also in the backaction of gravity *on* quantum matter, the classical gravitational field has to enter in some way in the evolution equation for the quantum mechanical state vector Ψ. If one tries to incorporate the classical gravitational field into the evolution of the quantum matter in the form of 1.1, one faces several difficulties. Since having a quantum average $\langle \hat{T}_{\mu\nu} \rangle$ in any deterministic dynamics such as the Schrödinger equation spoils the linearity of quantum mechanics, the resulting dynamics possess deep fundamental anomalies, one of the most spectacular ones being faster-than-light communication: As shown in [Gis89], if one wants to rule out superluminal signaling, this implies a linear evolution at the statistical level. These anomalies already appear in the Newtonian regime as they are essentially of quantum and not of relativistic nature. The proposal of Tilloy and Diósi is that these anomalies can be resolved in a specific framework called spontaneous collapse or continuous measurement models, which were originally developed in the context of the measurement problem in quantum mechanics. These frameworks provide a consistent way to get classical quantities out of quantum observables through the modelling of the measurement act. Most spontaneous collapse models introduce a small nonlinear stochastic term in the Schrödinger equation that produces a dynamical collapse of macroscopic superpositions. Tilloy and Diósi introduce gravity into these models and propose that it, among other things, is responsible for the absence of macroscopic superpositions. In the following we will give an introduction to quantum measurement theory, spontaneous collapse models and feedback models and their application to semiclassical gravity. We will develop the full formalism of semiclassical gravity in the framework of spontaneous collapse models as proposed by Tilloy and Diósi and then elaborate its predictions by application to specific examples.

2 A Brief Introduction to Quantum Measurement Theory

The contents of this section have been mostly compiled from the books "Quantum Measurement and Control" by Howard M. Wiseman and Gerard J. Milburn [WM09] and "The Theory of Open Quantum Systems" by Heinz-Peter Breuer and Francesco Petruccione [BP02].

2.1 Projective Measurements

In the formalism of traditional quantum mechanics, the measurement of a generic observable $\hat{\Lambda}$ is described by so-called projective measurements. Any observable $\hat{\Lambda}$ can be diagonalized as

$$\hat{\Lambda} = \sum_{\lambda} \lambda \hat{\Pi}_{\lambda} \tag{2.1}$$

where λ are the real eigenvalues of the observable which are assumed to be discrete for simplicity and $\hat{\Pi}_{\lambda}$ are the projection operators onto the subspace with eigenvalue λ. If the spectrum of the eigenvalue λ is non degenerate, the projection operator is simply $|\lambda\rangle\langle\lambda|$, where $|\lambda\rangle$ is the eigenvector of $\hat{\Lambda}$ with eigenvalue λ.

If we measure $\hat{\Lambda}$ at a time t, during a measurement time T short enough such that we can suppose that the evolution of the system is negligible during the duration T of the measurement, the probability of obtaining a particular eigenvalue λ is

$$\mathcal{P}_{\lambda}(t) = \langle\Psi|\hat{\Pi}_{\lambda}|\Psi\rangle \tag{2.2}$$

in the case of a pure state $|\Psi\rangle$, and

$$\mathcal{P}_{\lambda}(t) = \mathrm{Tr}[\hat{\Pi}_{\lambda}\rho(t)\hat{\Pi}_{\lambda}] \tag{2.3}$$

in the general case of a mixed state where ρ is the density matrix that describes the system. This is the von Neumann projection postulate of standard quantum mechanics. For transparency first consider a pure state, where we can expand the initial state as:

$$|\Psi(t)\rangle = \sum_{\lambda} c_{\lambda}|\lambda\rangle. \tag{2.4}$$

The state after a measurement with outcome λ is:

$$|\Psi(t+T)\rangle = |\lambda\rangle = \frac{\hat{\Pi}_{\lambda}|\Psi(t)\rangle}{c_{\lambda}} = \frac{\hat{\Pi}_{\lambda}|\Psi(t)\rangle}{\sqrt{\mathcal{P}_{\lambda}}} \tag{2.5}$$

in the general case of a mixed state, the conditional state of the system after the measurement is:

$$\rho_{\lambda}(t+T) = \frac{\hat{\Pi}_{\lambda}\rho(t)\hat{\Pi}_{\lambda}}{\mathcal{P}_{\lambda}} = \frac{\hat{\Pi}_{\lambda}\rho(t)\hat{\Pi}_{\lambda}}{\mathrm{Tr}[\hat{\Pi}_{\lambda}\rho(t)\hat{\Pi}_{\lambda}]}. \tag{2.6}$$

If we want to consider the case of unconditional measurement, that is one performs the measurement but ignores the result, the unconditional post-measurement state of the system is a mixed state which is the sum of all possible resulting post-measurement conditional states, weighted by the probability of each outcome:

$$\rho(t+T) = \sum_{\lambda} \mathcal{P}_{\lambda}\rho_{\lambda}(t+T) = \sum_{\lambda} \hat{\Pi}_{\lambda}\rho(t)\hat{\Pi}_{\lambda}. \tag{2.7}$$

This can also be interpreted as the density matrix describing a big number of individual equivalent systems, and the measurements are performed on each one of them.

2.2 Generalized Quantum Measurements

Consider now, instead of making completely sharp measurements with perfect accuracy, modelled by the sharp projection operators $\hat{\Pi}_{\lambda} = |\lambda\rangle\langle\lambda|$, an imperfect measurement which leaves some residual uncertainty about the state of the system, modelled by a gaussian distribution. If the result of the measurement gives λ, due to the unsharp nature of the measurement apparatus, the state has not collapsed completely to the eigenfunction $|\lambda\rangle$. Instead, it has collapsed primarily to $|\lambda\rangle$ but also to a series of eigenfunctions around it, weighted by the gaussian distribution:

$$|\Psi(t)\rangle = \sum_{\mu} c_{\mu}|\mu\rangle \rightarrow |\Psi(t+T)\rangle = \mathcal{N}\sum_{\mu} c_{\mu}e^{-(\mu-\lambda)^2/4\sigma^2}|\mu\rangle \tag{2.8}$$

where λ is the measurement outcome and $\mathcal{N}$ is a normalization constant. In the limit case $\sigma \rightarrow 0$, the gaussian becomes a delta-function and we recover the sharp measurement outcome.
The projection operators in the unsharp measurement case, called smooth operator effects are

$$\hat{\Omega}_{\lambda} = \mathcal{N}\sum_{\mu} e^{-(\mu-\lambda)^2/4\sigma^2}|\mu\rangle\langle\mu| \tag{2.9}$$

where λ is the measurement outcome, $\hat{\Omega}_{\lambda}$ are the smooth operator effects associated to each measurement outcome λ , μ and $|\mu\rangle$ are the eigenvalues and eigenstates of the measured observable $\hat{\Lambda}$ respectively and $\mathcal{N}$ is a normalization constant defined for a continuous spectrum as

$$\begin{aligned} &\mathcal{N}^2 \int \mathrm{d}\mu e^{-(\mu-\lambda)^2/2\sigma^2} = \mathcal{N}^2\sqrt{2\pi\sigma^2} = 1 \\ &\rightarrow \mathcal{N} = (2\pi\sigma^2)^{-1/4}. \end{aligned} \tag{2.10}$$

Using the expression for the observable $\hat{\Lambda}$ in terms of its eigenstates

$$\hat{\Lambda} = \sum_{\lambda} \lambda|\lambda\rangle\langle\lambda| = \sum_{\mu} \mu|\mu\rangle\langle\mu| \tag{2.11}$$

we can rewrite the smooth operator effects as

$$\hat{\Omega}_{\lambda} = \mathcal{N}e^{-(\hat{\Lambda}-\lambda)^2/4\sigma^2} \tag{2.12}$$

The conditional post-measurement state in the selective case is (taking into account that $\hat{\Omega}_\lambda$ is hermitian)

$$\rho \to \frac{\hat{\Omega}_\lambda \rho \hat{\Omega}_\lambda}{\mathcal{P}_\lambda} \tag{2.13}$$

where the probability $\mathcal{P}_\lambda$ is given by

$$\mathcal{P}_\lambda = \mathrm{Tr}[\hat{\Omega}_\lambda \rho \hat{\Omega}_\lambda] \tag{2.14}$$

In the non-selective case, the post-measurement state is given by

$$\rho \to \int \mathrm{d}\lambda \hat{\Omega}_\lambda \rho \hat{\Omega}_\lambda \tag{2.15}$$

Consider now continuously performing measurements of $\hat{\Lambda}$ on the system. This can be conceived as making periodic measurements with period τ, and taking the limit $\tau \to 0$. A dynamical equation for the state of the system ρ, independent of σ and τ can be derived if one takes the measurement error σ proportional to $\tau^{-1/2}$ in the limit $\tau \to 0$. Define in this limit

$$g \equiv \lim_{\sigma\to\infty,\tau\to 0} \frac{1}{\tau\sigma^2} \tag{2.16}$$

The evolution equation for the state of the system in the case in which no measurement is performed, is the usual Schrödinger-von Neumann equation:

$$\frac{d\rho}{dt} = -i\big[\hat{H}, \rho\big] \tag{2.17}$$

In our considered limit of continuous measurement it is possible to derive a modified Schrödinger-von Neumann equation that incorporates the dynamical consequences of the measurement process. In this continuous limit we write the smooth operator effects as [Dio88]:

$$\hat{\Omega}_\lambda = \mathcal{N} e^{-(\hat{\Lambda}-\lambda)^2/4\sigma^2} = \mathcal{N} e^{-g(\sqrt{\tau}\hat{\Lambda}-\sqrt{\tau}\lambda)^2/4} \tag{2.18}$$

and the post-measurement state in the non-selective case is

$$\begin{aligned}\rho \to \int \mathrm{d}\lambda \hat{\Omega}_\lambda \rho \hat{\Omega}_\lambda &= \mathcal{N}^2 \int \mathrm{d}\lambda e^{-g(\sqrt{\tau}\hat{\Lambda}-\sqrt{\tau}\lambda)^2/4} \rho e^{-g(\sqrt{\tau}\hat{\Lambda}-\sqrt{\tau}\lambda)^2/4} \\ &= \frac{\mathcal{N}^2}{\sqrt{\tau}} \int \mathrm{d}\phi e^{-g(\sqrt{\tau}\hat{\Lambda}-\phi)^2/4} \rho e^{-g(\sqrt{\tau}\hat{\Lambda}-\phi)^2/4}\end{aligned} \tag{2.19}$$

where we have performed a change of variable $\sqrt{\tau}\lambda \to \phi$. We now expand the exponentials around $\sqrt{\tau} = 0$:

$$\begin{aligned}e^{-g(\sqrt{\tau}\hat{\Lambda}-\phi)^2/4} &= e^{-g\phi^2/4}\, e^{-g\tau\hat{\Lambda}^2/4}\, e^{g\sqrt{\tau}\phi\hat{\Lambda}/2} \\ &= e^{-g\phi^2/4}\left(1 - \frac{g\tau\hat{\Lambda}^2}{4} + \mathcal{O}(\tau^2)\right)\left(1 + \frac{g\sqrt{\tau}\phi\hat{\Lambda}}{2} + \frac{g^2\tau\phi^2\hat{\Lambda}^2}{8} + \mathcal{O}(\tau^{3/2})\right) \\ &= e^{-g\phi^2/4}\left(1 + \frac{g\sqrt{\tau}\phi\hat{\Lambda}}{2} + \frac{g^2\tau\phi^2\hat{\Lambda}^2}{8} - \frac{g\tau\hat{\Lambda}^2}{4} + \mathcal{O}(\tau^{3/2})\right)\end{aligned} \tag{2.20}$$

inserting this in the formula for the post-measurement state we obtain (up to orders $\mathcal{O}(\tau^{3/2})$):

$$\begin{aligned}\rho \rightarrow & \frac{\mathcal{N}^2}{\sqrt{\tau}} \int \mathrm{d}\phi e^{\frac{-g\phi^2}{2}} \left(1 + \frac{g\sqrt{\tau}\phi\hat{\Lambda}}{2} + \frac{g^2\tau\phi^2\hat{\Lambda}^2}{8} - \frac{g\tau\hat{\Lambda}^2}{4}\right) \rho \left(1 + \frac{g\sqrt{\tau}\phi\hat{\Lambda}}{2} + \frac{g^2\tau\phi^2\hat{\Lambda}^2}{8} - \frac{g\tau\hat{\Lambda}^2}{4}\right) \\ = & \frac{\mathcal{N}^2}{\sqrt{\tau}} \int \mathrm{d}\phi e^{\frac{-g\phi^2}{2}} \left(\rho + \frac{g\phi\sqrt{\tau}}{2}(\hat{\Lambda}\rho + \rho\hat{\Lambda}) + \left(\frac{g^2\phi^2\tau}{8} - \frac{g\tau}{4}\right)(\hat{\Lambda}^2\rho + \rho\hat{\Lambda}^2) + \frac{g^2\phi^2\tau}{4}\hat{\Lambda}\rho\hat{\Lambda}\right) \\ = & \frac{\mathcal{N}^2}{\sqrt{\tau}} \left[\sqrt{\frac{2\pi}{g}}\rho + \left(\frac{g^2\tau\sqrt{\pi}}{16}\left(\frac{2}{g}\right)^{3/2} - \frac{g\tau}{4}\sqrt{\frac{2\pi}{g}}\right)(\hat{\Lambda}^2\rho + \rho\hat{\Lambda}^2) + \frac{g^2\tau\sqrt{\pi}}{8}\hat{\Lambda}\rho\hat{\Lambda}\right]\end{aligned} \tag{2.21}$$

substituting the value for $\mathcal{N}$ from 2.10, $\mathcal{N} = (2\pi\sigma^2)^{-1/4} = (\frac{2\pi}{g\tau})^{-1/4}$ we obtain:

$$\rho \rightarrow \rho + \left(\frac{g}{4}\hat{\Lambda}\rho\hat{\Lambda} - \frac{g}{8}(\hat{\Lambda}^2\rho + \rho\hat{\Lambda}^2)\right)\tau = \rho - \frac{g}{8}[\hat{\Lambda}, [\hat{\Lambda}, \rho]]\tau \tag{2.22}$$

Where the term proportional to $\sqrt{\tau}$ has vanished due to the gaussian integration. In the limit $\tau \rightarrow 0$ this expression can be rewritten as

$$\frac{d\rho}{dt}_{Measurement} = -\frac{g}{8}[\hat{\Lambda}, [\hat{\Lambda}, \rho]] \tag{2.23}$$

and the equation describing the full evolution of the state incorporating the dynamics caused by the measurement process is

$$\frac{d\rho}{dt} = -i[\hat{H}, \rho] - \frac{g}{8}[\hat{\Lambda}, [\hat{\Lambda}, \rho]] \tag{2.24}$$

An interesting point is that one expects the term proportional to $\sqrt{\tau}$ to be relevant in the limit $\tau \rightarrow 0$, however it has vanished due to the gaussian integration. The integration over ϕ in 2.21 can be interpreted as an average over all possible measurement outcomes which are weighted with the gaussian around $\phi = 0$, in the limit case of infinitely many measurements. The outcome of a single measurement however is an inherently stochastic event, and we can introduce a stochastic "Wiener" process W(t). The change of this process over a time interval τ is the Wiener increment $\Delta W = \sqrt{\tau}\phi = \sqrt{\frac{\tau}{g}}w$, where w is a random number with zero mean and unit variance. In the continuous limit we can write

$$\begin{aligned}\langle dW \rangle &= 0 \\ dW^2 &= \frac{d\tau}{g}\end{aligned} \tag{2.25}$$

