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Sistemas de distribución: avances en la gestión de inventarios

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A mis padres, Margarita y Felipe,
a mi hermana, Alicia
y a Roberto,
por su cariño, apoyo y compresión.

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B.A.J.B

Prólogo

La alta competitividad en el ámbito empresarial y los ajustados márgenes de beneficios en muchos sectores económicos hacen que una buena gerencia de los recursos disponibles sea esencial para aumentar los beneficios de las empresas. Así, el Control de Inventario se ha convertido en un factor crítico para el éxito de las compañías. Por otro lado, los inventarios también juegan un papel muy importante en la economía de un país. En general, los inventarios representan una cantidad importante en los balances de las empresas y, por lo tanto, los cambios que se producen en los inventarios están directamente relacionados con la economía de dicho país.

Sin embargo, el almacenamiento de mercancías ha sido siempre importante. Por ejemplo, en épocas de abundancia se guardaban alimentos y otros bienes de primera necesidad para así poder hacer frente a las épocas de escasez. Pero los conceptos de producción e inventario, tal y como se entienden hoy en día, no aparecen hasta la Revolución Industrial en el siglo dieciocho. Con la Revolución Industrial surgieron las primeras grandes máquinas y la división de los trabajos, lo que hizo que las compañías pudieran crecer considerablemente. Este crecimiento vino acompañado de grandes inversiones, lo que contribuyó a que se empezaran a desarrollar los primeros métodos para la planificación, organización, control de tareas, etc. Así surge lo que se conoce con el nombre de "gestión y administración científica". Los primeros progresos importantes se hicieron durante los primeros años del siglo veinte. En 1911, Frederick Taylor, que se considera el padre de la gestión científica, publicó uno de los primeros libros titulado "*The principles of Scientific Management*". Esta publicación ayudó a que las empresas aceptaran los principios de la gerencia científica y, en particular, se empezaron a utilizar sobre todo en la industria del automóvil. Además, es en este tipo de industrias donde también se introdujeron por primera vez las plantas de fabricación y ensamblaje, gracias a las cuales se consiguió disminuir considerablemente el tiempo de montaje y los costes. Así, las organizaciones comenzaron a darse cuenta de la relación entre la eficacia y la planificación de la producción y el almacenamiento de mercancías. Finalmente, surgen las primeras herramientas matemáticas y técnicas cuantitativas que permiten mejorar la toma de decisiones. Uno de los primeros modelos matemáticos en este campo fue publicado por F.W. Harris en 1913. Harris desarrolló una fórmula matemática para decidir qué cantidad se debe pedir en función de los costes de reposición y de mantenimiento. Este primer modelo y otras extensiones se recogen en el libro "*Quantity and Economy in Manufacture*" publicado por Raymond en 1931. En un principio, estos modelos no se utilizaron extensamente en la industria. Sin embargo, durante la Segunda Guerra Mundial resurgió el interés por estos modelos matemáticos y, se continuó desarrollando y refinando las herramientas cuantitativas para la toma de decisión. Excelentes revisiones de los modelos que se estudiaron hasta 1951 se encuentran en los trabajos de Whitin "*The Theory of Inventory Management*" y "*Inventory control research: A survey*" publicados en 1953 y en 1954, respectivamente. Durante la década de los sesenta y setenta, continúa el apogeo de estas técnicas aunque decae un poco en la década de los ochenta. Sin embargo, el uso de los ordenadores contribuyó a que resurgiera otra vez el interés por estos modelos. Así, en las últimas décadas son muchos los avances que se han conseguido en el control de inventario. Probablemente, uno de los progresos más importantes en la economía mundial que han tenido lugar recientemente ha sido el éxito de las firmas japonesas en los mercados occidentales. Este éxito se debe sobre todo a la capacidad que tienen estas empresas para trabajar con inventarios muy bajos. En efecto, teniendo en cuenta la alta competitividad en los actuales mercados, las compañías no pueden permitirse mantener altos niveles de inventarios. Sin embargo, también es importante que las empresas tengan los productos disponibles en el momento que los clientes lo solicitan. Tradicionalmente, para asegurar un alto nivel se servicio, las compañías solían mantener mucho inventario. Actualmente, las empresas se han dado cuenta que los costes se pueden reducir considerablemente con un buen control de los inventarios a lo largo de todas las instalaciones o organizaciones que forman parte de la cadena de suministro. Es importante observar que las acciones de un miembro de la cadena pueden afectar a todos los demás socios. Por lo tanto, es esencial que las diferentes instalaciones cooperen para conseguir controlar toda la cadena de suministro en todos los sentidos. En particular, uno de los problemas más importantes es el control del inventario en toda la cadena. Surge así lo que se conoce en la literatura como el control de inventario en los sistemas con múltiples instalaciones. Teniendo en cuenta que los costes de inventario representan normalmente una inversión importante, la reducción de dichos costes es fundamental para mejorar los beneficios de todas las empresas que forman parte de la cadena. Así, no es de extrañar que el control de inventario en los sistemas con múltiples instalaciones se haya convertido en un tema de investigación muy importante durante los últimos años. El objetivo de esta tesis es seguir avanzando en esta línea de investigación. Los sistemas de inventario con múltiples instalaciones son muy comunes tanto en los contextos de la distribución como en el de la producción de artículos. En concreto, nosotros nos centramos en el estudio de los sistemas de distribución con dos niveles donde un almacén suministra los artículos a un conjunto de minoristas.

Antes de analizar este tipo de modelos, en el Capítulo 1 introducimos los conceptos básicos del control de inventario. En este capítulo también resumimos los modelos más simples donde sólo interviene una instalación. El estudio de estos problemas es fundamental pues a partir de ellos se desarrollan los modelos con múltiples localizaciones.

En el Capítulo 2 presentamos los sistemas de inventario con múltiples instalaciones. En primer lugar, explicamos como surgen dichos sistemas en cualquier cadena de suministro y definimos las estructuras más comunes en la práctica. Finalmente, concluimos este capítulo con un resumen de nuestras contribuciones que están relacionadas con el problema conocido como el sistema con 1-almacén y N-minoristas.

En general, la forma de las políticas óptimas de reposición para los sistemas con 1-almacén y N-minoristas son muy complejas. Sin embargo, en muchos casos es posible utilizar estrategias más simples que, aunque no son óptimas, son bastantes efectivas. En particular, en el Capítulo 3 analizamos las políticas cíclicas que son una de las más simples que se pueden aplicar a este problema. En estas políticas se asume que el sistema de decisión es centralizado, es decir, el objetivo es calcular una política de reposición de manera que los costes totales medios del sistema se minimicen. En este capítulo, también estudiamos el problema asumiendo que cada localización del sistema toma decisiones por separado, es decir, cada instalación intenta minimizar sus costes de manera independiente sin tener en cuenta al resto de localizaciones del sistema. Finalmente, concluimos el capítulo con una comparación entre ambos tipos de políticas.

Las políticas cíclicas que se analizan en el Capítulo 3 pueden ser muy eficientes en muchas situaciones y, por supuesto, son muy fáciles de aplicar en la práctica lo que las hace muy atractivas. Sin embargo, cuando los costes de reposición son muy altos en comparación con las demandas, la efectividad de las políticas cíclicas disminuye considerablemente. Así, en el Capítulo 4 analizamos una clase de políticas centralizadas más generales y eficientes denominadas políticas de ratio-entero, las cuales también son comparadas con las estrategias descentralizadas introducidas en el Capítulo 3.

En los capítulos anteriores hemos asumido que el almacén suministra los artículos a los minoristas de forma instantánea. Sin embargo, el almacén también puede representar a una localización donde se fabrican los artículos a razón finita, es decir, la reposición no es instantánea. A pesar de que esta situación es muy común en la práctica, en la literatura no hay muchas referencias que aborden este problema. Además, la mayoría de ellas se centran en el caso en el que el almacén suministra solamente a un minorista. En el Capítulo 5 extendemos el estudio al caso en el que el almacén provee a múltiple minoristas. En particular, primero abordamos el problema asumiendo que el sistema de decisión es centralizado, y después, considerando que las diferentes localizaciones toman decisiones de manera independiente.

Resumen en español

Capítulo 1. Fundamentos del control de inventario

El objetivo de este capítulo es el estudio de los modelos de inventario donde las existencias se localizan en un único almacén. Aunque en la práctica son más comunes los sistemas de inventario con varias localizaciones o instalaciones, el estudio de los sistemas más simples referidos a un almacén ayuda a entender la esencia de los problemas de inventario y permite analizar con mayor facilidad los sistemas de inventario más complejos.

En este capítulo describimos las principales características o componentes que determinan la estructura de un sistema de inventario y estudiamos los principales modelos con demanda determinística constante. El modelo más básico y conocido es el modelo EOQ (Economic Order Quantity), desarrollado por Ford Harris en 1913.

Variables en un sistema de inventario

Demanda

Las suposiciones que se hacen respecto a la demanda son las más importantes, ya que suelen ser las que determinan la complejidad del modelo.

- Demanda determinística y estacionaria: La suposición más simple es asumir que la demanda es constante y conocida. Es decir, la demanda no cambia y puede ser fijada o estimada a priori. El modelo EOQ se basa en esta suposición.

- Demanda determinística variable en el tiempo: En este modelo, la cantidad demandada no es constante, sino que varía con el tiempo. El ejemplo real más conocido es el problema dinámico del tamaño del lote.

- Demanda incierta: Se dice que la demanda es incierta cuando no se pueden conocer a priori los valores exactos de la demanda, pero si se conoce la distribución de la demanda. Normalmente se dispone de una serie de valores de la demanda en el pasado y, a partir de ellos, se intenta estimar la distribución de la demanda y los valores de los parámetros que caracterizan dicha distribución. En algunas situaciones, debe considerarse demanda incierta, por ejemplo, en la salida al mercado de un nuevo producto.

- Demanda desconocida: Cuando tampoco es posible conocer la distribución de la demanda, se dice que la demanda es desconocida. En este caso se suele asumir una distribución a priori de la demanda, y después, cada vez que se dispone de una nueva observación de la demanda, se actualiza el parámetro estimado usando la regla de Bayes.

Costes

Dado que el objetivo normalmente consiste en minimizar los costes totales de inventario, las hipótesis que se hacen sobre la estructura de los costes también influyen en la complejidad del modelo. En general se suelen considerar los siguientes tipos de costes:

- Coste de mantenimiento: Representa el coste de almacenamiento de los productos. Hace referencia a los gastos generales del almacén, seguro, robos, objetos rotos, etc. También incluye el coste de oportunidad del dinero comprometido en inventario que se podría haber usado o invertido de otra manera.

- Coste de compra: En muchos modelos se supone que el precio por unidad de artículo es independiente del tamaño del pedido. Por este motivo este coste no se suele incluir en dichos modelos. Sin embargo, cuando el precio por unidad de producto depende de la cantidad pedida, el coste de compra se vuelve un factor importante. Por ejemplo, en los modelos con descuentos en el precio por volumen de compras se debe incluir este coste.

- Coste de reposición: Es el coste asociado a un pedido. La hipótesis más simple es suponer que este coste es lineal, es decir, si se desea reponer y unidades, entonces el coste es cy, para alguna constante c. Esta estructura se conoce como coste de reposición proporcional, y se suele asumir cuando la demanda es incierta. Sin embargo, es más real asumir que dicho coste está formado por dos componentes, uno fijo y otro variable. Es decir, en este caso el coste de reposición sería de la forma $cy + k\delta(y)$, donde $\delta(y)$ es una función Delta de Kronecker. Así, cada vez que se desea reponer se debe pagar una cantidad fija de k unidades monetarias más una cantidad proporcional a la cantidad solicitada. En muchos de los modelos de inventario determinísticos se asume que el coste de reposición es de la forma anterior. Sin embargo, en los modelos estocásticos, considerar este tipo de costes entraña una mayor dificultad.

- Coste de penalización o rotura: Algunos modelos determinísticos y muchos estocásticos, incluyen un coste de penalización o rotura, que se suele denotar por p, para los casos en los que no es posible satisfacer toda la demanda. En muchas ocasiones p es muy difícil de estimar, por lo que el coste de rotura se suele sustituir por un nivel de servicio. El nivel de servicio es una proporción aceptable del número de ciclos en los que se satisface toda la demanda.

La mayoría de los modelos de inventario asumen que los costes no varían con el tiempo. No obstante, en muchos casos se pueden considerar costes variables en el tiempo, sin que aumente por ello la complejidad del análisis.

Aspectos influyentes en el sistema

Dependiendo de ciertos condicionantes del sistema, se obtienen modelos más o menos complejos. Entre estos aspectos influyentes que condicionan el sistema destacan los siguientes:

- Periodo de retardo: El periodo de retardo se define como el tiempo que transcurre desde que se realiza el pedido hasta que se recibe. El valor del periodo de retardo es muy importante, ya que es una medida del tiempo de respuesta del sistema. La suposición más simple es que el periodo de retardo es cero, aunque esto no suele suceder en la práctica. Esta hipótesis sólo tiene sentido cuando el tiempo requerido para suministrar las reposiciones es pequeño en comparación con el tiempo entre reposiciones. Lo más común es suponer que el periodo de retardo es una constante fija. El análisis es mucho más complejo si se supone que el periodo de retardo es una variable aleatoria. Así, cuando el periodo de retardo es variable, pueden surgir complicaciones como que los pedidos no lleguen en el mismo orden en el que fueron solicitados.

- Roturas: También se deben realizar suposiciones acerca de cómo reacciona el sistema cuando la demanda excede la cantidad existente. Se puede asumir que todo el exceso es rotura, lo que implica un nivel de inventario negativo. También se puede suponer que todo el exceso es pérdida. Esta última situación se conoce como el caso de venta perdida, ya que el cliente no espera a que llegue el producto. Por el contrario, en el caso de rotura no se pierde la venta, ya que el cliente espera hasta que llegue el pedido.

- Proceso de revisión: La manera en la que se realiza la revisión del inventario es otro aspecto influyente a tener en cuenta. La revisión puede ser continua o periódica. Si la revisión es continua, en todo momento se conoce exactamente el nivel de inventario. Este es el caso de los supermercados que cuentan con un sistema de escáner en las cajas, a su vez conectado al ordenador que se utiliza para realizar las reposiciones de stock. Si la revisión es periódica, el nivel de inventario sólo se conoce en determinados puntos, cuando se realiza la revisión. Lo más común es suponer que la revisión es periódica, aunque a veces también se realizan aproximaciones a la revisión continua. Lógicamente, en los sistemas en los que se asume revisión continua, las reposiciones pueden realizarse en cualquier instante, mientras que en los sistemas donde se asume revisión periódica, las reposiciones sólo pueden tener lugar al principio de los periodos de reposición.

Modelo EOQ

Las hipótesis del modelo EOQ son las siguientes:

- La demanda es conocida y constante, a una razón de d unidades por unidad de tiempo.

- La cantidad a pedir puede ser un número no entero, y no hay restricciones sobre su tamaño.

- Los costes no dependen de la cantidad de reposición, es decir, no hay descuentos dependiendo del tamaño del lote.

- Los costes no varían con el tiempo. Existe un coste de reposición, k, por pedido, y un coste de mantenimiento, h, por unidad mantenida a lo largo de cierta unidad de tiempo.

- Las reposiciones son instantáneas, es decir, el periodo de reposición es cero.

- No se permiten roturas.

- Todo el pedido se entrega al mismo tiempo.

- El horizonte de planificación es muy largo, es decir, se asume que los parámetros toman el mismo valor durante un largo periodo de tiempo.

Como el periodo de retardo es cero y la demanda es conocida, es evidente que sólo se debe realizar un pedido cuando el nivel de inventario llega a cero. Un gráfico del nivel de inventario puede verse en la Figura 1.

Para este modelo la cantidad de reposición óptima, conocida como EOQ, (Economic Order Quantity), es

$$Q^* = \sqrt{\frac{2dk}{h}}$$

Esta fórmula es uno de los primeros resultados y el más conocido de la Teoría de Inventarios. Se conoce como la fórmula de Harris (1913) o de Wilson (1934), ya que estos autores fueron los primeros que recogieron en sus respectivos trabajos dicha fórmula.

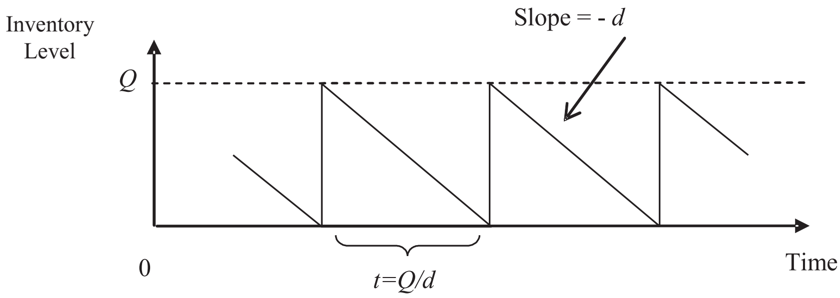

Figura 1: Nivel de inventario en el modelo EOQ

Modelo EPQ

Una extensión natural del modelo EOQ es el modelo EPQ (Economic Production Quantity). En el modelo EOQ, toda la cantidad pedida llega al mismo tiempo. Sin embargo, en el modelo EPQ el stock se produce a una razón finita de P unidades por unidad de tiempo, donde $P > d$. Entonces, el diagrama de la Figura 1 cambia a uno como el de la Figura 2.

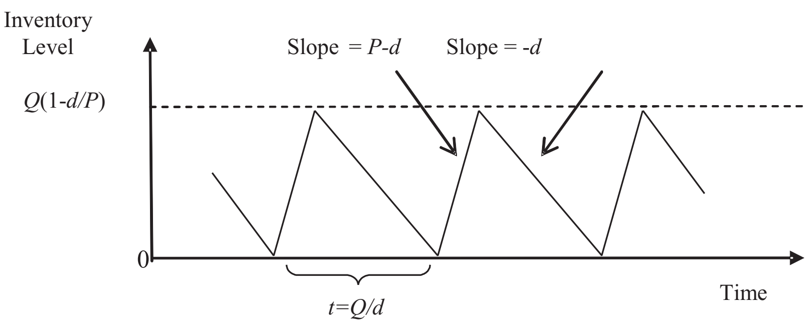

Figura 2: Nivel de inventario en el modelo EPQ

Modelos con demanda determinística variable en el tiempo

En los modelos anteriores la demanda es conocida y constante. Ahora asumimos que la demanda es conocida, pero permitimos que ésta varíe en el tiempo.

En general, cuando la demanda varía en el tiempo, la cantidad de reposición no tiene porque ser siempre la misma, lo que hace que el análisis sea un poco más complejo. Ahora, el diagrama del nivel de inventario no se ajusta a un patrón que se repite en el tiempo, como el de la Figura 1. Por lo tanto, no se pueden calcular los costes medios sobre un determinado periodo como se hace en el modelo EOQ. En este caso, para determinar la cantidad de reposición apropiada, se usa la información de la demanda sobre un periodo finito, conocido como periodo de planificación.

La demanda puede ser continua o discreta, pero esto no suele afectar a los métodos solución, ya que normalmente lo único que se necesita es conocer la demanda total en cada periodo de planificación. Un caso muy común es aquel en el que la demanda permanece constante en cada intervalo, y sólo cambia al pasar de un intervalo a otro. Una ilustración de esta situación puede verse en la Figura 3.

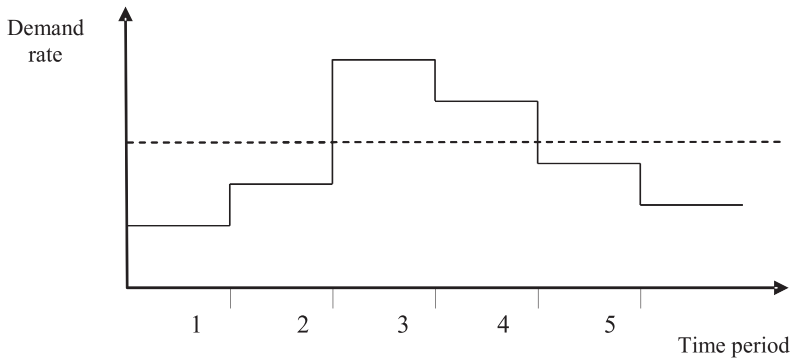

Figura 3: Patrón de demanda cuando la razón permanece constante en cada periodo

Un elemento que se debe tener en cuenta a la hora de calcular las cantidades de reposición, es si las reposiciones pueden planificarse en cualquier instante de tiempo, o si deben programarse en unos instantes determinados como, por ejemplo, considerar que sólo se pueden realizar los pedidos en instantes múltiplos de un periodo base. Cuando sólo hay un artículo, y la demanda es continua, es mejor permitir que las reposiciones puedan realizarse en cualquier instante de tiempo. Sin embargo, si son varios los artículos, tiene más sentido limitar el número de instantes donde se pueden realizar las reposiciones.

El método exacto más conocido para resolver este problema es el algoritmo de Wagner y Whitin.

Algoritmo de Wagner y Whitin

El algoritmo que proponen Wagner y Whitin (1958) para resolver el problema anterior es una aplicación de la programación dinámica. Sin embargo, en este caso, el esfuerzo computacional característico de la programación dinámica puede reducirse significativamente usando las siguientes dos propiedades que debe verificar la solución óptima:

- Una reposición sólo tiene lugar cuando el inventario es cero.

- Existe un límite superior para el número de unidades demandadas que pueden ser incluidas en una sola reposición. A medida que se incluyen más unidades en una sola reposición, los costes de mantenimiento aumentan, por lo que llegará un momento donde será mejor realizar otra reposición que seguir incluyendo unidades en la reposición anterior.

En concreto, el problema puede formularse como un problema de programación entera como sigue

$$\min \sum_{j=1}^{T} k\delta(q_j) + hI_j$$

s.a.

$$\begin{aligned}
I_j &= I_{j-1} + q_j - d_j, \quad j = 1, ..., T \\
q_j &\geq 0, \quad j = 1, ..., T \\
I_j &\geq 0, \quad j = 1, ..., T-1 \\
I_0 &= I_T = 0
\end{aligned}$$

donde $\delta(q_j) = \begin{cases} 1 & \text{si } q_j > 0 \\ 0 & \text{si } q_j = 0 \end{cases}$

Usando la programación dinámica, el problema puede reformularse de la siguiente manera

$$F(j) = \min \left\{ \min_{1 \leq t < j} \left[k + \sum_{g=t}^{j-1} \sum_{l=g+1}^{j} hd_l + F(t-1) \right], k + F(j-1) \right\}$$

donde $F(1) = k$ y $F(0) = 0$.

Es fácil comprobar que $\sum_{g=t}^{j-1} \sum_{l=g+1}^{j} hd_l$ representa la suma de los costes medios de mantenimiento para los periodos $t+1, ..., j$. Mientras que $k + F(j-1)$ representa la situación en la que se realiza una reposición en el periodo j. Por lo tanto, a la mejor solución encontrada hasta el periodo $j-1$ sólo hay que sumarle el coste de reposición k. Hay que elegir entre incluir la demanda del periodo j en las reposiciones anteriores, o realizar una nueva reposición en el periodo j.

Estos modelos son fundamentales para el desarrollo de los sistemas con múltiples instalaciones que estudiamos en los siguientes capítulos.

Capítulo 2. Sistemas de inventario con múltiples instalaciones

En el Capítulo 1 estudiamos los sistemas de inventario con una sola localización o instalación. Sin embargo, en la práctica son más comunes los sistemas de inventario con varias localizaciones. Por ejemplo, cuando una compañía distribuye productos sobre un área geográfica grande, normalmente, hace uso de un sistema de inventario formado por un almacén central, cercano a la fábrica que produce los artículos, y por un número determinado de minoristas cercanos a los clientes. También, es muy frecuente que en la producción de un artículo, éste no se fabrique en una sola localización, sino que distintos componentes del artículo se produzcan en diferentes localizaciones.

Para realizar un control efectivo de estos sistemas de inventario es necesario usar métodos especiales que tengan en cuenta las relaciones que existen entre las distintas localizaciones del sistema. El principal objetivo es coordinar las actividades entre dichas instalaciones para así conseguir que todo el sistema funcione de la manera más efectiva posible.

El sistema de inventario multinivel más simple es el que sólo tiene dos localizaciones. Un ejemplo puede verse en la Figura 4. La demanda de los clientes tiene lugar en la localización 1, la cual recibe los artículos de la localización 2. Un suministrador exterior satisface la demanda de la instalación 2. A este tipo de sistemas se les denomina sistemas en serie.

Figura 4: Sistema en serie con dos localizaciones

Los sistemas en serie pueden encontrarse tanto en la producción como en la distribución de artículos. Desde el punto de vista de un sistema de distribución, la localización 1 puede verse como un minorista que satisface la demanda de los clientes de una determinada área, y la instalación 2 como el almacén central cercano a la fábrica. El periodo de retardo de la localización 1 es, esencialmente, el tiempo de transporte desde la instalación 2 hasta la 1. En un sistema de producción, 1 representa la localización donde se obtiene el producto final, mientras que 2 es la localización donde se fabrica una parte del producto final. En este caso, el periodo de retardo en la instalación 2 coincide con el tiempo de producción en la localización

1.

Sistemas de Inventario/Producción

En general, en los sistemas de producción se fabrica un producto final a partir de una serie de componentes, cada uno de los cuales se produce en una localización. Por este motivo, los sistemas de producción suelen ser convergentes, es decir, al principio del sistema hay muchas instalaciones, y a medida que se avanza a lo largo del sistema el número de localizaciones disminuye. Teniendo en cuenta que normalmente los primeros componentes tienen menos valor que los últimos que están más cerca del producto final, es lógico que el coste de mantenimiento suela ser menor en los primeros niveles de la cadena de producción. Por lo tanto, suele ser más conveniente almacenar más stock en las primeras localizaciones del sistema que en las últimas. En la Figura 5 se representa un sistema de producción en el que cada localización tiene un sólo sucesor. A este tipo de sistemas se les denomina sistemas de ensamblaje.

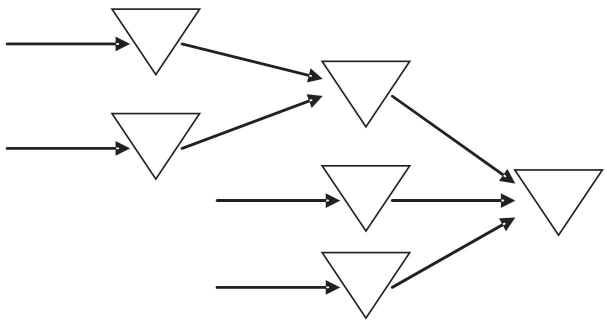

Figura 5: Sistema de ensamblaje

Sistemas de Inventario/Distribución

En los sistemas de distribución cada localización tiene un único predecesor que le suministra los artículos. A su vez, cada instalación satisface la demanda de las localizaciones inmediatamente sucesoras. Las localizaciones que no tienen sucesores son las encargadas de satisfacer la demanda exterior de los clientes y las que no tienen predecesores obtienen los artículos de un suministrador exterior. Un ejemplo de estos sistemas se muestra en la Figura 6.

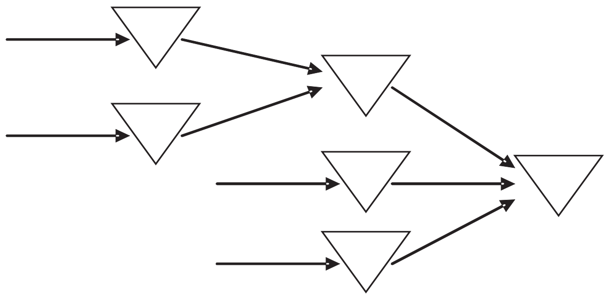

Figura 6: Sistema de distribución

Como se ve en la Figura 6, la estructura de los sistemas de distribución es divergente.

El sistema de distribución más simple es el que sólo tiene dos niveles, conocido como el sistema con 1-almacén y N-minoristas. Véase la Figura 7. En estos sistemas, los minoristas tienen que satisfacer la demanda de los clientes, y el almacén central, la demanda de todos los minoristas. Nótese que cuando $N = 1$ el sistema se reduce a un sistema en serie.

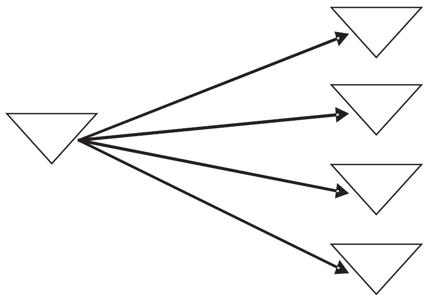

Figura 7: Sistema con 1-almacén y N-minoristas

Inventario nivelado (Echelon inventory)

El concepto de coste y stock nivelado fue introducido por primera vez por Clark y Scarf (1960). Para una localización j, el stock nivelado se define como el número de unidades del sistema que están o que han pasado por la localización j, pero que todavía no han sido demandadas por los clientes exteriores. Así, por ejemplo, para los sistemas en serie, el coste nivelado de la localización j, denotado por h'_j, se define como $h'_j = h_j - h_{j+1}$, donde h_j es el coste convencional de mantenimiento de la localización j.

La idea del stock nivelado es tener en cuenta el stock de todas las localizaciones sucesivas. Para un sistema en serie con dos instalaciones los niveles de inventarios convencionales y nivelados se muestran en la Figura 8 y Figura 9, respectivamente.

Es evidente, que el cálculo de los costes de mantenimiento es mucho más sencillo si se utilizan los inventarios y los costes nivelados. Así, en este capítulo se formulan los sistemas de inventario en serie, de ensamblaje y de distribución haciendo uso de los costes nivelados. En todos estos sistemas asumimos que la demanda, d, es constante y que no se permiten roturas. Además, en cada instalación hay un coste de mantenimiento y un coste fijo de reposición denotados por h_j y k_j, $\forall j$. El objetivo es determinar las cantidades de reposición óptimas, Q_j, $\forall j$.

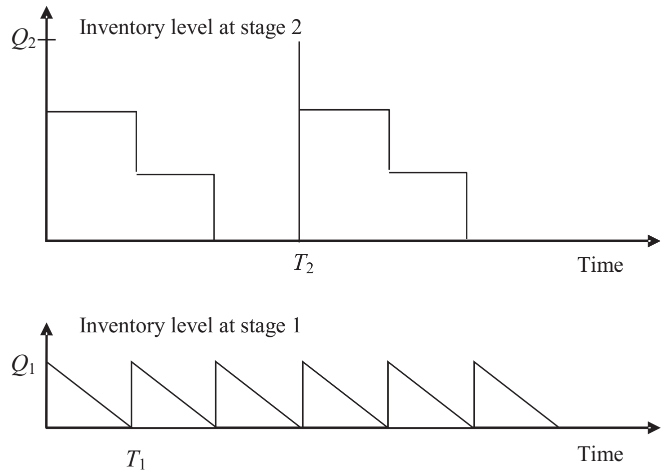

Figura 8: Inventarios convencionales para un sistema en serie con dos instalaciones

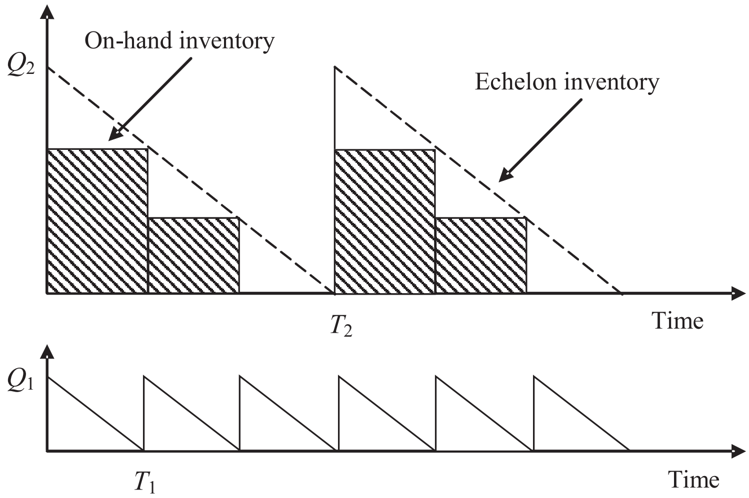

Figura 9: Inventarios nivelados para un sistema en serie con dos instalaciones

Sistemas en serie

Cuando se intenta extender el modelo básico EOQ para los sistemas en serie, surgen las políticas anidadas y estacionarias, que aunque no son siempre las óptimas, son muy importantes desde el punto de vista práctico. Una política se dice que es estacionaria si la cantidad de reposición es fija y el tiempo entre dos reposiciones consecutivas también es constante. Por otro lado, una política es anidada si cada vez que una localización realiza un pedido, todas las localizaciones sucesivas también realizan una reposición.

También hay autores que suelen asumir, sobre todo por motivos prácticos, que los periodos de reposición de cada localización son un múltiplo entero potencia de dos de un periodo base. Este tipo de políticas, conocidas como políticas potencias de dos fueron introducidas por Roundy (1985) el cual demostró que si se restringe el estudio a este tipo de políticas, el coste de la política que se obtiene es como mucho un 2% mayor que el coste de la solución óptima. Como veremos, este resultado se puede extender a los sistemas de ensamblaje.

Política óptima y costes nivelados

Las políticas estacionarias y anidadas son muy interesantes desde el punto de vista práctico, sobre todo porque permiten simplificar la planificación de la producción en cada localización. Por este motivo se aplican a todo tipo de sistemas multiniveles, aunque es importante recordar que estas políticas no son siempre las óptimas. Sólo se puede afirmar que la política óptima es estacionaria para los sistemas en serie con dos localizaciones. Sin embargo, para los sistemas en serie con más localizaciones, la política óptima puede que no sea estacionaria.

Lo que si se verifica siempre para cualquier sistema en serie, es que la política óptima tiene que ser anidada. Para comprobarlo supongamos un sistema en serie con dos localizaciones. Si la política no fuese anidada, la reposición en la localización 2 tendría lugar en un instante t en el cual la localización 1 no repone. Supongamos que $t' > t$ es la primera vez que la localización 1 realiza una reposición. Por lo tanto, el inventario de la localización 2 en el instante t se mantiene al menos hasta el instante t', momento en el que la localización 1 realiza un pedido. Consideremos ahora otra política alternativa en la que la instalación 2 no repone en t, sino que retrasa el pedido hasta el instante t' . En esta nueva política, el número de reposiciones en las dos localizaciones es el mismo, y el coste de mantenimiento es menor en la segunda planificación que en la primera, ya que la instalación 2 no realiza el pedido hasta t' evitando así mantener inventario innecesariamente. Por lo tanto, es evidente que es mejor que la localización 2 sólo reponga cuando la instalación 1 también repone, es decir, la política óptima debe ser anidada. Este mismo razonamiento se puede extender al caso general con N localizaciones, por lo que se puede establecer el siguiente Teorema.

Teorema 2.1

En un sistema en serie con N instalaciones las políticas anidadas dominan a las no anidadas.

En particular, el problema de determinar una política potencias de dos anidada y estacionaria para un sistema en serie se puede formular haciendo uso de los costes nivelados como sigue

$$\min \sum_{j=1}^{N} \left(\frac{k_j}{t_j} + \frac{h'_j t_j d}{2}\right)$$

s.a.

$$\begin{aligned} t_j &\geq t_{j-1} \geq 0 \\ t_j &= 2^{l_j} T_L, \quad l_j \in \{0, 1, ...\} \end{aligned}$$

Se trata de un problema de programación no lineal entera donde las variables de decisión enteras son l_j, $j = 1, 2, ..., N$. Este problema está muy relacionado con su problema relajado

$$\min \sum_{j=1}^{N} (\frac{k_j}{t_j} + \frac{h'_j t_j d}{2})$$

s.a.

$$t_j \geq t_{j-1} \geq 0$$

Roundy (1985) primero caracteriza la solución del problema relajado, y después, analiza su relación con el problema original. Es fácil comprobar que resolver el problema relajado es equivalente a dividir el sistema en serie en grupos, de manera que todas las localizaciones que pertenecen al mismo grupo tienen el mismo periodo de reposición. Después, a partir de estos periodos de reposición, se calcula la solución potencia de dos, $t_j = 2^{l_j} T_L$, donde l_j es el menor entero que verifica

$$2^{l_j - 1} \leq \frac{t_j^*}{\sqrt{2} T_L} \leq 2^{l_j}$$

Además, Roundy (1985) demostró que este procedimiento calcula en O(N log N) una política óptima potencias de dos cuyo coste es como mucho un 6% mayor que el coste de la política óptima. Y aún más, si se considera T_L como otra variable, este margen del 6% se reduce a un 2%. Es decir, siempre es posible encontrar una política potencias de dos cuyo coste es como mucho un 2% mayor que el coste óptimo. Sin embargo, en muchos sistemas T_L no puede tratarse como una variable, ya que éste viene determinado por los instantes en los que se obtiene la información del sistema.

Sistemas de ensamblaje

En los sistemas de ensamblaje se fabrica un producto final a partir de un conjunto de componentes, cada uno de los cuales se produce en una localización.

Como vimos anteriormente, las políticas óptimas para los sistemas en serie son anidadas. Cada localización en un sistema de ensamblaje, o bien tiene un único sucesor y no tiene demanda externa, o bien, tiene demanda externa y no tiene sucesor. Por lo tanto, se puede aplicar el mismo razonamiento que en los sistemas en serie, para demostrar que las políticas óptimas para los sistemas de ensamblaje también tienen que ser anidadas. Así, se puede establecer el siguiente Teorema.

Teorema 2.2.

En un sistema de ensamblaje las políticas anidadas dominan a las no anidadas.

Crowston, Wagner y Williams (1973) fueron los pioneros en estudiar los sistemas de ensamblaje, suponiendo que la demanda es constante y conocida, las reposiciones instantáneas y que las roturas no están permitidas. Ellos afirmaron que el tamaño del lote para cada localización, debe ser igual a un múltiplo entero del tamaño del lote de la instalación que le sucede. Utilizando esta propiedad, conocida como "ratio entera", Crowston, Wagner y Williams (1973) desarrollaron un algoritmo de programación dinámica mediante el cual se puede calcular los tamaños del lote óptimos.

En 1975, Schwarz y Schrage, haciendo uso de la propiedad "ratio entera", resolvieron el problema aplicando un algoritmo de ramificación y poda. Sin embargo, Williams (1982) advirtió que la propiedad que Crowston, Wagner y Williams (1973) habían demostrado sólo era cierta para los sistemas de ensamblaje puros, es decir, para los sistemas de ensamblaje con dos niveles. Para comprobarlo, Williams (1982) propone un contraejemplo para un sistema en serie con tres localizaciones, y determina una política que no verifica la propiedad "ratio entera", la cual es mejor que cualquier política que si verifica dicha propiedad.

A pesar de esto, el método que proponen Schwarz y Schrage (1975) es muy interesante, ya que calcula la mejor política que verifica la propiedad "ratio entera", que aunque no sea la óptima en general, en muchos casos es cercana a la óptima y puede resultar muy práctica. Sin embargo, las políticas que más se suelen aplicar en este tipo de sistemas son las ya mencionadas políticas potencias de dos. El problema de determinar una política potencias de dos anidada y estacionaria para este tipo de sistemas se puede formular de forma muy similar a los sistemas en serie

$$\min \sum_{j=1}^{N} \left(\frac{k_j}{t_j} + \frac{h'_j t_j d}{2}\right)$$

s.a.

$$\begin{aligned} t_j &\geq t_{s_j} \geq 0, \ \forall j \\ t_j &= 2^{l_j} T_L, \quad l_j \in \{0, 1, ...\} \end{aligned}$$

donde s_j es el sucesor directo de la instalación j y T_L es el periodo base. Para este tipo de sistemas también existe un procedimiento $O(N \log N)$ que determina políticas que son un 96% o un 98% efectivas, dependiendo de si T_L se asume fijo o variable.

Sistemas de distribución

La estructura de los sistemas de distribución es justamente la contraria a la de los sistemas de ensamblaje. En la práctica, las localizaciones de un sistema de distribución representan tanto a la fábrica central de un producto, como a almacenes regionales y locales y/o a minoristas. En particular, nosotros nos centramos en los sistemas de distribución con dos niveles, es decir, en los sistemas con 1-almacén y N-minoristas. Para los sistemas en serie y de ensamblaje con dos niveles hemos visto que las políticas óptimas son anidadas y estacionarias. Sin embargo, para los sistemas de distribución con dos niveles la política óptima no tiene porque ser anidada ni estacionaria. En particular, la política óptima para este tipo de sistemas puede ser muy complicada, tanto que ni siquiera sería posible aplicarla en la práctica. Por este motivo, normalmente, se analizan otras políticas más simples cercanas a las óptimas, como las políticas anidadas y estacionarias.

En los siguientes capítulos de esta memoria introducimos diferentes tipos de políticas que se pueden aplicar a los sistemas con 1-almacén y N-minoristas, y desarrollamos nuevos procedimientos para calcular políticas eficientes que son comparados con los más referenciados en la literatura.

Capítulo 3. Sistemas con 1-almacén y N-minoristas: políticas cíclicas frente políticas descentralizadas

Muchas compañías o industrias planifican sus actividades en función de la reposición de sus productos, de forma que los pedidos mantienen su frecuencia cada cierto periodo de tiempo. Esto se traduce en que las reposiciones de los artículos tienen lugar de forma estacionaria. Esta estacionariedad se refleja, por ejemplo, cuando un minorista dedicado a la venta de un determinado artículo, realiza semanalmente un mismo pedido al almacén central. Debido a que esta situación se presenta con frecuencia en muchas empresas, algunos autores han propuesto distintos métodos para calcular políticas estacionarias lo más efectivas posibles.