In the case of a big, but finite number of measurements, the integral over the measurement outcomes ϕ in 2.21 becomes a sum over each measurement outcome represented by the stochastic variable W. The term proportional to $\sqrt{\tau}$ now does not vanish, but instead yields the stochastic term

$$\frac{g}{2}(\hat{\Lambda}\rho + \rho\hat{\Lambda})dW \tag{2.26}$$

which has zero mean as expected. The now stochastic dynamics caused by the measurement process are then described by the equation

$$d\rho = -i\big[H,\rho\big]dt - \frac{g}{8}\big[\hat{\Lambda},[\hat{\Lambda},\rho]\big]dt + \frac{g}{2}(\hat{\Lambda}\rho + \rho\hat{\Lambda})dW \tag{2.27}$$

However this equation is not trace-preserving. One has to subtract the change in trace to keep the state normalized. This leads to the final stochastic master equation (SME) describing the stochastic dynamics of the system affected by the measurement process

$$d\rho = -i\big[H,\rho\big]dt - \frac{g}{8}\big[\hat{\Lambda},[\hat{\Lambda},\rho]\big]dt + \frac{g}{2}(\hat{\Lambda}\rho + \rho\hat{\Lambda} - 2\rho\langle\hat{\Lambda}\rangle)dW \tag{2.28}$$

Which is trace preserving as is easily checked. The outcome of the time-continuous measurement of the observable $\hat{\Lambda}$ is a time-dependent (classical) signal Λ_t which fluctuates around the quantum-mechanical average due to the stochastic nature of the measurements on the quantum system

$$\Lambda_t = \langle\hat{\Lambda}\rangle_t + \delta\Lambda_t \tag{2.29}$$

we can relate the Wiener increment to the time-dependent signal writing heuristically

$$\frac{dW}{dt} = \delta\Lambda_t \tag{2.30}$$

which gives a correlation for $\delta\Lambda_t$ as

$$\langle\delta\Lambda_t\delta\Lambda_\tau\rangle = \frac{1}{g}\delta(t-\tau) \tag{2.31}$$

Defining

$$\mathcal{H}\big[\hat{\Lambda}\big]\rho = \big\{\hat{\Lambda} - \langle\hat{\Lambda}\rangle, \rho\big\} = \hat{\Lambda}\rho + \rho\hat{\Lambda} - 2\rho\langle\hat{\Lambda}\rangle \tag{2.32}$$

one can rewrite the SME 2.28, in Itô sense as

$$\frac{d\rho}{dt} = -i\big[H,\rho\big] - \frac{g}{8}\big[\hat{\Lambda},[\hat{\Lambda},\rho]\big] + \frac{g}{2}\mathcal{H}\big[\hat{\Lambda}\big]\rho\delta\Lambda_t \tag{2.33}$$

which describes the dynamics of the density operator, incorporating the effects of the continuous measurement process.

2.3 Feedback

The time-dependent signal Λ_t obtained from the continuous measurement can be used to control the subsequent evolution of the system through feedback [WM09]. In a Markovian feedback scheme, this is done by applying a potential proportional to the signal:

$$\hat{V}_t = \Lambda_t\hat{B} \tag{2.34}$$

where $\hat{B}$ is another observable that can be chosen freely. In the feedback scheme, the feedback potential induces a unitary evolution $\exp(-i\hat{V}_t dt)$ an infinitesimal amount of time dt after the free evolution:

$$\rho + d\rho \rightarrow e^{-i\hat{V}_t dt}\left(\rho + d\rho^{free}\right)e^{i\hat{V}_t dt} \tag{2.35}$$

where $d\rho^{free}$ is given by the SME 2.28. Inserting expression 2.29 for the signal Λ_t and taking into account the relation 2.30 between the signal noise and the Wiener increment gives

$$\begin{aligned} \rho + d\rho \rightarrow & e^{-i(\langle\hat{\Lambda}\rangle_t + \delta\Lambda_t)\hat{B}dt}(\rho + d\rho^{free})e^{i(\langle\hat{\Lambda}\rangle_t + \delta\Lambda_t)\hat{B}dt} \\ = & e^{-i\langle\hat{\Lambda}\rangle_t\hat{B}dt}e^{-i\hat{B}dW}(\rho + d\rho^{free})e^{i\langle\hat{\Lambda}\rangle_t\hat{B}dt}e^{i\hat{B}dW} \end{aligned} \tag{2.36}$$

Expanding the exponentials up to first order in dt taking into account that the square variance of the Wiener increment from 2.25 is $dW^2 = dt/g$:

$$\begin{aligned} \rho + d\rho \rightarrow & (1 - i\langle\hat{\Lambda}\rangle_t\hat{B}dt)(1 - i\hat{B}dW - \frac{1}{2g}\hat{B}^2dt)(\rho + d\rho^{free}) \\ & \times (1 + i\langle\hat{\Lambda}\rangle_t\hat{B}dt)(1 + i\hat{B}dW - \frac{1}{2g}\hat{B}^2dt) \\ = & \rho - i[H,\rho] - \frac{g}{8}[\hat{\Lambda},[\hat{\Lambda},\rho]]dt + \frac{g}{2}\mathcal{H}[\hat{\Lambda}]\rho dW - i\langle\hat{\Lambda}\rangle[\hat{B},\rho]dt - i[\hat{B},\rho]dW \\ & - \frac{i}{2}[\hat{B},\mathcal{H}[\hat{\Lambda}]\rho]dt - \frac{1}{2g}\{\hat{B}^2,\rho\}dt + \frac{1}{g}\hat{B}\rho\hat{B}dt \\ = & \rho - i[H,\rho]dt - \frac{g}{8}[\hat{\Lambda},[\hat{\Lambda},\rho]]dt + \frac{g}{2}\mathcal{H}[\hat{\Lambda}]\rho\delta\Lambda_t dt - i[\hat{B},\rho]\delta\Lambda_t dt \\ & - \frac{i}{2}[\hat{B},\{\hat{\Lambda},\rho\}]dt - \frac{1}{2g}[\hat{B},[\hat{B},\rho]]dt \end{aligned} \tag{2.37}$$

where in the last step we used the expression 2.32 for $\mathcal{H}[\hat{\Lambda}]\rho$. This gives the final stochastic master equation incorporating the dynamics induced by the measurement and the feedback:

$$\frac{d\rho}{dt} = -i[H + \hat{B}\delta\Lambda_t,\rho] - \frac{g}{8}[\hat{\Lambda},[\hat{\Lambda},\rho]] - \frac{i}{2}[\hat{B},\{\hat{\Lambda},\rho\}] - \frac{1}{2g}[\hat{B},[\hat{B},\rho]] + \frac{g}{2}\mathcal{H}[\hat{\Lambda}]\rho\delta\Lambda_t \tag{2.38}$$

2.4 Generalization to multiple observables

The equations obtained in continuous measurement and feedback theory can be generalized straightforwardly to multiple observables. In the case of the simultaneous continuous monitoring of a set of n observables $\{\hat{\Lambda}_\mu\}$, indexed by the subscript $\mu = 0, 1, ..., n$, there are n time-dependent continuous classical signals, which fluctuate around the quantum-mechanical average of the observable:

$$\Lambda_{\mu,t} = \langle\hat{\Lambda}_\mu\rangle_t + \delta\Lambda_{\mu,t} \tag{2.39}$$

In general, the signals associated to different observables $\hat{\Lambda}_\mu$ can be correlated, and the correlation is encoded in the correlation strength matrix g :

$$\langle\delta\Lambda_{\mu,t}\delta\Lambda_{\nu,t}\rangle = \left(g^{-1}\right)_{\mu\nu}\delta(t-\tau) \tag{2.40}$$

where g^{-1} is the inverse of the matrix g. The stochastic master equation 2.33 reads in this case of multiple monitored observables:

$$\frac{d\rho}{dt} = -i[H,\rho] - \frac{g_{\mu\nu}}{8}[\hat{\Lambda}_\mu,[\hat{\Lambda}_\nu,\rho]] + \frac{g_{\mu\nu}}{2}\mathcal{H}[\hat{\Lambda}_\mu]\rho\delta\Lambda_{\nu,t} \tag{2.41}$$

where the Einstein summation convention over repeated indices is understood. The feedback potential can be generalized in the case of multiple observables as

$$\hat{V} = \Lambda_\mu \hat{B}_\mu \tag{2.42}$$

Using this potential generalizes the SME for the complete evolution including feedback 2.38 to

$$\begin{aligned} \frac{d\rho}{dt} =& - i\big[H + \hat{B}_\mu \delta\Lambda_{\mu,t}, \rho\big] - \frac{g_{\mu\nu}}{8}\big[\hat{\Lambda}_\mu, [\hat{\Lambda}_\nu, \rho]\big] - \frac{i}{2}\big[\hat{B}_\mu, \{\hat{\Lambda}_\mu, \rho\}\big] \\ & - \frac{1}{2}(g^{-1})_{\mu\nu}\big[\hat{B}_\mu, \big[\hat{B}_\nu, \rho\big]\big] + \frac{g_{\mu\nu}}{2}\mathcal{H}\big[\hat{\Lambda}_\mu\big]\rho\delta\Lambda_{\nu,t} \end{aligned} \tag{2.43}$$

3 Application of quantum measurement theory to semiclassical quantum gravity

3.1 General case

Let us now go back to the original problem: to develop a semiclassical theory of quantum gravity which couples quantum matter to a classical gravitational field and describes the combined dynamics. The question persists, how does one couple a quantum system to a classical one? We have already seen in the introduction that the simplest approach, namely getting a classical quantity out of a quantum operator using the quantum mechanical expectation value is not satisfactory. The approach proposed by Tilloy and Diósi is to make use of the previously developed continuous monitoring models. This may seem *ad hoc* at first sight, however, by construction, they are very useful for consistently getting the corresponding classical quantities out of quantum observables and thus for coupling quantum matter to a classical spacetime, since they model the measurement process in which we get a classical signal from the measurement on a quantum system.

To construct the model of semiclassical gravity using the previously developed continuous monitoring models, consider the continuous measurement of the mass density operators $\hat{\varrho}_\sigma$ at each point in space $\mathbf{r}$ in the case of N particles:

$$\hat{\varrho}_\sigma(\mathbf{r}) = \sum_{n=0}^{N} m_n g_\sigma(\mathbf{r} - \hat{\mathbf{x}}_n) \tag{3.1}$$

where $\hat{\mathbf{x}}_n$ is the position operator for the n-th particle. We use a normalized gaussian g_σ of width σ instead of delta-functions, to keep the theory finite. The monitoring of these mass density operators amounts to a generalization of the monitoring of a finite and discrete set of n observables developed in the last section. Now we have an infinite set of observables $\{\hat{\varrho}_\sigma(\mathbf{r})\}$ and the previously discrete index μ becomes the continuous variable $\mathbf{r}$. The interpretation is also different: Although we use the formalism of continuous monitoring models, the monitoring of the mass density operators is not to be thought of as performed by any physical detectors but rather is postulated to be a fundamental feature of nature. Obviously, there are no physical detectors constantly measuring throughout all of space, but we postulate that somehow the continuous monitoring just "happens". This may as well be an emergent phenomenon, since we are only constructing a semiclassical approximation to quantum gravity, and this point may become clearer in later, more complete theories. Thus, the continuous measurement of the mass density operator at each point in space $\mathbf{r}$ gives the time-continuous signal

$$\varrho_t(\mathbf{r}) = \langle \hat{\varrho}_\sigma \rangle_t + \delta \varrho_t(\mathbf{r}) \tag{3.2}$$

This signal is analogous to the signal $\Lambda_{\mu,t}$ in the case of multiple observables in the last section but instead of a discrete index μ we now have the continuous index $\mathbf{r}$. The generalization from a discrete index to a continuous one is straightforward. The signal is correlated in an analogous way:

$$\langle \delta \varrho_t(\mathbf{r}) \delta \varrho_\tau(\mathbf{s}) \rangle = \left(g^{-1}\right)_{\mathbf{rs}} \delta(t-\tau) \tag{3.3}$$

where $g_{\mathbf{rs}}$ now instead of a matrix is a (continuous) non-negative integral kernel that encodes the correlation between the outcomes of the measurements of the mass density $\hat{\varrho}_\sigma$ at different points in space $\mathbf{r}$ and $\mathbf{s}$. To generalize the SME 2.33 describing the evolution of the density matrix under the influence of the continuous measurement we just replace the sums with integrals:

$$\frac{d\rho}{dt} = -i\left[H, \rho\right] - \int \mathrm{d}\mathbf{r}\mathrm{d}\mathbf{s} \frac{g_{\mathbf{rs}}}{8} \left[\hat{\varrho}_\sigma(\mathbf{r}), [\hat{\varrho}_\sigma(\mathbf{s}), \rho]\right] + \int \mathrm{d}\mathbf{r}\mathrm{d}\mathbf{s} \frac{g_{\mathbf{rs}}}{2} \mathcal{H}\left[\hat{\varrho}_\sigma(\mathbf{r})\right] \rho \delta \varrho_t(\mathbf{s}) \tag{3.4}$$

This equation describes the effect that monitoring the mass density operators at each point in space has on the evolution of the state ρ of the quantum system. Through the monitoring we have now obtained a classical quantity, the signal $\varrho_t(\mathbf{r})$ out of the quantum operators $\hat{\varrho}_\sigma$ in a consistent way. Recall that in this interpretation there are no actual detectors measuring the mass density, instead the monitoring of the mass density operators is given as an inherently natural process. Following this line of thought we promote the signal $\varrho_t(\mathbf{r})$ to a real, physical quantity which is the source of the classical Newtonian gravitational field Φ_t. We source the gravitational field via the standard gravitational (classical) Poisson equation from the signal $\varrho_t(\mathbf{r})$:

$$\nabla^2 \Phi_t(\mathbf{r}) = 4\pi G \varrho_t(\mathbf{r}) \tag{3.5}$$

This implements the backaction of quantum matter on gravity. We will often suppress the subindex t for magnitudes related to the signal $\varrho_t(\mathbf{r})$. Notice that the Newton gravitational field Φ_t, since it depends on the stochastic signal $\varrho_t(\mathbf{r})$, is now also a stochastic quantity. The Poisson equation can also be written as

$$\Phi_t(\mathbf{r}) = -G \int \mathrm{d}\mathbf{s} \frac{\varrho_t(\mathbf{s})}{|\mathbf{r}-\mathbf{s}|} \tag{3.6}$$

To implement the backaction of gravity *on* quantum matter, we use the previously developed feedback scheme. As the feedback potential proportional to the signal $\varrho_t(\mathbf{r})$ we use the stochastic semiclassical Newton interaction potential $\hat{V}_{Gscl}$:

$$\hat{V}_{Gscl} = \int \mathrm{d}\mathbf{r} \varrho_t(\mathbf{r}) \hat{\Phi}_\sigma(\mathbf{r}) = \int \mathrm{d}\mathbf{r} \Phi_t(\mathbf{r}) \hat{\varrho}_\sigma(\mathbf{r}) \tag{3.7}$$

Where Φ_t is the classical gravitational field sourced from the signal, and $\hat{\Phi}_\sigma$ is the gravitational field operator sourced from the mass density operator via the operator