Schwarz (1973) fue uno de los primeros en proponer un procedimiento para calcular una política estacionaria y anidada para el problema de 1-almacén y N-minoristas. Su aportación más importante es que comprobó el conjunto de propiedades que debe verificar una política óptima. Él demostró que al menos una política óptima para el problema con 1-almacén y N-minoristas puede encontrarse dentro del conjunto de políticas denominadas políticas básicas. Una política básica es una política factible que cumple las siguientes propiedades: a) el almacén sólo repone cuando tiene inventario cero, y al menos uno de los minoristas también tiene inventario cero; b) el minorista realiza un pedido sólo cuando tiene inventario cero; c) todas las reposiciones que se hacen a un minorista entre dos reposiciones consecutivas al almacén, son de igual tamaño. También demostró que para el caso en el que todos los minoristas son iguales, la política óptima es una política estacionaria y anidada. Sin embargo, cuando los minoristas son diferentes, aunque estas políticas pueden ser bastante efectivas no tienen porque ser las óptimas. El método propuesto por Schwarz (1973) para calcular políticas estacionarias y anidadas, también conocidas como políticas cíclicas, proporciona buenas soluciones cuando el número de minoristas es pequeño, pero no para el caso general. De ahí que posteriormente, Graves y Schwarz (1977) intenten mejorar el método propuesto por Schwarz (1973). Dichos autores desarrollaron un algoritmo de ramificación y poda para obtener políticas cíclicas óptimas. El inconveniente de este método es que el esfuerzo computacional aumenta exponencialmente con el número de minoristas y, por lo tanto, sólo es efectivo para problemas pequeños. Muckstadt y Roundy (1993) proponen otro método para calcular políticas estacionarias y anidadas. En particular, se centran en las políticas cíclicas potencias de dos, es decir, se restringen a aquellas políticas donde tanto los minoristas como el almacén central realizan un pedido cada cierto periodo de tiempo t_j, que es un múltiplo potencia de dos de un periodo base T_L. Es decir, $t_j = 2^{l_j} \cdot T_L, \forall j, l_j = 0, 1, \ldots$. Este método propuesto por Muckstadt y Roundy (1993) calcula en $O(N \log N)$ una política óptima estacionaria y anidada potencias de dos cuyo coste es como mucho un 2% mayor que el coste de la política estacionaria y anidada óptima.

Cada uno de los procedimientos propuestos por Schwarz (1973), Graves y Schwarz (1977) y Muckstadt y Roundy (1993) para calcular políticas cíclicas tienen sus ventajas e inconvenientes. El método de Schwarz (1973) no proporciona buenas soluciones cuando el número de minoristas es elevado, pero la ventaja consiste en que el esfuerzo computacional es mínimo. Por el contrario, el método de Graves y Schwarz (1977) computa la solución óptima, pero el esfuerzo computacional es tan grande que es intratable cuando el número de minoristas aumenta. El procedimiento de Muckstadt y Roundy (1993) es muy efectivo computacionalmente pero no siempre da la solución óptima, pues al tener que redondear a potencias de dos, la solución puede alejarse de la óptima. Así, estas soluciones propuestas por el método de Muckstadt y Roundy se pueden mejorar eliminando la restricción potencias de dos. Evidentemente, estas mejoras nunca serán mayores del 2%. Sin embargo, teniendo en cuenta que los costes de inventario representan normalmente una cantidad considerable, tales mejoras pueden significar en muchos casos un ahorro importante.

En concreto, la formulación del problema es la siguiente

$$\min \; C_T = \sum_{j=0}^{N} \left(\frac{k_j}{t_j} + \frac{h'_j d_j t_j}{2}\right)$$

s.a.

$$n_1 t_1 = n_2 t_2 = \cdots = n_N t_N = t_0$$

$$n_j \geq 1, \text{ entero}$$

donde $d_0 = \sum_{j=1}^{N} d_j$.

En este capítulo proponemos una nueva heurística también de $O(N \log N)$ para calcular políticas estacionarias y anidadas cercanas a las óptimas. Además, esta heurística se compara con cada uno de los métodos propuestos por Schwarz (1973), Graves y Schwarz (1973) y Muckstadt y Roundy (1993). De la experiencia computacional se obtienen los siguientes resultados. En el 35% de los ejemplos que se resolvieron, la solución obtenida coincide con la política cíclica óptima. En el 74% de los casos, los costes de las soluciones calculadas con la heurística son menores que los costes de las soluciones obtenidas mediante el procedimiento de Muckstadt y Roundy, y en el 9% de los ejemplos generados, ambos métodos proporcionan las mismas soluciones.

Además, en la segunda parte del Capítulo 3 resolvemos el problema de 1-almacén y N-minoristas asumiendo que el sistema de decisión es descentralizado. Es decir, cada minorista toma decisiones por separado, sin tener en cuenta al resto de instalaciones. En este caso, proponemos un método solución que comprende dos fases. Primero, se calcula la política óptima para cada minorista de manera individual aplicando el conocido modelo EOQ. Después, una vez que se determinan las cantidades y los periodos de reposición óptimos para los minoristas, se calcula el vector de demandas para el almacén. Para encontrar la mejor política de reposición para el almacén, se puede usar el algoritmo de Wagner y Whitin (1958) o el de Wagelmans (1992). Estas políticas son comparadas con las políticas estacionarias y anidadas que se obtienen mediante la nueva heurística. La conclusión más importante es que a medida que el número de minoristas aumenta, resulta más conveniente aplicar las políticas descentralizadas. Sin embargo, la utilización de una política u otra también depende de los valores de los parámetros.

Capítulo 4. Sistemas con 1-almacén y N-minoristas: políticas de ratio-entero

Las políticas estacionarias y anidadas analizadas en el capítulo anterior son, en muchos casos, muy efectivas y muy fáciles de llevar a la práctica lo que las hace muy atractivas. Sin embargo, Roundy (1983) demostró que en algunas ocasiones estas políticas pueden ser muy costosas. Por ejemplo, cuando los costes de reposición son muy altos en comparación con las demandas, la efectividad de las políticas cíclicas disminuye considerablemente. Por este motivo, Roundy (1985) introduce un conjunto de políticas más generales llamadas políticas de ratio-entero las cuales no son ni estacionarias ni anidadas. Las políticas de ratio-entero se caracterizan porque el intervalo de reposición de cada minorista es un múltiplo entero del intervalo de reposición del almacén, o viceversa. Note que en las políticas potencias de dos estudiadas en el Capítulo 3 se exige que $t_j = 2^{l_j} \cdot T_L$, $\forall j$, $l_j = 0, 1, ...$, y también que $t_j \leq t_0, \forall j$. Sin embargo, ahora se permite tanto $t_j \leq t_0$ como $t_0 \leq t_j$. Por lo tanto, las políticas de ratio-entero son mucho más generales que las cíclicas.

En este caso, el problema se formula como sigue

$$\begin{aligned} \min C_T = \ & \{\frac{1}{t_0}(k_0 + \sum_{e \in E} k_e) + \frac{t_0}{2}(h'_0 \sum_{i \in E \cup L} d_i + \sum_{e \in E} h'_e d_e) + \\ & + \sum_{l \in L}(\frac{k_l}{t_l} + \frac{t_l h'_l d_l}{2}) + \sum_{g \in G}(\frac{k_g}{t_g} + \frac{t_g h_g d_g}{2})\} \end{aligned}$$

s.a.

$$\frac{t_j}{t_0} \text{ o } \frac{t_0}{t_j} \text{ es un entero positivo } j = 1, .., N$$

donde $G = \{j | t_j > t_0\}$, $E = \{j | t_j = t_0\}$ y $L = \{j | t_j < t_0\}$.

En particular, Roundy (1985) se restringe al caso donde t_j/t_0 o t_0/t_j es un entero potencia de dos y propone un procedimiento que determina una política óptima de ratio-entero potencia-de-dos en $O(N \log N)$. Además, Roundy (1985) demuestra que estas políticas son un 98% efectivas, es decir, el coste de una política óptima de ratio-entero potencia-de-dos es como mucho un 2% mayor que el coste de una política global óptima. Cuando analizamos las políticas estacionarias y anidadas en el Capítulo 3, vimos que el coste de la política que se obtiene es como mucho un 2% mayor que el coste de una política estacionaria y anidada óptima. En este caso se puede afirmar que los costes de las políticas que se obtienen son cercanos a los costes de las políticas óptimas. Sin embargo, al igual que en el capítulo anterior, este margen del 2% se puede reducir aún más eliminando la restricción de potencia de dos. Así, en este capítulo proponemos otra heurística para calcular políticas de ratio-entero. Además, realizamos un estudio computacional donde se comparan las políticas obtenidas con la nueva heurística con las políticas de ratio-entero potencias de dos. Las conclusiones que se obtienen son las siguientes. En el 85% de los ejemplos que se generaron, los costes de las políticas que se calculan con la heurística son menores que los costes de las políticas obtenidas con el procedimiento de Roundy, y en el 12%, los dos procedimientos calculan la misma solución.

Por último, en este capítulo también comparamos las políticas de ratio-entero con las políticas descentralizadas introducidas en el Capítulo 3. En este caso, para la gran mayoría de los ejemplos generados, resulta mucho más conveniente aplicar la política de ratio-entero que la descentralizada. Recuerde que las políticas de ratio-entero son más generales que las cíclicas al permitir que el periodo de reposición de los minoristas sea mayor que el del almacén. Por lo tanto, tiene sentido que ahora el número de ejemplos donde las políticas de ratio-entero son mejores que las descentralizadas sea mayor que el que se obtiene para el caso de las políticas cíclicas.

Capítulo 5. Sistemas con un vendedor y múltiples compradores con razón de producción finita

En los capítulos anteriores hemos analizado los sistemas de inventario con 1-almacén y N-minoristas asumiendo que la razón de producción es infinita, es decir, que el almacén suministra los artículos de forma instantánea. Sin embargo, en muchas ocasiones, el almacén puede representar a una instalación donde se fabrican los artículos a razón finita, P. Este problema, donde el almacén produce los artículos y después los envía a los minoristas, se conoce como el problema de 1-proveedor y N-compradores. Existen muchas referencias en la literatura que analizan este problema, pero la mayoría de ellas se centran en el caso 1-proveedor y 1-comprador. No obstante, en la práctica es más común que un proveedor suministra a un conjunto de compradores. El objetivo de este capítulo es analizar este caso general con múltiples compradores. Primero resolvemos el problema asumiendo que el sistema de decisión es centralizado y usando las políticas de ratio-entero. Después, planteamos el problema considerando que cada minorista toma decisiones de manera individual.

Veamos la formulación del problema usando las políticas de ratio-entero. Sin pérdida de generalidad, asumimos que $t_1 \leq t_2 \leq ... \leq t_N$ y además que los pedidos van a ser anidados. Diremos que los pedidos son anidados si cada vez que un comprador con intervalo de reposición t_j realiza un pedido, entonces, el resto de compradores con intervalo de reposición $t_i \leq t_j$ también necesitan reponer. Siguiendo la idea propuesta por Roundy (1985), para formular el problema lo primero que hacemos es clasificar los compradores en tres conjuntos, que denotaremos por G, L y E. Para Roundy (1985), aquellos compradores con $t_i > t_v$ forman el conjunto G, donde t_v es el tiempo de reposición del almacén. En el conjunto E están aquellos compradores con $t_i = t_v$, y por último, los compradores con $t_i < t_v$ pertenecen al conjunto L. En el trabajo de Roundy (1985), al ser la razón de producción del proveedor infinita, se puede trabajar con un intervalo de reposición constante para el proveedor. Por el contrario, en el caso que nos ocupa, en el que la razón de producción es finita, el intervalo de reposición del proveedor puede no ser constante. Por lo tanto, no podemos usar exactamente la misma definición que Roundy (1985) para los conjuntos G, L y E. Sin pérdida de generalidad, podemos asumir que cuando el proveedor tiene que satisfacer la demanda de los compradores del conjunto $L \cup E$ y de algunos compradores del conjunto G, éste produce primero las unidades que serán enviadas a los compradores del conjunto G, y después fabrica el pedido de los compradores del conjunto $L \cup E$. Entonces, bajo esta suposición, si sólo consideramos los compradores que pertenecen al conjunto $L \cup E$, el tiempo que transcurre entre dos activaciones consecutivas de la producción en el proveedor si es constante. Por este motivo, a partir de ahora vamos a trabajar con este intervalo de tiempo y lo vamos a denotar por t_0. Además, a partir de t_0 se puede determinar el tiempo entre dos activaciones consecutivas de la producción, considerando a todos los compradores, incluso a los del conjunto G. Teniendo esto en cuenta, ahora si podemos definir los conjuntos G, L y E de manera similar a Roundy (1985), pero considerando como referencia t_0 en vez de t_v. Es decir, $G = \{i|t_i > t_0\}$, $E = \{i|t_i = t_0\}$ y $L = \{i|t_i < t_0\}$.

El motivo por el cual en el caso en el que la razón de producción es finita el intervalo de reposición del proveedor no tiene porqué mantenerse constante es el siguiente. Cuando se considera una política de ratio-entero para un sistema con razón de producción infinita, el proveedor realiza un pedido a la vez que todos los compradores que pertenecen al conjunto G. Como consecuencia, el proveedor no tiene que mantener inventario para ninguno de los compradores del conjunto G. Sin embargo, en los sistemas con razón de producción finita, el proveedor fabrica primero los productos y después los envía. Por lo tanto, el proveedor en este caso si mantiene inventario para todos los compradores, incluso para aquellos que pertenecen al conjunto G. Además, en algunos casos el proveedor tiene que suministrar a todos los compradores, y en otros sólo a algunos de ellos. Entonces, es evidente, que en aquellas ocasiones en las que el proveedor sólo suministra a algunos compradores, la producción debe empezar más tarde que en los casos en el que el proveedor tiene que satisfacer a todos los compradores. De aquí que el tiempo entre dos activaciones consecutivas de la producción no tenga que ser siempre constante.

En las figuras 10-11 mostramos los patrones de inventario asumiendo que el tiempo entre dos activaciones consecutivas de la producción es constante y que puede variar, respectivamente.

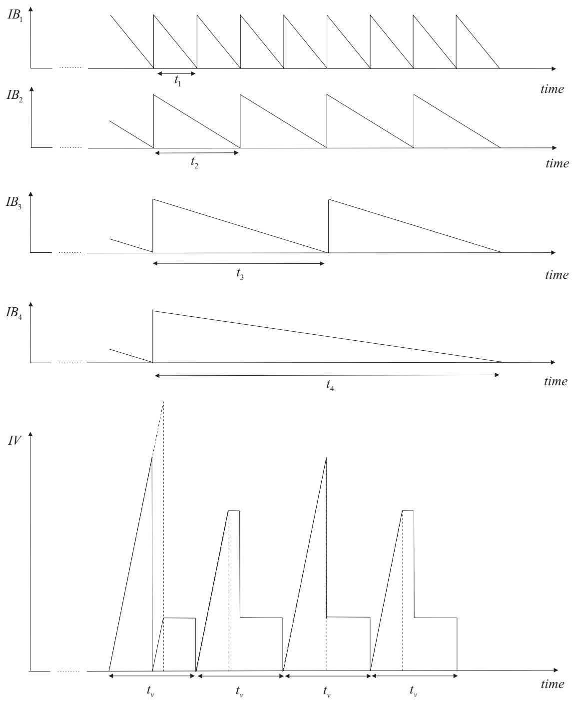

Figura 10: Fluctuaciones de los inventarios en el proveedor y en los compradores considerando que t_v es constante. Los compradores 1 y 2 pertenecen al conjunto $E \cup L$, y los compradores 3 y 4 están en el conjunto G. La línea discontinua representa la razón de producción

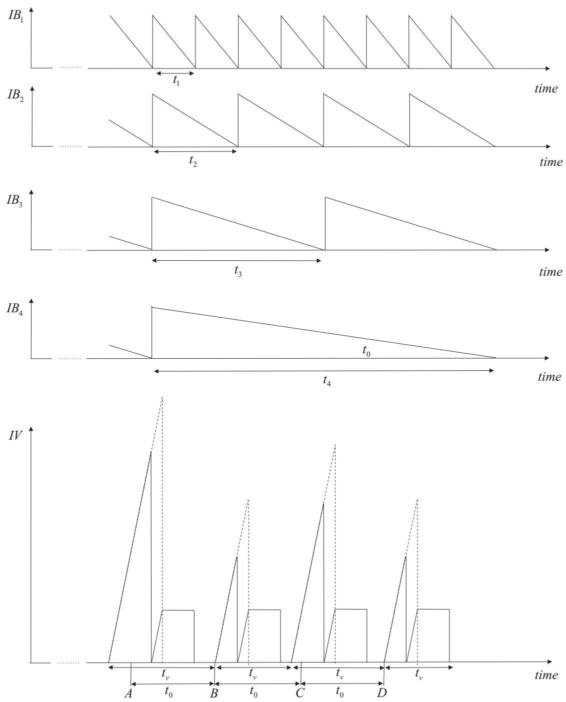

Figura 11: Fluctuaciones de los inventarios en el proveedor y en los compradores considerando que t_v no es constante. Los compradores 1 y 2 pertenecen al conjunto $E \cup L$, y los compradores 3 y 4 están en el conjunto G. La línea discontinua representa la razón de producción. En los instantes A, B, C y D el proveedor empieza a producir las unidades que serán enviadas a los compradores 1 y 2

Es evidente, que los costes de inventario en el almacén son menores cuando se permite que el tiempo entre dos activaciones consecutivas de la producción varíe. También, se puede observar en la Figura 11 que los inventarios medios en los compradores son muy fáciles de calcular ya que siguen un patrón EOQ. Por el contrario, determinar el inventario medio en el proveedor es un poco más complicado. Este problema también surge en los sistemas con un único comprador. Sin embargo, cuando $N = 1$, Hill (1999) demostró que el inventario medio total que hay en el sistema se puede obtener sin mucha dificultad. Así, el inventario medio en el proveedor se puede calcular como la diferencia entre el inventario medio total y el inventario medio en el comprador. Desafortunadamente, para el caso general con N compradores, el inventario medio total tampoco es muy sencillo de calcular directamente como puede verse en la Figura 12.

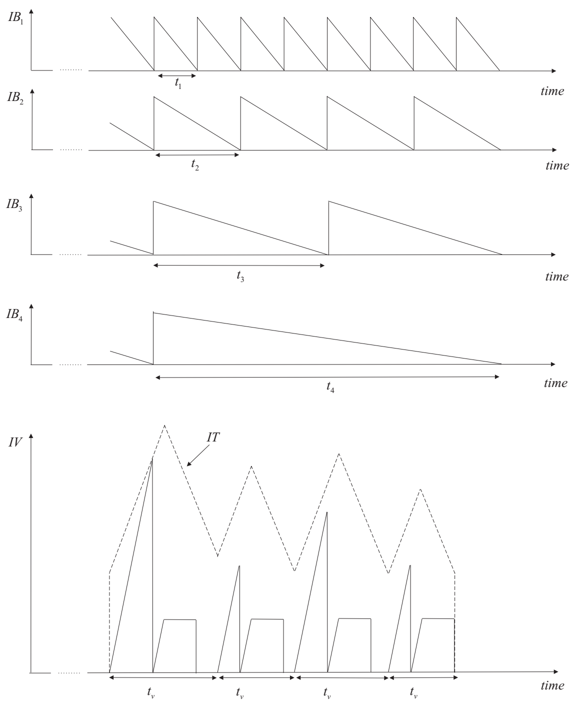

Figura 12: La línea discontinua representa las fluctuaciones de los inventarios totales del sistema

En este capítulo proponemos un método para determinar los costes de mantenimiento en el proveedor. Una vez calculados, vemos que el modelo se puede formular como el siguiente problema de programación no lineal mixta

$$\min C_T = \frac{K_0}{t_0} + \frac{t_0 H_0}{2} + \sum_{j \in L \cup G} \left[\frac{k_j}{t_j} + \frac{t_j H_j}{2} \right]$$

s.a.

$$\begin{aligned}
t_j &= r_{j-1} t_{j-1},\ r_{j-1} \text{ un entero positivo, } j \in \{2, .., l\} \\
t_0 &= r_l t_l,\ r_l \text{ un entero positivo} \\
t_{e+1} &= r_{e+1} t_0,\ r_{e+1} \text{ un entero positivo} \\
t_j &= r_j t_{j-1},\ r_j \text{ un entero positivo, } j \in \{e+2, .., N\} \\
\frac{\sum_{j=1}^{N} d_j t_j}{P} &\leq t_i, \quad i \in \{1, ..., N\}
\end{aligned}$$

donde

$$\begin{aligned}
K_0 &= k_0 + \sum_{j \in E} k_j \\
H_0 &= h_0 \left(1 - \frac{\sum_{j \in E \cup L} d_j}{P}\right) \sum_{j \in E \cup L} d_j + h_0 \frac{2 \sum_{j \in E} d_j}{P} \sum_{j=1}^{N} d_j + \sum_{j \in E} d_j (h_j - h_0) \\
H_j &= \begin{cases} d_j (h_j - h_0) + \frac{2 h_0 d_j}{P} \sum_{i=1}^{N} d_i & \text{si } j \in L \\ d_j h_j + \frac{2 h_0 d_j}{P} \left(\frac{d_j}{2} + \sum_{i=j+1}^{N} d_i\right) & \text{si } j \in G \end{cases}
\end{aligned}$$

Conclusiones y aportaciones

Los sistemas de inventario con varias localizaciones son muy comunes en la práctica y aparecen tanto en la producción como en la distribución de artículos. En los sistemas de producción se fabrica un producto final a partir de una serie de componentes, cada una de las cuales se produce en una localización. En los sistemas de distribución cada instalación tiene un único predecesor que le suministra los artículos. A su vez, cada localización satisface la demanda de las instalaciones inmediatamente sucesoras.

El control de este tipo de sistemas se puede llevar a cabo de una manera descentralizada, es decir, las decisiones en cada instalación se pueden basar exclusivamente en la información que se posee de esa localización. Sin embargo, actualmente, las empresas son cada vez más conscientes de que pueden reducir considerablemente sus costes de inventario con un buen control de los mismos a lo largo de todas las instalaciones que forman parte del sistema. Así, para realizar un control efectivo de estos sistemas de inventario, es necesario usar métodos que tengan en cuenta las relaciones que existen entre las localizaciones del sistema. El estudio de estos sistemas de inventario con múltiples instalaciones se ha convertido en los últimos años en un tema de investigación muy importante dentro de la Investigación Operativa.

En esta tesis nos hemos centrado en el análisis de un sistema de inventario con múltiples instalaciones muy frecuente en la práctica conocido como el problema de 1-almacén y N-minoristas. En estos sistemas existe un único almacén que distribuye un artículo a un conjunto de minoristas, por ejemplo, un almacén central que satisface la demanda de una cadena de tiendas.

En el Capítulo 2 presentamos las diferentes estructuras que aparecen en los sistemas multiniveles de inventario. En particular, analizamos los sistemas en serie, los sistemas de ensamblaje y los sistemas de distribución. Para cada uno de ellos hemos revisado los tipos de políticas de reposición que más se aplican en la práctica, y los diferentes algoritmos propuestos en la literatura para determinar políticas de reposición óptimas o casi-optimas. De la revisión bibliográfica realizada en este capítulo se concluye que para los sistemas en serie y los sistemas de ensamblaje la política óptima tiene que ser anidada y estacionaria. Esta propiedad facilita la búsqueda de soluciones óptimas. Así, ya existen en la literatura procedimientos de orden polinomial que determinan políticas óptimas para estos sistemas. Sin embargo, el problema de 1-almacén y N-minoristas es un problema NP-duro, es decir, no existe un algoritmo de orden polinomial que lo resuelva. De ahí que muchos autores se hayan centrado en clases de políticas más simples, y hayan desarrollado métodos para determinar la política óptima dentro de ese conjunto de políticas. En esta tesis presentamos procedimientos alternativos para calcular políticas de reposición para el problema de 1-almacén y N-minoristas. Además, para comprobar su efectividad los comparamos con los métodos más eficientes existentes en la literatura.

En particular, comenzamos analizando las políticas anidadas y estacionarias en el Capítulo 3. La principal aportación de este capítulo es el desarrollo de una nueva heurística de O(Nlog$N$) que para la mayoría de los casos proporciona políticas cíclicas más eficientes que las calculadas con los métodos heurísticos que existían hasta el momento en la literatura. Además, en este capítulo también se plantea y se resuelve el problema asumiendo que cada localización del sistema toma decisiones por separado, es decir, cada instalación intenta minimizar sus costes de manera independiente sin tener en cuenta al resto de localizaciones del sistema. Por último, se realiza un experimento computacional donde se comparan ambos tipos de estrategias. La conclusión más importante es que a medida que el número de minoristas aumenta, resulta más conveniente aplicar las políticas descentralizadas. Sin embargo, la utilización de una política u otra también depende de los valores de los parámetros. En concreto, los resultados que se presentan en este capítulo se recogen en Abdul-Jalbar et al. (2003, 2006).

En el Capítulo 4 se eliminan las restricciones de estacionariedad y anidamiento y se analiza un conjunto de políticas más generales llamadas políticas de ratio-entero. Para este caso también hemos propuesto un nuevo método heurístico de O(Nlog$N$) el cual es comparado con el método de Roundy (1985) que es el más referenciado en la literatura. Algunos de los aportaciones que se presentan en este capítulo ya han sido publicadas en Abdul-Jalbar et al. (2005).

Por último, en el Capítulo 5 se analiza el mismo problema pero considerando que la razón de producción en el almacén es finita. La mayoría de los autores que estudian este problema se centran en el caso en el que el almacén o proveedor suministra solamente a un comprador. Así, una contribución importante de este capítulo es la formulación del problema para la situación general con múltiples compradores asumiendo que el sistema de decisión es centralizado. En particular, primero estudiamos el problema asumiendo que el proveedor suministra a dos compradores, y después, extendemos el análisis al caso con más de dos compradores. Finalmente, también abordamos el problema suponiendo que el sistema de decisión es descentralizado y comparamos ambas estrategias, las centralizadas y las descentralizadas. De los resultados computacionales se puede concluir que ninguna de las dos clases de políticas domina a la otra. Las contribuciones que se incluyen en este capítulo se recogen en Abdul-Jalbar et al. (2004a, 2004b).

El problema que hemos analizado en esta tesis se puede extender a situaciones en las que, por ejemplo, se admiten roturas, demandas de los clientes variables en el tiempo o estocásticas, razones de producción finita en los minoristas, traspasos entre los minoristas, etc. Otra posible línea de investigación consiste en analizar sistemas de inventario más generales donde el grafo asociado sea cualquier grafo acíclico.

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Distribution Systems: Advances in Inventory Management

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Thesis for obtaining the doctoral degree
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Contents

Preface

There is no doubt of the importance of the inventory control in today's business environment. In practice, in many economic sectors the profit margins are tight, and hence, a good management of the available resources is essential for increasing benefits. Thus, inventory management has become a critical factor in the success and profitability of a firm. Even more, inventories play an extremely important role in a nation's economy. In fact, the total value of inventories represents, in general, an important quantity in the balance of firms. Consequently, changes in inventories are watched closely by economists since they are directly associated with the overall direction and health of the economy of the country.

However, the importance of storing goods has been recognized since ancient times. For example, in times of plenty, people stored food and other goods of first necessity to provide for times of scarcity. Nevertheless, the production and stocking systems from which the present ones are derived appear with the Industrial Revolution in the eighteenth century. In fact, the Industrial Revolution can be thought of as the most important pre-twentieth century influence on management. The use of machine powers, combined with the division labor made large factories possible. In addition, the huge investments of the new factories contributed to the development of procedures for planning, organizing, leading and controlling. In other words, scientific management began to be required.

The scientific management period brought widespread changes to the management of factories. Important developments in this field were made during the early part of the twentieth century. In 1911, the classic book *"The principles of Scientific Management"* was published by Frederick Taylor who is often referred to as the father of scientific management. This publication actually helped the scientific management principles to achieve wide acceptance in different industries. In particular, it is necessary to emphasize the use of the scientific management to the automobile industry. In addition, it was in the automobile industry where the moving assembly lines were first introduced. The use of these assembly lines allowed to decrease the assembly time and cost considerably. In addition, the advances achieved in the factory movement motivated that organizations realized of the relationship between efficiency, holding goods and production planning. Thus, it became necessary the use of mathematical tools and quantitative techniques to improve decision making of industries that were facing a combination of production scheduling problems and inventory problems.

One of the earliest applications of mathematical models for business decision making was published by F.W. Harris in 1913 who developed the Economic Order Quantity (EOQ) formula. He derived a mathematical formula in which the most economic choice for the order quantity depends on the cost factors. Although this formula was first developed by Harris, Andler (1929) and Wilson (1934) are also recognized in connection with the EOQ model. This model and many extensions were presented by Raymond in his book *"Quantity and Economy in Manufacture"* published in 1931. This is considered the first published book on inventory management. At first, these quantitative models were not widely used in industry and few papers were published in the next years. However, interest in mathematical inventory models resurfaced during the Second World War. In the war, specialists from many disciplines combined efforts to achieve advancements in the military and in manufacturing. After the war, many authors continued developing and refining quantitative tools for decision making. These efforts resulted in decision models for forecasting, inventory management and other areas of operations management. An excellent overview of the systems that were studied until 1951 is given by Whitin in *"The Theory of Inventory Management"* and in *"Inventory control research: A survey"* published in 1953 and in 1954, respectively.

During the 1960s and 1970s, management science techniques were highly regarded but in the 1980s, they lost some favor. However, the widespread use of computers and software contributed to a resurgence in the popularity of these techniques. Thus, many advances have been achieved in the last decades. Probably, one of the most important developments in the world economy in recent years has been the extraordinary success of japanese firms in western markets. Among the key factors underlying this phenomenon seems to be the ability of japanese firms to operate with substantially lower inventories than their western counterparts. Moreover, many of the success stories in retailing, automobiles, computers, and other industries are founded on operational capabilities that, among other things, keep inventories lean. Indeed, in today's increasingly competitive markets firms cannot afford to carry excessive levels of inventory. At the same time, it is also important to have the product available at the right place and at the right time. In order to balance these two issues it is critical for an organization to decide where inventory should be kept and in what quantities. Traditionally, when it was critical to have inventory available for customers, large amount of inventory was placed at all locations. This ensured a high service level, but could be enormously costly. Nowadays, firms have realized that costs can be reduced considerably with the management of the inventory throughout the different facilities in the supply chain. It is worth noting that actions of one member of the chain can affect the profitability of all partners of the supply chain. Consequently, firms are increasingly collaborating and integrating their processes with their partners to compete as part of a supply chain against other supply chains, rather than as a single entity. Supply chain management is the term used to describe the management of materials and information across the entire supply chain, from suppliers to customers. This topic covers multiple areas each one representing an issue within the supply chain. In particular, one of the most important problems in the management of supply chains is the control of inventory costs at the different locations throughout the system while satisfying the customer requirements. The area into supply chain management devoted to the effective management of inventory in a supply chain is what is known in the literature as *multi-echelon inventory theory*. Since inventory costs usually represent an important investment, the reduction of such costs is critical for the supply chain. Thus, it is not surprising that in recent years considerable efforts have been made to achieve advancements in this field.

In line with this interest, this dissertation represents an attempt to continue advancing in the deterministic multi-echelon inventory theory. The multi-echelon inventory systems are common in both distribution and production contexts. Specifically, we deal with a two-level distribution system where a warehouse supplies an item to multiple retailers. However, before introducing the multi-echelon inventory systems we present in Chapter 1 the fundamentals of inventory management. In addition, we also review the simplest inventory problems for a single location. These basics models are fundamental to all what follows since they form the building blocks from which more elaborated inventory models are constructed. Furthermore, such models are widely applied themselves.

In Chapter 2 we introduce the multi-echelon inventory systems where several installations are coupled to each other. We first illustrate how multi-echelon inventory systems arise in practical supply chain. Then, we describe the kind of structures that are most common in multi-echelon inventory systems. Finally, we conclude this chapter with a summary of our contributions which are related to the problem referred to as the one-warehouse multi-retailer system.

In general, the form of the optimal inventory policy for the one-warehouse multi-retailer system is very complex. However, it is often possible to use a simpler class of strategies. Accordingly, in Chapter 3 we restrict our attention to one of the simplest policies which can be applied to the one-warehouse multi-retailer problem. In such policies, known as single-cycle policies, it is assumed that the decision system is centralized. Thus, the goal consists of minimizing the average total cost, that is, the sum of the average cost at the warehouse and at the retailers. In this chapter we also study the problem assuming that at each location there is a decision maker. That is, each facility minimizes its total cost without taking the rest of the system into consideration. Finally, we compare the single-cycle policies with the decentralized strategies.

The single-cycle policies considered in Chapter 3 are very efficient in many situations and have clear managerial advantages. However, when relatively high order costs are combined with relatively low demand rates, the performance of these policies get worse. In order to achieve more effective strategies, we analyze in Chapter 4 a more general class of centralized policies known as integer-ratio policies. In this chapter, we also compare the integer-ratio solutions with the decentralized policies.

To this point it is assumed that the warehouse supplies the items to the retailers instantaneously. However, often the warehouse is also a manufacturing location and it produces at a finite rate. In spite of this, we find few references in the literature dealing with this problem. Besides, most of them focus on the case where the warehouse only supplies one retailer. Thus, in Chapter 5 we extend the analysis to the multiple retailer situation. That is, we study the one-warehouse multi-retailer system assuming finite production rate at the warehouse. In particular, we first formulate the problem in terms of integer-ratio policies. Finally, we also address the problem assuming that the decision system is decentralized.

Chapter 1
Fundamentals of Inventory Control

In this chapter we introduce the basic concepts in Inventory Control which will be used throughout this thesis. In addition, we also review the classical models for controlling inventory at a single installation. The study of these models are fundamental for a good understanding of the more complex inventory models where several locations are involved. In particular, we focus on the Economic Order Quantity (EOQ) model, on the Economic Production Quantity (EPQ) model and on the Wagner and Whitin problem.

1.1 Introduction

Inventories consist of physical goods and materials kept in stock to be used in a production process or to be sold to final customers. Usually, we think of inventory as a final product waiting to be sold to a customer. This is certainly one of its most important uses. However, especially in a manufacturing firm, inventory can take on forms besides finished goods as, for example, raw material and work-in-progress. Raw materials are used to make the components of the finishes products. Work-in-process inventories are partially completed final products that are still in the production process.

Another classification of inventories is based on their utility. Thus, working stocks, also known as cycle or lot size stocks, are those products arriving in large regular orders to meet smaller but more frequent customer demands. As we have commented, firms have to be protected from uncertainties of supply and demand. The inventories which are held to be used in these cases are refereed to as safety or fluctuation stocks. In addition, seasonal or anticipation stocks are inventories which are build up to maintain stable operations through seasonal variations in demand. The pipeline or transportation stocks are products which are currently being moved from one location to another. For example, inventories in a truck or inventories waiting to be processed are pipeline stocks. Finally, speculative or hedge stocks arise when firms purchase extra quantities at lower prices to achieve important savings.

Nowadays the time and the money needed to manage inventories represent a significant investment for the firms. Thus, firms cannot afford to have any money tied up in excess inventories and hence the objectives of good customer service and efficient production must be met at minimum inventory levels. Accordingly, an effective management of inventories keeps the business profitable. Therefore, the inventory control plays a critical role in successful management. Inventory control consists of all activities and procedures used to ensure that the right amount of each item is held in stock.

Since inventories normally represent a sizable investment, reasonable questions arise with respect to the convenience of inventories as well as the functions that they perform. Thus, inventories exist because supply and demand cannot be matched due to physical and economics causes. There are many reasons for which supply and demand frequently differ in the rates at which they respectively provide and require stock. According to Tersine (1994), these reasons can be explained by four factors: time, discontinuity, uncertainty and economy. The time factor refers to the time which takes the process of production and distribution before goods reach the final consumer. In general, customers want the product on time and they often disagree to wait. The discontinuity factor allows the common treatment of various dependent operations in an independent manner. Thus, the discontinuity factor permits firms to schedule processes such as purchasing, production, warehousing, distribution, and retailing with more freedom. Another reason for holding inventory is to protect firms from unexpected and unplanned events that modify their schedules. These events that are included in the uncertainty factor are errors in demand estimates, strikes, disasters, delays, etc. Finally, the economy factor relates primarily to cost reducing alternatives such as quantity discounts.

Depending on the type of organizations the inventory requirements are different. An organization can be a retailer, a warehouse, a factory, etc. The retailers are organizations that obtain products from warehouses or directly from factories and supply them to final customers. In contrast, the warehouses purchase large quantities of manufactured goods to be distributed to the retailers. In addition, these organizations do not usually provide goods to final consumers but supply the retailers. With respect to the factories, they are organizations which produce finished products from raw materials. Accordingly, inventories can be stocked at a single or at many locations or installations. Usually, smaller firms have single stocking points, whereas larger firms use to hold inventories at multiple locations. For example, when firms distribute products over large geographical areas, they often use an inventory system with a central warehouse close to the production facility and a number of retailers close to the customers in different areas. In the production context, it is also very common that raw materials, components, and finished products are stocked at different locations. In these inventory systems each stocking point is referred to as stage or location. Moreover, in such systems we can distinguish different hierarchy levels or echelons. Precisely, an echelon is a set of locations of the same level that are replenished by a common set of suppliers. In general, outside customer demands occur at the stages in the first level, that is, at the retailers. In turn, these locations replenish from the warehouses at the second level, which receive replenishments from the third level, etc.

It is worth noting that most inventory systems encountered in the real world are multi-echelon in nature. Hence, it is not surprising that in the last decades considerable efforts have been made to achieve advancements in this field. In line with this interest, this dissertation deals with deterministic multi-echelon inventory systems. Specifically, we focus on two-level distribution system where a warehouse supplies an item to multiple retailers. However, before addressing this main subject, we introduce some basic concepts in inventory management. In addition, we also review some of the more classical models and techniques for making cost-effective inventory decisions. These basic models are fundamental for the development of the multi-echelon inventory systems. In particular, we focus on the classical Economic Order Quantity (EOQ) model and on extensions of such a model that will be used throughout this thesis.

1.2 Elements of Inventory Management

Inventory problems come in all shapes, sizes and varieties but they usually are concerned with the making of decisions that minimize the total cost of an inventory system. Therefore, decisions that are made affect the costs, but such decisions can rarely be made directly in terms of costs. In fact, decisions are usually made in terms of time and quantity. Consequently, the time and quantity elements are usually the variables that are subject to control in an inventory system. Accordingly, the decisions to be made are basically when should a replenishment order be placed, and how much should the order quantity be. Such decisions will depend on different factors that can be recognized for every inventory systems. In particular, the main components which characterize an inventory system are the demand, the replenishment, the costs and the constraints. In general, the complexity of the models depends on the assumptions that one makes about these components. We will discuss each of these elements in turn.

1.2.1 Demand

The starting point for the management of inventory is to meet customer demands, hence a firm maintains inventory to cover such demands. In addition, the assumptions made about demand are usually the most important in determining the complexity of the model. In general, the following types of demand are considered:

1. *Deterministic and Stationary*: The simplest assumption is that the demand is constant and known. These are really two different cases: the demand is not anticipated to change, and the other case is that the demand can be predicted in advance. The simple EOQ model is based on constant and known demand.

2. *Deterministic and time varying*: Changes in demand may be systematic or unsystematic. Systematic changes are those that can be forecasted in advance. Lot sizing under time varying demand pattern is a problem that arises in the context of manufacturing final products from components and raw materials.

3. *Stochastic and uncertain*: The term uncertainty means that the distribution of demand is known, but the exact values of the demand cannot be predicted in advance. In most contexts, this means that there is a history of past observations from which we can estimate the form of the demand distribution and the values of the parameters. In some situations, such as when new products are produced, the uncertain demand could be assumed but some estimate of the probability distribution would be required.

4. *Stochastic and unknown*: In this case, the distribution of the demand is unknown. Here the approach consists of assuming some form of a distribution for the demand and updating the parameter estimates using Bayes rule each time a new observation becomes available.

1.2.2 Replenishment

The replenishments of inventory systems are usually controlled by decision-makers, and in general, they refer to the following elements.

1. *The scheduling period:* The scheduling period is the length of time between consecutive replenishments, and it is not always controllable. When the scheduling periods are not fixed, they are variables subject to control. In other case, they are given parameters and the only variables that can be controlled are the replenishment sizes.

2. *The replenishment size:* It represents the quantity scheduled for replenishment, and it is usually under control of the decision-maker. When the replenishment

size is the same for every scheduling period it is referred to as *lot size*. In addition, it is often assumed that when a replenishment is ordered the exact amount is delivered and added to stock.

3. *The replenishment period:* The replenishment period is the length of time during which the replenishment size is being added to inventory.

4. *The replenishment rate:* The average replenishment rate is the ratio between the replenishment size and the replenishment period. When the replenishment period is negligible we say that the corresponding rate is infinite and the replenishment is instantaneous.

5. *The lead time:* The lead time, often denoted by L, is defined as the amount of time that elapses from the point that a replenishment order is placed until it arrives. It is a very important quantity in inventory analysis since it is a measure of the system response time. Usually, it is assumed that the lead time is zero. This makes sense if the time required for replenishment is short compared with the time between replenishment decisions. Other common assumption is that the lead time is a fixed constant. In this case, the lead time can be incorporated in the model easily. However, the analysis becomes much more complicated if the lead time is assumed to be a random variable. Under this situation, issues such as order crossing, that is, orders not arriving in the same sequence that they were placed, must be considered.

6. *Inventory position and inventory level:* The purpose of an inventory control system is to determine when and how much to order. These decisions should be based on the stock situation, the anticipated demand, and different cost factors. When talking about the stock situation, it is natural to think of the physical stock on-hand. But an ordering decision cannot be based only on the stock on-hand. We must also include the outstanding orders that have not yet arrived, and backorders, i.e., units that have been demanded but not yet delivered. In inventory control the stock situation is characterized by the *inventory position*:

$$\text{inventory position} = \text{stock on-hand} + \text{outstanding orders} - \text{backorders}$$

Although the ordering decisions are based on the inventory position, holding and shortage costs will depend on the *inventory level*:

$$\text{inventory level} = \text{stock on-hand} - \text{backorders}$$

7. *Review process:* An inventory control system can be designed so that the inventory position is monitored continuously. As soon as the inventory position is sufficiently low, an order is triggered. We refer to this situation as *continuous*

review. Another situation consists of checking the inventory position only at certain given points in time. In general, the intervals between these reviews are constant and the review process is referred to as *periodic review*. Most systems are periodic review, although continuous review approximations are common. Both alternatives have advantages as well as disadvantages. In continuous review systems, generally the replenishment decisions can be made at any time. However, in periodic review they can be made only at predetermined time instants corresponding to the start of the periods. In contrast, periodic review has advantages especially when we want to coordinate orders for different items.

1.2.3 Costs

In most inventory problems the objective consists of minimizing the costs involved in the operations. Therefore, the assumptions about the cost structure are decisive. The basic costs associated with inventory are the following:

1. *Unit value cost*: The unit value of an inventory item is usually expressed in monetary unit per unit of item. For a vendor, the value cost is simply the price paid to the supplier plus any cost incurred to make it ready for sale. Depending on the size of the replenishment, this cost can vary as a result of quantity discounts. For a producer, the value cost is the unit production cost. The unit value is important for two reasons. First, the total acquisition costs per year clearly depend on its value. Second, the total cost of holding an item can also depend on this cost.

2. *Ordering or replenishment costs*: The ordering costs are related to the replenishments of inventory. The assumptions about the order cost function can represent a substantial difference in the complexity of the resulting model. The simplest assumption is that the replenishment cost is a linear function of the ordered quantity. This case is referred to as proportional replenishment cost and it is often assumed when demand is uncertain. However, it is more realistic to assume that the replenishment cost has both fixed and variable components.

Obviously, replenishment costs vary with the number of orders placed, as the number of orders increases so does the ordering cost. Replenishment costs can include charges as requisition and purchase orders, transportation and shipping, receiving, inspection, handling and storage, and accounting and auditing. Replenishment costs generally behave inversely to holding costs. As the size of orders increases, fewer orders are required, reducing ordering costs. However, ordering larger amounts results in higher inventory levels and hence higher holding costs.

3. *Holding costs*: The capital cost is usually the dominating part of the holding costs and it represents the return that could be obtained by investing elsewhere

the capital tied up in inventory. Other parts can be material handling costs, costs of storage, damage and obsolescence costs, insurance costs, and taxes. All those charges that vary with the inventory level should be included in the holding cost.

Holding costs are normally specified in one of the following ways. One way is to add all the individual charges mentioned above over a time period, such as a month or a year. Alternatively, holding costs are sometimes expressed as a percentage of the value of the item or as a percentage of average inventory value.

4. *Shortage costs*: A shortage occurs when the demand for a item cannot be satisfied on time. When this situation occurs the response of the customer can be different. There are situations where a customer agrees to wait while his order is backlogged, but in other cases the customer chooses some other supplier. If the customer order is backlogged there are often extra costs for administration, price discounts for late deliveries, material handling, and transportation. If the sale is lost, the contribution of the sale is also lost. In any case, shortages usually yield a loss of goodwill. Most of these costs are difficult to estimate, and the difficulty increases in production processes. Thus, if, for example, a component is missing, this can entail a chain of negative consequences such as delays, rescheduling, etc. There are also situations where shortage costs are easy to evaluate. Assume, for example, that a missing component can be bought at a higher price in a store next door. We can then use the additional cost as our shortage cost. Since shortage costs are so difficult to estimate, it is very common to replace them by a suitable service constraint. Obviously, it is also difficult to determine an adequate service level, but yet somewhat simpler in most practical situations.

1.2.4 Constraints

Constraints in inventory systems deal with various properties that in some way impose limitations on the components discussed in the previous sections. In general, we can consider the following constraints.

1. *Unit constraints*: The kind of mathematical analysis used in solving an inventory system depends on whether the units involved are continuous or discrete.

2. *Demand constraints*: Some constraints associated with demand are the *negative demand* and the *dependent demand structures*. For example, when returns by customers are allowed we have negative demand. A dependent demand structure occurs when the demand during any period may depend on the demand at the previous periods and on the inventory quantity in such periods.

3. *Replenishment constraints*: The replenishment constraints can be related to space constraints. In some systems the amount of space for storing inventories is

limited so that the inventory quantity at any time may not exceed some specified amount. We also have already mentioned that scheduling periods can be prescribed in some systems. Such prescriptions should be considered as other replenishment constraint.

4. *Cost constraints*: In some systems no shortages are allowed and hence the shortages costs are zero. In other models no inventory is carried, that is, the are not holding costs. In some situations, the replenishment costs are constant and then, they are not subject to control.

1.2.5 Inventory problems classification

Inventory problems can be classified in many ways. An admissible classification could distinguishes, at the first level, between single-echelon models and multi-echelon models. In the single-echelon systems the items are stocked at a single location. In contrast, in the multi-echelon systems there are several stocking points. In turn, both the single-echelon and the multi-echelon inventory problems can be subdivided into categories based on other characteristics. One of these characteristics is the number of different items which are considered in the system. Accordingly, we can discriminate between single-item and multi-item models. An additional subdivision includes finite and infinite replenishment rate. Recall that an infinite replenishment rate means that the replenishments are instantaneous. Inventory problems can also be subclassified according to the presence or absence of capacity constraints. Finally, we can identify different inventory problems depending on the characteristics of demand. Thus, we can differentiate between deterministic and stochastic demand, and between static and dynamic demand. Consequently, any inventory problem can be placed in one of the categories given in Figure 1.1.

As we have commented, the goal of this monograph is the study of the multi-echelon inventory problems. Most of these problems represent extensions of the basic single-echelon models. Therefore, the knowledge of such basic models is essential for a good understanding of the more complicated systems to be developed in the following chapters. Hence, we devote the remainder of this chapter to review the single location models that will be used throughout this work. We first focus on the Economic Order Quantity (EOQ) and on the Economic Production Quantity (EPQ) models where demand is deterministic and static. Finally, we analyze the problem assuming time-varying demand.

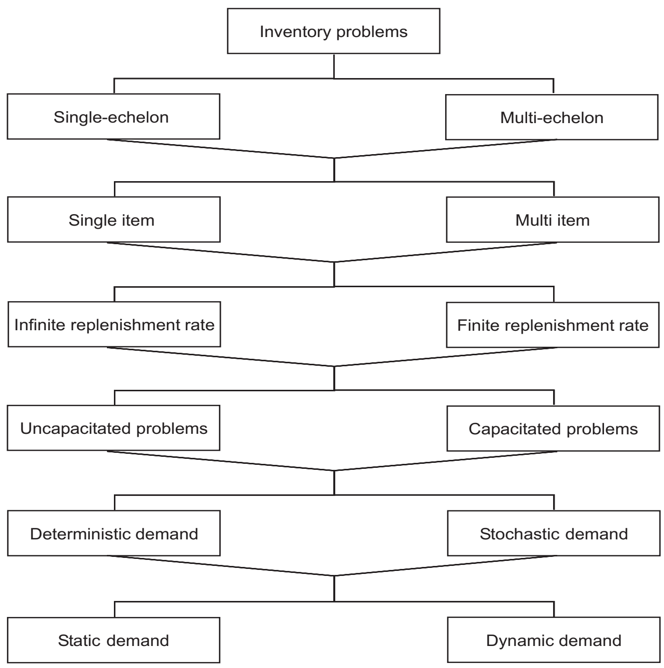

Figure 1.1: Inventory problems classification

1.3 Models with constant demand rates

The basic inventory models are based on the assumption that demand is deterministic and stationary. These assumptions may seem very unrealistic assumption, however, in general, are reasonable. First of all, there exist situations where a firm is really facing deterministic demand. An example is a firm which delivers items according to a long-range contract.

In addition, models requiring deterministic and stationary demand are also important for the following arguments. First, many results are quite robust with respect to the model parameters, such as the demand rate and costs. Second, the results obtained from these simple models are often good starting solutions for more complex models.

Finally, in case of stochastic demand it is often feasible to use deterministic lot sizing. A standard procedure is to first replace the stochastic demand by its mean and then, use a deterministic model to determine the order quantity Q.

1.3.1 The classical Economic Order Quantity (EOQ) model

The most well-known result in the whole inventory control area is the classical economic order quantity formula. This simple result has had and still has an enormous number of practical applications. It was first derived by Harris (1913), but also Wilson (1934) is recognized in connection with this model.

This model assumes that the demand is continuous at a constant rate of d item units per unit time and shortages are not allowed. In addition, the replenishments are instantaneous, that is, the entire order quantity is received at one time as soon as the order is released. The total cost consists of a holding cost per unit held per unit time, and a fixed ordering cost which is incurred with each replenishment. Usually, the holding cost is denoted by h and the replenishment cost by k.

Since the parameters involved are assumed to be constant with time, it is reasonable to think in terms of using the same order quantity, Q, each time that a replenishment is made. In fact, this yields a mathematical optimal solution. Furthermore, taking into account that demand is deterministic, the replenishment lead time is zero, and shortages are not allowed, it is obvious that each replenishment will be made when the inventory level is exactly at zero. The behavior of the inventory level is illustrated in Figure 1.2.

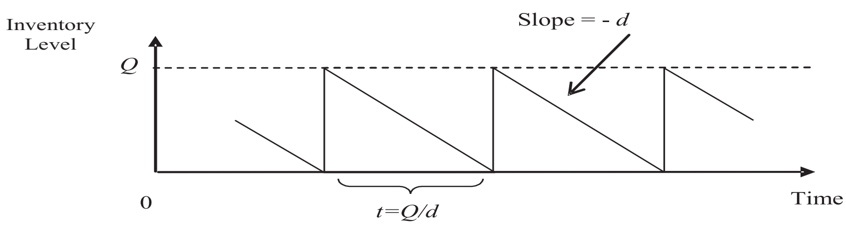

Figure 1.2: Behaviour of inventory level with time

The goal is to determine the optimal order size that minimizes the average total inventory cost. That is the sum of the average holding and replenishment costs. These two costs react inversely to each other. As the order size increases, fewer orders are required, causing the replenishment cost to decrease, whereas the average amount of inventory on-hand will increase, resulting in an increase in holding costs. Thus, in effect, the optimal order quantity represents a compromise between these two inversely related costs.

In Figure 1.3 we show the inverse relationship between replenishment and holding cost, resulting in a convex total cost curve. The optimal order quantity occurs at the point where the total cost curve is at a minimum, which coincides exactly with the point where the holding cost curve intersects the replenishment cost curve.

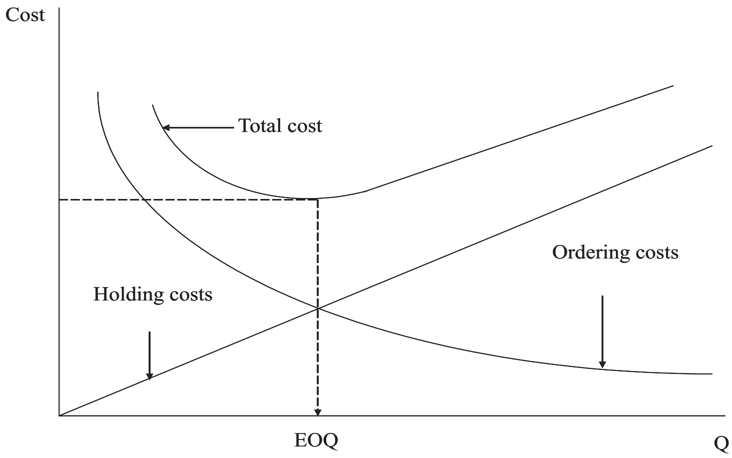

Figure 1.3: Costs related to the EOQ model

Note that the time between replenishments is given by $t = Q/d$, that is, the time necessary to deplete Q units at a rate of d units per unit time. Therefore, the number of replenishments per unit time is d/Q. Since there is a setup cost k associated with each replenishment, the replenishment costs per unit time are given by kd/Q. Regarding the holding costs, they are obtained as the average stock multiplied by the holding cost. From Figure 1.2, the average inventory levels can be easily obtained as $Q/2$, and then, the holding costs are given by $hQ/2$. Hence, the total cost per unit time is

$$C(Q) = k\frac{d}{Q} + h\frac{Q}{2} \tag{1.1}$$

The cost function C is obviously convex in Q, and we can therefore obtain the optimal Q by solving $dC/dQ = 0$.

Accordingly, the optimal solution Q^* is

$$Q^* = \sqrt{\frac{2dk}{h}} \tag{1.2}$$

This is the economic order quantity, EOQ, and it is one of the earliest and most well-known results in Inventory Theory.

Note that if we replace Q in (1.1) by Q^*, we have

$$C(Q^*) = \sqrt{\frac{dkh}{2}} + \sqrt{\frac{dkh}{2}} = \sqrt{2dkh} \tag{1.3}$$

That is, for the optimal order quantity the holding costs happen to be exactly equal to the replenishment costs.

Sensitivity analysis

The assumptions of the EOQ model are very restrictive and in the real world they are rarely fulfilled. However, despite several more sophisticated models are available, the EOQ formula is still widely used as an approximated solution in inventory control. The reasons of the success of the EOQ formula are the following. Firstly, it is very easy to implement and to apply in practice. Secondly, the model is robust with respect to its parameters. Referring back to Figure 1.3, note that the total cost curve is quite shallow in the neighborhood of the EOQ. This indicates that reasonable sized deviations from the EOQ will have little impact on the total cost.

Mathematically, suppose we use the order quantity Q instead of Q^*. Then, from (1.1)-(1.3) we obtain

$$\frac{C(Q)}{C(Q^*)} = \frac{1}{\sqrt{2dkh}}\left(k\frac{d}{Q} + h\frac{Q}{2}\right) = \frac{1}{2Q}\sqrt{\frac{2dk}{h}} + \frac{Q}{2Q^*} = \frac{1}{2}\left(\frac{Q^*}{Q} + \frac{Q}{Q^*}\right) \tag{1.4}$$

Then, it is evident that the relative cost increase, $C(Q)/C(Q^*)$, when using the batch quantity Q instead of Q^* is a simple function of Q/Q^*. It turns out that even quite large deviations from the optimal order quantity will give very limited cost increases. For example, if $Q/Q^* = 2$, then $C(Q)/C(Q^*) = 1.25$. This means that an error of 100% in Q results in an increase in the average cost of only 25%. Hence, if there are errors in the estimation of the cost and demand parameters, the corresponding error in Q does not yield a substantial cost penalty. Moreover, it is worth noting that the costs are even less sensitive to errors in the cost parameters. For example, if we use a value of the replenishment cost k which is 50% above the correct replenishment cost, we can see from (1.2) that the resulting relative error in the batch quantity is $Q/Q^* = \sqrt{3/2} = 1.225$ and the relative cost increase is only about 2%. Consequently, we can conclude that the choice of cost parameters when using the classical economic order quantity is not critical.

Powers-of-two restriction for the economic replenishment interval problem

Since the demand is known exactly, once the optimal order quantity is determined it is easy to compute the corresponding replenishment interval. Obviously, the optimal replenishment interval can be any positive real number, and hence, the solution is often impractical to implement. Typically, there are pragmatic reasons that yield orders can be placed only in certain time intervals which are multiples of a day, a week, etc. As a result, it is common to fix a minimum replenishment interval such as a day, a week, or other appropriate time period. This minimum replenishment interval is usually referred to as base planning period, and the replenishment intervals must be integer multiples of this period. In addition, if we force these integer multiples to be powers of two, then, the solutions so obtained are called powers-of-two policies.

In order to analyze the extension of the EOQ model to the case with powers-of-two restriction, we express the EOQ formula in terms of replenishment intervals instead of order quantities. Obviously, there is a relationship between the replenishment intervals and the order quantities. If t is the replenishment interval, then clearly $t = Q/d$. Hence, the average total cost (1.1) can be formulated in terms of t as follows

$$C(t) = \frac{k}{t} + hd\frac{t}{2} \tag{1.5}$$

This problem is referred to as the economic replenishment interval problem and it is easy to prove that

$$t^* = \sqrt{\frac{2k}{hd}}$$

Thus the optimal solution is to place an order every t^* time units. Since the lead time is zero, the optimal policy is obviously to place and receive an order only when the on-hand inventory level is zero.

As we have shown in the previous section, the average total cost is relatively robust to the choice of Q, or equivalently, to the choice of t. For example, if $t = 2t^*$, then, the corresponding average total cost exceeds the optimal cost by only 25%. The robustness of the cost to the value of t is an important factor affecting the usefulness of the powers-of-two policies.

Let us assume that the replenishment interval t must be a power of two multiple of the base planning period, T_L. That is,

$$t = 2^l T_L, \quad l \in \{0, 1, ...\}$$

As we show below, the average cost for this type of policies exceed the optimal cost at most 6%.

The powers-of-two replenishment interval problem can be formulated as follows

$$\min C(t) = \min \frac{k}{t} + \frac{hdt}{2} \tag{1.6}$$

s.t.

$$t = 2^l T_L, \quad l \in \{0, 1, ...\} \tag{1.7}$$

In order to solve problem (1.6)-(1.7) we define the following function

$$f(t) = \frac{k}{t} + \frac{hdt}{2}$$

Since $f(t)$ is a convex function, we solve problem (1.6)-(1.7) by finding the smallest nonnegative integer value l for which

$$f(2^{l-1}T_L) \geq f(2^l T_L) \tag{1.8}$$

and

$$f(2^{l+1}T_L) \geq f(2^l T_L) \tag{1.9}$$

This condition reduces to finding the smallest nonnegative integer l for which

$$\frac{1}{\sqrt{2}} t^* \leq 2^l T_L \leq \sqrt{2} t^* \tag{1.10}$$

where $t^* = \sqrt{2k/hd}$.

In fact, substituting $f(2^{l-1}T_L)$ and $f(2^l T_L)$ into (1.8) it follows

$$\begin{aligned} \frac{k}{2^{l-1}T_L} + \frac{hd2^{l-1}T_L}{2} &\geq \frac{k}{2^l T_L} + \frac{hd2^l T_L}{2} \iff \\ \frac{hd2^{l-1}T_L}{2} &\leq \frac{k}{2^l T_L} \iff 2^{2l}T_L^2 \leq \frac{2^2 k}{dh} \iff 2^l T_L \leq \sqrt{2} t^* \end{aligned}$$

Similarly, from (1.9) we have

$$\begin{aligned}\frac{k}{2^{l+1}T_L}+\frac{hd2^{l+1}T_L}{2} &\geq \frac{k}{2^lT_L}+\frac{hd2^lT_L}{2} \Longleftrightarrow \\ \frac{hd2^lT_L}{2} &\geq \frac{k}{2^{l+1}T_L} \Longleftrightarrow 2^{2l}T_L^2 \geq \frac{k}{dh} \Longleftrightarrow 2^lT_L \geq \frac{1}{\sqrt{2}}t^*\end{aligned}$$

Therefore, from (1.10) the optimal powers-of-two solution must be close to t^*. In particular, it must be at least 0.707 times t^* and at most 1.41 times t^*.

In addition, note that if $t^* = \sqrt{2k/hd}$ and $t = \alpha t^*$, then

$$\begin{aligned}f(t) &= \frac{k}{t}+\frac{hdt}{2}=\frac{k}{\alpha t^*}+\frac{hd\alpha t^*}{2}= \\ &= \frac{1}{\alpha}\sqrt{\frac{khd}{2}}+\alpha\sqrt{\frac{khd}{2}}= \\ &\quad \frac{1}{2}(\alpha+\frac{1}{\alpha})\sqrt{2khd}=\frac{1}{2}(\alpha+\frac{1}{\alpha})f(t^*)\end{aligned}$$

Hence, since $f(t)$ is convex and $2^lT_L \leq \sqrt{2}t^*$ we obtain

$$f(2^lT_L) \leq f(\sqrt{2}t^*) = \frac{1}{2}(\sqrt{2}+\frac{1}{\sqrt{2}})f(t^*) \simeq 1.06f(t^*)$$

Thus, the powers-of-two solution has an objective function value that must be very close to the optimal cost. In fact, the cost of the optimal powers-of-two policy is at most 6% above the cost of the optimal policy.

As we will see, these results hold in most of the models that will be analyzed in the following chapters. Therefore, the powers-of-two policies are widely implemented in practice.

1.3.2 The Economic Production Quantity (EPQ) model

The EOQ formulation assumes that the whole replenishment quantity arrives at the same time, that is, instantaneously. However, this assumption is often not realistic. Frequently, items are produced and added to inventory gradually rather than all at one. The Economic Production Quantity (EPQ) model revises the EOQ model to introduce this change. Now, if the replenishment quantity becomes available at a rate of P items per unit time, then the sawtoothed diagram of Figure 1.2 is modified to that of Figure 1.4. Remark that the inventory level never is as large as the lot size, since production and consumption occur simultaneously during the period of production.

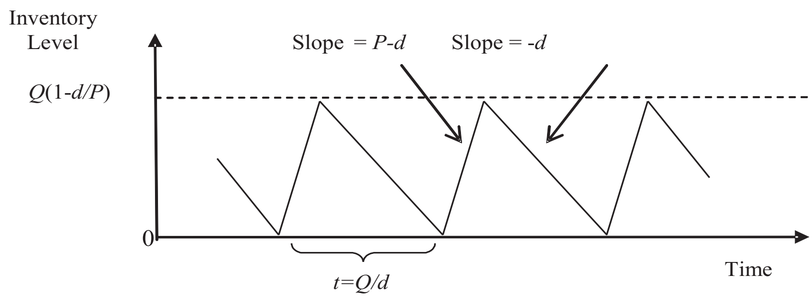

Figure 1.4: Behaviour of inventory level for the case with finite production rate

Now, while production of a batch is underway, stock is accumulating at a rate $P - d$ per unit time. The total time of production in a cycle is Q/P, so that, the peak level of inventory in a cycle is $Q(1 - d/P)$. Therefore, the average inventory level is $Q(1 - d/P)/2$. Then, the total relevant costs are given by

$$C(Q) = k\frac{d}{Q} + h\frac{Q(1 - d/P)}{2}$$

Proceeding as with the EOQ model, we obtain that the optimal lot size Q^*, which represents the Economic Production Quantity (EPQ), is given by

$$Q^* = \sqrt{\frac{2dk}{h(1 - d/P)}}$$

Note that $C(Q)$ and Q^* have the same form as in the EOQ model. The unique difference, is that the constant h is replaced by $h(1-d/P)$. That is, the EPQ is just the EOQ multiplied by a correction factor.

Remark that the relative cost increase given in (1.4) remains valid for this case. Also observe that if $P \to \infty$, then the EPQ model becomes the classical EOQ model.

1.4 Models with time-varying demand

In the previous models the demand occurs at a constant rate. These models are robust to variations from their underlying assumptions, including demand variations. However, there are situations where time variations in demand are so pronounced that the constant demand rate assumption is seriously violated. For example, when a firm delivers items which have a seasonal demand pattern.

When the demand rate varies with time we can no longer assume that the best strategy is always to use the same replenishment quantity. In fact, this will seldom be the case. Now, an exact analysis becomes complicated because the diagram of inventory level versus time, even for a constant replenishment quantity, is no longer the simple repeating sawtooth pattern as in Figure 1.2. This prevents us from using simple average costs over a typical unit period, as was possible in the EOQ derivation. Instead, under this situation we have to use the demand information over a finite period when determining the appropriate value of the current replenishment quantity. Such a period is known as the planning horizon and its length can have important effect on the total relevant costs of the selected strategy. The planning horizon must be divided into T periods and the demand at each period should be determined and fulfilled at the beginning of that period without shortages. This demand can be either continuous with time or can occur only at discrete equispaced points in time. However, in most approaches all that is needed is the total demand at each basic period. A common case is one in which the demand stays constant throughout a period, only changing from one period to another. An illustration is given in Figure 1.5. Besides, there is no initial stock and the whole batch is delivered at the same time. The holding and replenishment costs are constant over time. In addition, shortages are not allowed.

The goal consists of determining batch quantities so that the sum of the replenishment and holding costs is minimized. This problem is usually denoted as the *classical dynamic lot size problem.* However, since the original formulation of the problem is due to Wagner and Whitin (1958), it is also called the *Wagner and Whitin problem.* In particular, they develop an algorithm which is a dynamic programming approach. Next, we present the formulation of the problem and review some properties that the optimal solution must satisfy.

1.4.1 The Wagner and Whitin algorithm

Wagner and Whitin (1958) addressed the dynamic lot size problem assuming that replenishments are constrained to arrive at the beginnings of periods, lead time is

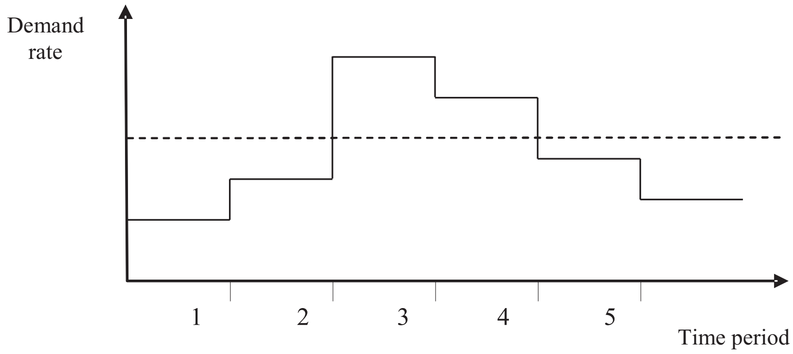

Figure 1.5: Demand pattern when the rate stays constant through each period

zero and shortages are not allowed. Taking this into account, an appropriate method of selecting replenishment quantities should lead to the arrival of replenishments only at the beginning of periods when the inventory level is exactly zero. This condition is usually referred to as ZIO (*Zero Inventory Ordering*) property and the solutions which hold this property are called ZIO policies. Based on this property, Wagner and Whitin (1958) developed an $O(T^2)$ algorithm to determine an optimal solution. This algorithm is an application of dynamic programming. However, the computational effort often prohibitive in dynamic programming formulation, is significantly reduced because of the use of the ZIO property. Another key property which is used in the algorithm is that the holding costs for a period demand should never exceed the setup cost. Eventually, the holding costs become so high that it is less expensive to made another replenishment at a given period than to include its requirements in a replenishment from many periods earlier.

In order to formulate the problem we introduce the following notation.

The demand at period $j = 1, 2, ..., T$, is given by d_j, and the holding cost per unit and time unit is denoted by h and the fixed setup cost by k. The ending inventory and the replenishment quantity in period j are represented by I_j and q_j respectively. Finally, we denote by $F(j)$ the total cost of the best replenishment strategy that satisfies the demand at periods $1, ..., j$.

Now, the problem can be formulated as an integer program in the following way

$$\min \sum_{j=1}^{T} k\delta(q_j) + hI_j$$

s.t.

$$\begin{aligned} I_j &= I_{j-1} + q_j - d_j, \quad j = 1, ..., T \\ q_j &\geq 0, \quad j = 1, ..., T \\ I_j &\geq 0, \quad j = 1, ..., T-1 \\ I_0 &= I_T = 0 \end{aligned}$$

where $\delta(q_j) = \begin{cases} 1 & \text{if } q_j > 0 \\ 0 & \text{if } q_j = 0 \end{cases}$

And the equivalent dynamic programming formulation is

$$F(j) = \min \left\{ \min_{1 \leq t < j} \left[k + \sum_{g=t}^{j-1} \sum_{l=g+1}^{j} hd_l + F(t-1) \right], k + F(j-1) \right\}$$

where $F(1) = k$ and $F(0) = 0$.

It is worth noting that $\sum_{g=t}^{j-1} \sum_{l=g+1}^{j} hd_l$ represents the inventory holding costs of periods $t+1$ through j. In contrast, $k + F(j-1)$ corresponds to the case in which a setup is performed in period j increased by the best solution through period $j-1$.

The Wagner and Whitin algorithm was generalized to the backlogging case by Zangwill (1966). Most recently, very efficient methods for solving the dynamic lot size problem have been provided by Federgruen and Tzur (1991, 1994a, 1995), Aggarwal and Park (1993) and Wagelmans et al. (1992).

The extension to the case with finite production rate has been addressed by Hill (1997a). In this work, Hill show how the finite production rate problem can be transformed into a discrete time lot-sizing problem that can be solved by the Wagner and Whitin algorithm or any of the other techniques currently available.

So far, we have analyzed inventory models for single locations. An exhaustive compilation of these inventory systems and their solution methods can be found, among others, in Naddor (1966), Tersine (1994), Plossl (1985), Narasimhan et al. (1995), Chikán (1990), Silver et al. (1998) and Axsäter (2000). In the next chapter we deal with inventory systems where several installations are coupled to each other.

Chapter 2
Multi-Echelon Inventory Systems

The control of inventories has been an important research topic in the field of Operations Research during the last century. Since Harris (1913) developed the classical EOQ model, several authors have addressed different extensions of this model. As we showed in Chapter 1, a large number of these extensions focus on determining the optimal order quantity at a single location under different initial assumptions. However, inventory systems in the real world usually involve multiple locations. In this chapter, we illustrate how such systems arise in practice and we review the most important references in the literature concerning these models. We finish this chapter with a summary of the contributions of this thesis.

2.1 Introduction

In practice, it is frequent to deal with inventory systems where a number of installations are coupled to each other. For example, a common situation is a chain of stores which are supplied by a single regional warehouse. This class of inventory systems usually arise in both distribution and production contexts. In distribution we meet such systems when products are distributed over large geographical areas. In this case, it is convenient to establish local stocking points close to the customers in different areas in order to provide good service. These local sites replenish from a central warehouse close to the production facility. In the production framework, stocks of raw materials, components and finished products are coupled to each other in a similar way.

It is worth noting that in these situations, decisions made by one member of the chain can affect to all other locations. Hence, it is necessary that all members of the supply chain collaborate and integrate their decision processes to achieve a more efficient control. Typically, a *supply chain* consists of suppliers, manufacturing centers, warehouses and retailers, with raw material at the beginning, work-in-progress

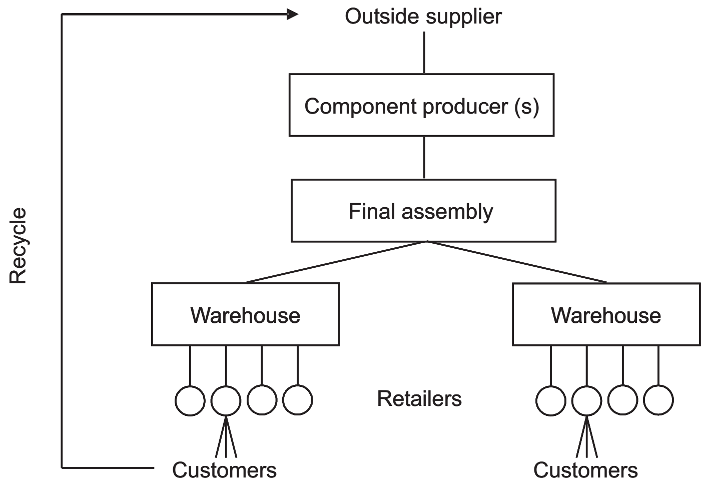

Figure 2.1: A scheme of a supply chain

at different stages of production and finished goods at the end. Figure 2.1 shows the components of a supply chain.

Silver et al. (1998) define *supply chain management* as the management of materials and information across the entire supply chain, from suppliers to customers. This topic covers multiple areas each one representing an issue within the supply chain. Since inventory costs usually represent an important investment, the reduction of such costs is one of the most critical issues. The area into supply chain management devoted to the effective management of inventory in a supply chain is what is known in the literature as *multi-echelon inventory theory*.

The first multi-echelon models were developed in the 1960s. The important works of Clark and Scarf (1960, 1962) got considerable interest from both academics and practitioners. The works of Hadley and Whitin (1963) and Veinott (1966) provide excellent summaries of many of these early modeling efforts. This interest on supply chain management and, in particular, on multi-echelon inventory systems has increased considerably in the last decades. Thus, we can find many works in the literature dealing with the problem of determining lot sizes in deterministic multi-echelon inventory systems. In this chapter we review the most important aspects of such works. However, we before illustrate how multi-echelon inventory systems occur in practical supply chains and, what kind of system structures are common in distribution and production contexts. Excellent reviews of multi-echelon inventory systems are given in Muckstadt and Roundy (1993), Silver et at. (1998), Axsäter

(2000) and Zipkin (2000), among others.

2.2 Production and Distribution Systems

Multi-echelon inventory systems appear in the production of the items and also when they are distributed. In both cases, the items and the relationships between them can be represented by a network, specifically by a directed graph. In particular, it can be distinguished the serial, the assembly and the arborescent structures.

The simplest structure is a *serial system* where the items represent the outputs of successive production stages or stocking points along a supply chain. Each product is used as input to make the next one, or each location supplies the next one. Only the first item receives supplies from outside the system, and only the last one meets customer demands. An illustration of such systems is depicted in Figure 2.2.

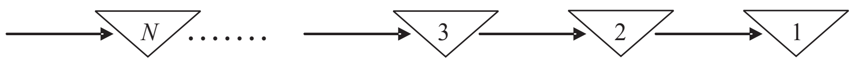

Figure 2.2: A serial system

The nodes represent the different stages and the arcs indicate the direction in which material flows through the system. The simplest serial systems are those with only two installations. In this case, customer demands take place at location 1 which is supplied by installation 2. Stage 2, in turn, replenishes from an outside supplier. In particular, from a distribution point of view, stage 1 can represent a retailer which satisfies customer demands in one area, while location 2 might be a central warehouse close to the factory. In contrast, in a production context, stock at stage 1 corresponds to the stock of a final product, and at stage 2 is the stock of a subassembly, which is used when producing the final product. It is worth noting that in both situations stage 1 can be seen as a customer of stage 2.

Next, we focus on the assembly and arborescent systems.

2.2.1 Production inventory systems

In a production environment, inventory systems usually consist of many stocking locations at the beginning of the material flow and successively fewer stocks at the end of the flow, that is, a convergent flow. In particular, if each installation has at most one immediate successor, as in Figure 2.3, we have an *assembly system*. In these systems there is only one finished product which is assembled from a set of components. Likewise, each component could be manufactured in several stages and could also be assembled from several other parts.

It should be remarked that raw materials and components at the beginning of the flow usually have much lower values than subassemblies and final products at the end of the flow. Therefore, the holding costs are lower at the first locations and hence, in general, for assembly systems it is better to keep stock early in the material flow.

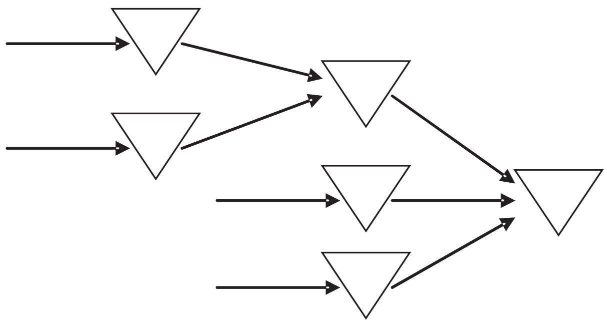

Figure 2.3: An assembly system

2.2.2 Distribution inventory systems

We now consider another special type of network structure corresponding to a distribution system. Figure 2.4 contains an illustration of this type of network. Notice that a distribution system looks like a backwards assembly system. Material access to the system at the first node and moves down through the different levels of the system until it is consumed by external demand. Thus, in general, distribution systems are divergent. In a pure distribution system each stage is supplied by a unique predecessor stage. These distribution systems are usually referred to as *arborescent systems*.

Although this structure is more typical in distribution systems, it can also appear in the production context. For example, this type of network is useful when there is one raw material and several final products. The raw material is successively specialized or refined as it moves down the production stages.

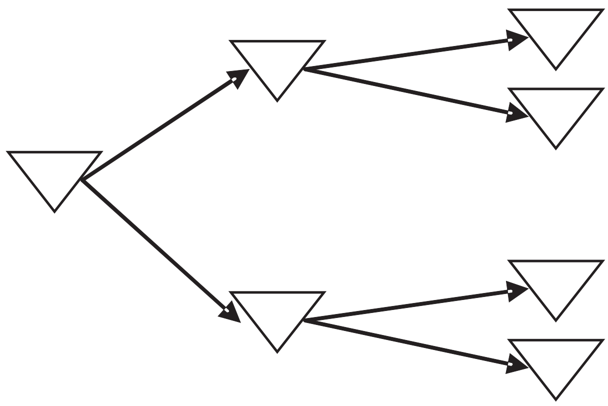

Figure 2.4: A distribution system

In practice, it is very common to deal with two-level distribution systems where a central warehouse supplies a number of retailers. Figure 2.5 is an example of this type of system. In this case, the retailers satisfy customer demands and similarly, the warehouse fulfills the demand at all retailers.

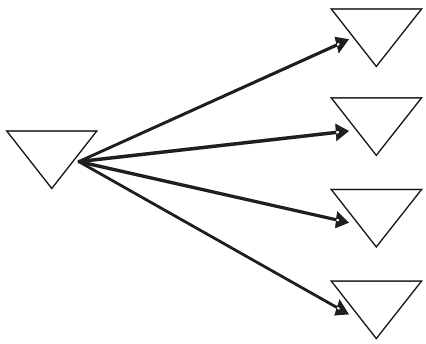

Figure 2.5: A two-level distribution inventory system

Obviously, a serial system is a special case of both an assembly and a distribution system.

So far, we have discussed why and how multi-echelon inventory systems appear in practice. In the next section, we address the problem of determining order quantities for the serial, assembly and distribution systems with constant demand rate. In particular, for each network structure, we will review some models and algorithms for finding effective inventory policies.

2.3 Serial systems

The serial systems represent the simplest extension to the single location models. In these systems, instead of considering a single installation, there are N stages and each produced unit must go through all stages, beginning at stage N and ending at stage 1.

In this section we analyze a serial system considering that the assumptions related to the parameters at each installation are identical to those in the EOQ model. Hence, the demand is constant and continuous and the lead-times are zero. The whole order quantity is delivered at the same time and shortages are not allowed. For each stage $j = 1, ..., N$, d_j represents the total demand rate for the units produced at stage j. Note that several units at stage j could be required to produce one unit at stage $j - 1$. Hence, d_j need not be the same for all stages. However, we can select the units of measure for inventory at the different stages so that $d_j = d$, $j = 1, ..., N$. Besides, at each stage $j = 1, ..., N$, there is a holding cost per unit stored per unit time, h_j, and a fixed replenishment cost, k_j. Finally, the order quantity and the replenishment interval at stage j are denoted by Q_j and t_j, respectively, for $j = 1, ..., N$.

Before introducing the formulation of the problem, we first discuss the form of the optimal policy for these systems.

2.3.1 Nested policies

When trying to extend the EOQ model to the serial systems it arises the stationary and nested policies. A *stationary* policy is one in which each facility orders the same quantity at equally-spaced points in time. A policy is said to be *nested* if each time a facility orders all its successors also order. It is worth noting that these policies can be easily applied and hence, many authors have analyzed their performance for several network structures. Although, in general, nested policies have not to be optimal when they are implemented in multi-echelon inventory systems, they are optimal for the serial systems. In order to prove this assertion we focus on a two-stage system as in Figure 2.6.

Figure 2.6: A two-stage serial system

Suppose that production occurs at time t at stage 2 while no production occurs at stage 1. Besides, let $t' > t$ be the earliest time after t where production occurs at stage 1. Then, the inventory produced at time t at stage 2 must be held until at least time t' before it is used at stage 1. Now, consider an alternative production plan in which the production at stage 2 starts at t' instead at t. In addition, all other production times remain unchanged. Then, it is obvious that the number of setups in the two plans is the same and, the holding costs are lower in the second one. Accordingly, it is preferable to produce at stage 2 only when production occurs at stage 1.

This proof can be easily extended for an N-stage system by following an argument similar to the previous one. Therefore, it can be stated the following theorem.

Theorem 2.1. For an N stage serial system every non-nested policy is dominated by a nested policy.

For a proof of this result see Love (1972) or Schwarz (1973).

However, notice that it is possible to have production at stage $j-1$ without having production at stage j. Thus, it holds that $t_j \geq t_{j-1}$, $j = 2, ..., N$.

In order to formulate the problem it is also important to introduce the echelon inventory which is a fundamental concept in multi-echelon systems.

2.3.2 Echelon inventory

For exposition purposes, we again restrict ourselves to the two-stage serial system. Then, the two decision variables are the order quantities Q_1 and Q_2. Note that stage 2 only satisfies the demand at stage 1 which always orders the same quantity Q_1. Hence, it is clear that Q_2 should always be an integer multiple of Q_1. Therefore, we can think of two alternative decision variables: Q_1 and n, where $Q_2 = nQ_1$, with n a positive integer.

Figure 2.7 shows the behavior of the inventory levels for both stages assuming that $Q_2 = 3Q_1$. Observe that the inventory fluctuations at stage 1 follow the usual sawtooth pattern that arises in the single stage systems. Hence, the average on-hand inventory is $Q_1/2$. However, the inventory fluctuations at stage 2 do not fit this form, and hence, the average on-hand inventory is not $Q_2/2$. Instead, in this case, the average on-hand inventory at stage 2 can be computed as follows

$$\frac{Q_1 t_1 \sum_{j=1}^{n-1} j}{n t_1} = \frac{Q_1 t_1 n(n-1)}{2 n t_1} = \frac{(n-1) Q_1}{2} \tag{2.1}$$

Taking this into account, the average holding costs can be determined using the on-hand inventories and the conventional holding costs.