Poisson equation $\nabla^2\hat{\Phi}_\sigma(\mathbf{r}) = 4\pi G\hat{\varrho}_\sigma(\mathbf{r})$. With this feedback potential, following the same steps as before in 2.37 and generalizing the SME 2.43 to continuous indices, the SME that describes the complete evolution gives:

$$\begin{aligned}\frac{d\rho}{dt} =& - i\Big[H + \hat{V}_{G,\sigma} + \int \mathrm{d}\mathbf{r}\delta\varrho_t(\mathbf{r})\hat{\Phi}_\sigma, \rho\Big] + \int \mathrm{d}\mathbf{r}\mathrm{d}\mathbf{s}\frac{g_{\mathbf{rs}}}{2}\mathcal{H}\big[\hat{\varrho}_\sigma(\mathbf{r})\big]\rho\delta\varrho_t(\mathbf{s}) \\ &- \int \mathrm{d}\mathbf{r}\mathrm{d}\mathbf{s}\left(\frac{g_{\mathbf{rs}}}{8}\big[\hat{\varrho}_\sigma(\mathbf{r}),\big[\hat{\varrho}_\sigma(\mathbf{s}),\rho\big]\big] + \frac{(g^{-1})_{\mathbf{rs}}}{2}\big[\hat{\Phi}_\sigma(\mathbf{r}),\big[\hat{\Phi}_\sigma(\mathbf{s}),\rho\big]\big]\right)\end{aligned} \tag{3.8}$$

With this equation we have recovered the Newton interaction potential that contributes to the deterministic Hamiltonian evolution of the system

$$\hat{V}_{G,\sigma} = \frac{1}{2}\int \mathrm{d}\mathbf{r}\hat{\varrho}_\sigma(\mathbf{r})\hat{\Phi}_\sigma(\mathbf{r}) \tag{3.9}$$

meaning that the gravitational interaction energy operator just adds to the Hamiltonian, as expected. The terms proportional to the signal noise $\delta\varrho_t$ implement the stochastic nature of the measurement process or analogously represent the quantum nature of the system. The term with the mass density operator $\hat{\varrho}_\sigma$ in the double commutator implements the decoherence caused by the continuous monitoring of the mass density and the term with the gravitational field operator $\hat{\Phi}_\sigma$ in the double commutator represents an additional decoherence caused by the gravitational backaction. Notice that the two decoherence terms are competing since one depends directly on the correlation strength $g_{\mathbf{rs}}$ and the other one on its inverse.
Let us summarize the model of stochastic semiclassical gravity that we have just developed (see figure 3.1): the mass density operators $\hat{\varrho}_\sigma$ are continuously monitored (not by physical detectors, rather by a process of nature itself). This continuous measurement of the mass density operators yields the classical stochastic signal ϱ_t containing the stochastic fluctuations $\delta\varrho_t$. This signal is promoted to a real physical entity, and to the source of the classical gravitational field Φ_t. This implements the backaction of the quantum matter, described by the quantum density matrix ρ on the classical gravitational field Φ_t. The continuous measurement of the mass density $\hat{\varrho}_\sigma(\mathbf{r})$ introduces a decoherence in the evolution of the density matrix, which tends to become diagonal in the position basis. The classical gravitational field Φ_t together with the quantum mass density operator $\hat{\varrho}_\sigma$ are then used to form the semiclassical gravitational interaction potential $\hat{V}_{Gscl}$. This potential is then used in a feedback scheme to contribute to the subsequent evolution of the quantum density matrix ρ. This implements the backaction of the classical gravitational field Φ_t on the quantum matter. The gravitational backaction adds to the unitary evolution of the state in the form of additional Hamiltonian terms, and also introduces an additional decoherence of the density matrix caused by the gravitational field. The non-Hamiltonian terms are responsible for the collapse of the state and the localization of the mass density, preventing large quantum fluctuations in mass and macroscopic superpositions.

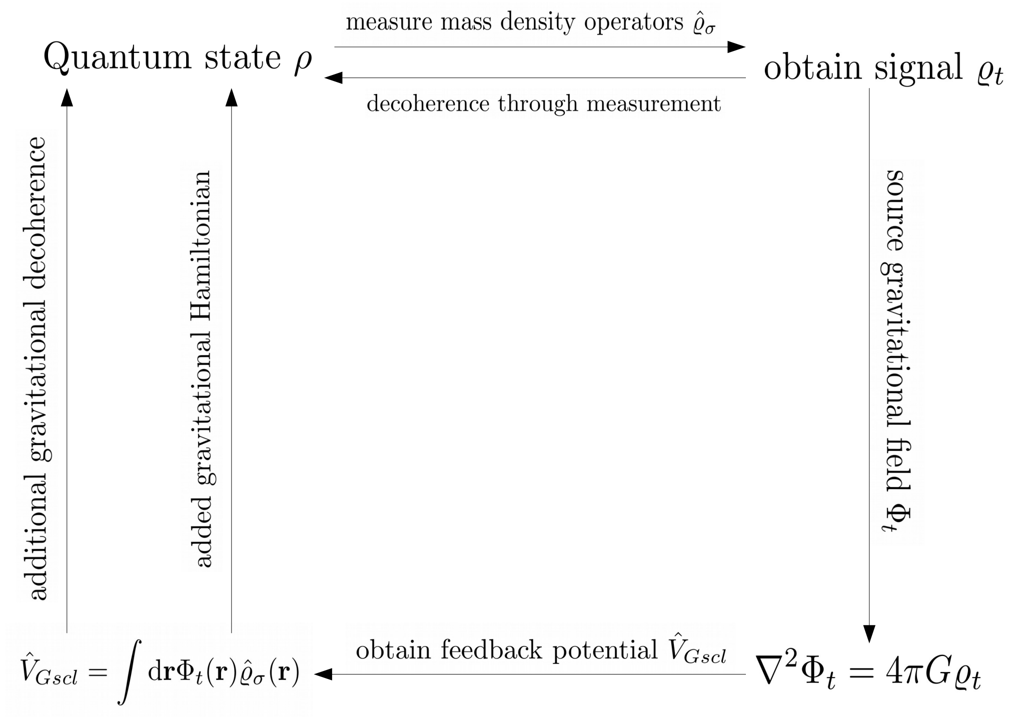

Figure 3.1: Diagram that illustrates the conceptual dependences and links between the four main objects in the previously developed theory of semiclassical gravity: The quantum state matrix ρ, the mass density signal ϱ_t, the gravitational field Φ_t and the semiclassical gravitational potential $\hat{V}_{Gscl}$.

3.2 DP - Gravity-related spontaneous collapse

This model, named after Diósi and Penrose (DP) uses the general formalism developed in the previous section but specifies the form of the spatial correlator $g_{\mathbf{rs}}$ as

$$g_{\mathbf{rs}} = \frac{\kappa G}{|\mathbf{r}-\mathbf{s}|} \tag{3.10}$$

where G is the Newton gravitational constant and κ is a dimensionless parameter. The localization strength is now directly linked to gravity and has a familiar form showing the typical behaviour of the gravitational field that decreases with the inverse of the distance. We will also need the inverse kernel $(g^{-1})_{\mathbf{rs}}$ so let us calculate it taking into account that it is an integral kernel which transforms a general function $f(\mathbf{r})$ into another function $h(\mathbf{s})$:

$$\begin{aligned} &\int \mathrm{d}\mathbf{r} g_{\mathbf{rs}} f(\mathbf{r}) = h(\mathbf{s}) \\ \text{inverse kernel}: &\int \mathrm{d}\mathbf{s} \left(g^{-1}\right)_{\mathbf{rs}} h(\mathbf{s}) = \int \mathrm{d}\mathbf{t}\mathrm{d}\mathbf{s} \left(g^{-1}\right)_{\mathbf{rs}} g_{\mathbf{ts}} f(\mathbf{t}) \stackrel{!}{=} f(\mathbf{r}) \\ &\Rightarrow \left(g^{-1}\right)_{\mathbf{rs}} g_{\mathbf{ts}} = \left(g^{-1}\right)_{\mathbf{rs}} \frac{\kappa G}{|\mathbf{t}-\mathbf{s}|} \stackrel{!}{=} \delta(\mathbf{r}-\mathbf{s})\delta(\mathbf{t}-\mathbf{s}) \end{aligned} \tag{3.11}$$

using the identity $\nabla^2 \frac{1}{|\mathbf{r}-\mathbf{s}|} = -4\pi\delta(\mathbf{r}-\mathbf{s})$ we can write the inverse kernel as

$$\left(g^{-1}\right)_{\mathbf{rs}} = -\frac{1}{4\pi\kappa G}\delta(\mathbf{r}-\mathbf{s})\nabla^2_{\mathbf{s}} \tag{3.12}$$

substituting this back into the integral and integrating twice by parts we get a nicer form:

$$\begin{aligned} \int \mathrm{d}\mathbf{s} \left(g^{-1}\right)_{\mathbf{rs}} h(\mathbf{s}) &= -\frac{1}{4\pi\kappa G}\int \mathrm{d}\mathbf{s}\delta(\mathbf{r}-\mathbf{s})\nabla^2_{\mathbf{s}} h(\mathbf{s}) = -\frac{1}{4\pi\kappa G}\int \mathrm{d}\mathbf{s}\nabla^2_{\mathbf{s}}\delta(\mathbf{r}-\mathbf{s})h(\mathbf{s}) \\ \Rightarrow \left(g^{-1}\right)_{\mathbf{rs}} &= -\frac{1}{4\pi\kappa G}\nabla^2_{\mathbf{s}}\delta(\mathbf{r}-\mathbf{s}) \end{aligned} \tag{3.13}$$

where we have neglected the surface terms since the delta-function and its derivative are zero at infinity. With this inverse kernel, together with relation 3.3 the correlation of the stochastic fluctuations in the signal yields

$$\langle \delta\varrho_t(\mathbf{r})\delta\varrho_\tau(\mathbf{s})\rangle = -\frac{1}{4\pi\kappa G}\nabla^2_{\mathbf{s}}\delta(\mathbf{r}-\mathbf{s})\delta(t-\tau) \tag{3.14}$$

and substituting the gravity-related kernel and inverse kernel into the general SME 3.8 gives the SME in this particular case:

$$\begin{aligned} \frac{d\rho}{dt} =& -i\left[H + \hat{V}_{G,\sigma} + \int \mathrm{d}\mathbf{r}\delta\varrho_t(\mathbf{r})\hat{\Phi}_\sigma, \rho\right] + \frac{\kappa G}{2}\int \frac{\mathrm{d}\mathbf{r}\mathrm{d}\mathbf{s}}{|\mathbf{r}-\mathbf{s}|}\mathcal{H}\left[\hat{\varrho}_\sigma(\mathbf{r})\right]\rho\delta\varrho_t(\mathbf{s}) \\ &- \int \mathrm{d}\mathbf{r}\mathrm{d}\mathbf{s}\left(\frac{1}{8}\frac{\kappa G}{|\mathbf{r}-\mathbf{s}|}\left[\hat{\varrho}_\sigma(\mathbf{r}), \left[\hat{\varrho}_\sigma(\mathbf{s}), \rho\right]\right] - \frac{1}{8\pi\kappa G}\nabla^2\delta(\mathbf{r}-\mathbf{s})\left[\hat{\Phi}_\sigma(\mathbf{r}), \left[\hat{\Phi}_\sigma(\mathbf{s}), \rho\right]\right]\right) \end{aligned} \tag{3.15}$$

where the deterministic backaction Hamiltonian as before is

$$\hat{V}_{G,\sigma} = \frac{1}{2}\int \mathrm{d}\mathbf{r}\hat{\varrho}_\sigma(\mathbf{r})\hat{\Phi}_\sigma(\mathbf{r}) = -\frac{G}{2}\int \mathrm{d}\mathbf{r}\mathrm{d}\mathbf{s}\frac{\hat{\varrho}_\sigma(\mathbf{r})\hat{\varrho}_\sigma(\mathbf{s})}{|\mathbf{r}-\mathbf{s}|} \tag{3.16}$$

The kernel $g_{\mathbf{rs}}$ that has been chosen lets us rewrite and unite the last two terms in the following form: the first one, taking into account the Poisson equation for the gravitational field $\hat{\Phi}_\sigma$ and integrating by parts:

$$
\begin{aligned}
&-\frac{\kappa G}{8}\int\frac{d\mathbf{r}d\mathbf{s}}{|\mathbf{r}-\mathbf{s}|}\big[\hat{\varrho}_\sigma(\mathbf{r}),\big[\hat{\varrho}_\sigma(\mathbf{s}),\rho\big]\big]=-\frac{\kappa G}{8}\left(\frac{1}{4\pi G}\right)^2\int\frac{d\mathbf{r}d\mathbf{s}}{|\mathbf{r}-\mathbf{s}|}\big[\nabla^2\hat{\Phi}_\sigma(\mathbf{r}),\big[\nabla^2\hat{\Phi}_\sigma(\mathbf{s}),\rho\big]\big]\\
&=-\frac{\kappa G}{8}\left(\frac{1}{4\pi G}\right)^2\int\left(\nabla^2\frac{d\mathbf{r}d\mathbf{s}}{|\mathbf{r}-\mathbf{s}|}\right)\big[\hat{\Phi}_\sigma(\mathbf{r}),\big[\nabla^2\hat{\Phi}_\sigma(\mathbf{s}),\rho\big]\big]\\
&=\frac{\kappa}{32\pi G}\int d\mathbf{r}d\mathbf{s}\delta(\mathbf{r}-\mathbf{s})\big[\hat{\Phi}_\sigma(\mathbf{r}),\big[\nabla^2\hat{\Phi}_\sigma(\mathbf{s}),\rho\big]\big]\\
&=\frac{\kappa}{32\pi G}\int d\mathbf{r}\big[\hat{\Phi}_\sigma(\mathbf{r}),\big[\nabla^2\hat{\Phi}_\sigma(\mathbf{r}),\rho\big]\big]=-\frac{\kappa}{32\pi G}\int d\mathbf{r}\big[\nabla\hat{\Phi}_\sigma(\mathbf{r}),\big[\nabla\hat{\Phi}_\sigma(\mathbf{r}),\rho\big]\big]
\end{aligned}
\tag{3.17}
$$

and the second one also integrating by parts:

$$
\begin{aligned}
&\frac{1}{8\pi\kappa G}\int d\mathbf{r}d\mathbf{s}\nabla^2\delta(\mathbf{r}-\mathbf{s})\big[\hat{\Phi}_\sigma(\mathbf{r}),\big[\hat{\Phi}_\sigma(\mathbf{s}),\rho\big]\big]=\frac{1}{8\pi\kappa G}\int d\mathbf{r}d\mathbf{s}\delta(\mathbf{r}-\mathbf{s})\big[\nabla^2\hat{\Phi}_\sigma(\mathbf{r}),\big[\hat{\Phi}_\sigma(\mathbf{s}),\rho\big]\big]\\
&=\frac{1}{8\pi\kappa G}\int d\mathbf{r}\big[\nabla^2\hat{\Phi}_\sigma(\mathbf{r}),\big[\hat{\Phi}_\sigma(\mathbf{r}),\rho\big]\big]=-\frac{1}{8\pi\kappa G}\int d\mathbf{r}\big[\nabla\hat{\Phi}_\sigma(\mathbf{r}),\big[\nabla\hat{\Phi}_\sigma(\mathbf{r}),\rho\big]\big]
\end{aligned}
\tag{3.18}
$$

the two terms can then be united as

$$
\begin{aligned}
&-\frac{\kappa G}{8}\int\frac{d\mathbf{r}d\mathbf{s}}{|\mathbf{r}-\mathbf{s}|}\big[\hat{\varrho}_\sigma(\mathbf{r}),\big[\hat{\varrho}_\sigma(\mathbf{s}),\rho\big]\big]+\frac{1}{8\pi\kappa G}\int d\mathbf{r}d\mathbf{s}\nabla^2\delta(\mathbf{r}-\mathbf{s})\big[\hat{\Phi}_\sigma(\mathbf{r}),\big[\hat{\Phi}_\sigma(\mathbf{s}),\rho\big]\big]\\
&=-\frac{1}{8\pi G}\left(\frac{\kappa}{4}+\frac{1}{\kappa}\right)\int d\mathbf{r}\big[\nabla\hat{\Phi}_\sigma(\mathbf{r}),\big[\nabla\hat{\Phi}_\sigma(\mathbf{r}),\rho\big]\big]
\end{aligned}
\tag{3.19}
$$

we can now fix the value of κ by requiring that the total decoherence be minimal in the DP-model:

$$
\frac{d}{d\kappa}\left(\frac{\kappa}{4}+\frac{1}{\kappa}\right)=\frac{1}{4}-\frac{1}{\kappa^2}=0\qquad\Rightarrow\kappa=2
\tag{3.20}
$$

where we have chosen the positive value for κ because the correlator $g_{\mathbf{rs}}$ is defined non-negative. The non-Hamiltonian stochastic term in 3.15 with the noise $\delta\varrho_t$ can also be rewritten with the same steps as before, using the Poisson equation and integrating by parts:

$$
\begin{aligned}
&\frac{\kappa G}{2}\int\frac{d\mathbf{r}d\mathbf{s}}{|\mathbf{r}-\mathbf{s}|}\mathcal{H}\big[\hat{\varrho}_\sigma(\mathbf{r})\big]\rho\delta\varrho_t(\mathbf{s})=\frac{\kappa G}{2}\int\frac{d\mathbf{r}d\mathbf{s}}{|\mathbf{r}-\mathbf{s}|}\left(\hat{\varrho}_\sigma(\mathbf{r})\rho+\rho\hat{\varrho}_\sigma(\mathbf{r})-2\rho\langle\hat{\varrho}_\sigma(\mathbf{r})\rangle\right)\delta\varrho_t(\mathbf{s})\\
&=\frac{\kappa G}{2}\frac{1}{4\pi G}\int\frac{d\mathbf{r}d\mathbf{s}}{|\mathbf{r}-\mathbf{s}|}\left(\nabla^2\hat{\Phi}_\sigma(\mathbf{r})\rho+\rho\nabla^2\hat{\Phi}_\sigma(\mathbf{r})-2\rho\langle\nabla^2\hat{\Phi}_\sigma(\mathbf{r})\rangle\right)\delta\varrho_t(\mathbf{s})\\
&=-\frac{\kappa}{2}\int d\mathbf{r}\left(\hat{\Phi}_\sigma(\mathbf{r})\rho+\rho\hat{\Phi}_\sigma(\mathbf{r})-2\rho\langle\hat{\Phi}_\sigma(\mathbf{r})\rangle\right)\delta\varrho_t(\mathbf{r})=-\frac{\kappa}{2}\int d\mathbf{r}\mathcal{H}\big[\hat{\Phi}_\sigma(\mathbf{r})\big]\rho\delta\varrho_t(\mathbf{r})
\end{aligned}
\tag{3.21}
$$

with $\kappa = 2$ we finally get the complete SME for stochastic semiclassical gravity in the DP-model:

$$\begin{aligned}\frac{d\rho}{dt} =& -i\left[H + \hat{V}_{G,\sigma} + \int \mathrm{d}\mathbf{r}\delta\varrho_t(\mathbf{r})\hat{\Phi}_\sigma, \rho\right] - \frac{1}{8\pi G}\int \mathrm{d}\mathbf{r}\left[\nabla\hat{\Phi}_\sigma(\mathbf{r}), \left[\nabla\hat{\Phi}_\sigma(\mathbf{r}), \rho\right]\right] \\ &- \int \mathrm{d}\mathbf{r}\mathcal{H}\left[\hat{\Phi}_\sigma(\mathbf{r})\right]\rho\delta\varrho_t(\mathbf{r})\end{aligned} \tag{3.22}$$

This equation is the central object of interest in the article [AT16] that we are reviewing. Observe that the backaction of gravity in the feedback scheme has just doubled the decoherence term with the double commutator of the initial continuous monitoring model without feedback. Notice also that the whole SME has become completely local in the final DP gravity-related model, in contrast with the general SME 3.8.

One can also ask what is the mean evolution of a system described by this equation, for example if one has a sample of many identical systems, each described by the SME 3.22. To answer this it suffices to observe that the stochastic noise terms in the SME have zero mean, so the master equation (ME) that describes the average dynamics is obtained by just eliminating the noise terms:

$$\frac{d\rho}{dt} = -i\left[H + \hat{V}_{G,\sigma}, \rho\right] - \frac{1}{8\pi G}\int \mathrm{d}\mathbf{r}\left[\nabla\hat{\Phi}_\sigma(\mathbf{r}), \left[\nabla\hat{\Phi}_\sigma(\mathbf{r}), \rho\right]\right] \tag{3.23}$$

This is the master equation that describes the average evolution of the system. The task now at hand is to interpret this equation and to visualize the dynamics it produces.

4 Dynamics of the Diósi-Penrose semiclassical gravity model

4.1 An analytical approach

In this section an analytical approximation will be developed for the decoherence term in the SME (2.62)

$$\hat{\mathcal{D}}\rho \equiv -\frac{1}{8\pi G}\int \mathrm{d}\mathbf{r}\big[\nabla\hat{\Phi}_\sigma(\mathbf{r}),\,\big[\nabla\hat{\Phi}_\sigma(\mathbf{r}),\rho\big]\big] \tag{4.1}$$

For the sake of simplicity consider the case of a single particle. The objects of interest are the smoothened mass density operator

$$\hat{\varrho}_\sigma(\mathbf{r}-\hat{\mathbf{X}}) = (2\pi\sigma^2)^{-3/2} m e^{-(\mathbf{r}-\hat{\mathbf{X}})^2/2\sigma^2} \equiv m g_\sigma(\mathbf{r}-\hat{\mathbf{X}}) \tag{4.2}$$

where $\hat{\mathbf{X}}$ is the position operator for the particle and we use the notation g_σ for a gaussian of width σ. And the Newton gravitational potential operator, which for a gaussian mass distribution is given by

$$\hat{\Phi}_\sigma(\mathbf{r}-\hat{\mathbf{X}}) = -\frac{Gm}{|\mathbf{r}-\hat{\mathbf{X}}|}\,\mathrm{erf}\,\frac{|\mathbf{r}-\hat{\mathbf{X}}|}{2\sigma^2} \tag{4.3}$$

where the error function $\mathrm{erf}(x)$ is defined as

$$\mathrm{erf}(x) \equiv \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt \tag{4.4}$$

the expression for $\hat{\Phi}_\sigma$ is obtained by integrating the Poisson equation for the gravitational potential ($\nabla^2\hat{\Phi}_\sigma = 4\pi G\hat{\varrho}_\sigma$) in spherical coordinates and it can easily be checked by explicitly evaluating the Laplacian.
Integrating by parts, the decoherence term 4.1 can be expanded as

$$\begin{aligned}\hat{\mathcal{D}}\rho &\equiv -\frac{1}{8\pi G}\int \mathrm{d}\mathbf{r}\big[\nabla\hat{\Phi}_\sigma,\,\big[\nabla\hat{\Phi}_\sigma,\rho\big]\big] = \frac{1}{8\pi G}\int \mathrm{d}\mathbf{r}\big[\nabla^2\hat{\Phi}_\sigma,\,\big[\hat{\Phi}_\sigma,\rho\big]\big]\\ &= \frac{1}{2}\int \mathrm{d}\mathbf{r}\big[\hat{\varrho}_\sigma,\,\big[\hat{\Phi}_\sigma,\rho\big]\big] = \frac{1}{2}\int \mathrm{d}\mathbf{r}\,\Big(\hat{\varrho}_\sigma\hat{\Phi}_\sigma\rho - \hat{\varrho}_\sigma\rho\hat{\Phi}_\sigma - \hat{\Phi}_\sigma\rho\hat{\varrho}_\sigma + \rho\hat{\Phi}_\sigma\hat{\varrho}_\sigma\Big)\end{aligned} \tag{4.5}$$

and the matrix elements of this operator in the position basis are given by

$$\begin{aligned}\langle\mathbf{r}|\,\hat{\mathcal{D}}\rho\,|\mathbf{s}\rangle = \mathcal{D}\rho(\mathbf{r},\mathbf{s}) = -\frac{1}{2}Gm^2\rho(\mathbf{r},\mathbf{s})\int \mathrm{d}\mathbf{u}\Big(\frac{g_\sigma(\mathbf{u}-\mathbf{r})}{|\mathbf{u}-\mathbf{r}|}\,\mathrm{erf}\,\frac{|\mathbf{u}-\mathbf{r}|}{\sqrt{2\sigma^2}} + \\ -\frac{g_\sigma(\mathbf{u}-\mathbf{r})}{|\mathbf{u}-\mathbf{s}|}\,\mathrm{erf}\,\frac{|\mathbf{u}-\mathbf{s}|}{\sqrt{2\sigma^2}} - \frac{g_\sigma(\mathbf{u}-\mathbf{s})}{|\mathbf{u}-\mathbf{r}|}\,\mathrm{erf}\,\frac{|\mathbf{u}-\mathbf{r}|}{\sqrt{2\sigma^2}} + \frac{g_\sigma(\mathbf{u}-\mathbf{s})}{|\mathbf{u}-\mathbf{s}|}\,\mathrm{erf}\,\frac{|\mathbf{u}-\mathbf{s}|}{\sqrt{2\sigma^2}}\Big)\end{aligned} \tag{4.6}$$

from symmetry considerations of the integrands this expression can be reduced to

$$\mathcal{D}\rho(\mathbf{r},\mathbf{s}) = -Gm^2\rho(\mathbf{r},\mathbf{s})\,(I_1 - I_2) \tag{4.7}$$

where

$$\begin{aligned}I_1 &= \int \mathrm{d}\mathbf{u}\frac{g_\sigma(\mathbf{u})}{|\mathbf{u}|}\,\mathrm{erf}\,\frac{|\mathbf{u}|}{\sqrt{2\sigma^2}}\\ I_2 &= \int \mathrm{d}\mathbf{u}\frac{g_\sigma(\mathbf{u}-\mathbf{d})}{|\mathbf{u}|}\,\mathrm{erf}\,\frac{|\mathbf{u}|}{\sqrt{2\sigma^2}} \qquad , \qquad \mathbf{d}\equiv\mathbf{r}-\mathbf{s}\end{aligned} \tag{4.8}$$

the first integral has spherical symmetry and can be evaluated analytically. Writing it in spherical coordinates gives

$$I_1 = \int \mathrm{d}\mathbf{u} \frac{g_\sigma(\mathbf{u})}{|\mathbf{u}|} \operatorname{erf} \frac{|\mathbf{u}|}{\sqrt{2\sigma^2}} = 4\pi(2\pi\sigma^2)^{-3/2}(2\sigma^2) \int_0^\infty \mathrm{d}r\, r\, e^{-r^2} \operatorname{erf}(r) \tag{4.9}$$

the radial integral can be easily evaluated integrating by parts, and I_1 yields

$$I_1 = \frac{1}{\sqrt{\pi}\sigma} \tag{4.10}$$

The second integral I_2 can not be solved analytically, but one can find asymptotic approximations. Noticing that $I_2 = I_2(\mathbf{d})$ and that inside the integral $\mathbf{d}$ only appears in the gaussian as

$$g_\sigma(\mathbf{u}-\mathbf{d}) = (2\pi\sigma^2)^{-3/2} e^{-(\mathbf{u}-\mathbf{d})^2/2\sigma^2} \tag{4.11}$$

we can consider two limit cases: $d \gg \sigma$ and $d \ll \sigma$, where $d = |\mathbf{r}-\mathbf{s}|$ measures the distance from the diagonal ($\mathbf{r} = \mathbf{s}$) of the density matrix in the position basis, and σ is the resolution of the mass density operator $\hat{\varrho}_\sigma$.

- **Case 1:** $d \gg \sigma$ (far from the diagonal)

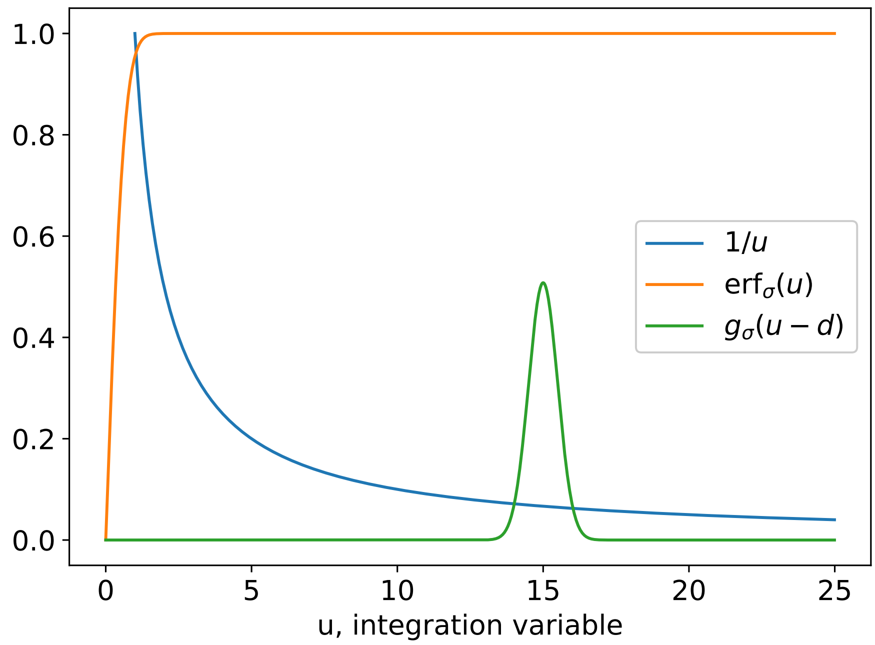

Figure 4.1: The three factors in the integrand of I_2 in the case $d \gg \sigma$ (simplified to an unidimensional case). For concreteness we have set $\sigma = 0.5$ and $d = 15$ in this plot. The dominant contribution to the integral comes from the region near the center of the gaussian.