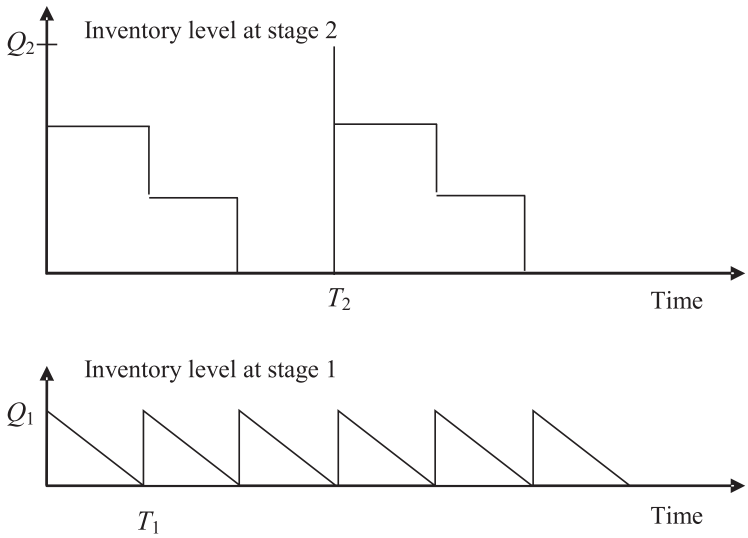

Figure 2.7: On-hand inventory levels in a two-stage serial system

However, it is easier to use a different inventory concept introduced by Clark and Scarf (1960), which is referred to as *echelon inventory.* They define the echelon inventory of echelon j as the number of units in the system that are at, or have passed through, echelon j but have as yet not been specifically committed to outside customers. For example, in Figure 2.8 we represent by rectangles the echelons of a four-stage system. As you can see, stage 1 is its own echelon. The external supplier and all prior stages can be viewed as stage 1's supply process. Similarly, echelon 2 consists of the last two stages. This is another subsystem, whose supply process includes the earlier stages $j > 2$. Thus, the entire system can be viewed as a hierarchy of nested subsystems, the echelons, each with a clearly defined supply process.

Now, consider a multistage production process where at each stage the material is transformed until the final item is produced. Notice that a unit of item $i < j$ comprises of one of item j and the echelon inventory allows to include downstream inventory of an item. This definition yields each echelon inventory to have sawtooth pattern with time and hence, the average value of an echelon inventory is easily obtained. However, the average total holding costs cannot be determined by simply multiplying each average echelon inventory by the standard holding costs and then, summing all of them to obtain the average total holding costs. The reason is that the same physical units of stock can appear in more than one echelon inventory. For example, in a two-stage system the echelon inventory at stage 2 includes the stock at stage 1. Therefore, the holding cost at stage 1 should only represent the value added when moving the product from stage 2 to stage 1. This incremental cost is exactly the echelon holding cost. Thus, in a two-stage system the echelon inventory at stage 2 is $h'_2 = h_2$, while the echelon inventory at stage 1 is $h'_1 = h_1 - h_2$. More generally, in any multi-echelon inventory system, the echelon holding cost h'_j at a particular stage j is given by $h'_j = h_j - \sum h_i$, where the summation is over all immediate predecessors, i.

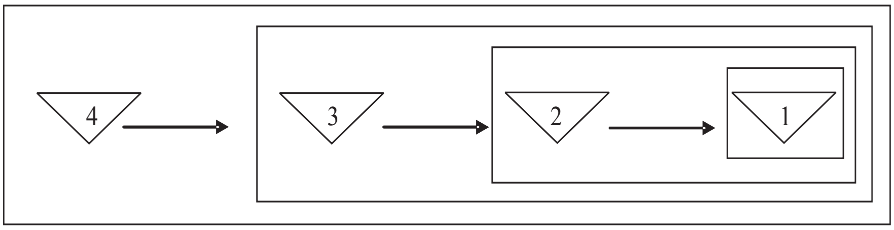

Figure 2.8: Echelons of a four-stage serial system

We provide now in Figure 2.9 the echelon inventory fluctuations in a two-stage system. Notice that the echelon stock at stage 1 coincides with its on-hand inventory. However, the echelon inventory at stage 2 is quite different. The echelon inventory at stage 2 consists of the stock on-hand at stage 2, the shaded area in Figure 2.9, plus the amount on-hand at stage 1, represented by the triangles. Therefore, the echelon inventory fluctuations at both stages follow the usual sawtooth pattern as in the EOQ model. Hence, the average echelon inventory at stage 1 and 2 is $Q_1/2$ and $Q_2/2$, respectively.

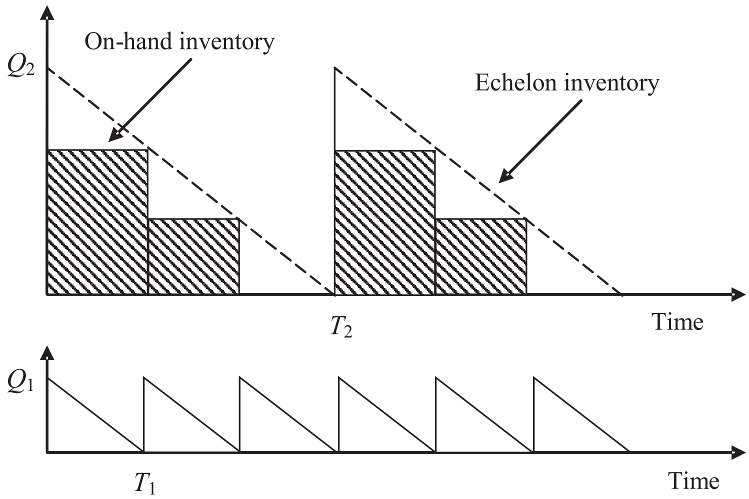

Figure 2.9: Echelon inventory levels in a two-stage serial system

Now, we show that indistinctly using the echelon or on-hand inventories the expression of the total holding costs is the same. Accordingly, we know that the conventional average holding costs at stages 1 and 2 are, respectively, $h_1Q_1/2$ and $h_2(n-1)Q_1/2$. Replacing now h_1 and h_2 by their expressions in terms of h_1' and h_2' we obtain

$$h_1Q_1/2 + h_2(n-1)Q_1/2 = (h_1' + h_2')Q_1/2 + h_2'nQ_1/2 - h_2'Q_1/2 = h_1'Q_1/2 + h_2'Q_2/2$$

This fact indicates that determining the average holding cost using the conventional way yields the same result as they were calculated applying the echelon inventories and the echelon holding costs.

Once we have defined the echelon inventory, we can focus on how to compute optimal policies for the serial systems. We first analyze the two-stage serial problem and then, we deal with the general case.

2.3.3 Two-stage serial system

Let us consider a simple serial system where the final product at stage 1 is produced from one unit of the component at stage 2, which, in turn, is obtained from an external supplier. Recall that we assume that the demand, d, for the item at stage 1 is constant and continuous. Besides, at each stage there is a holding cost per unit stored per unit time, h_j, and a fixed replenishment cost, k_j, $j = 1, 2$. In addition, when delivering a batch, the whole order quantity is supplied at the same time and shortages are not allowed. The goal is to determine the optimal order quantities, Q_1 and Q_2.

Since stage 1 faces constant continuous demand, the total average cost at this stage is easily obtained as

$$C_1 = k_1 \frac{d}{Q_1} + h_1 \frac{Q_1}{2} \tag{2.2}$$

An approach for solving the problem consists of applying the classical EOQ model to stage 1. Accordingly,

$$Q_1 = \sqrt{\frac{2dk_1}{h_1}}$$

and the corresponding cost is $C_1 = \sqrt{2dk_1h_1}$

Then, we proceed to compute the value of Q_2 which minimizes the total average cost at stage 2. Taking into account that the average conventional holding cost at stage 2 is $h_2(n-1)Q_1/2$, the total average cost at this stage is given by

$$C_2 = \frac{k_2 d}{Q_2} + \frac{h_2(n-1)Q_1}{2}$$

In addition, since it holds that $Q_2 = nQ_1$, with n a positive integer, C_2 can be formulated in terms of n and Q_1 as follows

$$C_2 = k_2 \frac{d}{nQ_1} + h_2 \frac{(n-1)Q_1}{2} \tag{2.3}$$

It is easy to see that (2.3) is convex in n and, if we disregard that n has to be an integer, we obtain

$$n^* = \frac{1}{Q_1}\sqrt{\frac{2k_2 d}{h_2}}$$

Then, if $n^* < 1$, it is optimal to choose $n = 1$. When $n^* > 1$, if $C_2(\lfloor n^* \rfloor) \leq C_2(\lceil n^* \rceil)$ we choose $n = \lfloor n^* \rfloor$, and we set $n = \lceil n^* \rceil$, otherwise. It is easy to see that $C_2(\lfloor n^* \rfloor) \leq C_2(\lceil n^* \rceil)$ if and only if $(n^*)^2 \leq \lfloor n^* \rfloor \lceil n^* \rceil$. Therefore, if $(n^*)^2 \leq \lfloor n^* \rfloor \lceil n^* \rceil$, it is optimal to choose $n = \lfloor n^* \rfloor$.

However, as we will show below this has not to be the optimal solution. Since the order quantity at stage 1 will affect the demand at stage 2, it is not feasible to optimize stage 1 separately without considering the consequences for stage 2. Therefore, for computing the optimal solution we have to minimize the total cost for the system.

Using the echelon holding costs, the average total cost is given by

$$C = k_1 \frac{d}{Q_1} + h'_1 \frac{Q_1}{2} + k_2 \frac{d}{Q_2} + h'_2 \frac{Q_2}{2} \tag{2.4}$$

Since $Q_2 = nQ_1$, (2.4) can be reformulated in terms of Q_1 and n as follows

$$C = \frac{d}{Q_1}(k_1 + \frac{k_2}{n}) + \frac{Q_1}{2}(h'_1 + nh'_2) \tag{2.5}$$

Now, for a given n, the optimal Q_1 is obtained from (2.5) by taking the derivative equal to zero

$$Q_1 = \sqrt{\frac{2d(k_1 + \frac{k_2}{n})}{(h'_1 + nh'_2)}} \tag{2.6}$$

Substituting (2.6) into (2.5) we obtain the optimal costs for a given n as

$$C(n) = \sqrt{2d(k_1 + \frac{k_2}{n})(h'_1 + nh'_2)} \tag{2.7}$$

Note that minimizing $C(n)$ is equivalent to minimizing $F(n) = (k_1 + k_2/n)(h'_1 + nh'_2)$ which is convex in n. Hence, the optimal integer n satisfies $F(n) \leq F(n+1)$ and $F(n) \leq F(n-1)$. In other words, $n(n+1) \geq k_2h'_1/k_1h'_2$ and $n(n-1) \leq k_2h'_1/k_1h'_2$. Therefore, the optimal value n is the smallest integer satisfying

$$n(n-1) \leq \frac{k_2h'_1}{k_1h'_2} \leq n(n+1)$$

Once the optimal value n is computed, the order quantity Q_1 is obtained from (2.6), and Q_2 from the relation $Q_2 = nQ_1$. In fact, it is worth noting that for this case the optimal policy is stationary and nested.

2.3.4 N-stage serial system

As we have showed previously, optimal policies for the two-stage serial systems are very easy to compute. However, optimal solutions for serial systems with N installations can be surprisingly complex. Crowston et al. (1973) were pioneer in the study of the serial system and they proved that in an optimal solution the order quantity at each stage must be an integer multiple of the order quantity at its successor stage. However, a few years later, Williams (1982) proved that this property does not hold in general. In particular, Williams (1982) showed that the proof of the property provided by Crowston et al. (1973) is defective at the point where they extended their results for two-level serial systems to more general serial systems. Therefore, the property is only valid for the two-level serial systems. Moreover, the optimal policies for an N-stage serial system can be very difficult to compute since the order quantity at one or more of the stages can vary with time. Accordingly, many authors have considered the possibility of restricting attention to a simpler class of strategies with a high guaranteed cost performance. In particular, Roundy (1985) introduced the powers-of-two policies which are 98% effective. That is, he proved that the cost of an optimal powers-of-two policy is at most 2% above the optimal cost.

Next, we formulate the problem in terms of powers-of-two policies and we describe the procedure given by Roundy (1985) for solving it.

Powers-of-two policies

In a powers-of-two policy the orders are placed at equal intervals of time which are powers of two multiples of a base planning period T_L. Therefore, in order to analyze these policies, it is better to formulate the model in terms of replenishment intervals instead of order quantities. In addition, if we use the echelon holding costs we can state the problem as follows

$$\min \sum_{j=1}^{N} \left(\frac{k_j}{t_j} + \frac{h'_j t_j d}{2}\right) \tag{2.8}$$

s.t.

$$t_j \geq t_{j-1} \geq 0 \tag{2.9}$$

$$t_j = 2^{l_j} T_L, \quad l_j \in \{0, 1, ...\} \tag{2.10}$$

Recall that by Theorem 2.1 optimal policies have to be nested. Note that constraints (2.9) and (2.10) force policies to be nested. As you can see, it results in a nonlinear integer programming problem where the integer decision variables are l_j, $j = 1, 2, ..., N$. However, due to its special structure, it can be easily solved. Problem (2.8)-(2.10) turns out to have a very close relationship to its relaxation without considering constraints given by (2.10).

Roundy (1985) first characterized the solution to problem (2.8)-(2.9) and then, he showed its relationship to problem (2.8)-(2.10).

All variables in the relaxed problem (2.8)-(2.9) are continuous, and the constraints are linear inequalities. Moreover, the objective function is strictly convex, so the optimal solution is unique. Thus, one way to solve problem (2.8)-(2.9) is to use a standard nonlinear-programming algorithm. However, we can exploit the special structure of the problem to develop a simple and fast method. The key idea consists of dropping constraints (2.9). Then, the optimal replenishment interval at stage j is $t_j^* = 2k_j/h_j d$, $j = 1, ..., N$. Consequently, if k_j/h_j increases with j, we have found the optimal solution since the resulting replenishment intervals will satisfy constraints (2.9). Suppose that for some j, $k_j/h_j < k_{j-1}/h_{j-1}$, or equivalently, that $t_j^* < t_{j-1}^*$. Also assume that in the optimal solution of problem (2.8)-(2.9) $t_j > t_{j-1}$. This implies that $t_j < t_j^*$, since we would otherwise reduce t_j, and that $t_{j-1} \geq t_{j-1}^*$, since we would otherwise increase t_{j-1}. But then, this means that $t_j < t_{j-1}$, which is a contradiction. Therefore, we can ensure that $t_j = t_{j-1}$ in the optimal solution of problem (2.8)-(2.9). Accordingly, we can aggregate location $j-1$ and location j into a single stage with replenishment cost $k_{j-1} + k_j$ and echelon holding cost $h'_{j-1} + h'_j$. Next, we consider the resulting reduced problem with $N-1$ stages and repeat the above procedure.

Therefore, we can conclude that solving problem (2.8)-(2.9) is equivalent to classifying the stages into groups or clusters. Besides, all stages in a cluster will use the same replenishment interval. In addition, local optimality implies that the replenishment interval for the stages in cluster C is $t(C) = \sqrt{2\sum_{j\in C} k_j/d\sum_{j\in C} h_j}$.

The O(N log N) algorithm for computing an optimal ordered partition for a serial system is given in Algorithm 2.1. This algorithm first computes the replenishment interval at each stage. Then, if for some j the constraint $t_j \geq t_{j-1} \geq 0$ does not hold, both installations are collapsed into the same cluster and both use the same replenishment interval. This procedure is repeated until all stages are assigned to a cluster. The replenishments intervals for these cluster satisfy $t(C^1) \leq t(C^2) \leq ... \leq t(C^R)$, where R is the number of clusters.

Once we have obtained the optimal partition of the serial system, the optimal solution for problem (2.8)-(2.9) is given by $t_j^* = t^*(C^i)$, $\forall\ j \in C^i$. Now, these

Algorithm 2.1 Procedure for computing an optimal partition for a serial system

Step 1

Set $C^i \leftarrow \{i\}$ and $\sigma(i) \leftarrow i-1$, $i = 1, ..., N$. Compute $t(C^i)$, $i = 1, ..., N$. Set $S \leftarrow \{1, 2, ..., n\}$ and $j \leftarrow 2$. Note that $\sigma(i)$ is the node that precedes i in the sequence S.

Step 2

If $t(C^j) \geq t(C^{\sigma(j)})$, go to Step 4. Otherwise, collapse $C^{\sigma(j)}$ into C^j, that is, $C^j \leftarrow C^{\sigma(j)} \cup C^j$. Set $\sigma(j) \leftarrow \sigma(\sigma(j))$ and compute $t(C^j)$. Go to Step 3.

Step 3

If $\sigma(j) > 0$, go to Step 2. Otherwise, go to Step 4.

Step 4

Set $j \leftarrow j+1$. If $j \leq N$, go to Step 2. Otherwise, let R be the number of clusters and go to Step 5.

Step 5

Reindex the clusters so that if $j \in C^i$ and $k \in C^l$, $j < k$, then $i < l$. Stop.

replenishment intervals are rounded off to powers of two multiples of a base planning period, T_L. In particular, for each $j \in C^i$ we choose $t_j = 2^{l_j} T_L$, where l_j is the smallest nonnegative integer value satisfying

$$2^{l_j - 1} \leq \frac{t_j^*}{\sqrt{2} T_L} \leq 2^{l_j} \tag{2.11}$$

It should be notice that if $t_i^* < t_j^*$, then, $t_i < t_j$, where l_j is computed using (2.11). Therefore, the above powers-of-two solution also satisfies constraints (2.9) and, then, it is an optimal solution for problem (2.8)-(2.10).

An analysis of the worst-case behavior of the above procedure is given in Muckstadt and Roundy (1993), where they compute an upper bound on the optimal objective function value to problem (2.8)-(2.10) which is compared with a lower bound. It is important to remark that the solution to problem (2.8)-(2.9) represents a lower bound on the average cost of any feasible policy for the original system, including policies that are neither powers-of-two nor stationary. Taking this into account, it can be proved that the cost of an optimal powers-of-two policy is within 6% of the solution of problem (2.8)-(2.10). Moreover, since nested policies are optimal for the serial systems, the above procedure is guaranteed to compute a policy whose cost is at most 6% above the cost of an optimal policy.

An alternative procedure for rounding off the replenishment intervals to powers of two multiples of a base planning period can be found in Roundy (1986). This procedure requires that T_L be treated as a variable and has the advantage of computing policies that are within 2% of optimal. However, since for many systems the base planning period is determined by the times at which information is reported, in general, T_L cannot be treated as a variable.

2.4 Assembly Systems

In this section we extend the analysis to assembly systems. Recall that in such systems a single finished product is assembled from a set of components. Each component could be manufactured in several stages and could also be assembled from several other parts. In addition, each installation has at most one downstream successor. For simplicity, we assume that item or component j is produced at stage j, which is consumed by its unique immediate successor stage. We also consider that the finished product is assembled at stage 1 which has no successor stage. In fact, stage 1 must satisfy the external demand, which remains to be constant, continuous, and met without backlogging. All other assumptions that have been considered for the serial systems hold here as well. That is, the lead-times are zero and the replenishments are instantaneous. For each stage $j = 1, ..., N$, k_j and h_j represent the fixed replenishment cost and the unit holding cost. The order quantity and the replenishment interval for stage j are denoted by Q_j and t_j, respectively, $j = 1, ..., N$. The goal consists of determining the order quantities or, equivalently, the replenishment intervals, which minimize the average total cost.

2.4.1 Nested policies

We have shown that nested policies are optimal for serial systems. Each stage of an assembly system has either no external demand and a unique successor, or no successor and external demand. Therefore, the same argument could be used to show that nested policies are optimal for assembly systems as Theorem 2.2 enunciates.

Theorem 2.2. For an assembly system every non-nested policy is dominated by a nested policy.

Crowston et al. (1973) also state that in an optimal policy for an assembly system the lot size at each stage must be an integer multiple of the lot size at its successor stage. However, recall that Williams (1982) proved that this property does not hold for the serial systems with more than two stages. Since a serial system is a special case of an assembly systems, it is obvious that this property is not verified for assembly systems with more than two levels.

In spite of this, many authors have let upstream order quantities be integer multiples of downstream order quantities. For example, Schwarz and Schrage (1975) developed a branch and bound procedure for computing an optimal policy verifying the above property. In this work, Schwarz and Schrage (1975) also studied the performance of other class of policies known as *myopic policies*. In a myopic policy the objective function is optimized with respect to any two stages and multistage interaction effects are ignored. These policies require less information and they are very easy to understand and to compute. However, the policies which are most applied in practice are the powers-of-two policies introduced by Roundy (1985). Next, we focus on how to compute an optimal powers-of-two policy for an assembly system.

Powers-of-two policies

Using the echelon holding costs, the problem of finding an optimal powers-of-two policy for an assembly system can be written as follows

$$\min \sum_{j=1}^{N} (\frac{k_j}{t_j} + \frac{h'_j t_j d}{2}) \tag{2.12}$$

s.t.

$$t_j \geq t_{s_j} \geq 0, \ \forall j \tag{2.13}$$

$$t_j = 2^{l_j} T_L, \quad l_j \in \{0, 1, ...\} \tag{2.14}$$

where s_j is the immediate successor of stage j and T_L is a base planning period.

The key ideas for solving problem (2.12)-(2.14) are similar to those introduced for the serial systems. First, we solve the problem dropping constraints (2.14). Then, the replenishment intervals thus obtained are rounded off to get a feasible nested powers-of-two policy. Analogously to the serial systems, it can be shown that the solution to problem (2.12)-(2.13) divides the assembly system into connected subgraphs or clusters. The nodes in these connected subgraphs are sets of stages whose costs induce them to place orders simultaneously. Thus, finding the clusters is equivalent to solving problem (2.12)-(2.13). It also can be proved that the solution of the relaxed problem is a lower bound on the average cost of any feasible solution to problem (2.12)-(2.14). Moreover, when the replenishment intervals are rounded off to powers of two multiples of a base planning period T_L, the cost of the resulting policy is proved to be within 6% of optimal, if T_L is fixed, or 2% if T_L is variable. In addition, the computational complexity of this approach is also O(N log N).

To this point, we have studied the serial and the assembly systems. In the next section, we focus on the distribution systems.

2.5 Distribution Systems

As we have commented, a distribution system looks like a backwards assembly system. Usually, the first node represents a central warehouse and the ending nodes correspond to retailers. The nodes in the middle are intermediate stocking points, such as regional warehouses. In what follows, we restrict our attention to the two-level distribution systems which are usually referred to as the *one-warehouse N-retailer problem*. We assume that customer demands occur at each retailer at a constant rate and shortages are not allowed. The costs defined for the serial and assembly systems also apply to this case. That is, there is a holding cost per unit stored per unit time and a fixed charge for each order placed at the warehouse and at each retailer. Accordingly, k_j and h_j represent the fixed replenishment cost and the holding cost per unit time at retailer $j = 1, ..., N$, respectively. Similarly, the fixed replenishment cost and the holding cost per unit time at the warehouse are denoted by k_0 and h_0, respectively.

Recall that for the two-level serial and assembly systems optimal policies have to be stationary and nested. However, these properties do not hold to two-level distribution systems. Thus, for these systems the computation of optimal policies becomes a more difficult task. Hence, it is not surprising that many authors have combined efforts to achieve advances in the control of inventories for these systems. This is also the main objective of this dissertation.

Many researchers have considered the possibility of focusing their attention on a simpler class of strategies. The simplest policies are those which are stationary. In addition, although they are not necessarily optimal, stationary policies are of significant practical importance. For example, companies often use this kind of policies to schedule their operations in regular intervals so that the same set of activities is repeated at constant time cycles. Thus, their replenishments are conducted in a stationary fashion.

We next review the most important contributions concerning stationary and nested policies for the two-level distribution systems. Such policies are also referred to as *single-cycle* policies.

2.5.1 Single-cycle policies

Schwarz (1973) was one of the first on studying the one-warehouse N-retailer problem. He developed a procedure to calculate stationary and nested policies for these systems. This author showed that such policies are optimal for the one-warehouse and N identical retailers system, and he provided a solution method to solve the problem. For the general case he developed a heuristic procedure. This heuristic yields good results when the number of retailers is smaller than 10. However, when the number of retailers increases the solutions obtained by the Schwarz heuristic are not very effective.

A few years later, Graves and Schwarz (1977) developed an exact procedure based on a branch and bound scheme to compute optimal single-cycle policies. Unfortunately, this approach can theoretically generate an infinite number of branches at each level of the enumeration tree. Hence, the computational effort increases exponentially with the number of retailers and the procedure is not suitable when the number of retailers is significantly large.

However, in Muckstadt and Roundy (1993) we can find an effective heuristic for computing stationary and nested policies. Again, the key idea is to focus on single-cycle policies which are also powers-of-two.

Single-cycle powers-of-two policies

The problem of finding an optimal stationary nested powers-of-two policy for the one-warehouse N-retailer problem can be formulated using the echelon holding costs as follows

$$\min \sum_{j=0}^{N} \left(\frac{k_j}{t_j} + \frac{h'_j t_j d_j}{2}\right) \tag{2.15}$$

s.t.

$$t_0 \geq t_j \geq 0, \ \forall j \tag{2.16}$$

$$t_j = 2^{l_j} T_L, \quad l_j \in \{0, 1, ...\} \tag{2.17}$$

where t_0 denotes the replenishment interval at the warehouse.

Similar to the serial and assembly systems, we first relax the constraints in (2.17) and solve the resulting problem. Then, we round off the replenishment intervals thus computed to powers of two multiples of T_L. It is worth noting that the only difference between problem (2.12)-(2.13) and problem (2.15)-(2.16) is the form of the constraint (2.16). Note that here the arcs are oriented away from the root, that is, from the warehouse, whereas in the assembly systems they are oriented towards the root. Hence, we can transform problem (2.15)-(2.16) into an instance of problem (2.12)-(2.13) by setting $u_j = 1/t_j$, which yields the following formulation

$$\min \sum_{j=0}^{N} (k_j u_j + \frac{h'_j d_j}{2u_j}) \tag{2.18}$$

s.t.

$$u_j \geq u_0 \geq 0, \; \forall j \tag{2.19}$$

Now, problem (2.18)-(2.19) is of the form of problem (2.12)-(2.13). Therefore, the O(N log N) approach introduced for the assembly systems can be also applied in this case. Consequently, we can compute an optimal powers-of-two policy which is within 6% of the cost of an optimal nested policy, or within 2% if the base period is treated as a variable. Observe that for the distribution systems we can only ensure that single-cycle powers-of-two policies are 98% effective whitin the class of nested policies. However, these policies are not always so efficient with respect to the global optimal policy.

In Chapter 3 we propose a new heuristic for computing near-optimal single-cycle policies for the one-warehouse N-retailer problem. This method is compared with the heuristics proposed by Schwarz (1973) and Muckstadt and Roundy (1993). We show that the new heuristic provides, on average, better single-cycle policies than the other approaches. In addition, we also illustrate how the problem should be addressed in case of each retailer orders independently, as does the warehouse. Under this situation, we propose a two-level optimization approach which consists of computing first the order quantities at the retailers, and then, determining the inventory policy for the warehouse.

The single-cycle policies can be very efficient and have clear managerial advantages. However, as Roundy (1983) showed, in some situations the optimal nested policies can have very low effectiveness. For that reason, Roundy (1985) dropped the assumption of nestedness and he analyzed a more general class of policies referred to as *integer-ratio* policies. We introduce such policies in the next subsection.

2.5.2 Integer-ratio policies

In these policies the warehouse orders at equally-spaced points in time and each retailer follows an EOQ pattern. Besides, the replenishment interval at the warehouse, t_0, and the replenishment interval at retailer j, t_j, must satisfy that either t_j/t_0 or t_0/t_j is a positive integer. Notice that following this policy, the warehouse has not necessarily to order the same quantity each time an order is placed. Therefore, integer-ratio policies are always stationary at the retailers but may not be at the warehouse. In particular, Roundy (1985) focused on the integer-ratio policies which are also powers-of-two, that is, t_j/t_0 or t_0/t_j is a power of two, proving that these policies are very effective. Specifically, Roundy (1985) showed that the cost of an optimal integer-ratio powers-of-two policy is at most 2% above the cost of an optimal policy.

Integer-ratio powers-of-two policies

Let now review Roundy's approach for computing an optimal powers-of-two policy. In order to determine the average holding costs, Roundy (1985) distinguishes between the retailers with replenishment interval greater and smaller than the replenishment interval at the warehouse.

Notice that if $t_j \geq t_0$, then $t_j = m_j t_0$ with m_j integer, and therefore, each time retailer j places an order, so does the warehouse. Hence, the warehouse has not to hold inventory for the retailers with replenishment interval greater than t_0. Taking this into account, at these retailers we should consider the conventional holding costs instead of the echelon holding costs. Thus, the average cost of holding all inventory in the system that is destined to retailer j, with $t_j \geq t_0$, is given by $h_j d_j t_j/2 = h'_j d_j t_j/2 + h_0 d_j t_j/2$.

However, when $t_j < t_0$, t_0 can be expressed as an integer multiple of t_j, that is, $t_0 = n_j t_j$. Under this situation it is convenient to use the echelon inventory to compute the holding costs. Hence, the average cost of holding all inventory in the system associated with retailer j, with $t_j < t_0$, is $h'_j d_j t_j/2 + h_0 d_j t_0/2$.

Therefore, the average total cost is given by

$$C_T = \sum_{j=0}^{N} \frac{k_j}{t_j} + \sum_{j=1}^{N} \left(\frac{h'_j d_j t_j}{2} + \frac{h_0 d_j \max\{t_j, t_0\}}{2}\right)$$

Since we are restricting to powers-of-two policies, we must consider the following constraints related to the replenishment intervals

$$t_j = 2^{l_j} T_L, \quad l_j \in \{0, 1, ...\}, \;\; j = 0, ..., N$$

Consequently, the problem of finding an optimal integer-ratio powers-of-two policy can be stated as follows

$$\min \sum_{j=0}^{N} \frac{k_j}{t_j} + \sum_{j=1}^{N} \left(\frac{h'_j d_j t_j}{2} + \frac{h_0 d_j \max\{t_j, t_0\}}{2} \right) \tag{2.20}$$

s.t.

$$t_j = 2^{l_j} T_L, \quad l_j \in \{0, 1, ...\}, \; j = 0, ..., N \tag{2.21}$$

Roundy (1985) solves this problem dropping constraints (2.21) and then, minimizing (2.20) for $t_j > 0$, $j = 0, ..., N$. After that, he rounds off the replenishment intervals thus computed to powers of two multiples of T_L. Again, the cost of the computed policy is within 6% of the optimal cost, if the base period T_L is fixed, and within 2%, if T_L is variable.

Moreover, it is possible to show that the solution which minimizes (2.20) is a lower bound on the average cost of any feasible policy. When we analyzed the single-cycle policies we showed that it was always possible to compute a single-cycle policy whose cost is close to the cost of an optimal nested policy. However, optimal nested policies can be far from the global optimal policies. In contrast, the optimal integer-ratio powers-of-two policies are close to the global optimal policies.

We outline now how (2.20) can be efficiently minimized. The key idea consists of clustering the retailers into three sets: G, L and E. Those retailers that place orders less frequently than the warehouse are in set G. In set L are the retailers placing orders more frequently than the warehouse. Finally, those retailers that place orders simultaneously with the warehouse belong to set E.

The procedure for computing sets G, L, and E is given in Algorithm 2.2.

Once the replenishment intervals t_j^*'s which minimize (2.20) are computed, they have to be rounded off to powers of two multiples of a base planning period T_L. If T_L is fixed, the rounded off replenishment interval for a given facility j is $t_j = 2^{l_j} T_L$ where l_j is the smallest nonnegative integer value satisfying $2^{l_j - 1} < t_j^* / \sqrt{2} T_L < 2^{l_j}$. If T_L is variable, the algorithm for computing optimal powers-of-two policies can be found in Roundy (1985). Moreover, it can be proved that an optimal integer-ratio powers-of-two policy can be computed in O($N \log N$).

In Chapter 4 we present an alternative approach for computing integer-ratio policies. This new approach is compared with the one proposed by Roundy (1985).

Algorithm 2.2 Procedure for computing sets G, L, and E

Step 1

Calculate the replenishment interval $\tau_j^{'} = \left(\frac{2k_j}{d_j(h_j+h_0)}\right)^{1/2}$ and $\tau_j = \left(\frac{2k_j}{d_j h_j}\right)^{1/2}$, and sort them to form a nondecreasing sequence of $2N$ numbers. This sequence is denoted by S. Label each replenishment interval with the value of j and with a flag indicating whether it is the conventional replenishment interval $\tau_j^{'}$ or the echelon replenishment interval τ_j.

Step 2

Set $E = G = \emptyset$, $L = \{1, ..., N\}$, $K = k_0$ and $H = h_0 \sum_{j=1}^{N} d_j/2$

Step 3

Let τ be the largest element in S. If $\tau^2 \geq \frac{K}{H}$ and $\tau = \tau_j$ is an echelon replenishment interval, remove τ from S and update E, L, K, and H as follows: $E \longleftarrow E \cup \{j\}$, $L \longleftarrow L \backslash \{j\}$, $K \longleftarrow K + k_j$, and $H \longleftarrow H + h_j d_j/2$. Then go to Step 3.
If $\tau^2 > \frac{K}{H}$ and $\tau = \tau_j^{'}$ is a conventional replenishment interval, remove τ from S and update E, G, K, and H as follows: $E \longleftarrow E\backslash\{j\}$, $G \longleftarrow G \cup \{j\}$, $H \longleftarrow H - d_j(h_j + h_0)/2$, and $K \longleftarrow K - k_j$, and go to Step 3. Otherwise, the current sets G, L, and E are optimal. Go to step 4.

Step 4

Set $t_0^* = \sqrt{\frac{K}{H}}$ and $t_j^* = t_0^*$ for all retailers $j \in E$. For retailers $j \in G$, consider $t_j^* = \sqrt{\frac{2k_j}{d_j(h_j+h_0)}}$, and for retailers $j \in L$ set $t_j^* = \sqrt{\frac{2k_j}{d_j h_j}}$. Stop.

The computational results will show that the new heuristic provides, on average, better policies than those given by the Roundy procedure.

2.6 Multi-echelon inventory systems with finite production rates

All previous models assume that the production or the replenishment occurs instantaneously, that is, the production rate is infinite. However, often a stage corresponds to a manufacturing operation where production occurs at a finite rate. Hence, several researches have extended the previous problems to include finite production rates. Accordingly, Schwarz and Schrage (1975) considered an assembly system under the assumption that material is transferred from one stage to another only after a batch is completed.

Szendrovits (1975) studied serial systems in which a single replenishment interval t is used at all stages. In addition, material is transferred between stages in batches of size b, with $1 \leq b \leq td$, where d is the demand rate.

Atkins, Queyranne and Sun (1992) obtained effective policies for assembly systems with finite production rates. Specifically, they assumed that the processing rate at the successor of node j is not less than node j's processing rate, for all j. They also proved that the class of powers-of-two policies introduced by Roundy (1985) extends to finite production rate assembly systems. In particular, they showed that an optimal powers-of-two policy can be derived in $O(N \log N)$ time and its cost is within 2% of the optimum in the worst case.

We can also find in the literature many works dealing with systems where a single vendor produces an item which is supplied to a buyer. For such systems, the total costs incurred by the vendor and the buyer can be reduced significantly by integrating the vendor's as well as the buyer's production/inventory problem. For that reason, most researches have conducted their efforts in studying integrated vendor-buyer inventory models. A large number of contributions on this model are confined to considering a single buyer. Although, in practice, the vendor usually supplies multiple buyers, we find few references in the literature addressing the multiple buyers case. Chapter 5 of this dissertation is devoted to the single-vendor multi-buyer problem.

2.7 Conclusions

Multi-echelon inventory systems are very common in practice. For example, consumers often do not purchase products directly from the producer. Instead, products are usually distributed through regional warehouses and local retailers to the consumer, that is, through a multi-echelon distribution system. In production, stocks of raw materials, components, and finished products are similarly coupled to each other. In this chapter we have introduced the typical structures that can appear in multi-echelon inventory systems: serial, assembly and distribution systems. Furthermore, we have reviewed the most important models and algorithms for solving lot sizing problems for these systems with constant demand rates.

For the two-level serial and assembly systems, optimal policies are stationary and nested which can be computed easily. When there are more than two levels the optimal policy remains to be nested although not necessarily stationary. In spite of this, if we restrict ourselves to nested, stationary and powers-of-two policies we can obtain a policy whose cost is at most 2% above the cost of an optimal policy. However, as we have pointed out, for distribution systems the form of the optimal policies can be very complex even when we restrict ourselves to the two-level distribution systems.

This dissertation is mainly concerned with such systems. In particular, Chapters 3 and 4 are devoted to the two-level distribution systems assuming that production is instantaneous. Finally, in Chapter 5 we address the problem considering finite production rate.

To conclude this section, we present in Tables 2.1 and 2.2 the current literature that have been summarized in this chapter. We also include the contributions of this dissertation which are to be developed in the next chapters.

Table 2.1: Literature review and contributions of the thesis

Chapter	Reference	N	Production Rate	Policy	Solution Method	Comment
3	Schwarz (1973)	≥ 1	Infinite	Stationary and nested.	Heuristic	Effective stationary nested policies only when $N < 10$.
3	Graves and Schwarz (1977)	≥ 1	Infinite	Stationary and nested.	Optimal	The computational effort increases exponentially with the number of retailers.
3	Muckstadt and Roundy (1993)	≥ 1	Infinite	Stationary, nested and powers-of-two.	Heuristic	The cost of such policy is within 2% of the cost of an optimal nested policy.
3	Abdul-Jalbar et al. (2006)	≥ 1	Infinite	Stationary and nested.	Heuristic	The new heuristic provides, on average, better stationary nested policies than the previous approaches.
3	Abdul-Jalbar et al. (2003)	≥ 1	Infinite	Stationary and nested Decentralized policies.	Heuristic	As N increases so does the number of instances where the decentralized policy is better.
4	Roundy (1985)	≥ 1	Infinite	Integer-ratio powers-of-two.	Heuristic	The cost of such policy is within 2% of the cost of the global optimum.
4	Abdul-Jalbar et al. (2005)	≥ 1	Infinite	Integer-ratio.	Heuristic	The new heuristic provides, on average, better integer-ratio policies than Roundy's procedure.

Table 2.2: Literature review and contributions of the thesis

Chapter	Reference	N	Production Rate	Policy	Solution Method	Comment
5	Banerjee (1986)	1	Finite	Lot-for-lot.	Optimal	He was a pioneer in studying an integrated inventory system. A joint optimal policy can be profitable for both parties.
5	Goyal (1988)	1	Finite	$Q_{vendor} = nQ_{buyer}$, n a positive integer.	Optimal	The vendor can supply the buyer only after completing the entire lot size.
5	Lu (1995)	≥ 1	Finite	Integer-ratio.	Optimal and heuristic	Equal sized shipments. The vendor can supply the buyers during production. Each buyer orders a different item.
5	Goyal (1995)	1	Finite	At each delivery all the available inventory is supplied to the buyer.	Optimal	Shipments can be made before the whole lot is produced. This policy can be better than the equal shipment size policy.
5	Hill (1997a)	1	Finite	Successive shipment sizes increase by a fixed factor.	Optimal	The Lu's and Goyal's policies represent special cases of this more general class of policies.
5	Hill (1999)	1	Finite	Combination of Goyal's and Lu's policies.	Optimal	Global optimal policy.
5	Yao and Chiou (2003)	≥ 1	Finite	Integer-ratio.	Heuristic	They improve Lu's heuristic.
5	Khouja (2003)	≥ 1	Finite	Equal cycle time, integer-ratio, powers-of-two.	Optimal	The whole lot has to be produced before delivering the batch.
5	Wee and Yang (2004)	≥ 1	Finite	$t_{vendor} = nt_{buyer}$, n a positive integer.	Optimal and heuristic	t_{buyer} cannot be greater than t_{vendor}, for any buyer.
5	Abdul-Jalbar et al. (2004a)	2	Finite	Integer-ratio.	Optimal	t_{buyer} can be greater than t_{vendor}, for any buyer.
5	Abdul-Jalbar et al. (2004b)	≥ 1	Finite	Integer-ratio Decentralized policies.	Heuristic	The integer-ratio policies are analyzed and compared with the decentralized policies. Depending on the parameter values, one strategy outperforms the other.

Chapter 3
The one-warehouse N-retailer problem: Single-cycle versus decentralized policies

This thesis is concerned to the study of the two-level distribution systems often referred to as the one-warehouse N-retailer systems. As we show in Chapter 2, we can easily obtain optimal policies for the two-level serial and assembly systems. However, the computation of optimal policies for the two-level distribution systems is much more complex. In particular, Arkin et al. (1989) proved that the one-warehouse N-retailer problem is an NP-hard problem, that is, it cannot be solved by polynomial time algorithms. Hence, in the last decades, extensive research efforts have been addressed to attempt efficient heuristics for solving the one-warehouse N-retailer problem. In this chapter we analyze the single-cycle policies which are one of the simplest policies that can be applied to these systems.

3.1 Introduction

The one-warehouse N-retailer problem represents a special category of inventory systems encountered frequently in practice. Due to their applicability in real world situations, these multi-echelon systems have caught many researchers' attention. Excellent reviews of such systems can be found in Muckstadt and Roundy (1993), Silver et at. (1998), Axsäter (2000) and Zipkin (2000), among others.

In this problem, the warehouse is the sole supplier of N retailers and customer demands occur at each retailer at a constant rate. This demand must be met as it occurs over an infinite horizon without shortages. Orders placed by retailers generate demands at the warehouse which in turn orders from an outside supplier. In addition, delivery of orders is assumed to be instantaneous, that is, lead times are assumed to be zero. The considered costs are a holding cost and a fixed charge for each replenishment placed at the warehouse and at each retailer.

The one-warehouse N-retailer system was examined by Schwarz (1973) who determined the necessary conditions of an optimal policy. He proved that an optimal policy can be found in the set of *basic policies.* A basic policy is any feasible policy where deliveries are made to the warehouse only when the warehouse has zero inventory and, at least one retailer has zero inventory. Moreover, deliveries are made to any given retailer only when that retailer has zero inventory. In addition, all deliveries made to any given retailer between successive deliveries to the warehouse are of equal size. However, Schwarz (1973) also showed that the form of the optimal policy can be very complex. In particular, it can require that the order quantity at one or more of the locations varies with time even though all relevant demand and cost factors are time invariant. Thus, many authors have considered the possibility of restricting attention to a simpler class of strategies, as the single-cycle policies.

In this chapter we review the different procedures proposed in the literature for computing single-cycle policies. In particular, we focus on the works of Schwarz (1973), Graves and Schwarz (1977) and Muckstadt and Roundy (1993). In addition, we propose an alternative heuristic for obtaining very effective single-cycle policies which is compared with the previous approaches.