The dominant contributions to the integral I_2 will come from the integrand in the region near the center of the gaussian (figure 4.1) : $\mathbf{u} \sim \mathbf{d}$. We then expand the integrand around $\mathbf{u} = \mathbf{d}$ up to second order, writing $u \equiv |\mathbf{u}|$, $d \equiv |\mathbf{d}|$, and $\mathbf{q} \equiv \mathbf{u} - \mathbf{d}$:

$$\frac{1}{u} \approx \frac{1}{d} - \frac{1}{d^3}\mathbf{q}\cdot\mathbf{d} + \frac{1}{2d^5}\left(3(\mathbf{q}\cdot\mathbf{d})^2 - q^2 d^2\right) + \mathcal{O}(q^3) \tag{4.12}$$

$$\text{erf}\,\frac{u}{\sqrt{2\sigma^2}} \approx \text{erf}\,\frac{d}{\sqrt{2\sigma^2}} + \frac{1}{\sqrt{2\pi\sigma^2}}\frac{1}{d}e^{-d^2/2\sigma^2}\left(2\mathbf{q}\cdot\mathbf{d} + q^2 - \frac{d^2+\sigma^2}{d^2\sigma^2}(\mathbf{q}\cdot\mathbf{d})^2\right) + \mathcal{O}(q^3) \tag{4.13}$$

We can then write the integral within this approximation as

$$I_2 \approx \int \mathrm{d}\mathbf{q}\; g_\sigma(\mathbf{q})\Bigg[\frac{1}{d}\,\text{erf}\,\frac{d}{\sqrt{2\sigma^2}} + \left(\frac{2\mathbf{q}\cdot\mathbf{d}}{d^2} + \frac{q^2}{d^2}\right)\left(\frac{1}{\sqrt{2\pi\sigma^2}}e^{-d^2/2\sigma^2} - \frac{1}{2d}\,\text{erf}\,\frac{d}{\sqrt{2\sigma^2}}\right) + \frac{(\mathbf{q}\cdot\mathbf{d})^2}{d^4}\left(\frac{3}{2d}\,\text{erf}\,\frac{d}{\sqrt{2\sigma^2}} - \frac{1}{\sqrt{2\pi\sigma^2}}\left(\frac{d^2+3\sigma^2}{\sigma^2}\right)e^{-d^2/2\sigma^2}\right) + \mathcal{O}(q^3)\Bigg] \tag{4.14}$$

Since the integrand only depends on the relative orientation of $\mathbf{q}$ and $\mathbf{d}$ through the scalar product $\mathbf{q}\cdot\mathbf{d}$ and not on the absolute orientation of $\mathbf{q}$, we can choose the z-axis of the $\mathbf{q}$-integration parallel to $\mathbf{d}$ without any loss of generality. We can then simply write $\mathbf{q}\cdot\mathbf{d} = qd\cos\theta$, where θ is the polar angle of the spherical coordinates. The integral I_2 then yields

$$I_2 \approx \frac{1}{d}\,\text{erf}\,\frac{d}{\sqrt{2\sigma^2}} - \frac{1}{\sqrt{2\pi\sigma^2}}e^{-d^2/2\sigma^2} \quad , \quad \text{for } d = |\mathbf{r}-\mathbf{s}| \gg \sigma \tag{4.15}$$

where the integrals proportional to $\mathbf{q}\cdot\mathbf{d}$ corresponding to the first order terms vanish because of their angular symmetry. The total decoherence term $\mathcal{D}\rho(\mathbf{r},\mathbf{s}) = -Gm^2\rho(\mathbf{r},\mathbf{s})\,(I_1 - I_2)$ far from the diagonal can then be written in an asymptotic approximation as

$$\mathcal{D}\rho(\mathbf{r},\mathbf{s}) \approx -Gm^2\left[\frac{1}{\sqrt{\pi}\sigma} - \frac{1}{d}\,\text{erf}\,\frac{d}{\sqrt{2\sigma^2}} + \frac{1}{\sqrt{2\pi\sigma^2}}e^{-d^2/2\sigma^2}\right]\rho(\mathbf{r},\mathbf{s}) \qquad \text{for d} \gg \sigma \tag{4.16}$$

Thus, for $d \to \infty$, the decoherence rate $\mathcal{D}\rho$ goes as $-\frac{Gm^2}{\sqrt{\pi}\sigma}$ which is constant and becomes bigger for bigger masses m and for a smaller/higher resolution σ of the mass density operator $\hat{\varrho}_\sigma$. For smaller d but still much bigger than σ, the decoherence rate gradually decreases.

- **Case 2:** $d \ll \sigma$ (near the diagonal)

  In this case the gaussian is centered very near the origin (figure 4.2). For $d = 0$, the integral I_2 becomes analytical and equal to I_1. The decoherence $\mathcal{D}\rho(\mathbf{r},\mathbf{s}) = -Gm^2\rho(\mathbf{r},\mathbf{s})\,(I_1 - I_2)$ in this case vanishes, meaning that decoherence of diagonal elements is zero. For d non-zero but very small ($d \ll \sigma$) we can expand the gaussian around $d = 0$:

$$\begin{aligned} g_\sigma(\mathbf{u}-\mathbf{d}) &= (2\pi\sigma^2)^{-3/2}e^{-(\mathbf{u}-\mathbf{d})^2/2\sigma^2} = (2\pi\sigma^2)^{-3/2}e^{-u^2/2\sigma^2}e^{-d^2/2\sigma^2}e^{\mathbf{u}\cdot\mathbf{d}/\sigma^2} \\ &\approx (2\pi\sigma^2)^{-3/2}e^{-u^2/2\sigma^2}e^{-d^2/2\sigma^2}\left(1 + \frac{\mathbf{u}\cdot\mathbf{d}}{\sigma^2} + \frac{(\mathbf{u}\cdot\mathbf{d})^2}{2\sigma^4} + \mathcal{O}(d^3)\right) \end{aligned} \tag{4.17}$$

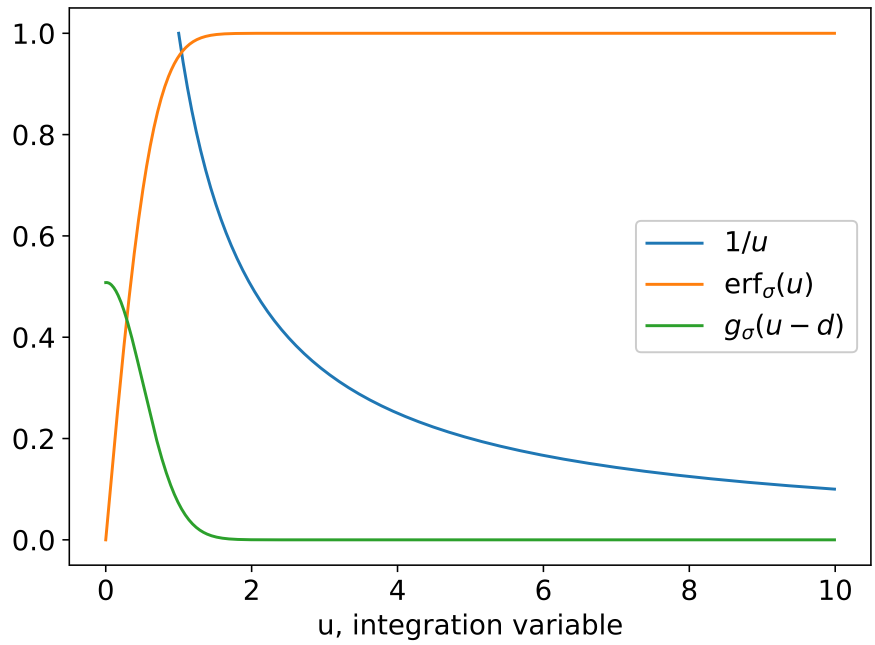

Figure 4.2: The three factors in the integrand of I_2 in the case $d \ll \sigma$. For concreteness we have set $\sigma = 0.5$ and $d = 0.01$ in this plot. We can expand the gaussian around $d = 0$.

With this approximation we can then write the integral I_2 as

$$I_2 \approx (2\pi\sigma^2)^{-3/2} e^{-d^2/2\sigma^2} \int \mathrm{d}\mathbf{u}\, \frac{1}{u}\, \mathrm{erf}\, \frac{u}{\sqrt{2\sigma^2}} e^{-u^2/2\sigma^2} \left(1 + \frac{\mathbf{u}\cdot\mathbf{d}}{\sigma^2} + \frac{(\mathbf{u}\cdot\mathbf{d})^2}{2\sigma^4} + \mathcal{O}(d^3)\right) \tag{4.18}$$

In the same way as before, the integrand only depends on the relative orientation of $\mathbf{u}$ and $\mathbf{d}$ and we can choose the z-axis of integration parallel to $\mathbf{d}$. We can then write $\mathbf{u}\cdot\mathbf{d} = ud\cos\theta$, where θ is the polar angle of the spherical coordinates. The integral I_2 then yields

$$I_2 \approx \frac{1}{\sqrt{\pi}\sigma} e^{-d^2/2\sigma^2} \left(1 + \frac{5}{12}\frac{d^2}{\sigma^2}\right) \quad , \qquad \text{for } d = |\mathbf{r} - \mathbf{s}| \ll \sigma \tag{4.19}$$

where again, the integrals proportional to $\mathbf{u}\cdot\mathbf{d}$ vanish because of their angular symmetry. The total decoherence term $\mathcal{D}\rho(\mathbf{r},\mathbf{s}) = -Gm^2\rho(\mathbf{r},\mathbf{s})\,(I_1 - I_2)$ near the diagonal can then be written in an asymptotic approximation as

$$\mathcal{D}\rho(\mathbf{r},\mathbf{s}) \approx -Gm^2 \frac{1}{\sqrt{\pi}\sigma}\left[1 - e^{-d^2/2\sigma^2}\left(1 + \frac{5}{12}\frac{d^2}{\sigma^2}\right)\right]\rho(\mathbf{r},\mathbf{s}) \qquad \text{for } d \ll \sigma \tag{4.20}$$

For $d = 0$, the decoherence rate is zero, meaning that decoherence of the diagonal elements is zero as predicted before evaluating the integral. For bigger d, but still much smaller than σ the decoherence rate rapidly increases.

In these two limit cases, very near and very far from the diagonal, an asymptotic analytical approximation can be given for the decoherence rate. In figure 4.3 the two approximations for the absolute value of the decoherence rate, $|\mathcal{D}|$ (without the factor $\rho(\mathbf{r},\mathbf{s})$) have been represented against the distance from the diagonal d.

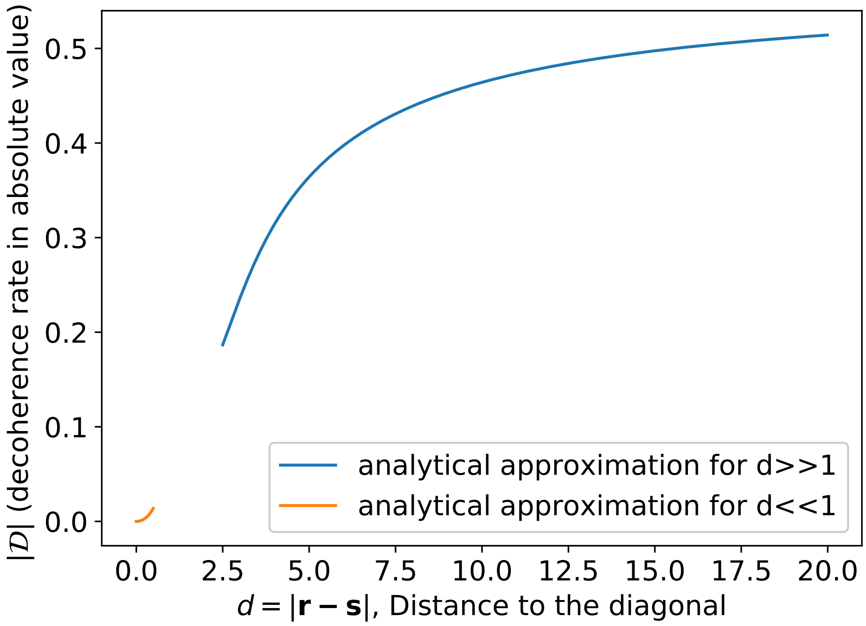

Figure 4.3: The two analytical approximations for the absolute value of the decoherence rate $|\mathcal{D}|$ in the two limit cases $d \gg \sigma$ and $d \ll \sigma$. For simplicity we have set $Gm^2 = \sigma = 1$. The decoherence of the diagonal elements (at $d = 0$) vanishes and grows steadily for larger values of d.

In figure 4.4 the analytical approximation for $d \gg \sigma$ is represented for very big d and it can be seen that the decoherence rate approaches the constant value $1/\sqrt{\pi}\sigma$.

On the other hand, one can also solve the integrals in equation 4.6 for the decoherence rate numerically. In this way one can obtain an approximation for the decoherence rate in the region $d \sim \sigma$ in which no analytical approximation is possible. We already know the exact result of the first integral $I_1 = 1/\sqrt{\pi}\sigma$ and we only have to solve the second integral I_2 numerically. To do this more efficiently we notice that the three-dimensional integral can be reduced to one dimension in the following way:

$$I_2 = \int \mathrm{d}\mathbf{u} \frac{g_\sigma(\mathbf{u}-\mathbf{d})}{|\mathbf{u}|} \operatorname{erf} \frac{|\mathbf{u}|}{\sqrt{2\sigma^2}} = (2\pi\sigma^2)^{-3/2} \int \mathrm{d}\mathbf{u} \frac{1}{u} e^{-u^2/2\sigma^2} e^{-d^2/2\sigma^2} e^{\mathbf{u}\cdot\mathbf{d}/\sigma^2} \operatorname{erf} \frac{u}{\sqrt{2\sigma^2}} \tag{4.21}$$

As before, this integral only depends on the relative orientation of $\mathbf{u}$ and $\mathbf{d}$ and we can thus choose the z-axis of integration parallel to $\mathbf{d}$. We can then write $\mathbf{u} \cdot \mathbf{d} = ud\cos\theta$, where θ is the polar angle of the spherical coordinates. Separating the radial and angular parts of the integral and integrating over the angular variables

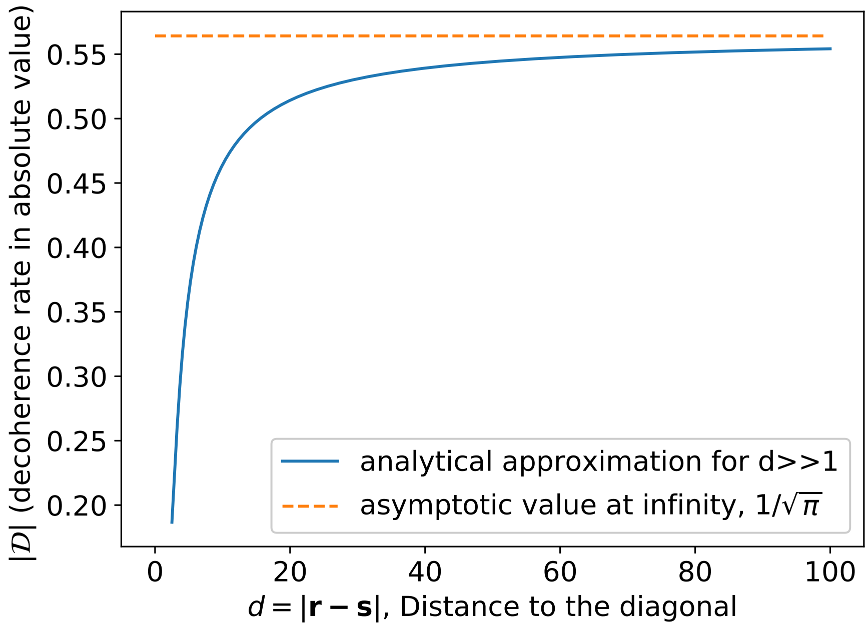

Figure 4.4: The analytical approximations for the absolute value of the decoherence rate $|\mathcal{D}|$ in the case $d \gg \sigma$. For simplicity we have set $Gm^2 = \sigma = 1$. The decoherence rate grows steadily with the distance from the diagonal d and approaches a constant value for very big values of d, very far from the diagonal.

yields

$$\begin{aligned} I_2 &= (2\pi\sigma^2)^{-3/2} \int \sin\theta \mathrm{d}\theta \mathrm{d}\varphi e^{ud\cos\theta/\sigma^2} \int u^2 \mathrm{d}u \frac{1}{u} e^{-u^2/2\sigma^2} e^{-d^2/2\sigma^2} \operatorname{erf} \frac{u}{\sqrt{2\sigma^2}} \\ &= (2\pi\sigma^2)^{-1/2} \frac{2}{d} e^{-d^2/2\sigma^2} \int_0^\infty \mathrm{d}u e^{-u^2/2\sigma^2} \sinh \frac{ud}{\sigma^2} \operatorname{erf} \frac{u}{\sqrt{2\sigma^2}} \end{aligned} \tag{4.22}$$

The radial integral can readily be solved numerically. In figure 4.5, the numerical approximation has been represented together with the analytical approximations in the two limit cases. The analytical approximations match the numerical solution very well, and in the intermediate region $d \sim \sigma$ the numerical solution smoothly connects the analytical approximations in the two limit cases. The overall behaviour of the decoherence rate is that it vanishes for the diagonal elements of the density matrix ($d = 0$), then grows very rapidly with d as one moves away from the diagonal. For very big values of d, very far from the diagonal, the decoherence rate continues growing very slowly and asymptotically approaches a constant value of $Gm^2/\sqrt{\pi}\sigma$.

Restoring the factors of $\hbar$ in the decoherence term one finds that the asymptotic value of the decoherence rate for $d \to \infty$ is $Gm^2/\sqrt{\pi}\hbar\sigma$. The resolution σ of the mass density operator can be taken as $\sim 10^{-12}cm$ in this model, which is about nuclear size, although this value has not been fixed completely. Significantly smaller values produce a faster decay of the coherences at microscopic levels which is not observed, and significantly bigger values fail to reproduce the decay of macroscopic coherences which was one of the original aims of this model. This means that "very big d" means $d \gg 10^{-12}cm$. These numerical values give an asymptotic decay rate

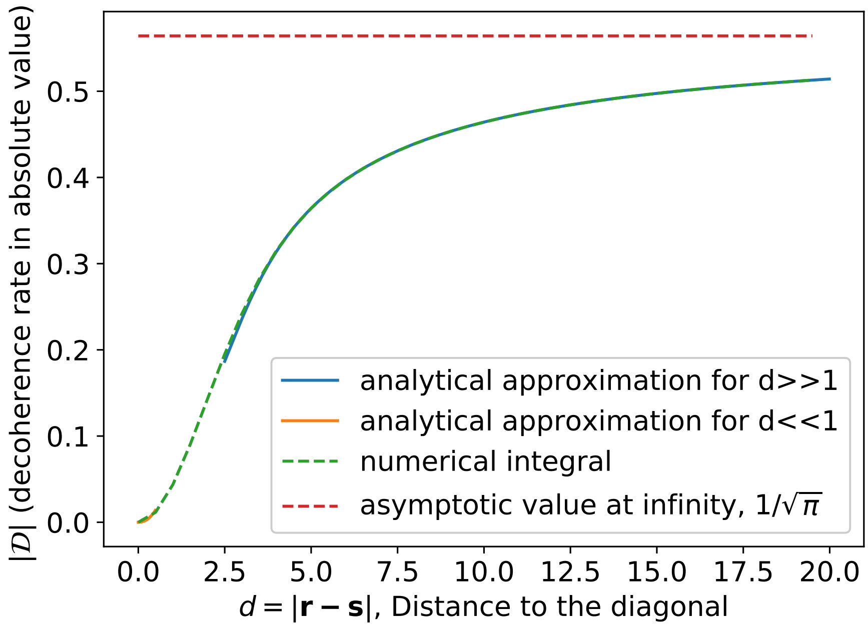

Figure 4.5: The analytical approximations for the absolute value of the decoherence rate $|\mathcal{D}|$ in the two limit cases $d \gg \sigma$ and $d \ll \sigma$ together with the numerical approximation. For simplicity we have set $Gm^2 = \sigma = 1$. The analytical and numerical approximations match very well in the two limit regions, and in the intermediate regions the numerical solution smoothly connects the two analytical approximations.

$Gm^2/\sqrt{\pi}\hbar\sigma \approx 3.571 \times 10^{37} m^2\ [kg^{-2}s^{-1}]$ which depends on the mass m of the object. Here again, we can consider two limit cases, microscopic and macroscopic objects:

- **Microscopic objects:** As a microscopic case, consider two entangled ions in a harmonic trap. For concreteness, we can take two Ca^+ ions of mass $40u$ in a harmonic trap of frequency 1 MHz. Calcium ions are commonly used because a big part of their electronic transitions lie in the visible range [Sta04]. The ground state energy of one ion in this harmonic trap then is $E = \frac{1}{2}\hbar\omega = 3.29 \times 10^{-10} eV$ and the corresponding magnitude of the Hamiltonian term in equation 3.23, $-\frac{i}{\hbar}[H, \rho]$ is about $5 \times 10^5 s^{-1}$.
Let us estimate the gravitational decoherence rate as a comparison: As shown before, the relevant parameter that gives the approximate order of magnitude of the gravitational decoherence rate is $Gm^2/\hbar\sigma$. In this case, for two Calcium ions of mass 40 uma, the gravitational decoherence rate will be of the order of $Gm^2/\hbar\sigma \approx 2.792 \times 10^{-13} s^{-1}$ which is absolutely negligible in comparison to the Hamiltonian evolution. We see thus that the motion of the entangled oscillators will stay coherent and that gravitational decoherence has a negligible influence on microscopic systems.

- **Macroscopic objects:** On a macroscopic scale however, consider masses of the order of 1 gram, the asymptotic decay rate is about $3.571 \times 10^{31}\ s^{-1}$. Consider that the particle is stationary and localized to about $\Delta x \sim 1\mu m$. We can estimate the momentum dispersion with the uncertainty principle as $\Delta p \sim \hbar/2\Delta x \approx 5.27 \times 10^{-29} kg\, m\, s^{-1}$. The Hamiltonian term for a free particle of mass $m = 1$ gram and momentum $p = 5.27 \times 10^{-29} kg\, m\, s^{-1}$ is about $1.318 \times 10^{-20} s^{-1}$, which is absolutely negligible in comparison to the gravitational decoherence. We can thus, as an approximation, neglect the Hamiltonian term in equation 3.23 together with the gravitational interaction energy $V_{G,\sigma}$, since we are considering only a single particle. (We will show later more in detail that this theory does not predict any self-interaction). The mean evolution equation for the density matrix in this case then becomes

$$\frac{d\rho}{dt} = \hat{\mathcal{D}}\rho \tag{4.23}$$

which has a formal solution

$$\rho(t) = e^{\hat{\mathcal{D}}t}\rho_0 \tag{4.24}$$

We can evaluate this expression in the position basis, and taking the asymptotic value for the decoherence rate $\mathcal{D} \sim -Gm^2/\sqrt{\pi}\hbar\sigma$ the solution for the time evolution of the off-diagonal (!) density matrix elements can be written as

$$\rho(\mathbf{r}, \mathbf{s}, t) = \exp\left(-\frac{Gm^2}{\sqrt{\pi}\hbar\sigma}t\right)\rho(\mathbf{r}, \mathbf{s}, t = 0)\ , \qquad |\mathbf{r} - \mathbf{s}| \gg \sigma \tag{4.25}$$

For the on-diagonal density matrix elements we do not have such a simple expression. This gives an exponential decay of the coherences with a halflife $\tau \equiv \frac{\sqrt{\pi}\hbar\sigma}{Gm^2} \approx 2.801 \times 10^{-32} s$, for a mass of 1 gram.

4.2 A numerical approach

In this section I will try to visualize the dynamics caused by the proposed DP master equation for semiclassical gravity with a simulation of a one-particle case. But first, as an illustrative example, let us consider the simple and well known case of a spin in a magnetic field but adding decoherence and feedback.

4.2.1 An illustration: A spin in a magnetic field

The Hamiltonian for a spin in an external magnetic field is:

$$H = -\boldsymbol{\mu} \cdot \mathbf{B} \tag{4.26}$$

for simplicity assume a particle of spin 1/2 with a gyromagnetic ratio of 1. Then $\boldsymbol{\mu} = \mathbf{S}$. For a magnetic field directed along the -x axis and of unit strength, $\mathbf{B} = -\hat{\mathbf{i}}$. Thus the Hamiltonian is simply:

$$H = S_x \tag{4.27}$$

The master equation describing the dynamics of the density matrix of the state in this case is

$$\frac{d\rho}{dt} = -i\big[H, \rho\big] = -i\big[S_x, \rho\big] \tag{4.28}$$

This equation obviously can be solved analytically, however we will solve it numerically and plot the time evolution of the matrix elements of the density operator.
If we place the inital state in an eigenstate of S_z,

$$|\Psi\rangle = |+\rangle_z\ ,\ \rho = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{4.29}$$

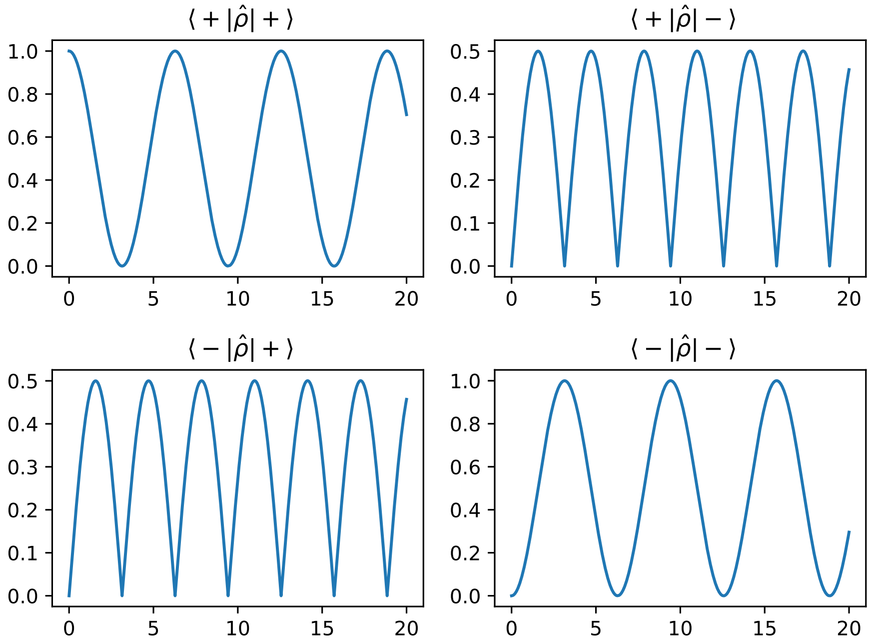

Figure 4.6: Time evolution of the matrix elements of the density operator in the z-basis of a particle in a magnetic field, initially in a spin-up state. The x-axis represents time in arbitrary units. The populations oscillate between 0 and 1, the coherences between 0 and 1/2. For simplicity we work with a magnetic field of unit strength, a particle with a unit gyromagnetic ratio and we have set $\hbar = 1$.

the matrix elements of the density operator in the z-basis will oscillate in time as can be seen in figure 4.6. If we now perform a continuous measurement of the observable S_z, the master equation is modified in the way developed in the previous sections. If we are only interested in the mean evolution of the system we can neglect the stochastic terms and the ME in this case is

$$\frac{d\rho}{dt} = -i\left[S_x, \rho\right] - \frac{g}{8}\left[S_z, \left[S_z, \rho\right]\right] \tag{4.30}$$

where g is the strength of the measurement. With $g = 1$ and the same initial state (spin-up in the z-basis), the populations in the z-basis perform a damped oscillation around the value 1/2, and the coherences perform a decaying oscillation as can be seen in figure 4.7. The measurement term in the ME thus introduces a decoherence in the evolution, inducing the decay of the coherences. The fact that we only consider the mean evolution amounts for the result that the populations are equally distributed in the end, i.e. half of the particles of the sample end up in a spin-up state and the other half in a spin-down state.

We can now introduce a feedback on the system. In the simplest way, this is done by applying a potential proportional to the continuously measured value of S_z:

$$\hat{V} = s_{z,t}\hat{B} \tag{4.31}$$

where $s_{z,t}$ is the time-dependent signal obtained from the continuous measurement of S_z and $\hat{B}$ is another observable that can be chosen freely. Applying this potential

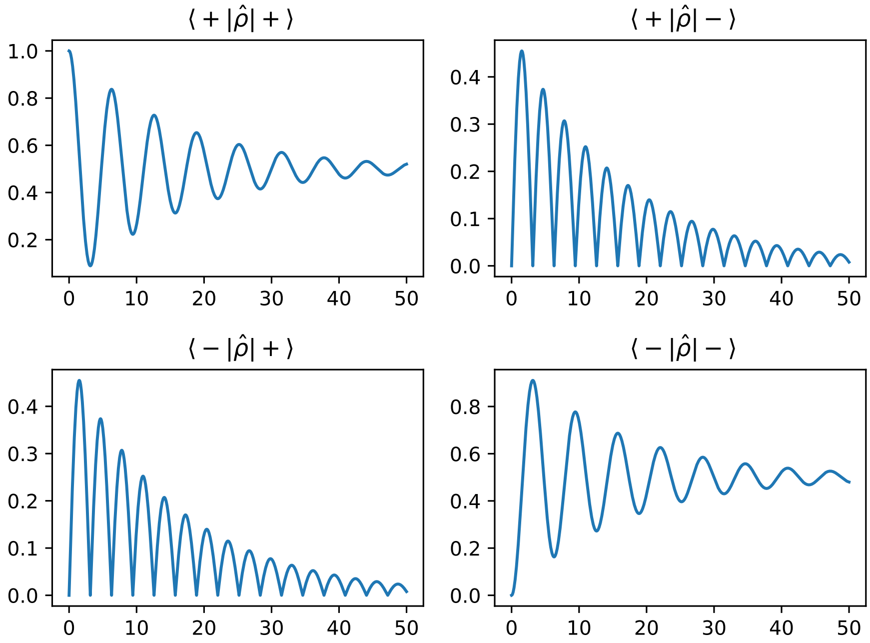

Figure 4.7: Time evolution of the matrix elements of the density operator in the z-basis of a particle in a magnetic field, initially in a spin-up state, with added decoherence through the measurement of S_z. The x-axis represents time in arbitrary units. The populations perform a damped oscillation around the value $1/2$, . For simplicity we work with a magnetic field of unit strength, a particle with a unit gyromagnetic ratio and we have set $\hbar = 1$ and the measurement strength $g = 1$.

in a feedback scheme, an infinitesimal amount of time dt after the free evolution in the same manner as developed in the last sections, the ME that governs the evolution of the system is given by:

$$\frac{d\rho}{dt} = -i[S_x, \rho] - \frac{g}{8}[S_z, [S_z, \rho]] - \frac{i}{2}[\hat{B}, \{S_z, \rho\}] - \frac{1}{2g}[\hat{B}, [\hat{B}, \rho]] \tag{4.32}$$

We can now choose the observable $\hat{B}$ that enters in the feedback potential. Choosing $\hat{B} = S_z$ just increases the decoherence caused by the measurement of S_z as can be seen in figure 4.8. The populations perform a damped oscillation that settles very rapidly to the value $1/2$ and the coherences oscillate decaying rapidly to 0. If instead we choose $\hat{B} = S_y$, the result is similar, but now the coherences asymptotically reach the value 0.4 instead of 0 (figure 4.9).

These examples should suffice to illustrate the effect of the previously introduced decoherence and feedback terms on the dynamics of a very simple system.