It is worth noting that all these procedures assume that the decision system is centralized. However, we also show how the problem should be addressed if the decision system is decentralized, that is, if each retailer orders independently, as does the warehouse. Under this situation, we propose a two-level optimization approach which consists of computing first the order quantities at the retailers, and then, determining the inventory policy for the warehouse.

The remainder of the chapter is organized as follows. In Section 3.2 we introduce the notation required to state the problem. In Section 3.3 we analyze the one-warehouse N-retailer problem assuming that the decision system is centralized. In particular, we focus on the simplest policies that can be applied under this situation. Such policies consists of forcing the retailers to place their orders at common time instants. In Section 3.4 we drop this assumption and we allow the retailers to place their orders at different time instants. Concretely, we analyze the single-cycle policies and we summarize the different procedures introduced by Schwarz (1973), Graves and Schwarz (1977) and Muckstadt and Roundy (1993). We also introduce a new heuristic for computing single-cycle policies. Section 3.5 deals with the decentralized situation, that is, when the retailers order independently. In Section 3.6 we solve a numerical example with the different procedures that have been introduced. Computational results are reported in Section 3.7. Finally, we draw some conclusions in Section 3.8.

3.2 Notation and Problem statement

Most of the notation required to formulate the one-warehouse N-retailer problem has been already introduced in Chapter 2. Recall that the input data associated with the retailers are d_j, k_j and h_j which represent the constant and continuous demand rate, the fixed replenishment cost and the holding cost per unit time at retailer $j = 1, ..., N$, respectively. The fixed replenishment cost and the holding cost per unit time at the warehouse are denoted by k_0 and h_0, respectively. We also introduced in Chapter 2 the echelon inventory concept which facilitates the computation of the average inventory at the warehouse. In particular, for the one-warehouse N-retailer problem, the echelon holding cost at retailer $j = 1, ..., N$ is $h'_j = h_j - h_0$, and the echelon holding cost at the warehouse is $h'_0 = h_0$.

When we analyze the problem assuming that the decision system is centralized, d_0 represents the demand per unit time at the warehouse and the decision variables are the replenishment intervals at the retailers, t_j, $j = 1, ..., N$, and at the warehouse, t_0. Obviously, since the demand is constant, once the replenishment intervals are established, it is easy to compute the corresponding order quantities at the retailers, denoted by Q_j, $j = 1, ..., N$, and at the warehouse, Q_0.

In case of independence among the warehouse and the retailers, we first compute the replenishment intervals at the retailers t_j, $j = 1, ..., N$, or equivalently, the order quantities Q_j, $j = 1, ..., N$, and then, we determine the shipment schedule at the warehouse. Accordingly, the warehouse behaves as an inventory system with time-varying demand. Under this situation, a time horizon at the warehouse, τ_0, is determined. Then, we compute the demand vector at the warehouse denoted by $\overline{d_0}$. Similarly, $\overline{t_0}$ represents a vector that contains the time instants where the retailers place their orders to the warehouse. Finally, we denote by $\overline{Q_0}$ the order quantities vector at the warehouse.

The total costs per unit time incurred by retailer $j = 1, ..., N$, the warehouse and the total system are C_j, C_0 and C_T, respectively. In addition, since the retailers follow an EOQ pattern the average total cost can be stated as follows

$$C_T = C_0 + \sum_{j=1}^{N} C_j = C_0 + \sum_{j=1}^{N} \left(\frac{k_j}{t_j} + \frac{h_j d_j t_j}{2}\right) \tag{3.1}$$

Depending on whether there exists dependence or not among the warehouse and the retailers, the average cost at the warehouse should be formulated in a different way.

First, let us analyze the problem assuming that the decision system is centralized.

3.3 Centralized policies with common replenishment intervals

In this section we assume that the decision system is centralized. This situation is very common in practice, for example, when the warehouse and the retailers belong to the same firm. In this case, the firm should pay all the costs and hence, the goal is to minimize the average total cost, that is, the average cost at the warehouse plus the average costs at the retailers. Since the firm is the unique decision-maker, it can force the retailers to place their orders at some time instants.

The simplest policy consists of forcing the retailers to place their orders at common time instants, say every t time units. Then, the average cost at each retailer $j = 1, ..., N$, is $C_j = h_j d_j t/2 + k_j/t$.

Let d be the sum of the demands at the retailers, i.e., $d = \sum_{j=1}^{N} d_j$. Since all retailers place an order at the same time, the one-warehouse N-retailer problem can be viewed as a one-warehouse one-retailer problem where the demand per unit time at the warehouse is $d_0 = d$. Besides, the new big retailer orders the sum of the quantities ordered by the original retailers, that is, it orders $Q = \sum_{j=1}^{N} Q_j$ units of item every t time units.

Recall that Crowston et al. (1973) and Williams (1982) proved that in an optimal solution the order quantity at the warehouse must be an integer multiple of the order quantity at the retailer. Therefore, in the previous situation it holds that $Q_0 = nQ$, with n a positive integer. Then, using (2.1) the average cost at the warehouse is given by $C_0 = h_0(n-1)td/2 + k_0/nt$.

Now, the average total cost can be written as follows

$$C_T = \frac{t}{2}\sum_{j=1}^{N}(h_j d_j + h_0(n-1)d) + \frac{1}{t}\sum_{j=1}^{N}(k_j + \frac{k_0}{n})$$

Note that the overall cost depends only on t and n. In order to calculate the optimal solution (t^*, n^*) we need Lemma 3.1.

Lemma 3.1 *If $h_0 < h_j$, $j = 1, ..., N$, and n is a continuous variable, then C_T is convex over the region: $\{R : 0 < n < \infty, 0 < t \leq B(n)\}$, and has its global minimum at (t^*, n^*), where* $B(n) = \frac{1}{n}\left[\frac{2k_0}{h_0 d_0}\left(2(1 + n\frac{\sum_{j=1}^{N} k_j}{k_0})^{1/2} - 1\right)\right]^{1/2}$ *and*

$$t^* = \left[\frac{2(\sum_{j=1}^{N} k_j + \frac{k_0}{n})}{\sum_{j=1}^{N} h_j d_j + h_0(n-1)d_0} \right]^{1/2} \tag{3.2}$$

$$n^* = \left[\frac{k_0(\sum_{j=1}^{N} h_j d_j - h_0 d_0)}{h_0 d_0 \sum_{j=1}^{N} k_j} \right]^{1/2} \tag{3.3}$$

Proof.

Assuming that n is a continuous variable and setting the first partial derivatives of C_T equal to zero, we obtain t^* by (3.2) and n^* by (3.3).

It is easy to see that the Hessian is positive definite at $t = t^*$ and $n = n^*$, therefore, C_T has a local minimum at (t^*, n^*).

The Hessian matrix is non-negative definite for any n and t in the region $\{R : 0 < n < \infty, 0 < t \leq B(n)\}$, and $(t^*, n^*) \in R$. Thus, C_T is convex on R with the global minimum at (t^*, n^*). ■

From value t^*, we can obtain the optimal order quantities at each retailer, that is,

$$Q_j^* = d_j t^*, \ j = 1, ..., N. \tag{3.4}$$

and

$$Q_0^* = n \sum_{j=1}^{N} Q_j^* \tag{3.5}$$

where n is the nearest integer to n^*.

Summarizing, if the firm forces the retailers to place their orders at the same time instants, the optimal solution is given by the formulae in Table 3.1.

However, due to some reasons such as logistics problems, it could be preferable to satisfy the demand at the retailers at different time instants. We address this case in the following section.

Table 3.1: Optimal solution assuming common replenishment intervals

	Time	Quantity
Retailer 1	$t_1 = t^* = \left[\frac{2\left(\sum_{j=1}^{N} k_j + \frac{k_0}{n}\right)}{\sum_{j=1}^{N} h_j d_j + h_0(n-1)D}\right]^{1/2}$	$Q_1^* = d_1 t^*$
Retailer 2	$t_2 = t^*$	$Q_2^* = d_2 t^*$
...	...	...
Retailer N	$t_N = t^*$	$Q_N^* = d_N t^*$
Warehouse	$t_0 = nt^*$, with n the nearest integer to n^*	$Q_0^* = n\sum_{j=1}^{N} Q_j^*$

3.4 Single-cycle policies

We continue assuming that the decision system is centralized. However, we now allow the retailers to place their orders at different time instants t_j, $j = 1, \ldots, N$. In particular, we deal with the class of single-cycle policies. Therefore, the unique condition that must be verified is that there must exist $n_1, n_2, ..., n_N \in \mathbb{N}$, such that, $n_1 t_1 = n_2 t_2 = \cdots = n_N t_N = t_0$. It is worth noting, that we can use either t_0 and t_j's or t_0 and n_j's as variables, where n_j represents the number of replenishment at retailer j during t_0, $j = 1, \ldots, N$.

The objective consists of minimizing the average total cost, that is, the sum of the average holding and replenishment costs at the retailers and at the warehouse. As we showed in Chapter 2, the problem of computing an optimal single-cycle policy using the echelon holding costs can be formulated as follows

$$\min\ C_T = \sum_{j=0}^{N} \left(\frac{k_j}{t_j} + \frac{h'_j d_j t_j}{2}\right) \tag{3.6}$$

$$\text{s.t. } n_1 t_1 = n_2 t_2 = \cdots = n_N t_N = t_0 \tag{3.7}$$

$$n_j \geq 1, \text{ integer} \tag{3.8}$$

where $d_0 = \sum_{j=1}^{N} d_j$.

As we have previously commented, many authors have addressed this problem. Next, we summarize the most important contributions in the literature.

3.4.1 The Schwarz heuristic

Schwarz (1973) proved that single-cycle policies are optimal for the one-warehouse one-retailer problem, and for the one-warehouse N-identical retailers system. For the general case with different retailers, he developed a heuristic procedure. Moreover, he proved that single-cycle policies obtained with this approach are effective when the number of retailers is smaller than 10. Otherwise, the policies provided by the Schwarz procedure are not very effective.

The Schwarz approach is based on the calculation of the replenishment interval at the warehouse, i.e., t_0. Once t_0 is known, the number of deliveries made to retailer j during t_0, i.e., n_j, $j = 1, ..., N$, is computed. This method is sketched in Algorithm 3.1.

3.4.2 The Graves and Schwarz procedure

Unfortunately, in general, the solutions provided by the Schwarz approach are not very effective. For that reason, Graves and Schwarz (1977) developed another procedure to calculate optimal single-cycle policies for the one-warehouse N-retailer problem. This procedure consists of searching, via a branch and bound scheme, the number of replenishments at each retailer during t_0, i.e., n_j, $j = 1, ..., N$. The search begins by generating a feasible initial solution $(x_1, x_2, ..., x_N)$. Notice that this initial solution can be considered as a heuristic solution. Then, a branch and bound scheme is used to improve the initial solution. There are N levels in the enumeration tree and each level corresponds to a different n_j. At level j, n_j is set to an integer value, namely, $n_j = x_j$. For a given branch at level j, a lower bound, LB, is determined by ignoring the integrality constraint. At level j, n_j is set to an integer value, namely I_j, with $I_j > x_j$ $(I_j < x_j)$ and the corresponding lower bound is computed. If such lower bound exceeds the cost of any known feasible solution, the investigation of all $n_j > I_j$ $(n_j < I_j)$ may be ignored. This is due to the fact that the function C is convex, and therefore, the corresponding lower bounds for all $n_j > I_j$ $(n_j < I_j)$ will also exceed the value of the feasible solution. This method is given in detail in Algorithm 3.2.

Algorithm 3.1 The Schwarz heuristic

Step 1

For each retailer $j = 1, ..., N$, calculate the optimal replenishment interval, that is,

$$t_j^* = \left[\frac{2k_j}{d_j h'_j}\right]^{1/2} \tag{3.9}$$

In addition, the optimal replenishment interval at the warehouse is computed as

$$t_0^* = \left[\frac{2k_0}{d_0 h'_0}\right]^{1/2} \tag{3.10}$$

Step 2

If $t_0^* \geq \max\{t_j^*\}$, set $t_0 = t_0^*$. Otherwise, define

$$J = \{j \mid t_0^* < t_j^*\} \tag{3.11}$$

and t_0 is given by

$$t_0 = \left[\frac{2(k_0 + \sum_{j \in J} k_j)}{h'_0 d_0 + \sum_{j \in J} h'_j d_j}\right]^{1/2} \tag{3.12}$$

Step 3

Set n_j equal to the integer value that minimizes the cost at retailer $j = 1, ..., N$. That is, choose n_j, $j = 1, ..., N$, such that

$$C_j = n_j \frac{k_j}{t_0} + \frac{h'_j d_j t_0}{2n_j} \tag{3.13}$$

is minimum.

Notice that n_j is the nearest integer to $n_j^* = t_0 \left[\frac{h'_j d_j}{2k_j}\right]^{1/2}$, $j = 1, ..., N$.

Algorithm 3.2 The Graves and Schwarz procedure

Step 1

Sort the retailers so that retailer $i <$ retailer j if, and only if, $h'_i d_i / k_i < h'_j d_j / k_j$.

Step 2

Initialize $i = N$.

Calculate the smallest integer m_j, $j = 1, ..., N$, satisfying

$$m_j(m_j + 1) \geq \frac{\widehat{k_0} h'_j d_j}{k_j \widehat{h'_0} d_0} \tag{3.14}$$

where

$$\widehat{k_0} = k_0 + \sum_{r=1}^{j-1} n_r k_r$$

and

$$\widehat{h_0} = h'_0 + \sum_{r=1}^{j-1} \frac{h'_r d_r}{n_r d_0}$$

Initialize $n_j = m_j$, $j = 1, ..., N$, and calculate the overall cost.

Step 3

If $i \geq 1$, branch at level i setting $n_i > m_i$ and $n_i < m_i$ until all the possible values for n_i have been examined. If the lower bound LB associated with one node at level i is smaller than the cost of the current feasible solution, then continue exploring the tree for levels $j = i{+}1, ..., N$. If a better feasible solution is obtained, e.g. $(n'_1, n'_2, ..., n'_N)$, the cost associated to this policy must be calculated and the values n_j's should be updated applying $n_j = n'_j$, $j = 1, ..., N$.

Set $i = i - 1$.

If $i = 0$, stop, the current n_j's are the optimal integer values. Otherwise, go to Step 2.

3.4.3 The Muckstadt and Roundy approach

Muckstadt and Roundy (1993) analyzed the single-cycle powers-of-two policies. Recall that in these policies the replenishment intervals are powers of two multiples of the base planning period T_L, that is, $t_j = 2^{l_j} T_L$, $j = 0, ..., N$ and $l_j = 1, 2, ...$ They proved that an optimal single-cycle powers-of-two policy can be computed in O($N \log N$) with at least 94% effectiveness. That is, when we restrict ourselves to such policies, we can guarantee that the cost of the optimal single-cycle powers-of-two policy is at most 6% above the cost of an optimal single-cycle policy. Even more, Muckstadt and Roundy (1993) showed that if T_L is considered as a variable the margin of 6% obtained when T_L is fixed is reduced to 2%.

Muckstadt and Roundy (1993) first solve the problem ignoring the powers of two constraints. They proved that solving this relaxed problem is equivalent to classifying the retailers and the warehouse in clusters, so that all locations in a cluster use the same replenishment interval. We summarize in Algorithm 3.3 the procedure for classifying the retailers and the warehouse in clusters.

Algorithm 3.3 Procedure for classifying the retailers and the warehouse in clusters

Step 1
Initialize $C^j = \{j\}$, $\forall j = 0, ..., N$, and $Q = \{1, 2, ..., N\}$.
Step 2
If $Q = \emptyset$, stop. The clusters C^j's are optimal. Otherwise, choose the retailer $i \in Q$ for which g_i/k_i is minimal, where $g_i = h'_i d_i/2$.
Step 3
Update $Q = Q - \{i\}$.
If $g_i/k_i < g_0/k_0$, where $g_0 = \sum_{j \in S_0} g_j$ and $k_0 = \sum_{j \in S_0} k_j$, then add retailer i to cluster C^0, that is, $C^0 = C^0 \cup \{i\}$ and $C^i = \emptyset$. Go to Step 2.
Otherwise, go to Step 4.
Step 4
$\forall$ $C^j \neq \emptyset$ calculate $t^*_j = (k_j/g_j)^{1/2}$ and $\forall$ $j \in C^0$ set $t^*_j = t^*_0$.

Once the replenishment intervals for the relaxed problem are computed, they have to be rounded off to get a feasible single-cycle powers-of-two policy. If the base planning period T_L is assumed fixed, for each $j \in C^i$ we should choose $t_j = 2^{l_j} T_L$, where l_j is the smallest nonnegative integer value satisfying $2^{l_j - 1} < t^*_j/\sqrt{2}T_L < 2^{l_j}$. In contrast, if T_L is variable we should use Algorithm 3.4 to compute the optimal single-cycle powers-of-two policy.

Algorithm 3.4 The Muckstadt and Roundy approach with T_L variable

Step 1
Let r_j be equal to t_j^*/t_0^*, where t_0^* and t_j^*, $j = 1, ..., N$, are the optimal replenishment intervals for the relaxed problem.
Let p_j be the smallest integer value satisfying $r_j \leq 2^{p_j} < 2r_j$, and set $q_j = r_j 2^{0.5-p_j}$ for all $j \notin C^0$.

Step 2
Let $q_{[s]}$ be the sth smallest value of the q_j, $j \notin C^0$, and set $q_{[0]} = 0.5^{1/2}$ and $q_{[R+1]} = 2^{1/2}$, where R is equal to the number of retailers that are not in C^0.

Step 3
For each s, $0 \leq s \leq R$, set $n_j = 2^{1-p_j}$ if $q_j \leq q_{[s]}$, or $n_j = 2^{-p_j}$ if $q_j > q_{[s]}$
Compute

$$k^s = k_0 + \sum_{q_j \leq q_{[s]}} k_j 2^{1-p_j} + \sum_{q_j > q_{[s]}} k_j 2^{-p_j}$$

and

$$g^s = g_0 + \sum_{q_j \leq q_{[s]}} g_j 2^{p_j - 1} + \sum_{q_j > q_{[s]}} g_j 2^{p_j}$$

Note that each s, $0 \leq s \leq R$, is related to a powers-of-two policy.

Step 4
For each s, $0 \leq s \leq R$, calculate the average cost incurred by the corresponding powers-of-two policy, that is,

$$C_s = \frac{k^s}{t_0} + g^s t_0 \tag{3.15}$$

Step 5
For fixed s, compute the value t_0 that minimizes C_s, i.e.,

$$t_0 = \left(\frac{k^s}{g^s}\right)^{1/2}$$

and the corresponding minimum cost $C_s = 2(k^s g^s)^{1/2}$.

Step 6
The optimal powers-of-two policy corresponds to the value s which minimizes C_s.

3.4.4 New heuristic $H(M)$

Previously, we have introduced different procedures for finding single-cycle policies. Although the Graves and Schwarz approach provides the optimal solution, the number of branches to be examined can be exponential. Thus, if the number of retailers is small this method is very efficient, but when N is significantly large this procedure should not be applied. In contrast, the Schwarz approach does not always provide very good solutions although the computational effort is minimum. The Muckstadt and Roundy procedure is computationally effective but in several problems the solution computed can be improved by dropping the powers-of-two constraint. It is obvious, that these improvements never can be greater than 2%. However, since inventory costs are usually a significant quantity, such improvements can represent, in many instances, an important saving.

We present a new heuristic for finding very effective single-cycle policies with minimum computational effort. In this new approach we force the quotients between the replenishment interval at the warehouse and the replenishment intervals at the retailers to be integers, but not necessarily powers of two. From a practical point of view, this assumption is very important since it provides a more unconstrained way to determine the replenishment intervals at the warehouse and at the retailers.

Consider problem (3.6)-(3.8) but dropping the integrality constraints, i.e., allowing the n_j's to be real values. Then, the optimal replenishment intervals that minimize (3.6) are

$$t_j = \left[\frac{2k_j}{h'_j d_j}\right]^{1/2}, \ j = 0, ..., N \tag{3.16}$$

Taking into account the constraints given in (3.7), it is easy to compute the optimal real values n_j^*'s as

$$n_j^* = \frac{t_0}{t_j}, \ j = 1, ..., N \tag{3.17}$$

Now, we propose a procedure to determine near optimal integers n_j's from the optimal real values n_j^*'s. The first step consists of sorting the retailers so that $i \leq j$ if, and only if, $n_i^* \leq n_j^*$. It is worth noting that this sorting process coincides with the relabeling method suggested by Graves and Schwarz (1977). Without loss of generality, we can assume that $n_1^* \leq n_2^* \leq ... \leq n_N^*$. Then, we can determine the number of deliveries made to retailer j during t_0, i.e., n_j's, $j = 1, ..., N$.

Suppose that n_1, n_2, ..., n_{j-1} have already been fixed to integer values satisfying

$n_i t_i = t_0$, $i = 1, ..., j-1$, and that the new value n_j has to be calculated. To that end, we proceed in the following way.

Let $C_{T;1,...,j-1}$ be the total cost for n_i's fixed, $i = 1, ..., j-1$. Then, taking into account that $n_i t_i = t_0$, for $i = 1, ..., j-1$, $C_{T;1,...,j-1}$ can be expressed as follows

$$C_{T;1,...,j-1} = \frac{1}{t_0}\sum_{i=0}^{j-1} n_i k_i + \frac{t_0}{2}\sum_{i=0}^{j-1}\frac{h'_i d_i}{n_i} + \sum_{i=j}^{N}\left(\frac{k_i}{t_i} + \frac{h'_i d_i t_i}{2}\right) \tag{3.18}$$

where $n_0 = 1$.

Let $t_{0;1,...,j-1}$ represent the optimal replenishment interval at the warehouse assuming that n_1, n_2, ..., n_{j-1} are known integer values. Minimizing (3.18) with respect to t_0 we obtain

$$t_{0;1,...,j-1} = \left[\frac{2\sum_{i=0}^{j-1} n_i k_i}{\sum_{i=0}^{j-1}\frac{h'_i d_i}{n_i}}\right]^{1/2} \tag{3.19}$$

Let n'_j denote the optimal real value that minimizes the total cost C_T assuming that n_1,n_2,...,n_{j-1} have already been fixed to integer values. Then, from (3.7), we can calculate the new optimal real value n'_j, as

$$n'_j = \frac{t_{0;1,...,j-1}}{t_j} \tag{3.20}$$

If $n'_j \geq 1$, we choose the value n_j to be either $n_j = \lfloor n'_j \rfloor$ or $n_j = \lceil n'_j \rceil$, $j = 1, ..., N$. If $n'_j < 1$ we set $n_j = 1$.

It is worth noting that the replenishment interval at the warehouse has changed and now, it depends on the integer values n_1, n_2, ..., n_{j-1}. Therefore, the replenishment intervals t_i's, $i = 1, ..., j-1$ have also changed. These new replenishment intervals can be computed from (3.7). On the other hand, taking into account that for retailers $i = j, ..., N$, the values n_i's have not been fixed to integer numbers, the replenishment intervals for these retailers are still given by (3.16). Thus, the cost function in (3.18) can be rearranged using (3.16) and (3.19) to give

$$C_{T;1,...,j-1} = \left[2\sum_{i=0}^{j-1} n_i k_i \sum_{i=0}^{j-1}\frac{h'_i d_i}{n_i}\right]^{1/2} + \sum_{i=j}^{N}\left[2h'_i d_i k_i\right]^{1/2} \tag{3.21}$$

Once we know how to calculate the new value n'_j, we can introduce the heuristic to compute the near optimal integer values n_j's, $j = 1, ..., N$. The key idea consists of determining, at each step of the algorithm, M values n_j's, that is, $(n_j, n_{j+1}, ..., n_{j+M-1})$. A scheme of the new heuristic denoted by $H(M)$ is given in Algorithm 3.5.

Algorithm 3.5 The heuristic $H(M)$

Step 1
Set $j = 1$.
If $n'_1 < 1$, set $n_1 = 1$, update $j = 2$ and go to Step 2.
Otherwise, go to Step 2.
Step 2
Compute n'_j using (3.20).
Since n_j can be equal either to $\lfloor n'_j \rfloor$ or $\lceil n'_j \rceil$, first set $n_j = \lfloor n'_j \rfloor$ and if $n_j \neq 0$ calculate n'_{j+1} from (3.20). Subsequently, set $n_j = \lceil n'_j \rceil$ and compute the new n'_{j+1}.
This procedure is repeated for each n_i, $i = j+1, ..., j+M-1$. Hence, we obtain 2^M M-tuples. Among these 2^M M-tuples, we choose the one with minimum cost, namely, $(n_j, n_{j+1}, ..., n_{j+M-1})$.
Update $j = j + M$. If $j + M - 1 \leq N$, then, go to Step 2. Otherwise, go to Step 3.
Step 3
If $j < N < j + M$, compute the 2^{N-j+1} $(N - j + 1)$-tuples, and choose the one with minimum cost, e.g., $(n_j, n_{j+1}, ..., n_N)$.

When the procedure concludes, the number of deliveries n_j that should be made to retailer $j = 1, ..., N$, during t_0 has been computed. Using these n_j's, the optimal replenishment interval at the warehouse can be calculated using the following expression

$$t_{0;1,...,N} = \left[\frac{2\sum_{i=0}^{N} n_i k_i}{\sum_{i=0}^{N} \frac{h'_i d_i}{n_i}} \right]^{1/2} \tag{3.22}$$

Thus, the single-cycle policy is completely determined. Once n_1, n_2, ..., n_N and $t_{0;1,...,N}$ are known, the replenishment interval at each retailer is computed using (3.7). Finally, the order quantity at each retailer can be calculated applying the relation $Q_j = d_j t_j$, $j = 1, ..., N$. It is also easy to see that the order quantity at the warehouse is given by $Q_0 = \sum_{j=1}^{N} n_j Q_j$.

An interesting case of the heuristic is obtained for $M = 2$. In this situation, at each step of the algorithm a couple of values (n_j, n_{j+1}) is computed and hence, the heuristic can be rewritten as in Algorithm 3.6.

Algorithm 3.6 The heuristic $H(2)$

Step 1
Set $j = 1$.
If $n'_1 < 1$, set $n_1 = 1$, update $j = 2$ and go to Step 2.
Otherwise, go to Step 2.
Step 2
If $j < N$, compute n'_j using (3.20).
Set $n_j = \lfloor n'_j \rfloor$. If $n_j \neq 0$, calculate n'_{j+1} from (3.20). Then, taking into account (3.21) determine

$$C_{T;1} = C_{T;1,\ldots,j+1}, \text{ assuming that } n_j = \lfloor n'_j \rfloor \text{ and } n_{j+1} = \lfloor n'_{j+1} \rfloor$$

and

$$C_{T;2} = C_{T;1,\ldots,j+1}, \text{ assuming that } n_j = \lfloor n'_j \rfloor \text{ and } n_{j+1} = \lceil n'_{j+1} \rceil.$$

Now, set $n_j = \lceil n'_j \rceil$ and compute the new n'_{j+1}. Then, calculate

$$C_{T;3} = C_{T;1,\ldots,j+1}, \text{ assuming that } n_j = \lceil n'_j \rceil \text{ and } n_{j+1} = \lfloor n'_{j+1} \rfloor$$

and

$$C_{T;4} = C_{T;1,\ldots,j+1}, \text{ assuming that } n_j = \lceil n'_j \rceil \text{ and } n_{j+1} = \lceil n'_{j+1} \rceil.$$

Choose n_j and n_{j+1} to be the values associated to the minimum of the four costs.
Notice that if $\lfloor n'_j \rfloor = 0$, only $C_{T;3}$ and $C_{T;4}$ are computed.
Update $j = j + 2$. If $j < N$, then go to Step 2. Otherwise, go to Step 3.
Step 3
If $j = N$, calculate n'_N using (3.20). Compute $C_{T;1} = C_{T;1,\ldots,N}$ with $n_N = \lfloor n'_N \rfloor$ and $C_{T;2} = C_{T;1,\ldots,N}$ with $n_N = \lceil n'_N \rceil$.
If $C_{T;1} < C_{T;2}$, set $n_N = \lfloor n'_N \rfloor$. Otherwise, $n_N = \lceil n'_N \rceil$. Stop.

As we will show in the computational results, the heuristic $H(2)$ computes very effective single-cycle policies. In addition, this procedure runs in $O(N \log N)$ time. Note that we first calculate the real values n^*_j, $j = 1, ..., N$, using (3.17). This can be done in $O(N)$ time. Once the real values n^*_j's have been obtained, they should be sorted. This operation is implemented in $O(N \log N)$ time. Also note that summations $\sum_{i=0}^{j-1} n_i k_i$, $\sum_{i=0}^{j-1} h'_i d_i / n_i$ and $\sum_{i=j}^{N} (2h'_i d_i k_i)^{\frac{1}{2}}$ in (3.21), can be stored in memory. Taking this into account, $C_{T;1,\ldots,j}$ can be calculated in $O(1)$. Hence, all operations that are carried out in the algorithm can be implemented in O(N) time. Therefore, the computational complexity of the heuristic $H(2)$ is O($N \log N$).

In the computational experience we will show that if we use greater values of M the solutions obtained can be better. However, as the value of M increases so does the running times of the heuristic.

3.4.5 Numerical example

In order to illustrate the different solution methods for computing single-cycle policies we are solving a one-warehouse five-retailer system with the data given in Table 3.2.

Table 3.2: Input data for an instance of the one-warehouse five-retailer problem

	d_j	k_j	h_j	h'_j
Retailer 1	993	202	183	172
Retailer 2	304	283	54	43
Retailer 3	542	144	389	378
Retailer 4	859	408	509	498
Retailer 5	478	84	452	441
Warehouse	3176	40	11	11

The Schwarz heuristic

The first step in the Schwarz heuristic consists of computing t_0. Following the scheme introduced in Algorithm 3.1 and using (3.9) and (3.10) we have $t_1^* = 0.0486$, $t_2^* = 0.2080$, $t_3^* = 0.0375$, $t_4^* = 0.0437$, $t_5^* = 0.0282$, and $t_0^* = 0.0478$. By virtue of (3.11) we obtain $J = \{1, 2\}$ and, from (3.12) it follows $t_0 = 0.0692$.

Once t_0 is obtained, we proceed to calculate the number of replenishments at each retailer during t_0, i.e., n_j, $j = 1, ..., 5$. Minimizing (3.13) for each retailer, the following n_j's are obtained

$$(n_1 = 2, n_2 = 1, n_3 = 2, n_4 = 2, n_5 = 3)$$

Taking into account that $n_j t_j = t_0$, the replenishment intervals at the retailers are easy to compute

$$(t_1 = 0.0346, t_2 = 0.0692, t_3 = 0.0346, t_4 = 0.0346, t_5 = 0.0230)$$

The average cost for this policy is 48080.3660 $\$/time\ unit$.

The Graves and Schwarz procedure

This procedure computes optimal single-cycle policies using a branch and bound search of the n_j's. First of all, the retailers must be sorted so that retailer $i <$ retailer j if, and only if, $h'_i d_i/k_i < h'_j d_j/k_j$. Thus, in this example, retailer 2 < retailer 1 < retailer 4 < retailer 3 < retailer 5.

The initial feasible solution is calculated from (3.14), and the following values are obtained

$$(n_1 = 2, n_2 = 1, n_3 = 3, n_4 = 2, n_5 = 3)$$

The average cost for this policy is 46518.7937 $\$/time\ unit$. It is worth noting that this initial solution can also be considered as a heuristic solution.

Now, Graves and Schwarz improve this solution using the branch and bound scheme. Accordingly, the first improved solution is

$$(n_2 = 1, n_1 = 2, n_4 = 3, n_3 = 3, n_5 = 4)$$

with average cost 46463.3164 $\$/time\ unit$.

And the second proposed solution is

$$(n_2 = 1, n_1 = 3, n_4 = 3, n_3 = 4, n_5 = 5)$$

with average cost 46336.4603 $\$/time\ unit$. Since there is no other solution better than this, the optimal single-cycle solution is

$$(n_1 = 3, n_2 = 1, n_3 = 4, n_4 = 3, n_5 = 5)$$

Substituting these n_j's into (3.22), it follows that the replenishment interval at the warehouse is $t_0 = 0.1359$ *time units*. Besides, the replenishment intervals at the retailers computed using (3.7) are

$$(t_1 = 0.0453, t_2 = 0.1359, t_3 = 0.0339, t_4 = 0.0453, t_5 = 0.0271)$$

The Muckstadt and Roundy approach

The Muckstadt and Roundy method is confined to single-cycle powers-of-two policies. Their procedure consists of first solving the problem ignoring the powers of two constraints, and then, rounding off the replenishment intervals to get a feasible, single-cycle powers-of-two policy. Recall that solving the relaxed problem is equivalent to classifying the retailers and the warehouse in clusters. Using Algorithm 3.3 the following clusters are obtained

$$\left(C^0 = \{0, 2\}, C^1 = \{1\}, C^3 = \{3\}, C^4 = \{4\}, C^5 = \{5\}\right)$$

and the replenishment intervals associated with each cluster are

$$(t_0^* = 0.1160, t_1^* = 0.0486, t_3^* = 0.0375, t_4^* = 0.0437, t_5^* = 0.0282)$$

Now, these times must be rounded off to powers of two multiples of the base planning period T_L. We assume that T_L is variable because the solutions obtained for this case are better than those computed when T_L is considered fixed. Then, following Algorithm 3.4 we can compute the optimal T_L. For this example, $T_L = 0.0929$, which yields the following replenishment intervals

$$(t_0 = 0.0929, t_1 = 0.0464, t_2 = 0.0929, t_3 = 0.0464, t_4 = 0.0464, t_5 = 0.0232)$$

Thus, the values n_j's have changed to be

$$(n_1 = 2, n_2 = 1, n_3 = 2, n_4 = 2, n_5 = 4)$$

and now the average total cost is 46664.2381 $\$/time\ unit$.

The new heuristic $H(2)$

We are solving the problem using the new heuristic $H(2)$. This approach starts computing the optimal real values n_j^*'s using (3.17). For this example the n_j^*'s are

$$(n_1^* = 0.9839, n_2^* = 0.2300, n_3^* = 1.2763, n_4^* = 1.0957, n_5^* = 1.6950)$$

Next, the retailers must be sorted so that $i \leq j$ if, and only if, $n_i^* \leq n_j^*$. Accordingly, retailer 2 $<$ retailer 1 $<$ retailer 4 $<$ retailer 3 $<$ retailer 5. Thus, we relabel the retailers as follows: retailer 1 = retailer 2, retailer 2 = retailer 1, retailer 3 = retailer 4, retailer 4 = retailer 3 and retailer 5 = retailer 5. Therefore, we obtain $n_1^* = 0.2300$, $n_2^* = 0.9839$, $n_3^* = 1.0957$, $n_4^* = 1.2763$ and $n_5^* = 1.6950$.

Since $n_1' = n_1^* < 1$, set $n_1 = 1$, $j = 2$ and go to Step 2 of Algorithm 3.6.

Using (3.20) we compute $n_2' = 2,3853$. Then, n_2 should be equal to 2 or 3. If we set $n_2 = 2$, the value n_3' calculated using (3.20) is 2.3906; otherwise, n_3' is equal to 3.0469. Now, the cost for the four couples of values are computed using (3.21).

$$\begin{aligned}
C_{T;1} &= C_{T;1,\ldots,3} \text{ (with } n_2 = 2, n_3 = 2) = 46370.1199\\
C_{T;2} &= C_{T;1,\ldots,3} \text{ (with } n_2 = 2, n_3 = 3) = 46449.3432\\
C_{T;3} &= C_{T;1,\ldots,3} \text{ (with } n_2 = 3, n_3 = 3) = 46280.2355\\
C_{T;4} &= C_{T;1,\ldots,3} \text{ (with } n_2 = 3, n_3 = 4) = 46575.9470
\end{aligned}$$

Taking these results into account, we set $n_2 = 3$ and $n_3 = 3$.

Repeating the above process for n_4 and n_5, we determine all values n_i's. The new value n_4' computed using (3.20) is 3.5180. Then, n_4 is equal to 3 or 4. If we set $n_4 = 3$, the new value n_5' is 4.5322; otherwise, n_5' is equal to 4.7879. Depending on these values we obtain the following costs

$$\begin{aligned}
C_{T;1} &= C_{T;1,\ldots,5} \text{ (with } n_4 = 3, n_5 = 4) = 46399.6333\\
C_{T;2} &= C_{T;1,\ldots,5} \text{ (with } n_4 = 3, n_5 = 5) = 46384.1940\\
C_{T;3} &= C_{T;1,\ldots,5} \text{ (with } n_4 = 4, n_5 = 4) = 46415.5558\\
C_{T;4} &= C_{T;1,\ldots,5} \text{ (with } n_4 = 4, n_5 = 5) = 46336.4603
\end{aligned}$$

Since C_4 is the smallest cost, we set $n_4 = 4$ and $n_5 = 5$. Hence, using the initial labels of the retailers, the single-cycle solution is

$$(n_1 = 3, n_2 = 1, n_3 = 4, n_4 = 3, n_5 = 5)$$

Substituting these n_j's into (3.22), it follows that $t_0 = 0.1359$ *time units*.

Finally, the replenishment intervals at the retailers computed using (3.7) are

$$(t_1 = 0.0453, t_2 = 0.1359, t_3 = 0.0339, t_4 = 0.0453, t_5 = 0.0271)$$

The average cost incurred by this policy is 46336.4603 $/*time unit*. Notice that for this example the new heuristic $H(2)$ provides a solution better than those given by the Schwarz heuristic and the Muckstadt and Roundy approach. In addition, this policy is also better than the initial solution of the Graves and Schwarz procedure.

3.4.6 Computational results: $H(M)$ versus previous methods

In order to analyze the effectiveness of the different procedures for computing single-cycle policies, we have tested each approach considering that the number of retailers, N, is multiple of 5 ranging in $[5, 100]$. The parameters k_0 and h_0 have been chosen from a uniform distribution varying on $[1, 100]$. Moreover, given h_0, the value h_j is selected from a uniform distribution on $[h_0, 500]$. Finally, we select k_j and d_j from uniform distributions on $[1, 500]$ and $[1, 1000]$, respectively.

For each N, we carry out 100 instances and we report the results in Tables 3.3-3.11. Regarding the Muckstadt and Roundy method, we compute the solutions assuming that T_L is variable since these solutions are more efficient than those generated when T_L is fixed.

For notation convenience, let C_S, C_{MR}, $C_{H(M)}$, $C_{GS(I)}$ and C^* denote the cost of the Schwarz approach, the cost of the Muckstadt and Roundy method considering that T_L is variable, the cost of the new heuristic $H(M)$, the cost of the initial solution of the Graves and Schwarz algorithm and the optimal cost computed by the Graves and Schwarz method, respectively.

The first computational results in Tables 3.3-3.9 are related to the heuristic $H(2)$. In Table 3.3, we compare the different approaches with the optimal Graves and Schwarz procedure. The first column represents the number of retailers. The next three columns contain the number of instances where the three approaches reach the optimal solution. The last column shows the number of instances where the initial solution provided by the Graves and Schwarz method coincides with the optimal. From the results in Table 3.3, we can conclude that, on average, the heuristic $H(2)$ provides the optimal solution in 34.35% of the instances. The initial solution of the Graves and Schwarz procedure coincides with the optimal in 31.90%. In contrast, the Schwarz procedure reaches the optimal policy only in 0.60% and the Muckstadt and Roundy approach in 10.90%.

In Table 3.4 the heuristic $H(2)$ is compared with the Schwarz method. The first column represents the number of retailers. The other columns contain the number of instances which satisfy the condition expressed in the top of each column. For example, the second column in Table 3.4 shows the number of problems where both, the new heuristic $H(2)$ and the Schwarz method, provide the same solution. All other columns must be interpreted in the same way.

Notice that the single-cycle policies provided by $H(2)$ are better than those given by the Schwarz heuristic in 99.30%. In addition, both methods compute the same solution in 0.7%.

Table 3.3: Comparison between the different heuristics and the optimal Graves and Schwarz procedure. In 34,35% of the examples the new heuristic $H(2)$ reaches the optimal solution

N	$C_S = C^*$	$C_{MR} = C^*$	$C_{H(2)} = C^*$	$C_{GS(I)} = C^*$
5	9	55	94	91
10	3	37	79	76
15	0	29	72	66
20	0	20	63	57
25	0	16	51	47
30	0	12	45	38
35	0	9	37	34
40	0	3	37	32
45	0	4	29	25
50	0	4	27	24
55	0	3	21	18
60	0	4	21	17
65	0	6	20	19
70	0	5	19	17
75	0	2	18	16
80	0	4	13	12
85	0	4	8	8
90	0	1	12	12
95	0	0	10	9
100	0	0	11	8
Average	0.60	10.90	34.35	31.90

Table 3.4: Comparison between the new heuristic $H(2)$ and the Schwarz method. In 99.3% of the instances the heuristic $H(2)$ provides better solutions than the Schwarz approach

N	$C_{H(2)} = C_S$	$C_{H(2)} < C_S$	$C_{H(2)} > C_S$
5	10	90	0
10	4	96	0
> 15	0	100	0
Average	0.70	99.30	0

Additionally, in Table 3.5 we compare the new heuristic $H(2)$ with the Muckstadt and Roundy approach. From the results, we can conclude that when N is small, in most cases both methods provide the same solution. Hence, we could use either the heuristic $H(2)$ or the Muckstadt and Roundy method to compute single-

Table 3.5: Comparison of the new heuristic $H(2)$ with the Muckstadt and Roundy approach. In 74,05% of the examples the heuristic $H(2)$ gives better solutions than the Muckstadt and Roundy approach

N	$C_{H(2)} = C_{MR}$	$C_{H(2)} < C_{MR}$	$C_{H(2)} > C_{MR}$
5	53	42	5
10	32	58	10
15	26	63	11
20	16	71	13
25	13	70	17
30	9	80	11
35	5	73	22
40	1	76	23
45	1	79	20
50	3	74	23
55	1	80	19
60	2	77	21
65	3	72	25
70	2	78	20
75	0	82	18
80	1	77	22
85	0	87	13
90	1	79	20
95	0	81	19
100	0	82	18
Average	8.45	74.05	17.50

cycle policies. However, as the number of retailers increases so does the number of instances where the heuristic provides better solutions than the Muckstadt and Roundy approach. Therefore, in this case, it seems more convenient to use the new heuristic $H(2)$ to solve the problem. In addition, the solutions provided by the Muckstadt and Roundy procedure are worse than those computed by the heuristic $H(2)$ in 74.05%, whereas in 8.45% both solutions are equal. Hence, the policies computed by $H(2)$ are worse than those given by the Muckstadt and Roundy approach only in 17.50%.