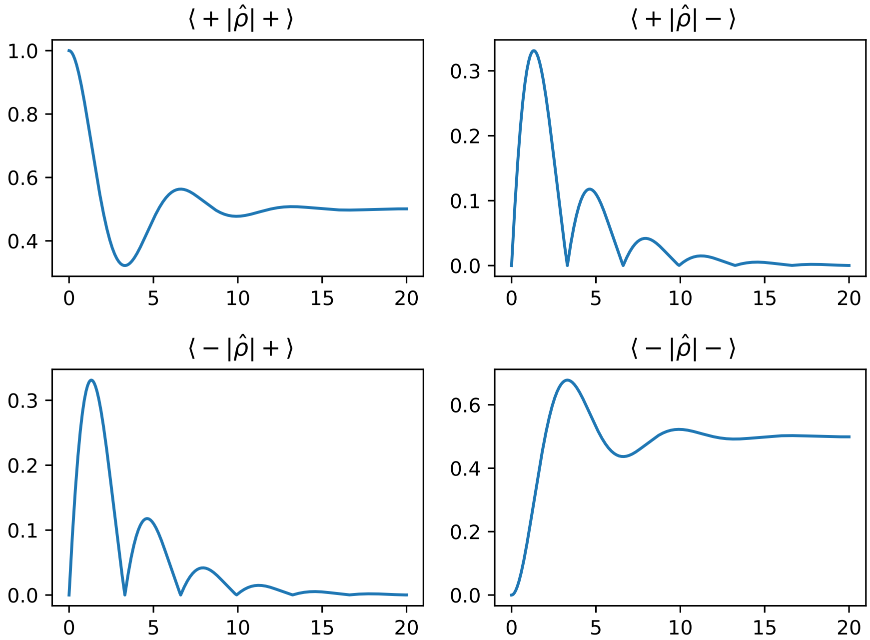

Figure 4.8: Time evolution of the matrix elements of the density operator in the z-basis of a particle in a magnetic field, initially in a spin-up state, with added decoherence through the measurement of S_z and with an applied feedback potential proportional to S_z. The x-axis represents time in arbitrary units. The populations perform a damped oscillation that settles very rapidly to the value $1/2$ and the coherences oscillate decaying rapidly to 0. For simplicity we work with a magnetic field of unit strength, a particle with a unit gyromagnetic ratio and we have set $\hbar = 1$ and the measurement strength $g = 1$.

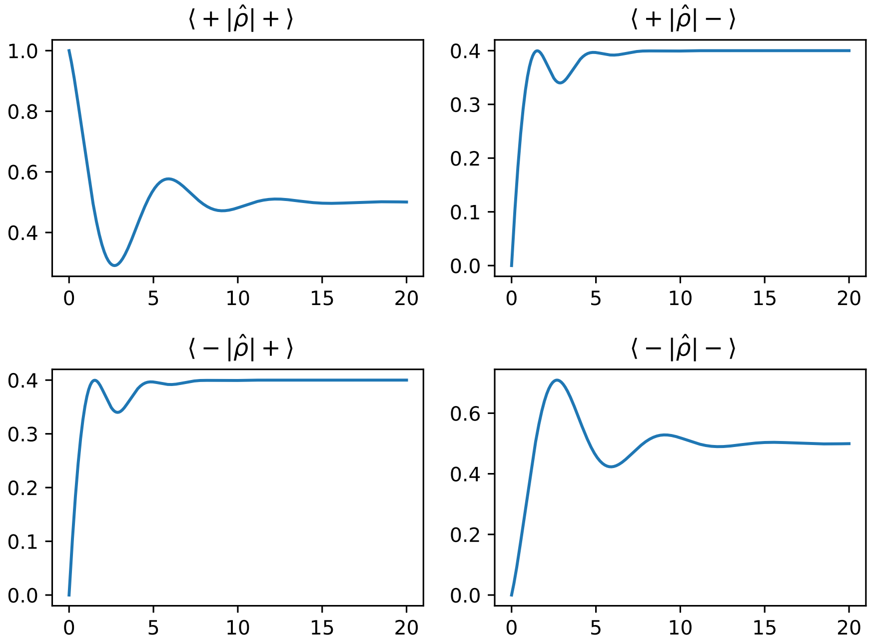

Figure 4.9: Time evolution of the matrix elements of the density operator in the z-basis of a particle in a magnetic field, initially in a spin-up state, with added decoherence through the measurement of S_z and with an applied feedback potential proportional to S_y. The x-axis represents time in arbitrary units. The populations perform a damped oscillation that settles very rapidly to the value $1/2$ and the coherences oscillate decaying rapidly to 0. For simplicity we work with a magnetic field of unit strength, a particle with a unit gyromagnetic ratio and we have set $\hbar = 1$ and the measurement strength $g = 1$.

4.2.2 Semiclassical gravity and gravitational decoherence

The main objective of this section is to visualize the dynamics of the equation proposed by Tilloy and Diósi [AT16] which we have developed in the last sections. We will focus our attention on the mean evolution of the system, neglecting the stochastic terms. The master equation which governs the system in this case is given by:

$$\frac{d\rho}{dt} = -i\big[H + V_{G,\sigma}, \rho\big] - \frac{1}{8\pi G}\int \mathrm{d}\mathbf{r}\big[\nabla\Phi_\sigma(\mathbf{r}), \big[\nabla\Phi_\sigma(\mathbf{r}), \rho\big]\big] \tag{4.33}$$

where Φ_σ is the gravitational potential field operator and $V_{G,\sigma}$ is the gravitational pair potential operator given by

$$\hat{V}_{G,\sigma} = \frac{1}{2}\int \mathrm{d}\mathbf{r}\hat{\varrho}_\sigma(\mathbf{r})\hat{\Phi}_\sigma(\mathbf{r}) = -\frac{G}{2}\int \mathrm{d}\mathbf{r}\mathrm{d}\mathbf{s}\frac{\hat{\varrho}_\sigma(\mathbf{r})\hat{\varrho}_\sigma(\mathbf{s})}{|\mathbf{r}-\mathbf{s}|} \tag{4.34}$$

We will consider only free particles, free meaning no interaction apart from gravity. In this case the free Hamiltonian is given by

$$\hat{H} = \frac{\hat{\mathbf{p}}^2}{2m} \tag{4.35}$$

The simplest case which gives interesting dynamics is the unidimensional case of a single localized particle in a spatial superposition, i.e. an initial state given by two gaussians centered at different positions:

$$\Psi(x) \propto e^{-(x-\mu)^2/4\varsigma^2} + e^{-(x+\mu)^2/4\varsigma^2} \tag{4.36}$$

Where we denote the width of the gaussian which describes the localization of the particle by the letter ς to prevent confusion with the sharpness of the mass density operator $\hat{\varrho}_\sigma$ which is denoted by σ. In this unidimensional case, the density operator in the position basis will be a function of two continuous variables: $\rho = \rho(x, y)$ and will have two gaussian peaks in the diagonal elements, and two off-diagonal gaussian peaks representing the coherences:

$$\langle x|\rho|y\rangle \equiv \rho(x, y) \propto \left(e^{-(x-\mu)^2/4\varsigma^2} + e^{-(x+\mu)^2/4\varsigma^2}\right)\left(e^{-(y-\mu)^2/4\varsigma^2} + e^{-(y+\mu)^2/4\varsigma^2}\right) \tag{4.37}$$

In the free evolution of the system, the peaks will just spread out evenly but maintain their relative heights as can be seen in figure 4.10. This behaviour of the wavefunction or density matrix of a localized free particle is widely known and can be proven analytically but we will not entertain us with this problem here and will just give the numerical simulation. Let us now add the gravity-related terms in equation 4.33. Consider first the gravitational pair potential operator $\hat{V}_{G,\sigma}$. For the case of a single particle, the mass density operator will be given by

$$\hat{\varrho}_\sigma(\mathbf{r}) = \hat{\varrho}_\sigma(\mathbf{r} - \hat{\mathbf{X}}) = mg_\sigma(\mathbf{r} - \hat{\mathbf{X}}) \tag{4.38}$$

where we use again the compact notation $g_\sigma(x)$ for a normalized gaussian of width σ. Let us evaluate the commutator of the gravitational pair potential with the density matrix in the position basis:

$$\begin{aligned}\langle\mathbf{x}|\big[V_{G,\sigma}, \rho\big]|\mathbf{y}\rangle &= -\frac{1}{2}Gm^2\int\frac{\mathrm{d}\mathbf{r}\mathrm{d}\mathbf{s}}{|\mathbf{r}-\mathbf{s}|}\big[g_\sigma(\mathbf{r}-\mathbf{x})g_\sigma(\mathbf{s}-\mathbf{x}) - g_\sigma(\mathbf{r}-\mathbf{y})g_\sigma(\mathbf{s}-\mathbf{y})\big]\rho(\mathbf{x},\mathbf{y}) \\ &= -\frac{1}{2}Gm^2\int\frac{\mathrm{d}\mathbf{r}\mathrm{d}\mathbf{s}}{|\mathbf{r}-\mathbf{s}|}\big[g_\sigma(\mathbf{r})g_\sigma(\mathbf{s}) - g_\sigma(\mathbf{r})g_\sigma(\mathbf{s})\big]\rho(\mathbf{x},\mathbf{y}) = 0\end{aligned} \tag{4.39}$$

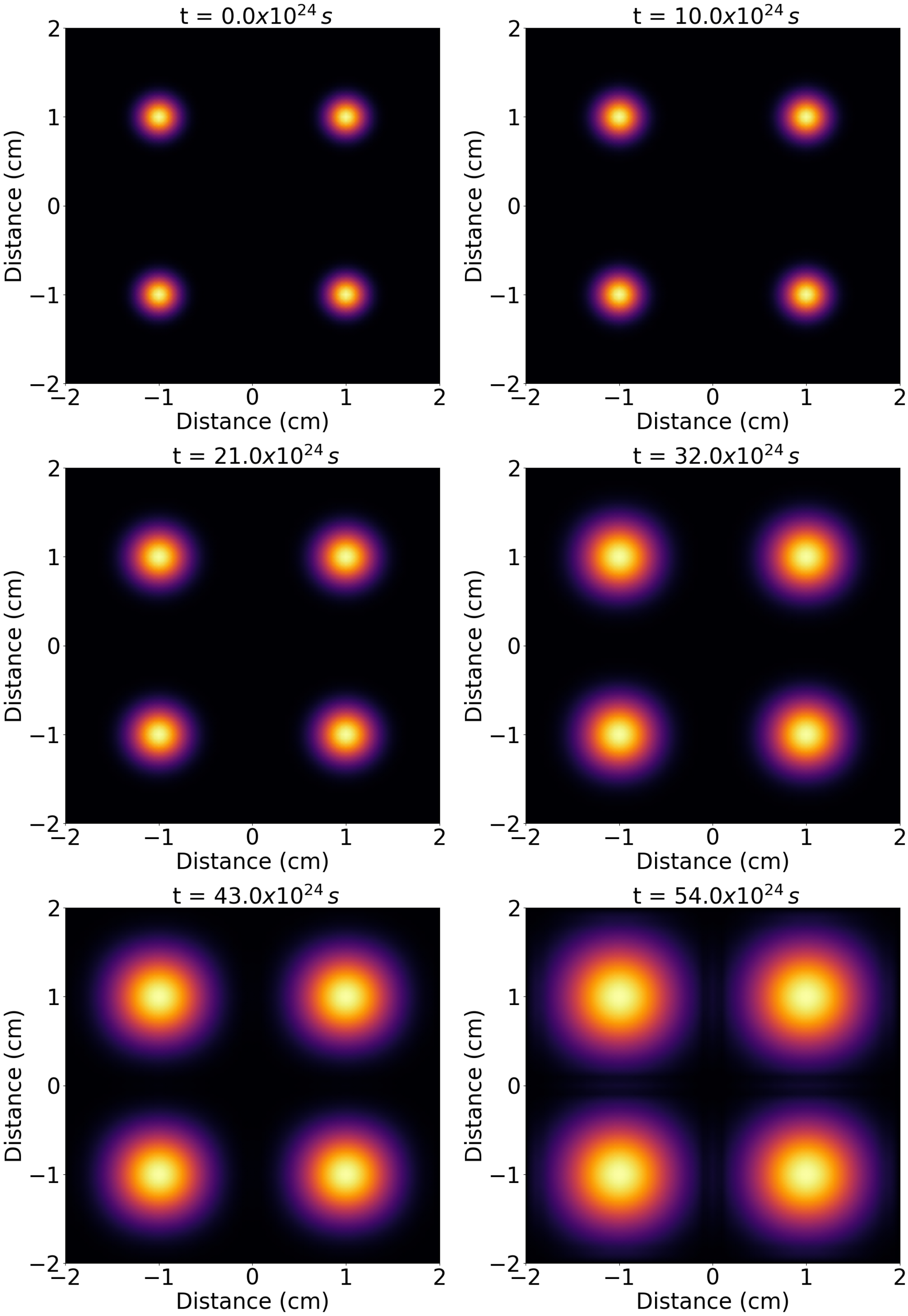

Figure 4.10: Free evolution of a particle of mass $m = 1\,g$ in a spatial superposition. The x and y axes are the two variables of the density matrix ρ in the position basis. The color represents the probability density or the value of the density matrix for a given x and y. The two peaks on the diagonal and the two off-diagonal peaks representing the coherences just spread out evenly without any further relative deformation. Notice the timescale: about $10^{25}s$!

where we performed a simple change of variables in the last step. We see that the commutator vanishes for the case of a single particle and thus this theory does not predict any gravitational self-energy. If we now include the gravitational decoherence term with the double commutator in equation 4.33, we expect the coherences to decay, and that the density matrix will gradually become diagonal in the position basis. As shown in the section 4.1, the asymptotic decoherence rate for matrix elements far from the diagonal is given by (4.16):

$$\mathcal{D}\rho(\mathbf{r},\mathbf{s}) \approx -\frac{Gm^2}{\sqrt{\pi}\hbar\sigma}\rho(\mathbf{r},\mathbf{s}) \qquad \text{for d} \gg \sigma \tag{4.40}$$

As commented before, this decoherence is absolutely negligible for microscopic objects so we will have to consider macroscopic situations to get an appreciable decoherence. Although the real proposed value for the mass density operator sharpness is $\sigma \approx 10^{-12}cm$, for a better visualization we will set here the mass density operator sharpness to $\sigma = 5 \times 10^{-2}cm$. Consider a macroscopic object of mass $m = 1g$ with a localization described by a gaussian of width $\varsigma = 0.3cm$ in a spatial superposition of separation $2\mu = 2cm$. Such a situation is described by a density matrix of the same form as in equation 4.37. In this case the two peaks that represent the coherences will be very far from the diagonal since $\mu \gg \sigma$ and our approximation that the decoherence is given by 4.40 holds. We thus expect the two off-diagonal peaks to decay at a nearly constant rate.

For the two peaks on the diagonal the situation is quite different: since the width of the peaks is much greater than the sharpness of the mass density operator ($\varsigma \gg \sigma$), the center of the peaks will be exactly on the diagonal satisfying the condition $d \ll \sigma$ (near the diagonal) while the tails of the gaussians will satisfy the condition $d \gg \sigma$ (far from the diagonal). Although we have no analytical expression for the intermediate region $d \sim \sigma$ we can anticipate that the center of the peaks will not suffer any decoherence while the rest of the gaussian which is not exactly on the diagonal will certainly decay. The result of solving just the decoherence term numerically can be seen in figure 4.11.

We can now solve the complete dynamics of the master equation 4.33 that describes the mean evolution of the system numerically. We expect the off-diagonal peaks to decay, together with the tails of the on-diagonal peaks which should only leave behind a very narrow region localized exactly on the diagonal. The results of solving the dynamics numerically and representing the density matrix elements at different instants of time can be seen in figure 4.12.