Finally, in Table 3.6 we compare the heuristic $H(2)$ with the initial solution of the Graves and Schwarz algorithm. Notice that in this case, the new heuristic $H(2)$ always provides solutions that are equal to or better than the initial solutions.

We also measure the difference between the costs of the policies provided by the Muckstadt and Roundy approach, C_{MR}, and those provided by the new heuristic, $H(2)$. Accordingly, in case of $C_{MR} < C_{H(2)}$ we compute

Table 3.6: Comparison of the new heuristic $H(2)$ with the initial solution of the Graves and Schwarz procedure. The heuristic $H(2)$ always provides policies that are equal to or better than the initial solutions of the Graves and Schwarz procedure

N	$C_{H(2)} = C_{GS(I)}$	$C_{H(2)} < C_{GS(I)}$
5	96	4
10	90	10
15	87	13
20	93	7
25	87	13
30	79	21
35	85	15
40	75	25
45	84	16
50	87	13
55	87	13
60	81	19
65	83	17
70	89	11
75	82	18
80	82	18
85	78	22
90	77	23
95	74	26
100	83	17
Average	83.95	16.05

$$Gap_{(H(2)-MR)} = \frac{C_{H(2)} - C_{MR}}{C_{MR}} \times 100$$

Otherwise, if $C_{MR} \geq C_{H(2)}$, we calculate

$$Gap_{(MR-H(2))} = \frac{C_{MR} - C_{H(2)}}{C_{H(2)}} \times 100$$

The results are shown in Table 3.7. The second and third columns contain for a given number of retailers, N, the average percentages $\overline{Gap}_{(H(2)-MR)}$ and $\overline{Gap}_{(MR-H(2))}$ of 100 instances, respectively. This difference is also computed between the cost of the initial solution of the Graves and Schwarz procedure, $C_{GS(I)}$, and the cost of the new heuristic $H(2)$, $C_{H(2)}$. The last column shows the average percentage $\overline{Gap}_{(GS(I)-H(2))}$. Notice that for all instances $C_{H(2)} \leq C_{GS(I)}$ and then, we always compute $Gap_{(GS(I)-H(2))} = (C_{GS(I)} - C_{H(2)})100/C_{H(2)}$.

Table 3.7: Percentage difference between the costs of the policies provided by the Muckstadt and Roundy approach and by the new heuristic $H(2)$, and between the costs of the initial solutions of the Graves and Schwarz procedure and the costs given by the heuristic $H(2)$. On average, the percentage improvement of the heuristic $H(2)$ versus the Muckstadt and Roundy approach is 0.275

N	$\overline{Gap}_{(H(2)\ -MR)}$	$\overline{Gap}_{(MR-H(2)\ )}$	$\overline{Gap}_{(GS(I)\ -H(2)\ )}$
5	0.330	0.403	0.324
10	0.191	0.380	0.308
15	0.180	0.304	0.136
20	0.203	0.309	0.106
25	0.142	0.328	0.220
30	0.170	0.274	0.064
35	0.135	0.263	0.191
40	0.178	0.253	0.197
45	0.139	0.270	0.122
50	0.157	0.290	0.041
55	0.167	0.238	0.063
60	0.172	0.278	0.050
65	0.150	0.246	0.031
70	0.155	0.242	0.053
75	0.129	0.210	0.075
80	0.167	0.253	0.049
85	0.122	0.254	0.035
90	0.147	0.225	0.058
95	0.105	0.234	0.048
100	0.083	0.254	0.053
Average	0.161	0.275	0.111

Notice that when the Muckstadt and Roundy approach gives better policies than the heuristic $H(2)$, the average of $Gap_{(H(2)\ -MR)}$ is equal to 0.161. However, when $C_{H(2)}\ < C_{MR}$, the average of $Gap_{(MR-H(2)\ )}$ increases to 0.275. Therefore, when the new heuristic $H(2)$ gives better policies than those provided by the Muckstadt and Roundy approach, the difference between the costs is greater than in the opposite case.

In order to compare the effectiveness of the different methods, we calculate for each instance the percentage difference between the cost provided by each heuristic and the optimal cost

$$E = \frac{(\text{Cost of the heuristic policy - } C^*)}{C^*} \times 100.$$

Table 3.8: Average percentage cost differences and maximum percentage differences

N	S		MR		$H(2)$		$GS(I)$	
	$\overline{E}$	$E_{\max}$	$\overline{E}$	$E_{\max}$	$\overline{E}$	$E_{\max}$	$\overline{E}$	$E_{\max}$
5	1.785	10.638	0.267	1.332	0.016	0.772	0.026	0.772
10	2.136	13.106	0.247	1.302	0.029	0.684	0.061	0.784
15	2.323	7.025	0.255	1.517	0.046	0.846	0.057	1.046
20	2.342	10.710	0.259	1.420	0.049	0.603	0.053	0.603
25	2.480	9.813	0.260	1.069	0.112	1.752	0.121	1.752
30	2.401	6.294	0.264	1.534	0.062	1.047	0.095	1.618
35	2.266	10.173	0.265	1.551	0.143	1.003	0.160	1.003
40	2.643	12.569	0.283	1.295	0.114	1.637	0.134	1.637
45	2.744	8.496	0.300	1.375	0.126	0.791	0.141	0.949
50	2.488	9.097	0.272	1.492	0.126	1.310	0.145	1.410
55	2.375	8.780	0.236	1.068	0.114	1.585	0.119	1.585
60	2.234	8.531	0.263	1.509	0.119	1.336	0.137	1.336
65	2.543	8.170	0.278	1.215	0.123	0.862	0.131	0.862
70	2.233	8.862	0.235	1.442	0.124	0.795	0.132	0.795
75	2.382	8.526	0.295	1.507	0.110	0.803	0.124	0.805
80	2.366	8.569	0.280	1.318	0.150	1.217	0.156	1.217
85	2.722	9.708	0.315	1.495	0.131	0.658	0.139	0.658
90	2.238	9.077	0.239	1.371	0.138	0.927	0.144	0.927
95	2.817	6.233	0.327	1.012	0.116	0.540	0.135	0.540
100	2.552	7.359	0.253	1.231	0.108	0.625	0.139	0.625

We report in Table 3.8 the average percentage differences, $\overline{E}$, and the maximum percentage difference, $E_{\max}$. These results show that the heuristic $H(2)$ provides, on average, closer solutions to the optimal than the initial solution provided by the Graves and Schwarz procedure and than the solution given by the Muckstadt and Roundy approach. Hence, using the new heuristic $H(2)$ seems to be an appropriate way to compute single-cycle policies.

To check the difference between the costs of each procedure, we show in Table 3.9 the values E_{MR}, $E_{H(2)}$ and $E_{GS(I)}$, given by $(C_{MR}-C^*)100/C^*$, $(C_{H(2)}-C^*)100/C^*$ and $(C_{GS(I)}-C^*)100/C^*$, for a collection of instances. It is important to emphasize that the computational running times for all instances are zero. However, for the Graves and Schwarz exact procedure these times are significantly large. In some cases the Graves and Schwarz procedure can take several minutes to solve the problem.

Table 3.9: Comparison among costs obtained using the different approaches

N	E_{MR}	$E_{H(2)}$	$E_{GS(I)}$	N	E_{MR}	$E_{H(2)}$	$E_{GS(I)}$
10	0.05	0.00	0.03	60	0.08	0.06	0.11
10	0.23	0.00	0.00	60	0.16	0.24	0.32
10	1.30	0.00	0.00	60	0.19	0.00	0.00
20	0.67	0.15	0.15	70	0.13	0.05	0.05
20	0.61	0.00	0.04	70	0.15	0.00	0.00
20	0.88	0.00	0.04	70	0.34	0.41	0.41
30	1.53	0.00	0.00	80	0.95	0.05	0.05
30	0.32	1.04	1.04	80	1.31	0.06	0.06
30	0.23	0.00	1.61	80	0.19	0.40	0.40
40	0.19	0.00	0.05	90	0.20	0.17	0.17
40	0.15	0.23	0.23	90	1.37	0.13	0.14
40	0.00	0.00	0.04	90	0.23	0.00	0.00
50	0.59	0.07	0.07	100	0.10	0.07	0.07
50	0.28	0.59	0.71	100	0.14	0.19	0.21
50	0.02	0.00	0.04	100	0.36	0.16	0.16

Results obtained for different values of M

In Tables 3.3-3.9 we only have considered the heuristic $H(2)$. In order to compare the policies provided by the heuristic for other values of M, we carry out 100 instances for each number of retailers $N = 10, 15, 20, 25, 30$ and 35, and we solve each problem for all possible values of M, that is, $M = 2, ..., N$. A summary of the results are shown in Tables 3.10 and 3.11.

The first column in Table 3.10 contains the value of M. For some values of N, we show in the next columns two values. Value "a" denotes the number of instances where the heuristic $H(M)$ gives better policies than the initial policy given by the Graves and Schwarz method. On the other hand, value "b" represents the number of instances where the heuristic $H(M)$ provides worse policies than the Muckstadt and Roundy procedure. As it is expected, the best results are obtained when M is near to N. However, taking into account that the computational effort increases exponentially with the value of M, when the number of retailers is considerable large it is not suitable to set $M = N$. Therefore, it is interesting to analyze if there is another value of M which provides so good results as those obtained when $M = N$. From Table 3.10, it seems that apart from the values near to N, the best value of M belongs to the set $\{\lfloor N/2 \rfloor - 1, \lfloor N/2 \rfloor, \lfloor N/2 \rfloor + 1\}$. Remark that when the value of M increases, the solutions obtained are not always better. For example, for $N = 30$, on average, it is preferable to set $M = 15$ than $M = 20$.

Table 3.10: Results obtained with the heuristic for different values of M. Value "a" denotes the number of instances where the new heuristic gives better policies than the initial policy given by the Graves and Schwarz method. Letter "b" represents the number of instances where the heuristic provides worse policies than the Muckstadt and Roundy procedure

M	$N=10$		$N=15$		$N=20$		$N=25$		$N=30$		$N=35$	
	a	b	a	b	a	b	a	b	a	b	a	b
2	10	6	10	13	9	11	10	23	18	21	9	15
3	14	3	21	10	15	12	25	15	26	20	24	15
4	17	5	22	10	21	7	28	16	36	17	27	13
5	18	4	23	7	21	8	26	17	41	15	25	14
6	18	3	28	6	26	7	33	10	47	15	34	12
7	21	2	28	6	35	4	35	11	46	8	37	9
8	19	3	26	7	29	3	37	7	54	5	39	8
9	21	0	26	6	33	8	36	9	48	12	44	7
10	21	0	27	4	31	5	34	11	46	11	40	10
11			28	3	32	2	40	2	51	6	44	6
12			31	1	31	4	44	1	57	4	43	5
13			35	1	34	4	44	3	55	3	44	2
14			38	0	36	4	42	8	54	2	45	4
15			38	0	34	3	38	6	56	3	44	6
16					33	2	39	7	56	3	49	4
17					36	2	39	11	56	5	52	2
18					38	2	37	9	51	11	52	3
19					39	0	39	9	49	10	52	3
20					39	0	37	10	50	10	46	5
21							42	4	51	6	45	5
22							45	2	55	6	45	3
23							48	2	54	6	44	2
24							49	0	59	3	44	5
25							49	0	59	3	44	4
26									61	2	46	2
27									61	1	47	4
28									61	1	48	3
29									62	1	46	4
30									62	1	47	5
31											48	4
32											52	2
33											52	3
34											54	0
35											54	0

However, when N is a large value, it is not computationally efficient to set

$M \simeq N/2$. Indeed, from the computational experience, we can conclude that when $N > 50$, M should be a value below 25. Furthermore, if M is set to be greater than 25, reducing costs is not significant, whereas the running times increase drastically.

Finally, in order to test differences between the costs obtained using the heuristic $H(M)$ for different values of M, we present in Table 3.11 a collection of several instances generated for $M = 2, 8, 14, 20$. We also include the initial solution of the Graves and Schwarz procedure. In the last column we show the reduction percentage obtained using $C_{H(M)}$ instead of $C_{H(2)}$, that is, $\% = (C_{H(M)} - C_{H(2)})100/C_{H(2)}$, with M the maximum value studied. Notice that the maximum difference between the cost of the policy provided by the heuristic $H(2)$ and the cost of the policy given by the heuristic $H(M)$ with M equals to the maximum value, is only 0.51%. In addition, as the value of M increases so does the running times of the heuristic $H(M)$. In particular, for the instances in Table 3.11 the computational running times vary from 0 to 73.46 seconds.

Table 3.11: Costs obtained using the initial solution of the Graves and Schwarz procedure and the heuristic for different values of M. The reduction percentage obtained using the maximum value of M studied instead of $M = 2$ is denoted by %

N	$C_{GS(I)}$	$C_{H(2)}$	$C_{H(8)}$	$C_{H(14)}$	$C_{H(20)}$	%
10	121019.96	120937.18	120846.68			0.07
10	109992.96	109841.40	109841.40			0.00
20	187543.14	187439.78	187234.00	187234.00	187234.00	0.10
20	153747.17	153747.17	153747.17	153747.17	153747.17	0.00
30	330635.93	330318.87	328921.59	328921.59	329702.87	0.18
30	261138.40	261134.87	260647.68	260647.68	260647.68	0.18
40	280155.78	280155.78	280155.78	280155.78	280141.96	0.004
40	332610.25	332584.59	332610.25	332584.59	332584.59	0.00
50	456090.62	456090.62	456090.62	456090.62	455923.12	0.03
50	482807.781	482743.43	482743.43	482743.43	482743.43	0.00
60	393708.03	393679.31	393570.93	393290.71	393283.37	0.10
60	724859.37	724859.37	723057.37	723057.37	722897.75	0.27
70	595168.62	594742.68	594196.81	594742.68	594196.81	0.09
70	619112.81	619112.81	619112.81	617641.00	615957.62	0.51
80	624209.60	624209.60	624036.25	623308.37	623352.93	0.13
80	661880.56	661822.31	659931.12	660286.56	659931.12	0.28
90	945160.18	944772.75	944441.31	944441.31	944303.87	0.04
90	708407.43	706931.93	706220.93	705641.12	705623.93	0.18
100	691559.75	691559.75	690786.50	690584.43	689041.68	0.36
100	910268.31	910085.25	909703.75	909719.06	905639.06	0.49

To this point we have addressed the one-warehouse N-retailer problem assuming that the decision system is centralized. In particular, we have focused on the centralized policies with common replenishment intervals at the retailers and on the single-cycle policies. However, the problem can be also solved assuming that the decision system is decentralized. The following section is devoted to this case, that is, we assume that each retailer orders independently, as does the warehouse.

3.5 Decentralized policies

Let now suppose that there is independence among the warehouse and the retailers, that is, at each location there is a decision-maker. For example, if each retailer belongs to a different firm, each one is interested in minimizing its own cost independently. In this case, we first determine the order quantities at the retailers, and then, we compute the shipment schedule at the warehouse. Accordingly, since the retailers follow an EOQ pattern, the total cost at retailer j can be easily obtained as $C_j = k_j/t_j + h_j d_j t_j/2$. Moreover, the optimal order quantities, replenishment intervals, and costs are given by the following expressions

$$t_j^* = \sqrt{\frac{2k_j}{d_j h_j}},\ j = 1, ..., N$$
$$Q_j^* = d_j t_j^*,\ j = 1, ..., N$$
$$C_j^* = \sqrt{2D_j k_j h_j},\ j = 1, ..., N$$

Since each retailer places orders according to an EOQ pattern, the replenishment intervals are not related. Therefore, the warehouse behaves as an inventory system with time-varying demand. When the demand rate varies with time, we can no longer assume that the best strategy is to always order the same order quantity. In fact, this will seldom be the case. Hence, the warehouse does not follow the classical sawtooth pattern of the EOQ model. Indeed, we now have to use the demand information at the retailers, over a finite planning horizon, to determine the appropriate order quantities at the warehouse.

Following Schwarz (1973), deliveries should be made to the warehouse only when the warehouse and at least one retailer have zero inventory. Note that the optimal replenishment intervals at each retailer are real values. Therefore, we cannot assure that a point in time exists where all retailers order simultaneously. In this case, the number of periods of the demand vector at the warehouse is not finite and then, the problem cannot be solved by the Wagner and Whitin (1958) or the Wagelmans et al. algorithm (1992). Under this assumption, we propose an approach to overcome this problem. The idea consists of either truncating or rounding up to rational times the real replenishment intervals. It is clear that the solution provided by this method is not the real optimal plan but it is quite a good approximation. Furthermore, in practice, it does not make sense to work with irrational times.

Let B be the set of rational times where any retailer orders to the warehouse, that is, $B = \{t \in \mathbb{Q} \mid t = nt_j, \text{ for some } n \in \mathbb{N} \text{ and } j \in (1, 2, \ldots, N)\}$, where each $t_j = a_j/b_j$, $j = 1, ..., N$, is obtained by rounding or truncating the optimal replenishment interval at each retailer. Moreover, following the characterization of the *basic policies* stated by Schwarz (1973), each value in B represents a candidate instant where the warehouse can place an order.

Since the optimal replenishment intervals have been transformed into rational values, a set $S = \{n_1, n_2, \ldots, n_N\}$ of integer values can always be found such that $n_1 t_1 = n_2 t_2 = \cdots = n_N t_N = \tau_0$, or, in other words

$$n_1 \frac{a_1}{b_1} = n_2 \frac{a_2}{b_2} = \cdots = n_N \frac{a_N}{b_N} = \tau_0 \tag{3.23}$$

Recall that τ_0, or an integer multiple of τ_0, represents the time horizon for the warehouse.

Note that (3.23) represents a linear equations system with n variables and $n-1$ equations. In order to ensure the integrality of the n_j's, set

$$n_N = b_N a_{N-1} a_{N-2} \cdots a_2 a_1 \tag{3.24}$$

Therefore, the remaining integer values in (3.23) are obtained by

$$n_j = n_N \frac{a_N}{b_N} \frac{b_j}{a_j}, \; j = 1, \ldots, N-1 \tag{3.25}$$

Finally, each n_j's is divided by the $m.c.d.(n_1, n_2, \ldots, n_N)$. Then, the values thus obtained are considered as the new values n_j's and τ_0 can be calculated by (3.23). Also, these values can be used to determine the number P_w of different instants in time over τ_0 where the warehouse receives an order from some retailer. First of all, the values n_j's must be clustered in the following way. Those n_i's that are powers of some value n_j are included in a cluster. That is, those n_i's which verify $n_i = n_j^k$, for some k integer belong to the same cluster. If there are not values n_i's that are powers of some n_j, then this cluster contains only the value n_j. Let R be the number of clusters. For each cluster j, we choose the highest power value n'_j as the representative element. That is, $n'_j = n_j^k$ being k the highest power. Then, set $m_j = n'_j - 1$ for $j = 1, \ldots, R$. The integer m_j represents the number of equidistant points over τ_0 needed to get n'_j intervals. Theorem 3.1 states when orders are placed to the warehouse. The proof of Theorem 3.1 requires the following lemma.

Lemma 3.2 *Let t_1 and t_2 be two rational numbers and let n_1 and n_2 be integer numbers such that $n_1 t_1 = n_2 t_2 = \tau_0$. Then, the number of points in $(0, \tau_0)$ where $it_1 = jt_2$ for $i = 1, \ldots, n_1$ and $j = 1, \ldots, n_2$ is given by the m.c.d.$(n_1, n_2) - 1$.*

The proof of this lemma is straightforward.

Theorem 3.1 *The number P_w of different instants in time where the warehouse receives an order from some retailer is*

$$P_w = \sum_{i=1}^{R} m_i - \sum_{i=1}^{R-1} \sum_{j=i+1}^{R} (m.c.d.(n'_i, n'_j) - 1) \tag{3.26}$$

Proof

By Lemma 3.2, the double summation in (3.26) represents the points in $(0, \tau_0)$ which have been considered more than once in the first summation. Therefore, P_w stands for the number of different instants in $(0, \tau_0)$ where the warehouse receives an order from some retailer. ■

Once the number of points P_w is obtained, we can generate the demand vector at the warehouse of dimension $P_w + 1$ or a multiple of $P_w + 1$. Since the replenishment intervals have been rounded, the order quantity at each retailer, Q_j^*, has changed to be $Q_j = t_j d_j$. Let J_j be the set of retailers ordering from the warehouse in period j. This set can be used to determine the quantity to be satisfied by the warehouse in period $j = 0, 1, ..., P_w$, in the following way $d_0[j] = \sum_{i \in J_j} Q_i$. This demand vector represents the quantities that the warehouse has to supply. To solve this problem, the Wagner and Whitin algorithm or any of the other techniques currently available can be applied.

In the next section we present a numerical example to illustrate the different approaches introduced in this chapter.

3.6 Numerical example

Let now consider a one-warehouse three-retailer system with the input data given in Table 3.12.

We proceed to calculate the costs provided by the three class of policies introduced in the previous sections: centralized policy with common replenishment intervals at the retailers, single-cycle policy and decentralized policy.

Table 3.12: Input data for an instance of the one-warehouse three-retailer system

	d_j	k_j	h_j	h'_j
Retailer 1	75	42	48	40
Retailer 2	79	100	21	13
Retailer 3	97	28	52	44
Warehouse		37	8	8

3.6.1 Centralized policy with common replenishment intervals

In this case, the retailers place their orders at the same time. Using (3.2) and (3.3), we have $t^* = 0.2004$ time units and $n^* = 0.9482$ and, therefore, $n = 1$. Thus, the retailers and the warehouse place their orders once every $t^* = 0.2004$ time units. The order quantities at the retailers are calculated using (3.4). Accordingly, $Q_1^* = 15.03$ units of item, $Q_2^* = 15.83$ units of item and $Q_3^* = 19.44$ units of item. Then, the order quantity at the warehouse can be computed from (3.5) to give $Q_0^* = 50.30$ units of item. Following this policy the overall cost is 2065.2947 \$/*time unit*.

3.6.2 Single-cycle policy

Now, the retailers can place their orders at different time instants t_j, $j = 1, 2, 3$, subject to the constraint $n_1 t_1 = n_2 t_2 = n_3 t_3 = t_0$, where $n_1, n_2, n_3 \in \mathbb{N}$. Both, the new heuristic and the Muckstadt and Roundy procedure can be applied. We show in Table 3.13 the optimal single-cycle powers-of-two policy given by the Muckstadt and Roundy procedure, and the policy provided by the heuristic $H(3)$. It is worth noting that $H(M)$ provides the best policy when $M = N$. Recall that as the value of M increases so does the running times of the heuristic. However, since now $N = 3$ there is no problem in using $H(3)$.

When the Muckstadt and Roundy procedure is used, the overall cost is 1922.1409 \$/ *time unit.* In contrast, when the new heuristic $H(3)$ is applied, the overall cost is 1906.3500 \$/ *time unit.* Therefore, the single-cycle policy obtained using the heuristic $H(3)$ is better than Muckstadt and Roundy's solution. Obviously, the single-cycle solution is also better than the centralized policy assuming common replenishment intervals.

Table 3.13: Single-cycle policies for the numerical example

	MR			$H(3)$		
	n_i	t_i	Q_i	n_i	t_i	Q_i
Retailer 1	2	0.1441	10.8075	2	0.1608	12.0600
Retailer 2	1	0.2882	22.7678	1	0.3216	25.4064
Retailer 3	2	0.1441	13.9777	3	0.1072	10.3984
Warehouse	1	0.2882	72.3382	1	0.3216	80.7216

3.6.3 Decentralized policy

Using the classical EOQ expressions, we calculate the optimal order quantities and the replenishment intervals at the retailers. Such values are given in Table 3.14.

Table 3.14: Optimal order quantities and replenishment intervals

	Q_i^*	t_i^*
Retailer 1	$\sqrt{\frac{2\cdot 75\cdot 42}{48}} \simeq 11.4564$	$\frac{\sqrt{\frac{2\cdot 75\cdot 42}{48}}}{75} \simeq 0.1527$
Retailer 2	$\sqrt{\frac{2\cdot 79\cdot 100}{21}} \simeq 27.4295$	$\frac{\sqrt{\frac{2\cdot 79\cdot 100}{21}}}{79} \simeq 0.3472$
Retailer 3	$\sqrt{\frac{2\cdot 97\cdot 28}{52}} \simeq 10.2206$	$\frac{\sqrt{\frac{2\cdot 97\cdot 28}{52}}}{97} \simeq 0.1053$

Remark that the replenishment intervals are not rational numbers. For that reason, we round the t_i^*'s to obtain the following values: $t_1 = 0.2 = 2/10$, $t_2 = 0.3 = 3/10$ and $t_3 = 0.1 = 1/10$.

Now, the values n_i's can be calculated using (3.24) and (3.25) to give $n_1 = 60 \cdot 1/10 \cdot 10/2 = 30$, $n_2 = 60 \cdot 1/10 \cdot 10/3 = 20$ and $n_3 = 10 \cdot 3 \cdot 2 = 60$.

Then, we divide the values n_i's by the $m.c.d.(n_1, n_2, n_3) = 10$ obtaining the following results: $n_1 = 3$, $n_2 = 2$ and $n_3 = 6$. After that, the different clusters are calculated. In this case there are three clusters, one for each n_j. Hence, $n'_j = n_j$, $j = 1, 2, 3$.

Using the new replenishment intervals, the order quantities and the costs at each retailer are given in Table 3.15.

Table 3.15: Order quantities and costs at each retailer for the rounded replenishment intervals

	Q_i	C_i
Retailer 1	15.0	570.0000
Retailer 2	23.7	582.1833
Retailer 3	9.7	532.2000

The time horizon at the warehouse is $\tau_0 = n_i t_i = 0.6$. Besides, the number P_w of instants where the warehouse receives an order is

$$P_w = \sum_{i=1}^{3} m_i - \sum_{i=1}^{3-1} \sum_{j=i+1}^{3} (m.c.d.(n_i^{'}, n_j^{'}) - 1) = 5$$

and the time vector $\overline{t_0}$ is

0.0	0.1	0.2	0.3	0.4	0.5

The demand vector at the warehouse $\overline{d_0}$ is given by

48.4	9.7	24.7	33.4	24.7	9.7

Once the demand vector is obtained, the Wagner and Whitin algorithm (1958) or the Wagelmans et al. procedure (1992) provide the optimal order planning for the warehouse. That is,

$$\overline{Q_0} = \boxed{58.1 \mid 0.0 \mid 58.1 \mid 0.0 \mid 34.4 \mid 0.0}\ .$$

Now, the cost at the warehouse is 255.4 $\$/\textit{time unit}$. The overall cost including the costs at the retailers and at the warehouse is 1939.7833 $\$/\textit{time unit}$.

Notice that for this example the single-cycle solution is better than the decentralized policy. However, this is not always the case. In fact, there are instances where the best solution is obtained when the retailers make decisions independently. In particular, the computational experience developed in the next subsection shows that as the number of retailers increases so does the number of instances where the decentralized policies are better than the single-cycle policies.

3.7 Computational results: Single-cycle versus decentralized policies

Before starting with the comparison analysis between the single-cycle policies and the decentralized ones, we should choose the approach to be implemented for computing single-cycle policies. In Section 3.4 we showed that $H(2)$ provides, on average, better policies than the Muckstadt and Roundy approach. Furthermore, the solutions obtained by the heuristic $H(M)$ can be improved by using greater values of M. Concretely, the heuristic $H(M)$ provides the best policies when $M = N$. The inconvenient of setting M equals to N is that the running times of the heuristic increase. However, now the main goal is to compare the costs of the policies provided by the decentralized method with the costs of the single-cycle policies. For that reason, for this computational experience we have used the heuristic $H(N)$.

Next, we proceed to compare the heuristic $H(N)$ with the decentralized approach proposed in Section 3.5. In this analysis, the number of retailers N takes the following values: 2, 3, 4, 5, 6, 7, 8, 9, 10, 15, 20, 25 and 30. The parameters d_j, k_0, h_0 and k_j have been chosen from two different uniform distributions varying on $[1, 100]$ and on $[1, 10]$, respectively. Moreover, given h_0, the value h_j is selected from a uniform distribution on $[h_0 + 1, 101]$ and on $[h_0 + 1, 11]$, respectively. For each problem, one hundred instances were carried out and the results are shown in Table 3.16. The first column represents the number of retailers. The results in the second and third columns are obtained when d_j, k_0, h_0 and k_j are selected from a uniform distribution on $[1, 100]$ and h_j from a uniform distribution on $[h_0{+}1, 101]$. In contrast, the results in the fourth and fifth columns are obtained when d_j, k_0, h_0 and k_j are selected from a uniform distribution on $[1, 10]$ and h_j from a uniform distribution on $[h_0 + 1, 11]$. We denote by C_D the cost provided by the decentralized approach. In particular, the second column collects the number of instances where the decentralized approach provides better solutions than the heuristic $H(N)$. Similarly, the third column shows the number of instances where the single-cycle policies are better than the decentralized strategies. When parameters range in $[1, 100]$, the average number of instances where it is preferable to apply the single-cycle policies is around 45%. On the other hand, when parameters vary on $[1, 10]$, the average number of instances where it is better to follow a single-cycle policy is around 52%. However, these percentages change depending on the number of retailers. For example, for $N = 2$ and considering the first interval, the single-cycle policies are better in 87% of instances. Nevertheless, for $N = 20$ and considering the same interval, the best solution is always given by the decentralized approach.

From Table 3.16, it is easy to see that as the number of retailers increases so does the number of instances where the decentralized policy is better. However, the gap between this number and the one corresponding to the single-cycle policies decreases when the parameters vary in the interval $[1, 10]$. In our opinion, this fact can be explained since the variability of the parameters is reduced from [1,100] to [1,10]. Thus, the reduction of the interval leads the demands and costs at the retailers to be quite similar. For that reason, in some instances the single-cycle policies are better than the decentralized solutions even when $N = 20$.

Table 3.16: Comparison between decentralized and single-cycle policies

N	$d_j, k_0, h_0, k_j \sim U[1, 100]$ $h_j \sim U[h_0 + 1, 101]$		$d_j, k_0, h_0, k_j \sim U[1, 10]$ $h_j \sim U[h_0 + 1, 11]$	
	$C_D < C_{H(N)}$	$C_D > C_{H(N)}$	$C_D < C_{H(N)}$	$C_D > C_{H(N)}$
2	13	87	10	90
3	20	80	17	83
4	43	57	30	70
5	37	63	43	57
6	33	67	37	63
7	50	50	47	53
8	50	50	50	50
9	63	37	37	63
10	43	57	53	47
15	67	33	47	53
20	100	0	70	30
25	100	0	80	20
30	100	0	100	0
Average	55.30	44.70	47.77	52.23

In order to analyze the effect of the parameters on the cost of the policies, a more detailed analysis is required. Accordingly, the number of retailers is fixed to 10 and the parameters are chosen from different uniform distributions, as it is shown in Table 3.17. For each combination, ten problems are tested. The fourth, sixth and eighth columns in Table 3.17, contain the number of instances where the decentralized approach provides better strategies than the single-cycle policies. In contrast, the fifth, seventh and ninth columns show the number of instances where the single-cycle policies are better than the decentralized strategies.

Table 3.17 shows that as the interval of the replenishment cost at the warehouse increases so does the number of instances where the single-cycle policies are better than the decentralized strategies. On the other hand, when the costs at the retailers are significantly greater than the costs at the warehouse, it is preferable that the retailers make decisions independently.

Table 3.17: Comparison between decentralized and single-cycle policies when k_0, h_0 are selected from the uniform distributions: $U_1 \equiv U[1, 10]$, $U_2 \equiv U[10, 100]$, and $U_3 \equiv U[100, 1000]$. In addition, the values h_j's are selected from the uniform distributions: $U_4 \equiv U[h_0 + 1, 101]$, $U_5 \equiv U[h_0 + 1, 1001]$, and $U_6 \equiv U[h_0 + 1, 10001]$

k_0	h_0	h_j	$k_j \sim U_1$		$k_j \sim U_2$		$k_j \sim U_3$	
			$C_D < C_{H(N)}$	$C_D > C_{H(N)}$	$C_D < C_{H(N)}$	$C_D > C_{H(N)}$	$C_D < C_{H(N)}$	$C_D > C_{H(N)}$
U_1	U_1	U_4	5	4	4	6	10	0
U_1	U_2	U_5	5	10	10	0	7	3
U_1	U_3	U_6	7	6	6	4	10	0
U_2	U_1	U_4	0	6	6	4	8	2
U_2	U_2	U_5	0	5	5	5	7	3
U_2	U_3	U_6	0	1	1	9	5	5
U_3	U_1	U_4	0	0	0	10	3	7
U_3	U_2	U_5	0	0	0	10	1	9
U_3	U_3	U_6	0	0	0	10	2	8

In Tables 3.16 and 3.17, we have only shown the ratio where either the decentralized or the single-cycle policies are better, but nothing is said about the difference between the costs of both strategies. In Table 3.18, we report a collection of 25 instances, where parameters d_j, k_0, h_0 and k_j vary in $[1, 100]$ and h_j in $[h_0 + 1, 101]$. The first column represents the number of retailers with $N = 3$, 5, 10, 15 and 20. For the decentralized case, the cost of each instance is shown in the second column. The next column contains the costs for the single-cycle policies. For each instance, ^{s} indicates the smallest cost. In the last column, the gap (%) represents the quotient between the difference of both costs and the minimum of them.

Table 3.18: Comparison among costs using the different policies for several instances. ^{s} indicates the smallest cost. The gap (%) represents the quotient between the difference of the costs in the second and fourth column and the minimum of them

N	C_D	$C_{H(N)}$	Gap (%)
3	4381.57	4252.46^s	3
3	3719.15	3547.20^s	4
3	1523.50^s	1600.41	5
3	3763.06	3663.96^s	2
3	1573.08	1540.28^s	2
5	6739.30	6512.45^s	3
5	4185.26	4009.00^s	4
5	5277.57^s	5678.70	7
5	3695.20^s	3881.59	2
5	5732.79^s	5918.62	3
10	8064.20	7470.76^s	7
10	8669.57^s	8867.98	2
10	6419.84	6176.58^s	3
10	7564.83	7322.83^s	3
10	7083.12^s	7225.19	2
15	13955.70^s	15015.90	7
15	9433.32^s	9904.33	5
15	13483.20^s	14064.90	8
15	16597.80	16415.10^s	1
15	8337.82	8172.11^s	2
20	14427.30^s	15623.77	8
20	11082.80^s	11932.30	7
20	13419.60^s	14335.00	6
20	9719.26^s	10484.90	7
20	14801.40^s	16893.00	14

3.8 Conclusions

In this chapter we have studied the one-warehouse N-retailer problem where stocking decisions have to be adopted to achieve effective strategies. We have focused our attention on two different policies. Firstly, it has been assumed that the decision system is centralized, that is, there is only a decision-maker and the goal is to minimize the average total costs. Secondly, we have addressed the problem considering that at each location there is a decision-maker, that is, the decision system is decentralized.

In turn, for the centralized situation we first have analyzed a simple class of policies where all retailers order at the same time instants. Then, we have studied the single-cycle policies which represent a more general class of policies. In particular, we have introduced a new heuristic to determine single-cycle policies, which has been compared with the procedures proposed by Schwarz (1973), Graves and Schwarz (1977) and Muckstadt and Roundy (1993). Both the advantages and difficulties of each approach have been discussed.

It is worth noting that each step of the heuristic deals with only M of the N retailers. An interesting case corresponds to $M = 2$ since, in this situation, the computational complexity is $O(N \log N)$. Notice that both, the heuristic $H(2)$ and the Muckstadt and Roundy procedure, compute single-cycle policies in $O(N \log N)$ time. However, the computational experience shows that the policies given by the new heuristic $H(2)$ are equal to or better than those provided by the Muckstadt and Roundy approach in 82%. With respect to the initial solution provided by the Graves and Schwarz procedure, the heuristic $H(2)$ always provides policies that are equal to or better than such initial solution.

When the decision system is decentralized, we propose a two-level optimization approach which consists of computing first the order quantities at the retailers, and then, determining the inventory policy for the warehouse. It is important to remark that for the warehouse the problem becomes a time-varying demand inventory system.

Finally, we have compared the single-cycle policies with the decentralized strategies. From the computational results it can be concluded that as the number of retailers increases so does the number of instances where the decentralized policies are better. In addition, given a number of retailers, we have carried out an analysis of sensitivity of the parameters. This analysis suggests that, under specific conditions of the unit replenishment and holding costs at the warehouse, the decentralized policies can provide better solutions.

In spite of this, nowadays it is very common that different firms work together in order to improve the coordination of the total material flow. For that reason, most efforts are devoted to develop more efficient centralized policies.

In this chapter we have focused on centralized policies which are both stationary and nested. Next, we drop these assumptions and analyze a more general class of centralized strategies known as integer-ratio policies.

Chapter 4
The one-warehouse N-retailer problem: Integer-ratio policies

The single-cycle policies considered in Chapter 3 are very efficient in many situations and they have clear managerial advantages. However, in some cases, as for example when relatively high replenishment costs are combined with relatively low demand rates, the performance of these policies get worse. In order to achieve more effective strategies we analyze in this chapter a more general class of centralized policies known as integer-ratio policies. We propose a heuristic procedure for computing near-optimal integer-ratio policies which is compared with the most efficient technique reported in the literature. In addition, the integer-ratio solutions provided by the new heuristic are also compared with the decentralized policies introduced in Chapter 3. The computational results show that usually the integer-ratio policies are more effective than the decentralized strategies. However, the values of the parameters also influence in which policy is better.

4.1 Introduction

Optimal policies for the one-warehouse N-retailer problem can be very complex, and this complexity would make them unattractive even if they could be computed efficiently. Hence, many authors have restricted the class of admissible policies with the goal of finding good approximations to optimal policies without excessive computation. One of the simplest class of policies are the stationary and nested policies considered in Chapter 3. However, as Roundy (1983) showed, the optimal nested policies can have very low effectiveness. For example, consider a system where a warehouse supplies two retailers. Also assume that the first retailer has very low demand and a very high replenishment cost. Then, one would expect this retailer to order less often than the warehouse. However, if a nested policy is followed, each retailer must order every time that the warehouse does. Thus, a nested policy would incur either high holding costs at the warehouse due to a long replenishment interval, or high replenishment costs at the first retailer due to a low replenishment interval, or both. Therefore, nested policies become seriously suboptimal when relatively high replenishment costs are combined with relatively low demand rates. For that reason, Roundy (1985) introduced the more general class of integer-ratio policies dropping the assumptions of stationarity and nestedness.

An integer-ratio policy is one where the warehouse orders at equally spaced points in time and each retailer follows an EOQ pattern. Besides, the replenishment interval at the warehouse t_0, and the replenishment interval at retailer j, t_j, must satisfy that either t_j/t_0 or t_0/t_j is a positive integer. Recall that in the single-cycle policies the replenishment interval at the retailers cannot be longer than the replenishment interval at the warehouse. Moreover, the warehouse always orders from the supplier the same replenishment quantity. In contrast, when an integer-ratio policy is applied, a retailer can order less frequently than the warehouse. In addition, the warehouse has not necessarily to order always the same quantity. Therefore, integer-ratio policies are stationary at the retailers but may not be at the warehouse. Hence, it is obvious that the class of single-cycle policies are contained into the class of integer-ratio policies.

In particular, Roundy (1985) focused on the integer-ratio policies which are also powers-of-two. In such policies orders are placed at equal intervals of time which are powers of two multiples of some base planning period T_L. Roundy showed that if T_L is fixed the cost of an optimal powers-of-two policy is at most 6% above the optimal cost. Besides, if T_L is variable this procedure provides policies that are within 2% of optimality. Moreover, he showed that an optimal powers-of-two policy can be computed in $O(N \log N)$. However, from a practical point of view, integer-ratio policies are very important since they provide a more unconstrained way to determine the replenishment intervals at the warehouse and at the retailers. Notice that integer-ratio policies do not restrict the replenishments intervals to be powers of two multiples of the base planning period.

In this chapter we develop an $O(N \log N)$ heuristic for computing near-optimal integer-ratio policies and we compare it with the procedure proposed by Roundy (1985). The computational results show that the integer-ratio policies generated by the new heuristic are, on average, more effective than those computed by the Roundy procedure. It is worth noting that the decision system in the integer-ratio policies is centralized. Therefore, it also makes sense to compare the integer-ratio policies with the decentralized strategies introduced in Chapter 3.

We complete this section with an outline of the chapter. Section 4.2 introduces and states the problem. In Section 4.3 we outline the algorithm given by Roundy (1985) to determine integer-ratio powers-of-two policies. Section 4.4 presents the new heuristic developed to compute integer-ratio policies. We introduce in Section 4.5 a numerical example which is solved using both the Roundy procedure and the new heuristic. Computational results are reported in Section 4.6. Finally, our conclusions are drawn in Section 4.7.

4.2 Problem statement

In this section we formulate the one-warehouse N-retailer problem in terms of integer-ratio policies. To that end, we use the same notation introduced in Chapter 3. That is, the input data associated with the retailers are d_j, k_j and h_j which represent the constant and continuous demand rate, the fixed replenishment cost and the holding cost per unit time at retailer $j = 1, ..., N$, respectively. The values k_0 and h_0 represent the fixed production replenishment cost and the holding cost per unit time at the vendor, respectively. In addition, the decision variables are the replenishment intervals at the retailers t_j, $j = 1, ..., N$, and at the warehouse t_0. Finally, recall that C_j, C_0 and C_T denote the total costs per unit time incurred by retailer $j = 1, ..., N$, the warehouse and the total system, respectively.