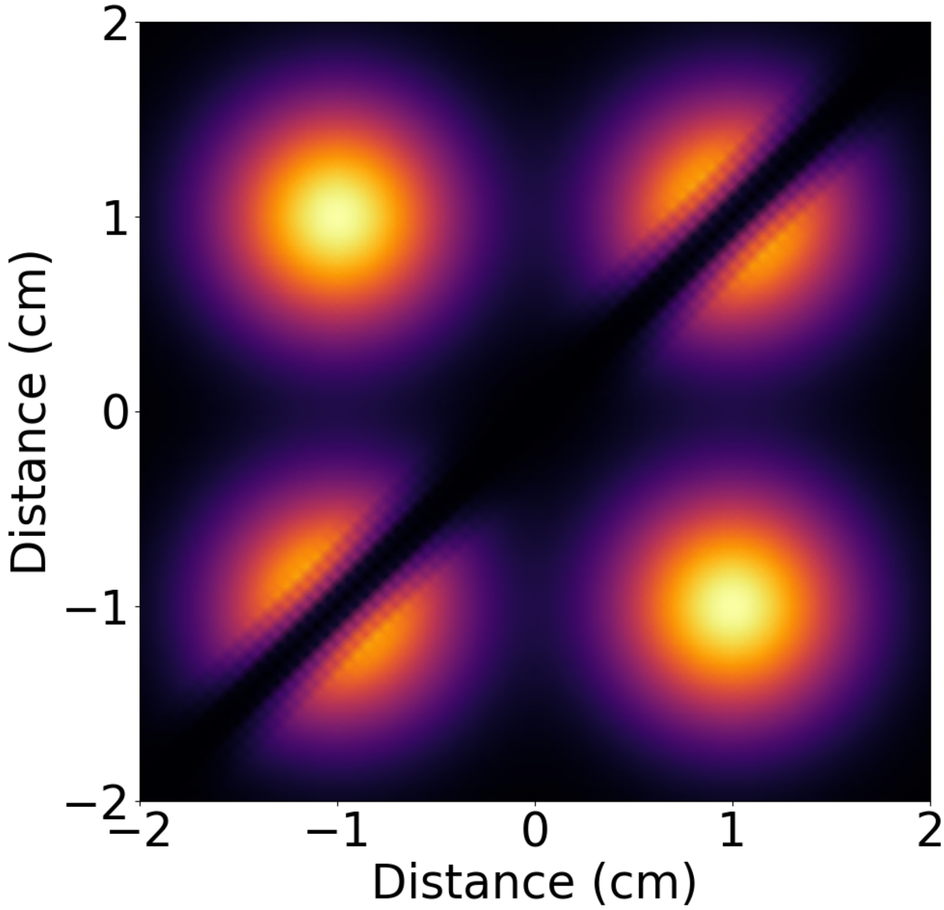

Figure 4.11: Decoherence rate density $\mathcal{D}\rho(\mathbf{r}, \mathbf{s})$ (in absolute value) for a single particle in a spatial superposition. One can clearly see that the decoherence rate for the two off-diagonal peaks is constant, while for the peaks on the diagonal only the parts which are significantly distant from the diagonal suffer decoherence. The decoherence exactly on the diagonal is zero. In this example we have used a particle of mass $m = 1g$ with a gaussian localization of width $\varsigma = 0.3cm$ in a spatial superposition of separation $2\mu = 2cm$. For a better visualization of the "diagonal cut" through the peaks we have set the mass density operator sharpness $\sigma = 0.05cm$.

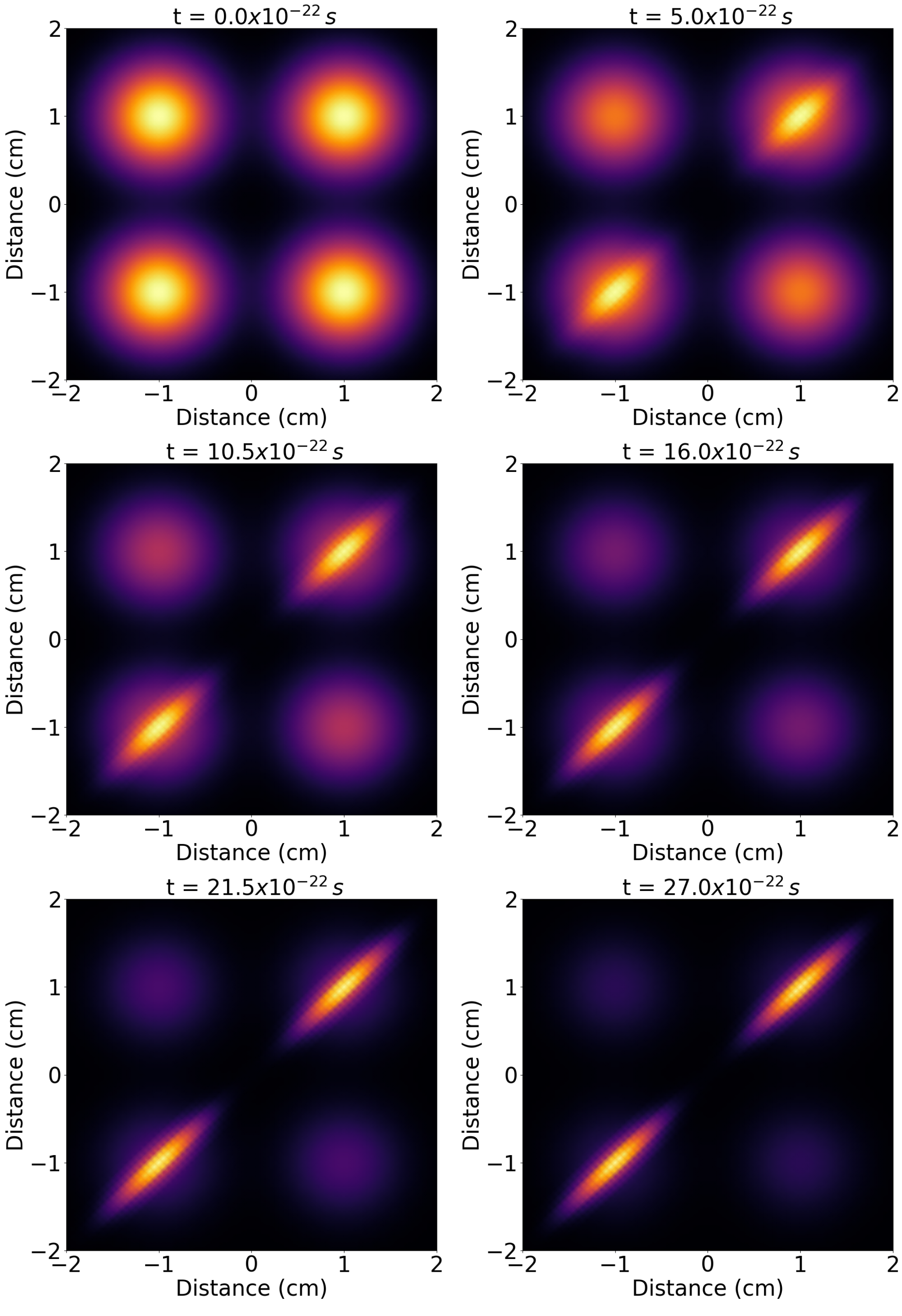

Figure 4.12: Evolution of a particle in a spatial superposition with gravitational decoherence. We have used a particle of mass $m = 1g$ with a gaussian localization of width $\varsigma = 0.3cm$ in a spatial superposition of separation $2\mu = 2cm$. The mass density operator sharpness is $\sigma = 0.05cm$. One can clearly see the off-diagonal peaks (coherences) decaying and the on-diagonal peaks leaving only a narrow region behind, exactly on the diagonal, where decoherence is zero. Notice the difference in the timescale in comparison with the free evolution, which is now about $10^{-21}s$. This agrees with the estimated timescale from equation 4.25: $\tau = \frac{\sqrt{\pi}\hbar\sigma}{Gm^2} \approx 1.4 \times 10^{-21}s$.

5 Results and Conclusions

We have seen that the model proposed by Tilloy and Diósi provides a consistent framework for semiclassical gravity. We have managed to source a classical gravitational field from quantum matter and to compute its backaction on it. In doing so we have made use of continuous monitoring or spontaneous localization models which were originally developed in the context of the measurement problem in quantum mechanics. In our context, it provides a consistent framework for a classical-quantum coupling, specifically for coupling the classical gravitational field to quantum matter. The formalism is analogous to continuously measuring the smoothened mass density operators at every point in space and using the resultant (classical) signal which carries the information about the outcome of the measurement of the mass density to source the gravitational field. The monitoring of the mass density however is not interpreted as being performed by actual physical detectors and is rather seen as a fundamental process of nature. This signal is inherently stochastic which reflects the stochastic nature of the measurements of the quantum system, and consequently the gravitational field sourced from the signal is stochastic too. Additionally, the signal at different points in space can be correlated. In the Diósi-Penrose model the correlator is directly related to gravity through the Newton gravitational constant G and is proportional to the inverse of the distance of the two points in space. The continuous measurement of the mass density leads to a decoherence and to a localization of the matter.

To implement the backaction of gravity on the quantum matter, we apply a potential proportional to the gravitational field in a feedback scheme. This feedback potential is chosen to be the semiclassical Newton interaction potential. The backaction of gravity leads to a doubling of the decoherence and introduces an additional Hamiltonian term in the evolution which is just the Newton potential energy. In this way we obtained a consistent stochastic master equation which describes the dynamics of the density matrix of the system under the influence of gravity.

To solve the dynamics of this equation we have tried an analytical approach which gave us the decoherence rates of the matrix elements very near and very far from the diagonal. We obtained that the decoherence vanishes on the diagonal, as expected, and asymptotically approaches a constant value as you move away from the diagonal. The asymptotic value of the decoherence rate at infinity is proportional to the Newton gravitational constant G and to the square of the mass m and inversely proportional to the sharpness σ with which the mass density is measured. This decoherence rate is absolutely negligible for microscopic objects but produces a very fast decoherence for macroscopic masses. We have also seen that the theory does not predict any self interaction of the particles.

We have then performed a completely numerical approach to solve the master equation that describes the dynamics of the system. We have recovered the fact that decoherence is zero on the diagonal and rapidly settles to a constant value as you move farther away. We have then simulated the time evolution of the density matrix elements for a macroscopic superposition and have observed that the coherences decay while the populations leave behind only a narrow, localized strip exactly on the diagonal.

We can thus assert that Tilloy and Diósi have constructed a theory and formalism that successfully couples classical gravity to quantum matter and describes the resulting combined dynamics. An additional Hamiltonian term equal to the Newton interaction energy emerges naturally from this theory and additionally it predicts a decoherence term which produces a localization of the quantum matter and leads to a rapid decay of any macroscopic superpositions.

6 Resúmenes en español de cada capítulo

6.1 Capítulo 1: Introducción

Dado que después de medio siglo de esfuerzo en crear una teoría unificada de la gravedad cuántica aún no se ha obtenido ningún resultado completamente satisfactorio, uno puede considerar la posibilidad de aproximarse a la teoría completa de la gravedad cuántica mediante una teoría semiclásica. Discutimos previos intentos de construir una teoría semiclásica de la gravedad y sus limitaciones. Introducimos la idea de Tilloy y Diósi de utilizar los modelos de medida continua para construir un nuevo formalismo de gravedad semiclásica.

6.2 Capítulo 2: Una breve introducción a la teoría cuántica de la medida

Empezamos con un repaso del formalismo de medida y colapso de función de onda en mecánica cuántica tradicional. A continuación lo generalizamos a medidas imperfectas que dejan una incertidumbre residual sobre el estado post-medida del sistema. Tomando el límite en el que las medidas se realizan continuamente en el tiempo obtenemos una ecuación maestra para la evolución del operador densidad del sistema que incorpora los efectos dinámicos de la medida continua. Extendemos el formalismo añadiendo la posibilidad de incorporar una realimentación al sistema y obtenemos otra ecuación maestra que describe la dinámica del operador densidad bajo la influencia de la medida continua y de la realimentación. Finalmente generalizamos todo el formalismo al caso en el que se mide continuamente un conjunto de n observables.

6.3 Capítulo 3: Aplicación de la teoría cuántica de la medida a la gravedad semiclásica

Se presenta la idea de Tilloy y Diósi de utilizar el modelo de medida continua y realimentación previamente introducido para construir un nuevo formalismo de gravedad semiclásica. En concreto, se particulariza el modelo previamente desarrollado al caso en el que se mide continuamente la densidad de masa en todo el espacio. El resultado de la medida es una señal clásica que contiene la información sobra la distribución de la masa. Se emplea esta señal clásica para obtener el campo gravitatorio clásico. A partir de este campo gravitatorio clásico obtenemos el potencial gravitatorio semiclásico que le aplicamos en el esquema de realimentación al sistema cuántico. De esta forma se obtiene un formalismo que describe la influencia de la materia cuántico sobre el campo gravitatorio clásico y la influencia del campo gravitatorio clásico sobre la materia cuántica. El modelo Diósi-Penrose (DP) particulariza la forma en la que las medidas de la densidad de masa en distintos puntos del espacio están correlacionados y lo relaciona con la constante gravitatoria G. Con esta correlación particular las ecuaciones se simplifican y se vuelven completamente locales.

6.4 Capítulo 4: Dinámica del modelo Diósi-Penrose de la gravedad semiclásica

Intentamos analizar la dinámica del modelo DP de la gravedad semiclásica previamente desarrollado. Para ello empezamos con una aproximación analítica que nos lleva a importantes conclusiones sobre la dependencia de la dinámica del sistema de los parámetros de la teoría. Vemos que para sistemas microscópicos la decoherencia gravitatoria es despreciable frente a la evolución hamiltoniana, en cambio para sistemas macroscópicos domina claramente la decoherencia gravitatoria. A continuación optamos por una resolución completamente numérica de la dinámica de las ecuaciones. Ilustramos primero los efectos de medida y realimentación en el caso sencillo de un espín en un campo magnético externo. Después resolvemos las ecuaciónes de Diósi-Penrose de la gravedad semiclásica para el caso concreto de una partícula localizada y en superposición espacial. Comprobamos que para un caso macroscópico domina la decoherencia gravitatoria que causa un decaimiento de las coherencias, rompe la superposición inicial e induce una localización de la partícula.

6.5 Capítulo 5: Resultados y conclusiones

Discutimos el modelo de gravedad semiclásica previamente desarrollado y los resultados obtenidos en los capítulos anteriores. El modelo se basa en modelos de medida continua particularizados a la medida de la densidad de masa. La ecuación final que obtuvimos describe la dinámica del sistema cuántico acoplado al campo gravitatorio clásico y predice una decoherencia gravitatoria que es despreciable a escalas microscópicas pero domina a escalas macroscópicas. Esto lo hemos comprobado con una aproximación analítica y lo hemos visualizado con una simulación numérica. Concluimos finalmente que Tilloy y Diósi han construido una teoría y un formalismo que permite acoplar consistentemente el campo gravitatorio clásico y un sistema cuántico.

References

[AT16] Lajos Diósi Antoine Tilloy. Sourcing semiclassical gravity from spontaneously localized quantum matter. *Physical Review D*, 2016.

[BP02] Heinz-Peter Breuer and Francesco Petruccione. *The theory of open quantum systems*. Oxford University Press on Demand, 2002.

[Dio88] L. Diosi. Continuous quantum measurement and Ito formalism. *Phys. Lett. A*, 129(8-9):419–423, 1988.

[Gis89] Nicolas Gisin. Stochastic quantum dynamics and relativity. *Helv. Phys. Acta*, 62(4):363–371, 1989.

[Mø62] C. Møller. Les theories relativistes de la gravitation. *Nucl. Phys.*, 40:353, 1962.

[Ros63] L. Rosenfeld. On quantization of fields. *Nucl. Phys.*, 40:353, 1963.

[Sta04] Peter Staanum. Quantum optics with trapped calcium ions. http://phys.au.dk/fileadmin/site_files/publikationer/phd/Peter_Staanum.pdf, 2004.

[WM09] Howard M Wiseman and Gerard J Milburn. *Quantum measurement and control*. Cambridge university press, 2009.