Before proceeding with the formulation of the problem it is useful to remark the following fact. Since we focus on integer-ratio policies, for each retailer j, either t_j/t_0 or t_0/t_j must be a positive integer. Notice that if $t_j \leq t_0$, t_0 can be expressed as an integer multiple of t_j, that is, $t_0 = n_j t_j$. Hence, as we showed in Chapter 2, under this situation it is better to use the echelon holding costs. Recall that if h_j is the holding cost at retailer j and $h_0 < h_j$ is the holding cost at the warehouse, then, $h'_j = h_j - h_0$ and $h'_0 = h_0$ are the echelon holding costs at retailer j and at the warehouse, respectively. However, if $t_j > t_0$, then $t_j = m_j t_0$ with m_j integer, and therefore, each time retailer j places an order so does the warehouse. Consequently, the warehouse has not to hold inventory for the retailers with replenishment interval greater than t_0. Therefore, in this case, we only have to consider the holding costs incurred at the retailers. Accordingly, we should use the holding cost rates at these retailers instead of the echelon holding cost rates.

Taking the above argument into account, it is obvious that holding costs at retailer j must be computed in a different way depending on if $t_j > t_0$ or $t_j \leq t_0$. Hence, in order to formulate the problem we have to classify the retailers in sets according to their replenishment intervals. Concretely, we use the sets G, L and E defined by Roundy (1985) and introduced in Chapter 2. Recall that in set G are the retailers j with $t_j > t_0$, L denote the set of retailers j with $t_j < t_0$, and E is the set of retailers j with $t_j = t_0$.

Now, using these sets it is easy to see that the cost at the warehouse and at the retailers are given by the following expressions

$$C_0 = \frac{k_0}{t_0} + \frac{t_0 h_0 \sum\limits_{i \in E \cup L} d_i}{2} \tag{4.1}$$

$$C_e = \frac{k_e}{t_0} + \frac{t_0 h'_e d_e}{2}, \ e \in E \tag{4.2}$$

$$C_l = \frac{k_l}{t_l} + \frac{t_l h'_l d_l}{2}, \ l \in L \tag{4.3}$$

$$C_g = \frac{k_g}{t_g} + \frac{t_g h_g d_g}{2}, \ g \in G \tag{4.4}$$

Notice that as we pointed out, the costs at retailers in set $E \cup L$ are formulated in terms of the echelon holding costs. In contrast, the costs at retailers in set G depend on the conventional holding costs.

Therefore, the total cost for the system is given by

$$\begin{aligned} C_T &= C_0 + \sum_{e \in E} C_e + \sum_{l \in L} C_l + \sum_{g \in G} C_g = \\ & \frac{1}{t_0}(k_0 + \sum_{e \in E} k_e) + \frac{t_0}{2}(h_0 \sum_{i \in E \cup L} d_i + \sum_{e \in E} h'_e d_e) + \\ & \sum_{l \in L}(\frac{k_l}{t_l} + \frac{t_l h'_l d_l}{2}) + \sum_{g \in G}(\frac{k_g}{t_g} + \frac{t_g h_g d_g}{2}) \end{aligned}$$

Then, we can formulate the problem of computing an integer-ratio policy for the one-warehouse N-retailer system as follows

$$\begin{aligned} \min \ & \{\frac{1}{t_0}(k_0 + \sum_{e \in E} k_e) + \frac{t_0}{2}(h_0 \sum_{i \in E \cup L} d_i + \sum_{e \in E} h'_e d_e) + \\ & + \sum_{l \in L}(\frac{k_l}{t_l} + \frac{t_l h'_l d_l}{2}) + \sum_{g \in G}(\frac{k_g}{t_g} + \frac{t_g h_g d_g}{2})\} \end{aligned} \tag{4.5}$$

s.t.

$$\frac{t_j}{t_0} \text{ or } \frac{t_0}{t_j} \text{ is a positive integer } j = 1, .., N \tag{4.6}$$

This problem has been already solved by Roundy (1985) but he restricts himself to the integer-ratio policies which are also powers-of-two. That is, he forces the quotients t_j/t_0 or t_0/t_j to be powers-of-two. The approach provided by Roundy (1985) is the technique more referenced in the literature as a very efficient method for computing integer-ratio policies. Thus, in order to check the performance of the new heuristic we will compare it with the Roundy approach.

4.3 The Roundy procedure

Roundy (1985) solved problem (4.5)-(4.6) considering powers-of-two policies. That is, for each facility he assumes that the replenishment intervals are of the form

$$t_j = 2^{l_j} T_L,\ j = 0, .., N,\ \text{where } l_j \text{ is integer, } T_L > 0 \text{ is a base planning period} \quad (4.7)$$

As we commented in Chapter 2, to solve this problem Roundy (1985) first relaxes (4.7) and minimizes (4.5) for $t_j > 0$, $j = 0, ..., N$. Once the values t_j^*'s are computed they are rounded-off to powers of two multiples of the base planning period T_L. If T_L is fixed, the rounded-off replenishment interval for a given facility j is $t_j = 2^{l_j} T_L$ where $2^{l_j - 1} < t_j^*/\sqrt{2}T_L < 2^{l_j}$. Otherwise, if T_L is variable, the way to compute optimal powers-of-two policies is not as direct as before and it can be found in Roundy (1985). It is important to remark that Roundy (1985) proved that the cost of an optimal powers-of-two policy is at most 6% or 2% above the cost of an optimal policy when T_L is fixed or variable, respectively. Moreover, Roundy (1985) showed that such a policy can be computed in O($N \log N$) time. Therefore, powers-of-two policies are easy to compute and they are very effective. However, in the following section we develop a new approach to compute integer-ratio policies which are, on average, more effective than those given by the Roundy procedure. Moreover, this approach is very easy to implement and its computational complexity is also O(N $\log N$).

4.4 New heuristic H_{IR}

Unlike the Roundy procedure, we admit that for each retailer j, either t_j/t_0 or t_0/t_j is a positive integer, but not necessarily a power of two integer. The way to compute these integer ratios is as follows. We first apply Algorithm 2.2 given in Chapter 2 to obtain sets G, L and E. Then, following the idea introduced in Graves and Schwarz (1977), we sort the retailers belonging to sets G or L so that retailer $i <$ retailer j if and only if $h_i' d_i / k_i < h_j' d_j / k_j$. In case of tie, we sort the retailers according to their indexes, namely, if $i < j$, then retailer $i <$ retailer j. Without loss of generality, we can assume that retailer 1 < retailer 2 < ... < retailer N. Next, we compute the optimal replenishment intervals at the warehouse and at the retailers.

Adding (4.1) and (4.2) for each retailer $e \in E$, and taking the derivative with respect to t_0 equal to zero we have that the optimal replenishment interval is

$$t_0 = \sqrt{\frac{2(k_0 + \sum_{e \in E} k_e)}{(h_0 d_0 + \sum_{e \in E} d_e h'_e)}} \tag{4.8}$$

where $d_0 = \sum_{i \in E \cup L} d_i$.

From (4.3) the optimal replenishment interval at any retailer $l \in L$ is given by

$$t_l = \sqrt{\frac{2k_l}{h'_l d_l}} \tag{4.9}$$

Similarly, from (4.4) we obtain that the optimal replenishment interval at any retailer $g \in G$ is

$$t_g = \sqrt{\frac{2k_g}{h_g d_g}} \tag{4.10}$$

Taking into account that for all retailers $l \in L$, $t_0 = n_l^* t_l$, and for all retailers $g \in G$, $t_g = m_g^* t_0$, the optimal real values n_l^*'s and m_g^*'s can be computed as follows

$$n_l^* = \frac{t_0}{t_l}, \; l \in L \tag{4.11}$$

$$m_g^* = \frac{t_g}{t_0}, \; g \in G \tag{4.12}$$

Now, we should adjust these real values to integer values according to the following procedure.

This procedure consists of $N - |E|$ steps and in each iteration either a n_l or a m_g integer value is computed as $\lceil n_l^* \rceil$ or $\lfloor n_l^* \rfloor$, or $\lceil m_g^* \rceil$ or $\lfloor m_g^* \rfloor$, respectively. Suppose that we are evaluating step i, then, for each retailer $i < j$ either its n_i, if retailer $i \in L$, or its m_i, if retailer $i \in G$, has been already calculated. Besides, for any retailer $k \geq j$, we consider that the values n_k's and m_k's might be real values. Therefore, each retailer $k \geq j$, can order at its optimal replenishment interval, t_k, since it is always possible to find a real value that satisfies $t_0 = n_k^* t_k$, if retailer

$k \in L$, or $t_k = m_k^* t_0$, if retailer $k \in G$. Thus, we obtain that $t_k = \sqrt{2k_k/h_k d_k}$, if retailer $k \in G$ or $t_k = \sqrt{2k_k/h'_k d_k}$, if retailer $k \in L$, and the cost incurred by these retailers is

$$\sum_{i=j}^{N} C_i = \sum_{g \in \{G \cap \{j,\ldots,N\}\}} \sqrt{2d_g k_g h_g} + \sum_{l \in \{L \cap \{j,\ldots,N\}\}} \sqrt{2d_l k_l h'_l}$$

Let $C_{T;\,1,\ldots,j-1}$ be the total cost when n_l's and m_g's have been already determined, with $l, g < j$. Then, the total cost (4.5) can be reformulated to give

$$C_{T;\,1,\ldots,j-1} = \sum_{g \in \{G \cap \{j,\ldots,N\}\}} \sqrt{2d_g k_g h_g} + \sum_{l \in \{L \cap \{j,\ldots,N\}\}} \sqrt{2d_l k_l h'_l} + \\ + \frac{K_{1,\ldots,j-1}}{t_0} + \frac{t_0}{2} H_{1,\ldots,j-1} \quad (4.13)$$

where

$$K_{1,\ldots,j-1} = k_0 + \sum_{e \in E} k_e + \sum_{l \in \{L \cap \{1,\ldots,j-1\}\}} n_l k_l + \sum_{g \in \{G \cap \{1,\ldots,j-1\}\}} \frac{k_g}{m_g}$$

and

$$H_{1,\ldots,j-1} = h_0 d_0 + \sum_{e \in E} d_e h'_e + \sum_{l \in \{L \cap \{1,\ldots,j-1\}\}} \frac{h'_l d_l}{n_l} + \sum_{g \in \{G \cap \{1,\ldots,j-1\}\}} m_g h_g d_g$$

Let $t_{0;\,1,\ldots,j-1}$ be the optimal replenishment interval at the warehouse for known n_l's and m_g's, with $l, g < j$. Then, taking the derivative of (4.13) with respect to t_0 equal to zero we have

$$t_{0;\,1,\ldots,j-1} = \sqrt{\frac{2K_{1,\ldots,j-1}}{H_{1,\ldots,j-1}}} \quad (4.14)$$

Thus, the cost function in (4.13) can be rearranged using (4.14) to give

$$C_{T;\,1,\ldots,j-1} = \sum_{g \in \{G \cap \{j,\ldots,N\}\}} \sqrt{2d_g k_g h_g} + \sum_{l \in \{L \cap \{j,\ldots,N\}\}} \sqrt{2d_l k_l h'_l} \\ + \sqrt{2K_{1,\ldots,j-1} H_{1,\ldots,j-1}} \quad (4.15)$$

At this point, we can compute the new optimal real value n_j or m_j as follows

$$n'_j = \frac{t_{0;\,1,\ldots,j-1}}{t_j}, \text{ if } j \in L, \tag{4.16}$$

$$m'_j = \frac{t_j}{t_{0;\,1,\ldots,j-1}}, \text{ if } j \in G \tag{4.17}$$

In case that retailer $j \in L$, we set $n_j = \lfloor n'_j \rfloor$ if $C_{T;\,1,\ldots,j-1,j}(n_j = \lfloor n'_j \rfloor) < C_{T;\,1,\ldots,j-1,j}(n_j = \lceil n'_j \rceil)$; otherwise, $n_j = \lceil n'_j \rceil$. On the contrary, if retailer $j \in G$, we set $m_j = \lfloor m'_j \rfloor$ if $C_{T;\,1,\ldots,j-1,j}(m_j = \lfloor m'_j \rfloor) < C_{T;\,1,\ldots,j-1,j}(m_j = \lceil m'_j \rceil)$; otherwise $m_j = \lceil m'_j \rceil$.

Once either n_j or m_j has been obtained, we proceed in the same way to compute n_{j+1} or m_{j+1}. Finally, in step $N-|E|$ all n_j's and m_j's have been calculated. Observe that each iteration of the algorithm only checks two values $\lceil n'_j \rceil$ and $\lfloor n'_j \rfloor$, if retailer $j \in L$, or $\lceil m'_j \rceil$ and $\lfloor m'_j \rfloor$, if retailer $j \in G$. For that reason, we cannot guarantee that this approach always computes the optimal integer-ratio policy. However, from the computational results it can be concluded that in most cases the heuristic H_{IR} is more effective than the Roundy procedure.

The computational complexity of the algorithm is determined as follows. It is easy to prove that sets G, L and E are computed in $O(N \log N)$ (Roundy (1985)). Moreover, the sort of the retailers can also be implemented in $O(N \log N)$ time. Remark that the first two summations in (4.15) are calculated in $O(N)$ time. This operation is carried out when $j = 1$. In the remaining iterations a constant number of additions and subtractions are made. Therefore, the computational complexity of the new heuristic H_{IR} is $O(N \log N)$.

4.5 Numerical example

In this section we solve a one-warehouse and five-retailer problem with the data given in Table 4.1 using both the Roundy procedure and the heuristic H_{IR}.

4.5.1 Roundy's powers-of-two policy

Applying Algorithm 2.2, we obtain $E = \{5\}$, $L = \{2, 3\}$, $G = \{1, 4\}$, and the following replenishment intervals $t^*_0 = 0.0991$, $t^*_1 = 0.1634$, $t^*_2 = 0.0405$, $t^*_3 = 0.0356$, and $t^*_4 = 0.2277$. Once we have determined the t^*_i's, they are rounded-off to optimal powers of two multiples of the base planning period T_L. For this example, we

Table 4.1: Input data for an instance of the one-warehouse five-retailer problem

	d_j	k_j	h_j	h'_j
Retailer 1	36	102	212	181
Retailer 2	336	118	458	427
Retailer 3	430	116	456	425
Retailer 4	101	453	173	142
Retailer 5	100	74	165	134
Warehouse		124	31	31

obtain that the optimal base planning period is $T_L = 0.0862$, and the replenishment intervals are $t_0 = t_5 = 0.0862 = 2^0 T_L$, $t_1 = 0.1724 = 2^1 T_L$, $t_2 = 0.0431 = 2^{-1} T_L$, $t_3 = 0.0431 = 2^{-1} T_L$, and $t_4 = 0.1724 = 2^1 T_L$. The cost for this policy is 21874.2 $\$/time\ unit$.

4.5.2 Integer-ratio policy provided by the heuristic H_{IR}

We first sort the retailers belonging to G or L so that retailer $i <$ retailer j if and only if $h'_i d_i / k_i < h'_j d_j / k_j$. For this example we obtain that retailer 4 < retailer 1 < retailer 2 < retailer 3. So, we relabel the retailers in the following way: retailer 1 = retailer 4, retailer 2 = retailer 1, retailer 3 = retailer 2 and retailer 4 = retailer 3.

Next, we compute the optimal replenishment interval for each retailer taking into account whether the retailer belongs to set $G = \{1, 2\}$ or to set $L = \{3, 4\}$. From (4.10) it follows that $t_1 = 0.2277$ and $t_2 = 0.1634$. From (4.9) we have that $t_3 = 0.0405$ and $t_4 = 0.0356$.

We now apply the procedure introduced in Section 4.4 to compute the near-optimal integers m_1, m_2, n_3 and n_4.

First, the optimal replenishment interval at the warehouse is computed. Using (4.14), for $j = 1$, it follows that $t_0 = 0.0991$. Since retailer $1 \in G$, we use (4.17) to calculate $m'_1 = 2.2976$. Therefore, we should choose between $m_1 = 2$ or $m_1 = 3$. If $C_{T;\,1,1}(m_1 = 2) < C_{T;\,1,1}(m_1 = 3)$ we set $m_1 = 2$, otherwise, $m_1 = 3$.

Taking into account that

$$C_{T;\,1,1} = \sqrt{2 d_2 k_2 h_2} + \sqrt{2 d_3 k_3 h'_3} + \sqrt{2 d_4 k_4 h'_4} + \sqrt{2 K_{1,1} H_{1,1}}$$

it follows that $C_{T;\,1,1}(m_1 = 2) = 21567.9311$ and $C_{T;\,1,1}(m_1 = 3) = 21620.4243$.

Thus, we set $m_1 = 2$.

Notice that if m_1 changes so does the replenishment interval at the warehouse. The new replenishment interval at the warehouse computed using (4.14) is equal to $t_{0;\ 1,1} = 0.1062$.

We now should determine value m_2. Since retailer $2 \in G$, $t_2 = m_2 t_{0;\ 1,1}$ and the new m_2' is equal to $t_2/t_{0;\ 1,1} = 1.5386$, hence either $m_2 = 1$ or $m_2 = 2$. If we compute the total cost for both values, we obtain that the lower cost is achieved when $m_2 = 2$. Repeating this process in each step the following integer values n_l's and m_g's are obtained: $m_1 = 2$, $m_2 = 2$, $n_3 = 3$ and $n_4 = 3$. From the previous values and using (4.14) we obtain that $t_{0;\ 1,4} = t_5 = 0.1087$. From (4.16) and (4.17) we can compute the replenishment interval at each retailer to give $t_1 = 0.2174$, $t_2 = 0.2174$, $t_3 = 0.0362$, and $t_4 = 0.0362$. The cost for this policy is 21658.9046 $/*time unit*.

Remark that for this example the integer-ratio policy provided by the new heuristic H_{IR} is better than the optimal powers-of-two policy. The computational results show that, in general, the policies generated by the heuristic H_{IR} are more effective than those given the Roundy procedure.

4.6 Computational results

This section provides a set of randomly generated numerical examples to illustrate the average effectiveness of the integer-ratio policies provided by the new heuristic H_{IR}. First, we compare the Roundy procedure with the heuristic H_{IR}. Then, we contrast the costs of the integer-ratio policies against the costs obtained by the decentralized approach.

4.6.1 The new heuristic H_{IR} versus the Roundy procedure

We generate 100 instances for each N multiple of 5 varying from 5 to 100. The values of h_0, k_0 and k_j are taken from a uniform distribution $U[1, 500]$. The values h_j and d_j are taken from uniform distributions $U[h_0, 500]$ and $U[1, 1000]$, respectively. For each instance we compute both the optimal powers-of-two policy given by the Roundy procedure and the integer-ratio policy provided by the new heuristic H_{IR}. For notation convenience, let C_R and $C_{H_{IR}}$ denote the cost of the policy computed by the Roundy procedure and by the heuristic H_{IR}, respectively. The results are summarized in Table 4.2.

In Table 4.2 the first column represents the number of retailers. In the second column we show the number of problems where both the new heuristic H_{IR} and

Table 4.2: Comparison between the optimal powers-of-two policy given by the Roundy procedure and the integer-ratio policy provided by the heuristic H_{IR}

N	$C_{IR} = C_R$	$C_{IR} < C_R$	$C_{IR} > C_R$
5	49	41	10
10	20	62	18
15	3	79	18
20	3	70	27
25	0	73	27
30	0	76	24
35	0	86	14
40	0	82	18
45	0	90	10
50	0	89	11
55	0	92	8
60	0	91	9
65	0	95	5
70	0	93	7
75	0	94	6
80	0	92	8
85	0	96	4
90	0	93	7
95	0	98	2
100	0	97	3
Average	3.75	84.45	11.8

the Roundy procedure provide the same solution. The third and the last columns contain the number of instances where the new heuristic computes better and worse policies than the Roundy procedure, respectively.

Notice that in 84.45% of the instances the new heuristic H_{IR} provides better solutions than those given by the Roundy procedure, and in 3.75% both methods compute the same solution. Therefore, only in 11.8% our method provides worse solutions. Hence, we can conclude that usually the heuristic H_{IR} is more effective than the Roundy procedure.

Roundy (1985) proved that the minimum cost of problem (4.5) is a lower bound of the cost of any feasible policy. We use this lower bound, LB, to compare the effectiveness of the policies provided by the Roundy procedure and by the new heuristic. In particular, for each instance we compute the ratio between the cost of the corresponding policy and the lower bound. In Table 4.3, we report for each N the average percentages of these ratios. It is worth noting that the new heuristic H_{IR} provides closer solutions to the lower bound than the Roundy procedure.

Table 4.3: Comparison between both heuristics for computing integer-ratio policies and the lower bound given by Roundy

N	$Gap(\%)$ $C_{H_{IR}}$ vs. LB	$Gap(\%)$ C_R vs. LB
5	0.72	0.78
10	0.96	1.04
15	1.11	1.29
20	1.15	1.30
25	1.29	1.44
30	1.35	1.48
35	1.25	1.52
40	1.20	1.56
45	1.26	1.57
50	1.17	1.58
55	1.21	1.65
60	1.16	1.59
65	1.14	1.62
70	1.13	1.64
75	1.18	1.66
80	1.15	1.65
85	1.07	1.66
90	1.18	1.58
95	0.97	1.66
100	1.02	1.69

In Table 4.4 we report a collection of the instances generated. The first column represents the number of retailers. Columns two and three contain the costs of the policies computed by the Roundy procedure and by the heuristic H_{IR}, respectively. In column four, the value $Gap_{R-H_{IR}}(\%)$ is given by $|C_R - C_{H_{IR}}| \,/\, \min\{C_R,\, C_{H_{IR}}\}$ and, ^{h} represents that for such instance the new heuristic is better than the Roundy procedure.

4.6.2 Integer-ratio versus decentralized policies

The decision system in the integer-ratio policies is centralized. That is, there is a unique decision maker and the goal is to minimize the average total cost. However, as we shown in Chapter 3, the problem can be also addressed assuming that the decision system is decentralized. That is, each retailer orders independently, as does the warehouse. Recall that under this situation, we have proposed a two-level optimization approach which consists of computing first the order quantities at the

Table 4.4: Comparison between costs obtained using the Roundy procedure and the heuristic H_{IR}

N	C_R	$C_{H_{IR}}$	$Gap_{R-H_{IR}}$ (%)
5	83845.71	83845.71	0.00
5	84061.64	83510.25	0.66^h
5	106199.10	106199.10	0.00
10	173067.18	173250.10	0.10
10	135338.48	134312.87	0.76^h
10	88282.90	88282.90	0.00
20	234442.84	233807.93	0.27^h
20	258449.20	258669.18	0.08
20	265508.62	262211.03	1.25^h
30	368350.06	367242.46	0.30^h
30	260362.26	260491.04	0.04
30	490240.03	483312.93	1.43^h
40	600655.56	595761.25	0.82^h
40	572237.62	570751.75	0.26^h
40	416977.04	410407.05	1.60^h
50	577839.75	570333.00	1.31^h
50	655718.00	650201.75	0.84^h
50	496238.93	494919.00	0.26^h
60	599231.25	599763.12	0.08
60	509232.46	501726.50	1.49^h
60	496108.62	489862.84	1.27^h
70	701994.06	691055.31	1.58^h
70	660135.87	657934.93	0.33^h
70	1055170.37	1048056.50	0.67^h
80	693146.62	693316.00	0.02
80	1042022.37	1027320.50	1.43^h
80	1025367.31	1023174.62	0.21^h
90	1422478.75	1399963.37	1.60^h
90	1415412.00	1411788.87	0.25^h
90	1275187.12	1262969.37	0.96^h
100	1588765.50	1562112.12	1.70^h
100	986661.31	983054.56	0.36^h
100	1595893.25	1580128.12	0.99^h

retailers and then, determining the inventory policy for the warehouse. In the previous chapter we compare the single-cycle policies with the decentralized strategies. Here, we proceed in a similar way for the integer-ratio policies. Accordingly, for the set of instances generated in Chapter 3 we have computed the integer-ratio policies and we have compared them with the decentralized strategies. The results are summarized in tables 4.5-4.9. Note that in these tables we have also included the results obtained for the single-cycle policies. Thus, we can easily realize of the changes in the results due to use the integer-ratio policies instead of the single-cycle solutions.

Table 4.5: Comparison between decentralized and single-cycle policies, and between decentralized and integer-ratio policies when $d_j, k_0, h_0, k_j \sim U[1, 100]$ and $h_j \sim U[h_0 + 1, 101]$

N	$C_D < C_{H(N)}$	$C_D > C_{H(N)}$	$C_D < C_{H_{IR}}$	$C_D > C_{H_{IR}}$
2	13	87	1	99
3	20	80	2	98
4	43	57	1	99
5	37	63	3	97
6	33	67	5	95
7	50	50	3	97
8	50	50	0	100
9	63	37	0	100
10	43	57	0	100
15	67	33	0	100
20	100	0	0	100
25	100	0	0	100
30	100	0	0	100
Average	55.3	44.7	1.15	98.85

From Table 4.5, it is worth noting that when parameters range in $[1, 100]$, the number of instances where the single-cycle centralized policies are better than the decentralized strategies is around 45%. In contrast, the number of instances where it is preferable to apply the integer-ratio policy instead of the decentralized solution is almost 99%.

On the other hand, when parameters vary on $[1, 10]$, we can conclude from Table 4.6 that in 52% of the instances the single-cycle policies are more efficient than the decentralized ones. With respect to the integer-ratio policies, note that only in 1.2% it is better to use the decentralized strategies instead of the integer-ratio solutions. This is due to the fact that the class of integer-ratio policies is much more general than the class of single-cycle policies. Recall that in an integer-ratio policy the replenishment intervals at the retailers can be longer than the replenishment interval at the warehouse. However, in a single-cycle policy the replenishment intervals at the retailers are always equal to or smaller than the replenishment interval at the warehouse. Therefore, it is not surprising that the integer-ratio policies outperform the decentralized strategies in much more cases than the single-cycle policies.

As in Chapter 3, here we also analyze the effect of the parameters on the cost of

Table 4.6: Comparison between decentralized and single-cycle policies, and between decentralized and integer-ratio policies when $d_j, k_0, h_0, k_j \sim U[1, 10]$ and $h_j \sim U[h_0 + 1, 11]$

N	$C_D < C_{H(N)}$	$C_D > C_{H(N)}$	$C_D < C_{H_{IR}}$	$C_D > C_{H_{IR}}$
2	10	90	0	100
3	17	83	0	100
4	30	70	0	100
5	43	57	3	97
6	37	63	2	98
7	47	53	3	97
8	50	50	4	96
9	37	63	2	98
10	53	47	2	98
15	47	53	0	100
20	70	30	0	100
25	80	20	0	100
30	100	0	0	100
Average	47.77	52.23	1.23	98.77

the policies. To that end, we have fixed the number of retailers to 10 and we have chosen the parameters from different uniform distributions. For each combination, ten problems are tested and the results are summarized in tables 4.7-4.9. As in tables 4.5 and 4.6 we show both results, those obtained using the single-cycle policies and those achieved with the integer-ratio solutions.

From tables 4.7-4.9 we can conclude that the effect of the parameters on the costs of the policies is similar to that obtained when the single-cycle policies were compared with the decentralized strategies. That is, as the interval of the replenishment cost at the warehouse increases so does the number of instances where the integer-ratio policies are better than the decentralized strategies. On the other hand, when the costs at the retailers are significantly greater than the costs at the warehouse, it could be preferable that the retailers make decisions independently. However, as we expected, the number of instances where the decentralized policies are more efficient than the integer-ratio solutions decreases considerably with respect to the case where the single-cycle policies were considered.

Table 4.7: Comparison between decentralized and single-cycle policies, and between decentralized and integer-ratio policies when $k_j \sim U_1 \equiv U[1,10]$, k_0, $h_0 \sim U_1 \equiv U[1,10]$, $U_2 \equiv U[10,100]$ and $U_3 \equiv U[100,1000]$, and $h_j \sim U_4 \equiv U[h_0+1,101]$, $U_5 \equiv U[h_0+1,1001]$, and $U_6 \equiv U[h_0+1,10001]$

k_0	h_0	h_j	$C_D < C_{H(N)}$	$C_D > C_{H(N)}$	$C_D < C_{H_{IR}}$	$C_D > C_{H_{IR}}$
U_1	U_1	U_4	5	5	1	9
U_1	U_2	U_5	5	5	0	10
U_1	U_3	U_6	7	3	4	6
U_2	U_1	U_4	0	10	0	10
U_2	U_2	U_5	0	10	0	10
U_2	U_3	U_6	0	10	0	10
U_3	U_1	U_4	0	10	0	10
U_3	U_2	U_5	0	10	0	10
U_3	U_3	U_6	0	10	0	10

Table 4.8: Comparison between decentralized and single-cycle policies, and between decentralized and integer-ratio policies when $k_j \sim U_2 \equiv U[10,100]$, k_0, $h_0 \sim U_1 \equiv U[1,10]$, $U_2 \equiv U[10,100]$ and $U_3 \equiv U[100,1000]$, and $h_j \sim U_4 \equiv U[h_0+1,101]$, $U_5 \equiv U[h_0+1,1001]$, and $U_6 \equiv U[h_0+1,10001]$

k_0	h_0	h_j	$C_D < C_{H(N)}$	$C_D > C_{H(N)}$	$C_D < C_{H_{IR}}$	$C_D > C_{H_{IR}}$
U_1	U_1	U_4	4	6	5	5
U_1	U_2	U_5	10	0	6	4
U_1	U_3	U_6	6	4	1	9
U_2	U_1	U_4	6	4	1	9
U_2	U_2	U_5	5	5	2	8
U_2	U_3	U_6	1	9	0	10
U_3	U_1	U_4	0	10	0	10
U_3	U_2	U_5	0	10	0	10
U_3	U_3	U_6	0	10	0	10

Table 4.9: Comparison between decentralized and single-cycle policies, and between decentralized and integer-ratio policies when $k_j \sim U_3 \equiv U[100, 1000]$, k_0, $h_0 \sim U_1 \equiv U[1, 10]$, $U_2 \equiv U[10, 100]$ and $U_3 \equiv U[100, 1000]$, and $h_j \sim U_4 \equiv U[h_0 + 1, 101]$, $U_5 \equiv U[h_0 + 1, 1001]$, and $U_6 \equiv U[h_0 + 1, 10001]$

k_0	h_0	h_j	$C_D < C_{H(N)}$	$C_D > C_{H(N)}$	$C_D < C_{H_{IR}}$	$C_D > C_{H_{IR}}$
U_1	U_1	U_4	10	0	9	1
U_1	U_2	U_5	7	3	4	6
U_1	U_3	U_6	10	0	7	3
U_2	U_1	U_4	8	2	4	6
U_2	U_2	U_5	7	3	3	7
U_2	U_3	U_6	5	5	7	3
U_3	U_1	U_4	3	7	1	9
U_3	U_2	U_5	1	9	1	9
U_3	U_3	U_6	2	8	2	8

4.7 Conclusions

For the one-warehouse N-retailer systems the powers-of-two policies are very effective. Besides they are easy to implement and it is always possible to compute a powers-of-two policy in $O(N \log N)$ whose cost is at most 2% above the cost of an optimal policy. However, integer-ratio policies have the advantage of giving more freedom in determining the times at which the orders will be placed, since they do not restrict the replenishments intervals to be powers of two multiples of some base planning period.

In this chapter we develop an $O(N \log N)$ heuristic for computing near-optimal integer-ratio policies. The computational results show that the integer-ratio policies generated by the new heuristic H_{IR} are, on average, more effective than those computed by the Roundy procedure.

We also have compared the integer-ratio policies with the decentralized strategies introduced in Chapter 3. Recall that the class of integer-ratio policies are more general than the class of single-cycle policies. Hence, it is not surprising that the number of instances where it is preferable to use an integer-ratio policy instead of a decentralized strategy has increased considerably with respect to the single-cycle policies.

To this point, we have focused on the one-warehouse N-retailer problem assuming infinite production rate. However, often the warehouse represents to a manufacturing location where production occurs at a finite rate. This problem, where the warehouse produces an item which is supplied to the retailers, is usually referred to as the single-vendor multi-buyer system. It is worth noting that most contributions on this model are confined to considering a single buyer. However, in practice, the vendor usually supplies multiple buyers. In spite of this, we find few references in the literature dealing with the multiple buyers case. The analysis of the single-vendor multi-buyer systems is the goal of the next chapter.

Chapter 5
Single-vendor multi-buyer systems with finite production rate

In practice is very common to deal with inventory systems where a single vendor produces an item at a finite rate which is supplied to multiple buyers. In this chapter we analyze this problem and we formulate it in terms of integer-ratio policies. We first focus on the single-vendor two-buyer problem which is the simplest case within the single-vendor multi-buyer systems. Then, we extend the analysis to the multi-buyer problem. Finally, we also show how the problem should be addressed in case of independence among the vendor and the buyers. In addition, we compare the integer-ratio policies with the decentralized policies, and a sensitivity analysis of parameters is also reported.

5.1 Introduction

An interesting decision problem arises whenever a product needs to be supplied by a vendor to multiple buyers. This inventory/distribution system has been analyzed extensively in the literature. However, most work to date is confined to the case where the production is instantaneous, namely, infinite production rate. Besides, most contributions concerning finite production rate are related to the single-vendor single-buyer system.

Banerjee (1986) was one of the first in analyzing the integrated single-vendor single-buyer model where the vendor produces the items at a finite rate. He examined a lot-for-lot model in which the vendor manufactures each buyer shipment as a separate batch.

Goyal (1988) further generalized Banerjee's model by relaxing the assumption of lot-for-lot policy for the vendor. He showed that manufacturing a batch which is made up of an integral number of equal shipments, generally produces a lower cost solution. Goyal's model is based on the assumption that the vendor can supply the buyer only after completing the entire lot size. This assumption was relaxed by Lu (1995) who gave an optimal solution to the single-vendor single-buyer problem assuming equal shipments. This strategy is an improvement over the policies earlier proposed by Banerjee (1986) and Goyal (1988). A review of contributions related to vendor-buyer coordination models is given in Goyal and Gupta (1989).

Goyal (1995) also allowed the first shipment to be made before the whole lot is produced and he incorporated an alternative policy in which the quantity which is delivered to the buyer is not identical at every replenishment. Instead, at each delivery all available inventory is supplied to the buyer. This policy was based on a preceding argument proposed by Goyal (1976) for solving a single-vendor single-buyer system with infinite production rate at the vendor. This new policy involves successive shipment sizes within a batch which are increased by a factor equal to the ratio between the vendor's production rate and the demand rate at the buyer. Goyal (1995) used the numerical example given in Lu (1995) to show that this policy can be more effective than the equal shipment size policy.

Viswanathan (1998) identified problem parameters under which the equal shipment size policy of Lu (1995) and the increasing shipment policy of Goyal (1995) are optimal. He found that Goyal's policy has lower cost only if the holding cost at the buyer is not much larger than the holding cost at the vendor. This is because as the holding cost at the buyer increases, it is better to hold inventory at the vendor than at the buyer. In addition, Viswanathan (1998) also showed that the production rate at the vendor has a significant influence on which policy is better. In particular, if the production rate decreases relative to the demand rate, the policy provided by Goyal (1995) becomes more attractive. This is due to the fact that as the production rate decreases, the vendor would find more difficult to cope up with the demand, and, therefore, it makes more sense to deliver whatever inventory is available at each replenishment.

Hill (1997b) showed that neither the equal shipment size policy of Lu (1995) nor the increasing shipment size policy of Goyal are always optimal. He took Goyal's idea a stage further by considering successive shipment sizes which are increased by a general fixed factor. This factor ranges from 1 to an upper bound, which coincides with the quotient between the production rate and the demand rate. Therefore, both the equal shipment size policy and Goyal's policy represent special cases of this more general class of policies. The equal shipment size policy is obtained when the factor is equal to 1. Additionally, when the factor coincides with the quotient between the production rate and the demand rate, the policy given by Goyal (1995) is achieved. Hence, it is not surprising that the policies provided by Hill (1997b) outperform those obtained by Lu (1995) and Goyal (1995).

Most papers on this topic have focused on finding the optimal solution for a given class of policies. More recently, Hill (1999) determined the form of the globally optimal batching and shipping policy by combining the policy provided by Goyal (1995) and an equal shipment policy. However, although the problem has already been optimality solved, we can find recent papers dealing with the single-vendor single-buyer problem. For example, Goyal (2000) suggested an improvement over the policy of Hill (1997b) consisting of modifying the size of the shipments obtained by the procedure of Hill (1997b). Moreover, Goyal and Nebebe (2000) introduced a policy in which the batch quantity is received by the buyer in different shipments. The first delivery is of small size, and the next ones are equal to the size of the first shipment multiplied by the ratio between the production rate and the demand rate.

It is worth noting that most contributions on this model are confined to considering a single buyer. However, in practice, the vendor usually supplies multiple buyers. In spite of this, we find few references in the literature dealing with the multiple buyers case. Lu (1995) was a pioneer in studying the single-vendor multi-buyer inventory model, where it is assumed that the vendor manufactures at a finite rate. In Lu's model it is assumed that each buyer orders a different item to the vendor and the objective consists of minimizing the total cost at the vendor subject to the maximum cost that the buyers may be prepared to incur.

Yao and Chiou (2003) also proposed a heuristic for the single-vendor multi-buyer problem considering the same assumptions that those in Lu's model. In addition, they developed a numerical study in order to compare the new approach with the heuristic provided by Lu (1995). From the results, they concluded that the new approach outperforms Lu's heuristic.

Other related papers have been developed by Khouja (2003) and Wee and Yang (2004). In particular, Khouja (2003) study a three-stage system with multiple firms at each stage which can supply two or more customers. Furthermore, he assumes that the whole lot has to be produced before delivering the batch. Wee and Yang (2004) also analyze a three-stage system with a single producer and multiple distributors and retailers. They assume that the replenishment interval at each retailer is smaller than the replenishment interval at its distributor, which in turns, is smaller than the replenishment interval at the producer.

This chapter focuses on the single-vendor multi-buyer problem assuming that all buyers order the same item to the vendor which manufactures the item at a finite rate. Moreover, we assume that the shipments can be made before the whole lot is produced and we allow the replenishment interval at any buyer to be greater than the replenishment interval at the vendor.

The rest of this chapter is organized as follows. Section 5.2 introduces the notation and assumptions required to state the problem. In Section 5.3 we deal with the single-vendor two-buyer system and, in Section 5.4, we extend the analysis to the multi-buyer case. We discuss in Section 5.5 how to solve the problem if the vendor and the buyers are treated separately. In Section 5.6 the computational results and sensitivity analysis of the parameters are reported. Finally, we conclude with some final remarks in Section 5.7.

5.2 Notation and assumptions

We consider the single-vendor multi-buyer problem assuming that the production rate at the vendor is finite. It is assumed that customer demands occur at each buyer at a constant rate. There is a holding cost per unit stored per unit time at the vendor and at each buyer. The vendor incurs a fixed setup cost associated with each shipment. Additionally, when a buyer places an order it incurs a fixed ordering cost.

Similar to the previous chapters, the input data associated with the buyers are d_j, k_j and h_j which represent the constant and continuous demand rate, the fixed replenishment cost and the inventory holding cost per unit time at buyer $j = 1, ..., N$, respectively. The fixed production setup cost and the inventory holding cost per unit time at the vendor are denoted by k_0 and h_0, respectively. Additionally, P represents the continuous production rate for the vendor.

The decision variables are the replenishment intervals at the buyers, t_j, $j = 1, ..., N$, and the time interval between two consecutive setups at the vendor, t_v. The total costs per unit time incurred by buyer $j = 1, ..., N$, the vendor and the system are C_j, C_0 and C_T, respectively.

In the formulation of the problem we also use the following notation. Let I_0^t be the inventory at the vendor at instant t and $\overline{I_0}$ the average inventory during a cycle at the vendor. Similarly, we denote by ${I_j}^t$ and $\overline{I_j}$ the inventory at buyer j at instant t and the average inventory during a cycle at buyer j, respectively. Finally, IT^t represents the total inventory in the system at instant t and $\overline{IT}$ the average total inventory during a cycle in the system.

The stock value normally increases as a product moves down the distribution chain, and therefore, the associated holding costs also increase. Taking this into account, it is reasonable to assume that $h_j \geq h_0$, $j = 1, ..., N$. The consequence of this is that stock should be retained by the vendor until a buyer needs another shipment.

5.3 The single-vendor two-buyer problem

Many authors have addressed the single-vendor multi-buyer systems. However, they main focused on the case with infinite production rate. See for example Schwarz (1973), Graves and Schwarz (1977), Williams (1981, 1983) and Roundy (1985). The work of Roundy (1985) is specially relevant since he showed that optimal policies for this problem can be very complex. Moreover, he pointed out that even if we are able to compute an optimal policy it may be very difficult to apply in practice. Hence, he looked for a simple class of policies which facilitates both computation and implementation. As we showed in Chapter 3, Roundy (1985) introduced the integer-ratio policies, where the replenishment interval at the vendor, t_v, and the replenishment interval at buyer l, t_l, $l = 1, 2$, satisfy that either t_l/t_v or t_v/t_l is a positive integer. In particular, he focused on a subclass into the integer-ratio policies called powers-of-two policies, where the replenishment intervals are powers of two multiples of a base planning period. Recall that Roundy (1985) proved that the cost of an optimal powers-of-two policy is at most 2% above the cost of an optimal policy. Taking into account the good performance of such policies for Roundy's model, some authors have studied their behavior in other multi-echelon problems.

Accordingly, in this section we use the integer-ratio policies to formulate the single-vendor two-buyer system with finite production rate at the vendor. In particular, we confine ourselves to those integer-ratio policies where the quotient t_j/t_i corresponds to a positive integer, $i, j \in \{1, 2\}$, $i \neq j$. That is, if buyer j places an order, then, buyer i also orders. This class of policies can be profitable in practice since the transportation cost per cycle of the vendor is reduced by forcing the buyers to order simultaneously in some instants. Thus, there will be points in time where the vendor should supply both buyers simultaneously, and others where it only supplies buyer i. Therefore, in the latter case, the vendor should start the production later than when the vendor supplies both buyers. Hence, we let the time interval between two consecutive setups to be non-constant. Remark that this class of policies is a generalized case of the powers-of-two policies proposed by Roundy (1985).

Let us consider the example illustrated in Figures 5.1 and 5.2. The inventory fluctuations at the buyers and at the vendor, when we force the time interval between two consecutive setups to be constant, are depicted in Figure 5.1. In this case, the vendor anticipates the production that will be withdrawn later, and hence it unnecessarily holds inventory. However, as it can be seen in Figure 5.2, if we allow the time interval between two consecutive setups to be non-constant the holding cost at the vendor can be reduced. Nevertheless, if we only consider buyer 1, and we suppose that the vendor produces first the units to be sent to buyer 2, then the time interval between two consecutive setups is constant. In what follows we will denote this time interval by t_0. That is, in general, t_0 represents the time interval between two consecutive setups when it is only considered the buyers with replenishment interval $t_i \leq t_v$. Moreover, if buyer j with $t_j > t_v$ is also considered, then the time interval between two consecutive setups can be easily obtained from t_0.

Once t_0 has been established, we can use a definition similar to that in Roundy (1985) for the integer-ratio policies. The only difference is that we use the value t_0 instead of value t_v. That is, we will say that a policy is integer-ratio with respect to t_0 if either t_l/t_0 or t_0/t_l is a positive integer, with $l = 1, 2$.

Now, we focus on computing the average inventory at the vendor and at the buyers. For the single-vendor single-buyer problem, both inventories are easily computed. However, obtaining the average inventory at the vendor involves more complexity for the case with two buyers.

The expression of the average inventory at the vendor is different depending on if $t_i \leq t_j \leq t_0$ or $t_i \leq t_0 < t_j$, $i, j \in \{1, 2\}$, $i \neq j$. Hence, we should analyze both possibilities.

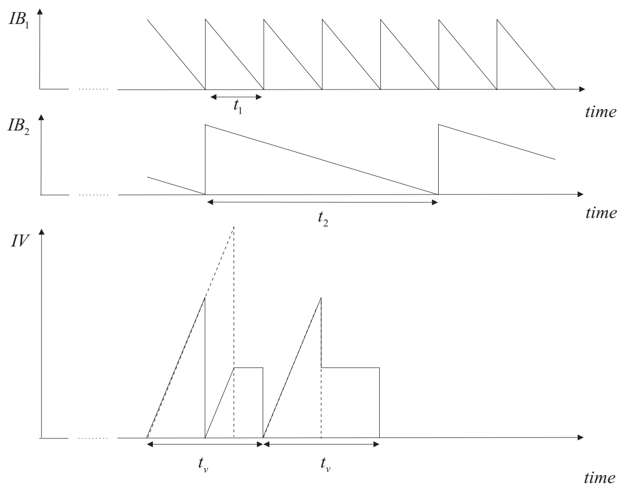

Figure 5.1: Inventory fluctuations at the vendor and at the buyers considering that the replenishment interval t_v is constant. The dotted line represents the production

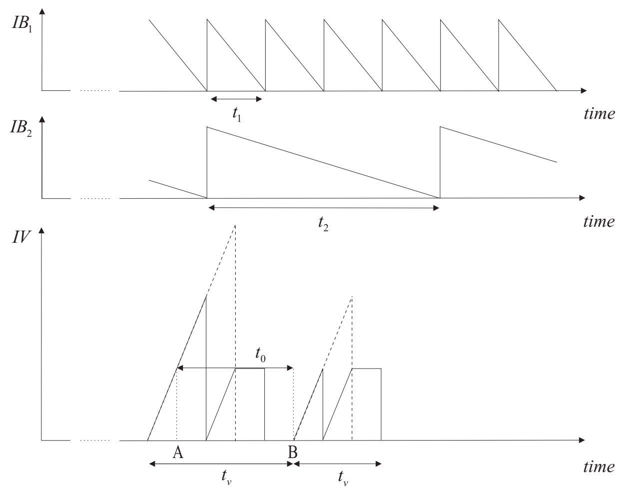

Figure 5.2: Inventory fluctuations at the vendor and at the buyers considering that the replenishment interval t_v is non-constant. At instants A and B the vendor begins the production of the items which will be shipped to buyer 1. The dotted line represents the production

Case 1: $t_i \leq t_j \leq t_0$, $i, j \in \{1, 2\}$, $i \neq j$

Figure 5.3 illustrates the inventory pattern within a cycle t_0 for the vendor, for the buyers and for the total system. When the production starts IT^0 is minimum, and the inventory at the vendor is equal to zero. Moreover, the inventories at the buyers are just enough to satisfy their demands until the next deliveries arrive. The quantity ordered by buyer l is $d_l t_l$, $l = 1, 2$, and hence, the time required to produce the quantities ordered by buyers 1 and 2 initially is $\sum_{l=1}^{2} d_l t_l / P$. Therefore, when the production starts, inventory levels I_k^0, $k = 1, 2$, and IT^0 should be $d_k \sum_{l=1}^{2} d_l t_l / P$ and $\sum_{k=1}^{2} d_k \sum_{l=1}^{2} d_l t_l / P$, respectively. Then, from the latter value IT^0, IT keeps on increasing at a rate of $P - \sum_{l=1}^{2} d_l$ during the time needed to manufacture $t_0 \sum_{l=1}^{2} d_l$ units. This value represents the sum of the quantities ordered by both buyers during a cycle t_0. We can also see in Figure 5.3 that IT reaches its maximum at $t' = t_0 \sum_{l=1}^{2} d_l / P$, which represents the instant when the production finishes. Thus, the maximum value for IT is

$$IT^{t'} = \frac{\sum_{k=1}^{2} d_k \sum_{l=1}^{2} d_l t_l}{P} + (1 - \frac{\sum_{l=1}^{2} d_l}{P}) t_0 \sum_{l=1}^{2} d_l$$

Accordingly, $\overline{IT}$ can be written as follows

$$\overline{IT} = \frac{1}{t_0} \left[\frac{\sum_{k=1}^{2} d_k \sum_{l=1}^{2} d_l t_l}{P} t_0 + t_0 (1 - \frac{\sum_{l=1}^{2} d_l}{P}) \frac{t_0 \sum_{l=1}^{2} d_l}{2} \right] = \frac{\sum_{k=1}^{2} d_k \sum_{l=1}^{2} d_l t_l}{P} + (1 - \frac{\sum_{l=1}^{2} d_l}{P}) \frac{t_0 \sum_{l=1}^{2} d_l}{2}$$

Observe that this expression is very similar to the average cost in the classical EPQ model. It only differs from the typical EPQ expression in the first term which corresponds to the value of IT^0 when production starts.

Once $\overline{IT}$ is obtained, we can compute $\overline{I_0}$ in the following way

$$\overline{I_0} = \overline{IT} - \sum_{l=1}^{2} \overline{I_l} = \frac{\sum_{k=1}^{2} d_k \sum_{l=1}^{2} d_l t_l}{P} + (1 - \frac{\sum_{l=1}^{2} d_l}{P}) \frac{t_0 \sum_{l=1}^{2} d_l}{2} - \sum_{l=1}^{2} \frac{d_l t_l}{2} \quad (5.1)$$

Now, we can compute the average total cost. Since the buyers follow an EOQ pattern, the average cost for a buyer l, $l = 1, 2$, is easily obtained from the following expression

$$C_l = \frac{k_l}{t_l} + \frac{h_l d_l t_l}{2}$$

On the other hand, the average cost for the vendor is the sum of the average holding cost, that is, $h_0 \overline{I_0}$, where $\overline{I_0}$ is given by (5.1), plus the average setup cost, that is, k_0/t_0. Accordingly, C_0 can be expressed as follows

$$C_0 = \frac{k_0}{t_0} + h_0 \left[\frac{\sum_{k=1}^{2} d_k \sum_{l=1}^{2} d_l t_l}{P} + (1 - \frac{\sum_{l=1}^{2} d_l}{P}) \frac{t_0 \sum_{l=1}^{2} d_l}{2} - \sum_{l=1}^{2} \frac{d_l t_l}{2} \right]$$

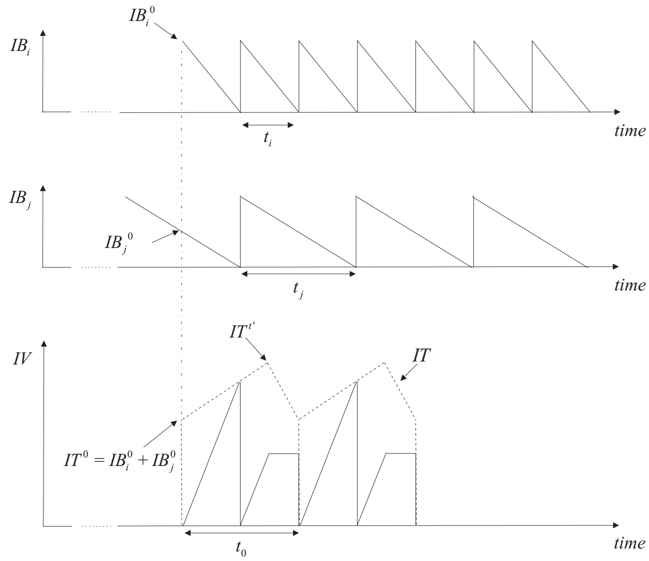

Figure 5.3: Inventory fluctuations at the vendor and at the buyers when $t_i \leq t_j \leq t_0$, $i, j \in \{1, 2\}$, $i \neq j$

Then, the total cost per unit time is

$$
\begin{aligned}
C_T &= C_0 + \sum_{l=1}^{2} C_l = \\
&= \frac{k_0}{t_0} + h_0 \left[\frac{\sum_{k=1}^{2} d_k \sum_{l=1}^{2} d_l t_l}{P} + (1 - \frac{\sum_{l=1}^{2} d_l}{P}) \frac{t_0 \sum_{l=1}^{2} d_l}{2} - \sum_{l=1}^{2} \frac{d_l t_l}{2} \right] + \sum_{l=1}^{2} (\frac{k_l}{t_l} + \frac{h_l d_l t_l}{2})
\end{aligned}
$$

For notation convenience we define

$$
H_0 = h_0 (1 - \frac{\sum_{l=1}^{2} d_l}{P}) \sum_{l=1}^{2} d_l
$$

$$H_l = d_l(h_l - h_0) + \frac{2h_0 d_l}{P}\sum_{k=1}^{2} d_k,\ \ l = 1,2$$

Then, the total cost C_T can be reformulated to give

$$C_T = \sum_{l=0}^{2}(\frac{k_l}{t_l} + \frac{t_l H_l}{2}) \qquad (5.2)$$

Case 2: $t_i \leq t_0 < t_j$, $i, j \in \{1, 2\}$, $i \neq j$

It is important to note that in this case the vendor only holds inventory for buyer j during the production time, but not after the shipment. Accordingly, we distinguish two types of inventories at the vendor at instant t: the inventory $I_{0,i}^t$ which will be used to satisfy the demand at buyer i, and the inventory $I_{0,j}^t$ which will be shipped to buyer j. That is, $I_0^t = I_{0,i}^t + I_{0,j}^t$. In addition, we denote by IT_i^t and $\overline{IT_i}$ the total inventory in the system for buyer i at instant t, and the average total inventory for buyer i during a cycle, respectively. Similarly, $\overline{I_{0,l}}$ represents the average inventory at the vendor for buyer l during a cycle, with $l = 1, 2$. In Figure 5.4, we show the inventories IT_i, $I_{0,i}$, $I_{0,j}$, I_i and I_j.

Notice that the inventory pattern for IT_i coincides with the form of IT in Case 1. Hence, $\overline{IT_i}$ is given by the following expression

$$\overline{IT_i} = \frac{1}{t_0}\left[\frac{d_i d_i t_i}{P}t_0 + t_0(1 - \frac{d_i}{P})\frac{t_0 d_i}{2}\right] = \frac{d_i d_i t_i}{P} + (1 - \frac{d_i}{P})\frac{t_0 d_i}{2}$$

and then, $\overline{I_{0,i}}$ can be computed as follows

$$\overline{I_{0,i}} = \overline{IT_i} - \overline{I_i} = \frac{d_i d_i t_i}{P} + (1 - \frac{d_i}{P})\frac{t_0 d_i}{2} - \frac{d_i t_i}{2}$$

Now we proceed to determine $\overline{I_{0,j}}$. Since buyer j always orders $d_j t_j$ units every t_j units of time, the vendor requires $d_j t_j / P$ units of time to produce them. Moreover, recall that each time buyer j orders, buyer i also places an order. Therefore, once the vendor has produced the units to be sent to buyer j, they are holding during the time that the vendor needs to produce the units ordered by buyer i, that is, during $d_i t_i / P$ units of time. Taking this into account, the average inventory at the vendor which will be shipped to buyer j is given by the following expression

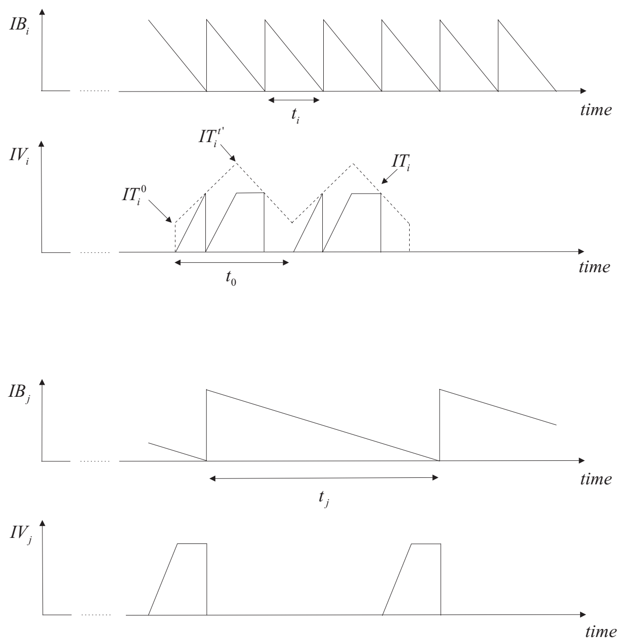

Figure 5.4: Inventory at buyer i and at buyer j, inventory at the vendor which will be shipped to buyer i and to buyer j, that is, IV_i and IV_j, and total inventory in the system for buyer i, i.e., IT_i, when $t_i \leq t_0 < t_j$, $i, j \in \{1, 2\}$, $i \neq j$

$$\overline{I_{0,j}} = \frac{1}{t_j}\left[\frac{d_j t_j}{P}\frac{d_j t_j}{2} + \frac{d_j t_j d_i t_i}{P}\right] = \frac{d_j d_j t_j}{2P} + \frac{d_j d_i t_i}{P}$$

Thus, the average total inventory at the vendor is

$$\overline{I_0} = \overline{I_{0,i}} + \overline{I_{0,j}} = \frac{d_i d_i t_i}{P} + (1 - \frac{d_i}{P})\frac{t_0 d_i}{2} - \frac{d_i t_i}{2} + \frac{d_j d_j t_j}{2P} + \frac{d_j d_i t_i}{P} =$$
$$= \frac{d_i t_i}{P}\sum_{l=1}^{2} d_l + \frac{d_j t_j}{P}\frac{d_j}{2} + (1 - \frac{d_i}{P})\frac{t_0 d_i}{2} - \frac{d_i t_i}{2}$$

Once we have determined $\overline{I_0}$, the total cost per unit time is easily derived

$$
\begin{aligned}
C_T &= C_0 + \sum_{l=1}^{2} C_l = \\
&= \frac{k_0}{t_0} + h_0 \left[\frac{d_i t_i}{P} \sum_{l=1}^{2} d_l + \frac{d_j d_j t_j}{2P} + (1 - \frac{d_i}{P}) \frac{t_0 d_i}{2} - \frac{d_i t_i}{2} \right] + \sum_{l=1}^{2} (\frac{k_l}{t_l} + \frac{h_l d_l t_l}{2})
\end{aligned}
$$

Moreover, if we define

$$
\begin{aligned}
H_0 &= h_0(1 - \frac{d_i}{P}) d_i \\
H_i &= d_i(h_i - h_0) + \frac{2h_0 d_i}{P} \sum_{l=1}^{2} d_l \\
H_j &= d_j h_j + \frac{h_0 d_j d_j}{P}
\end{aligned}
$$

the average total cost can be expressed as in (5.2).

Therefore, an optimal integer-ratio policy for the single-vendor two-buyer system with finite production rate can be found by solving the following problem

$$
\text{(P)} \qquad \min C_T = \sum_{l=0}^{2} (\frac{k_l}{t_l} + \frac{t_l H_l}{2}) \tag{5.3}
$$

s.t.

$$
\begin{aligned}
t_0 &= n_i t_i \text{ or } t_i = n_i t_0,\ n_i \text{ a positive integer} && (5.4) \\
t_0 &= n_j t_j \text{ or } t_j = n_j t_0,\ n_j \text{ a positive integer} && (5.5) \\
t_j &= m_i t_i,\ m_i \text{ a positive integer},\ i, j \in \{1, 2\}, i \neq j && (5.6) \\
\frac{\sum_{l=1}^{2} d_l t_l}{P} &\leq t_k,\ \ k = 1, 2 && (5.7) \\
t_k &> 0,\ \ k = 1, 2 && (5.8)
\end{aligned}
$$

where

$$
\begin{aligned}
H_0 &= h_0\left(1 - \frac{\sum\limits_{l\in\{k|t_k\le t_0\}} d_l}{P}\right) \sum_{l\in\{k|t_k\le t_0\}} d_l \\
H_i &= d_i(h_i - h_0) + \frac{2h_0 d_i}{P}\sum_{l=1}^{2} d_l \\
H_j &= \begin{cases} d_j(h_j - h_0) + \frac{2h_0 d_j}{P}\sum\limits_{l=1}^{2} d_l, & \text{if } t_j \le t_0 \\ \\ d_j h_j + \frac{h_0 d_j d_j}{P}, & \text{if } t_j > t_0 \end{cases}
\end{aligned}
$$

Constraints (5.4) and (5.5) yield integer-ratio policies. In addition, constraint (5.6) guarantees that each time buyer j places an order buyer i also orders, $i, j \in \{1, 2\}$, $i \neq j$. Finally, the constraint set in (5.7) ensures the feasibility of the solution, i.e., the vendor delivers the orders on time. Notice that the vendor supplies both buyers only at the beginning of each cycle. Then, we should guarantee that the time needed to produce $\sum_{l=1}^{2} d_l t_l$ units of item is smaller than the replenishment interval at each buyer.

Next, we develop a procedure for solving problem P.

5.3.1 Solution procedure

The single-vendor two-buyer problem with finite production rate given in (5.3)-(5.8) corresponds to a nonlinear mixed integer programming problem. We propose a solution method which combines the Karush-Kuhn-Tucker (KKT) conditions with a branch and bound scheme for determining the optimal integer values n_i, n_j and m_i.

Initially, we formulate the relaxed problem incorporating constraints given in (5.7) into the objective function (5.3) via the Lagrange multipliers technique and then, we obtain a feasible real-valued solution. That is, we solve the following relaxed problem

$$\text{(P1)} \qquad \min C_T = \sum_{l=0}^{2}(\frac{k_l}{t_l} + \frac{t_l H_l}{2}) \tag{5.9}$$

s.t.

$$\frac{\sum_{l=1}^{2} d_l t_l}{P} \leq t_k, \; k = 1,2 \tag{5.10}$$

$$t_k > 0, \; k = 1,2 \tag{5.11}$$

Once problem P1 is solved, we apply a branch and bound scheme to obtain an optimal solution for problem P.

First of all, we must verify that function C_T is convex.

Lemma 5.1 *C_T is convex over the region $R = \{(t_0, t_1, t_2) : t_0 > 0, t_1 > 0, t_2 > 0\}$*

Proof.

Let us denote by $H(t_0, t_1, t_2)$ the Hessian matrix. Then,

$$H(t_0,t_1,t_2) = \begin{pmatrix} \frac{\partial^2 C_T}{\partial t_0^2} & \frac{\partial^2 C_T}{\partial t_0 \partial t_1} & \frac{\partial^2 C_T}{\partial t_0 \partial t_2} \\ \frac{\partial^2 C_T}{\partial t_1 \partial t_0} & \frac{\partial^2 C_T}{\partial t_1^2} & \frac{\partial^2 C_T}{\partial t_1 \partial t_2} \\ \frac{\partial^2 C_T}{\partial t_2 \partial t_0} & \frac{\partial^2 C_T}{\partial t_2 \partial t_1} & \frac{\partial^2 C_T}{\partial t_2^2} \end{pmatrix} = \begin{pmatrix} \frac{2k_0}{t_0^3} & 0 & 0 \\ 0 & \frac{2k_1}{t_1^3} & 0 \\ 0 & 0 & \frac{2k_2}{t_2^3} \end{pmatrix}$$

It is easy to see that the Hessian matrix is positive definite in the region R. Thus, C_T is strictly convex on R. ■

Since function C_T is convex and the constraints given in (5.10) are linear, the KKT conditions are both necessary and sufficient. That is, any feasible solution for problem P1 satisfying the KKT conditions is an optimal solution.

Let λ_i and λ_j be the Lagrange multipliers related to constraints $\sum_{l=1}^{2} d_l t_l / P \leq t_i$ and $\sum_{l=1}^{2} d_l t_l / P \leq t_j$, respectively, with $i, j \in \{1, 2\}$, $i \neq j$. Then, the Lagrangian function can be stated as follows

$$L(t_0, t_i, t_j, \lambda_i, \lambda_j) = \sum_{l=0}^{2}(\frac{k_l}{t_l} + \frac{t_l H_l}{2}) - \lambda_i \left[(P - d_i)t_i - d_j t_j\right] - \lambda_j \left[(P - d_j)t_j - d_i t_i\right]$$

where λ_i and λ_j are nonnegative, $i, j \in \{1, 2\}$, $i \neq j$.

Consequently, the KKT conditions yield the following equations

$$\frac{\partial L}{\partial t_0} = -\frac{k_0}{t_0^2} + \frac{H_0}{2} \geq 0 \tag{5.12}$$

$$t_0 \frac{\partial L}{\partial t_0} = t_0 \left[-\frac{k_0}{t_0^2} + \frac{H_0}{2} \right] = 0 \tag{5.13}$$

$$\frac{\partial L}{\partial t_i} = -\frac{k_i}{t_i^2} + \frac{H_i}{2} - \lambda_i(P - d_i) + \lambda_j d_i \geq 0 \tag{5.14}$$

$$t_i \frac{\partial L}{\partial t_i} = t_i \left[-\frac{k_i}{t_i^2} + \frac{H_i}{2} - \lambda_i(P - d_i) + \lambda_j d_i \right] = 0 \tag{5.15}$$

$$\frac{\partial L}{\partial t_j} = -\frac{k_j}{t_j^2} + \frac{H_j}{2} - \lambda_j(P - d_j) + \lambda_i d_j \geq 0 \tag{5.16}$$

$$t_j \frac{\partial L}{\partial t_j} = t_j \left[-\frac{k_j}{t_j^2} + \frac{H_j}{2} - \lambda_j(P - d_j) + \lambda_i d_j \right] = 0 \tag{5.17}$$

$$\frac{\partial L}{\partial \lambda_i} = -(P - d_i)t_i + d_j t_j \leq 0 \tag{5.18}$$

$$\lambda_i \frac{\partial L}{\partial \lambda_i} = \lambda_i \left[-(P - d_i)t_i + d_j t_j \right] = 0 \tag{5.19}$$

$$\frac{\partial L}{\partial \lambda_j} = -(P - d_j)t_j + d_i t_i \leq 0 \tag{5.20}$$

$$\lambda_j \frac{\partial L}{\partial \lambda_j} = \lambda_j \left[-(P - d_j)t_j + d_i t_i \right] = 0 \tag{5.21}$$

Since $t_l > 0$, for $l = 0, 1, 2$, from (5.13), (5.15) and (5.17) we obtain the following replenishment intervals

$$t_0 = \sqrt{\frac{2k_0}{H_0}} \tag{5.22}$$

$$t_i = \sqrt{\frac{2k_i}{H_i - 2\lambda_i(P - d_i) + 2\lambda_j d_i}} \tag{5.23}$$

$$t_j = \sqrt{\frac{2k_j}{H_j - 2\lambda_j(P - d_j) + 2\lambda_i d_j}} \tag{5.24}$$

In order to find those values t_i, t_j, λ_i and λ_j that satisfy the KKT conditions, we must consider the following three cases.

Case 1: $\lambda_i = \lambda_j = 0$, $i, j \in \{1, 2\}$, $i \neq j$.

From (5.23) and (5.24), we obtain that $t_i = \sqrt{2k_i/H_i}$ and $t_j = \sqrt{2k_j/H_j}$. If these replenishment intervals satisfy the KKT conditions (5.18) and (5.20), then, they represent an optimal solution for problem P1, and hence, we do not need to analyze more cases.

Case 2: $\lambda_i = 0$ and $\lambda_j > 0$, $i, j \in \{1, 2\}$.

Now, using (5.23) and (5.24), we have that

$$t_i = \sqrt{\frac{2k_i}{H_i + 2\lambda_j d_i}} \tag{5.25}$$

$$t_j = \sqrt{\frac{2k_j}{H_j - 2\lambda_j (P - d_j)}} \tag{5.26}$$

Moreover, since $\lambda_j > 0$, (5.21) yields $d_i t_i = (P - d_j)t_j$, that is, $t_i = (P - d_j)t_j/d_i$. Now, substituting (5.25) and (5.26) into $t_i = (P - d_j)t_j/d_i$, we can isolate the Lagrange multiplier λ_j to give

$$\lambda_j = \frac{k_i d_i^2 H_j - k_j (P - d_j)^2 H_i}{2d_i(P - d_j)\left[k_j(P - d_j) + k_i d_i\right]} \tag{5.27}$$

Once λ_j is determined, the values t_i and t_j are calculated from (5.25) and (5.26). If this solution satisfies the KKT condition (5.18) we have an optimal solution for problem P1.

Case 3: $\lambda_i > 0$ and $\lambda_j > 0$, $i, j \in \{1, 2\}$, $i \neq j$.

In this case, (5.19) and (5.21) yield the following equations

$$-(P - d_i)t_i + d_j t_j = 0 \tag{5.28}$$

$$d_i t_i - (P - d_j)t_j = 0 \tag{5.29}$$

Consequently, if $d_j d_i \neq (P - d_i)(P - d_j)$ the only solution is $t_i = t_j = 0$, which is not a feasible solution. In other case, if $d_j d_i = (P - d_i)(P - d_j)$, then, it follows that $P = d_i + d_j$. Recall that we are assuming that $P > d_i + d_j$. If $P = d_i + d_j$, from (5.29) we have that $t_i = t_j = t$. In addition, the cost function C_T is reduced to

$$C_T = C_0 + \sum_{l=1}^{2} C_l = \frac{k_0}{t_0} + \frac{\sum_{l=1}^{2} k_l}{t} + \frac{t}{2}\sum_{l=1}^{2}(h_0 + h_l)d_l$$

and then, the solution is trivial.

Once the relaxed problem P1 is solved, the optimal real-valued n_i^*, n_j^* and m_i^* can be obtained from (5.4)-(5.6). However, in most cases these values are not integer. Hence, we develop the following algorithm to determine the optimal integer values n_i, n_j and m_i. This algorithm is based on a branch and bound scheme where at each branch we analyze a possible integer value for n_i, n_j or m_i.

Depending on how the replenishment intervals t_0, t_i and t_j are related, the total cost C_T and the constraints given by (5.4)-(5.6) will be expressed in a different way. Thus, we must consider the following cases.

Case 1: $t_0 = n_i t_i$, $t_0 = n_j t_j$ and $t_j = m_i t_i$, $i, j \in \{1, 2\}, i \neq j$.

In this case, it suffices to consider the constraints $t_0 = n_j t_j$ and $t_j = m_i t_i$, since $t_0 = n_i t_i$ can be obtained from the other two. Hence we only need to determine n_j and m_i.

First, we compute the optimal real values m_i^* and n_j^* as t_j/t_i and t_0/t_j, respectively, where t_0, t_i and t_j are the replenishment intervals which solve problem P1. The next step consists of using a branch and bound algorithm to search the optimal integer values m_i and n_j. Accordingly, we choose as initial feasible solution, i.e., as the initial upper bound, UB, the feasible solution with lower cost among the following candidates: ($m_i = \lfloor m_i^* \rfloor$, $n_j = \lfloor n_j^* \rfloor$), ($m_i = \lfloor m_i^* \rfloor$, $n_j = \lceil n_j^* \rceil$), ($m_i = \lceil m_i^* \rceil$, $n_j = \lfloor n_j^* \rfloor$), and ($m_i = \lceil m_i^* \rceil$, $n_j = \lceil n_j^* \rceil$). Note that the enumeration tree generated by the branch and bound only has two levels. In addition, at the first level we analyze all possible values for m_i. For a fixed m_i, we isolate t_i as a function of both t_j and m_i, that is, $t_i = t_j/m_i$, and we formulate the cost function as follows

$$C_T = \frac{k_0}{t_0} + \frac{t_0 H_0}{2} + \frac{1}{t_j}\left(m_i k_i + k_j\right) + \frac{t_j}{2}\left(\frac{H_i}{m_i} + H_j\right) \tag{5.30}$$

Taking the derivative of (5.30) with respect to t_j and setting it equal to zero, we obtain the expression of the replenishment interval t_j depending on value m_i

$$t_j(m_i) = \sqrt{\frac{2(m_i k_i + k_j)}{\frac{H_i}{m_i} + H_j}} \tag{5.31}$$

Now, it is easy to see that C_T in (5.30) can be rewritten as follows

$$C_T(m_i) = \sqrt{2k_0H_0} + \sqrt{2(m_ik_i + k_j)(\frac{H_i}{m_i} + H_j)} \tag{5.32}$$

Notice that $C_T(m_i)$ in (5.32) corresponds to a lower bound for the branch that we are inspecting, which is associated with value m_i.

It is worth noting that for this case the feasibility constraints in (5.10) can be expressed as follows

$$\begin{aligned} \frac{d_i}{m_in_j} + \frac{d_j}{n_j} &\leq \frac{P}{m_in_j} \\ \frac{d_i}{m_in_j} + \frac{d_j}{n_j} &\leq \frac{P}{n_j} \end{aligned}$$

Obviously, the second feasibility constraint is redundant, and hence it is enough to check the following inequality

$$\frac{d_i}{m_i} + d_j \leq \frac{P}{m_i} \tag{5.33}$$

Taking this into account, it can be proved that the value m_i satisfying (5.33) for which the function $C_T(m_i)$ in (5.32) is minimum, coincides with the integer nearest to $m_i^* = t_j/t_i$ which satisfies (5.33). We denote such a value by m_i'.

Thus, if the cost associated with a value $m_i = \widehat{m_i}, \widehat{m_i} > m_i'$ or $\widehat{m_i} < m_i'$, exceeds the cost of the current feasible solution, then, the analysis of all other values $m_i > \widehat{m_i}$ or $m_i < \widehat{m_i}$, can be discarded since their corresponding costs will also exceed the cost of the current feasible solution. This fact allow us to deal with an enumeration tree which theoretically has an infinite number of branches at each level.

Similarly, at the second level of the tree we study all possible values for n_j. To that end, we isolate t_j as a function of t_0 and n_j, that is, $t_j = t_0/n_j$, and we rewrite the cost C_T in (5.30) in the following way

$$C_T = \frac{1}{t_0}\left[k_0 + n_j(m_ik_i + k_j)\right] + \frac{t_0}{2}[H_0 + \frac{1}{n_j}(\frac{H_i}{m_i} + H_j)] \tag{5.34}$$

Taking the derivative of (5.34) with respect to t_0 and setting it equal to zero, we obtain

$$t_0(n_j) = \sqrt{\frac{2\,[k_0 + n_j(m_i k_i + k_j)]}{H_0 + \frac{1}{n_j}(\frac{H_i}{m_i} + H_j)}} \tag{5.35}$$

Substituting (5.35) into (5.34), we have that the total cost in (5.34) can be stated as follows

$$C_T(n_j) = \sqrt{2\,[k_0 + n_j(m_i k_i + k_j)]\,[H_0 + \frac{1}{n_j}(\frac{H_i}{m_i} + H_j)]} \tag{5.36}$$

Now, it is easy to see that the value n_j which minimizes (5.36) coincides with the integer nearest to $t_0/t_j(m_i)$, where $t_j(m_i)$ is given by (5.31). In what follows, we denote this value for n'_j. Notice that when we are inspecting any branch at the second level of the tree, the value m_i is fixed. Also note, that in this case the feasibility constraints hold for all values n_j.

As at the first level, if the cost associated with a value $n_j = \widehat{n_j}$, $\widehat{n_j} > n'_j$ or $\widehat{n_j} < n'_j$, is greater than the cost of the current feasible solution, then, it is not necessary to analyze all other values $n_j > \widehat{n_j}$ or $n_j < \widehat{n_j}$.

Finally, each time a new feasible solution for problem P with smaller cost than the current upper bound, UB, is found, such upper bound should be updated.

Once values m_i and n_j have been computed, the replenishment interval t_0 is obtained from (5.35). Then, using that $t_0 = n_j t_j$ and $t_j = m_i t_i$, the replenishment intervals t_j and t_i can be computed as $t_j = t_0/n_j$ and $t_i = t_j/m_i$, respectively.

Case 2: $t_0 = n_i t_i$, $t_j = n_j t_0$ and $t_j = m_i t_i$, $i, j \in \{1, 2\}, i \neq j$.

Under this situation, the last constraint can be obtained from the other two. Hence we only need to determine n_j and n_i. In this case, the optimal real values n^*_j and n^*_i can be computed as t_j/t_0 and t_0/t_i, respectively, where t_0, t_i and t_j are the replenishment intervals obtained by solving problem P1.

As in Case 1, in order to determine an initial feasible solution, we analyze the following solutions: ($n_j = \lfloor n^*_j \rfloor$, $n_i = \lfloor n^*_i \rfloor$), ($n_j = \lfloor n^*_j \rfloor$, $n_i = \lceil n^*_i \rceil$), ($n_j = \lceil n^*_j \rceil$, $n_i = \lfloor n^*_i \rfloor$), and ($n_j = \lceil n^*_j \rceil$, $n_i = \lceil n^*_i \rceil$). The combination which yields the best feasible solution is chosen as an upper bound for problem P given in (5.3)-(5.8). Then, at the first level of the tree, we analyze all possible values for n_j. Taking into account that $t_j = n_j t_0$ we can write the cost function as follows

$$C_T = \frac{1}{t_0}(k_0 + \frac{k_j}{n_j}) + \frac{t_0}{2}(H_0 + n_j H_j) + \frac{k_i}{t_i} + \frac{t_i H_i}{2} \tag{5.37}$$

Taking the derivative of (5.37) with respect to t_0 and setting it equal to zero, we have

$$t_0(n_j) = \sqrt{\frac{2(k_0 + \frac{k_j}{n_j})}{H_0 + n_j H_j}} \tag{5.38}$$

Then, the total cost C_T in (5.37) can be reformulated to give

$$C_T(n_j) = \sqrt{2(k_0 + \frac{k_j}{n_j})(H_0 + n_j H_j)} + \sqrt{2k_i H_i} \tag{5.39}$$

Now, the feasibility constraints in (5.10) can be expressed as follows

$$d_i t_i + d_j n_j t_0(n_j) \leq P t_i \tag{5.40}$$

$$d_i t_i + d_j n_j t_0(n_j) \leq P n_j t_0(n_j) \tag{5.41}$$

Therefore, the value n_j satisfying (5.40) and (5.41) for which the function $C_T(n_j)$ in (5.39) is minimum coincides with the integer nearest to $n_j^* = t_j/t_0$ which satisfies (5.40) and (5.41). We denote such a value by n_j'.

Again, if the cost associated with a value $n_j = \widehat{n}_j$, $\widehat{n}_j > n_j'$ or $\widehat{n}_j < n_j'$, exceeds the cost of the current feasible solution, then, the analysis of all other values $n_j > \widehat{n}_j$ or $n_j < \widehat{n}_j$, can be discarded since their corresponding costs will also exceed the cost of the current feasible solution.

Next, at the second level of the tree we study all possible values for n_i. To that end, we isolate t_i as a function of t_0 and n_i, that is, $t_i = t_0/n_i$, and we formulate the cost C_T in (5.37) as follows

$$C_T = \frac{1}{t_0}(k_0 + \frac{k_j}{n_j} + n_i k_i) + \frac{t_0}{2}(H_0 + n_j H_j + \frac{H_i}{n_i}) \tag{5.42}$$

Taking the derivative of (5.42) with respect to t_0 and setting it equal to zero, we obtain

$$t_0(n_i) = \sqrt{\frac{2(k_0 + \frac{k_j}{n_j} + n_i k_i)}{H_0 + \frac{H_i}{n_i} + n_j H_j}} \tag{5.43}$$

Then, using (5.43), the total cost (5.42) can be rewritten as follows

$$C_T(n_i) = \sqrt{2(k_0 + \frac{k_j}{n_j} + n_i k_i)(H_0 + \frac{H_i}{n_i} + n_j H_j)} \tag{5.44}$$

At this level, the feasibility constraints in (5.10) are reduced to

$$\frac{d_i}{n_i} + d_j n_j \leq \frac{P}{n_i} \tag{5.45}$$

Now, the value n_i satisfying (5.45) for which the function $C_T(n_i)$ in (5.44) is minimum coincides with the integer nearest to $t_0(n_j)/t_i$ which satisfies (5.45), where $t_0(n_j)$ is given by (5.38). Notice that, at this point, n_j is fixed.

Once again, if the cost associated with a value $n_i = \widehat{n_i}$, $\widehat{n_i} > n_i'$ or $\widehat{n_i} < n_i'$, is greater than the cost of the current feasible solution, then, it is not necessary to analyze all other values $n_i > \widehat{n_i}$ or $n_i < \widehat{n_i}$.

Similar to Case 1, each time we find a feasible solution with an associated cost smaller than the upper bound, UB, we should update UB and proceed with the branch and bound.

Once values n_j and n_i have been computed, the replenishment interval t_0 is obtained from (5.43). Moreover, t_i and t_j can be easily determined from equations $t_0 = n_i t_i$ and $t_j = n_j t_0$.

5.3.2 Numerical example

Now, we present an instance of the single-vendor two-buyer system in order to illustrate the solution method developed in the previous subsection. The parameter values are given in Table 5.1.

Table 5.1: Input data for an instance of the single-vendor two-buyer problem

	d_j	k_j	h_j	P
Buyer 1	10	10	12	
Buyer 2	12	1000	20	
Vendor		50	10	50

In subsection 5.3.1 we consider the following two cases: Case 1: $t_0 = n_i t_i$, $t_0 = n_j t_j$ and $t_j = m_i t_i$, $i, j \in \{1, 2\}, i \neq j$, and Case 2: $t_0 = n_i t_i$, $t_j = n_j t_0$ and

$t_j = m_i t_i,\ i, j \in \{1, 2\}, i \neq j$. It is obvious, that each case involves two situations depending on if $i = 1$ and $j = 2$, or viceversa. Therefore, we actually distinguish the following four cases.

Case 1: $t_0 = n_i t_i$, $t_0 = n_j t_j$ and $t_j = m_i t_i$, with $i = 1$ and $j = 2$.

For this case, we obtain the following values

$$
\begin{aligned}
H_0 &= h_0(1 - \frac{d_1 + d_2}{P})(d_1 + d_2) = 123.2 \\
H_1 &= d_1(h_1 - h_0) + \frac{2h_0 d_1}{P}(d_1 + d_2) = 108 \\
H_2 &= d_2(h_2 - h_0) + \frac{2h_0 d_2}{P}(d_1 + d_2) = 225.6
\end{aligned}
$$

First, we have to solve the relaxed problem P1. Accordingly, we must analyze the different possibilities for the Lagrange multipliers.

Case 1: $\lambda_1 = \lambda_2 = 0$.

If both Lagrange multipliers are equal to zero, we have that $t_1 = \sqrt{2k_1/H_1} = 0.4303$ and $t_2 = \sqrt{2k_2/H_2} = 2.9774$. However, these replenishment intervals do not satisfy conditions (5.18) and (5.20). Therefore, we have to analyze other values for λ_1 and λ_2.

Case 2: $\lambda_i = 0$ and $\lambda_j > 0$.

Here, we should in turn distinguish two situations, namely, if $i = 1$ and $j = 2$ and, on the contrary, if $i = 2$ and $j = 1$. In the former case, (5.27) implies $\lambda_2 < 0$, which violates the initial assumptions.

In the second situation, from (5.27) we obtain $\lambda_1 = 1.0032$. Now, (5.25) and (5.26) yield $t_1 = 0.8490$ and $t_2 = 2.8302$, respectively. In addition, we can compute t_0 from (5.22) to give $t_0 = 0.9009$. Since these replenishment intervals satisfy the KKT conditions, Case 3 is omitted.

Once problem P1 is solved, the next step consists of applying the branch and bound scheme introduced in subsection 5.3.1 to determine the integer values n_2 and m_1. Note that n_1 can be computed from n_2 and m_1 as $n_1 = n_2 m_1$. Thus, we compute the optimal real values m_1^* and n_2^* as $m_1^* = t_2/t_1 = 2.8302/0.8490 = 3.3335$ and $n_2^* = t_0/t_2 = 0.9009/2.8302 = 0.3183$. Then, we analyze the following solutions: ($m_1 = 3$, $n_2 = 1$) and ($m_1 = 4$, $n_2 = 1$). For each combination we should check if the feasibility constraint (5.33) is satisfied. Since the values ($m_1 = 4$, $n_2 = 1$) not yield a feasible solution, we only compute the cost associated with the values ($m_1 = 3$, $n_2 = 1$). From (5.36) we have that such a cost is 911.6841 $/*time unit*, which represents an initial upper bound for problem P given in (5.3)-(5.8). Next, we start with the branch and bound procedure to find the optimal integer values m_1 and n_2. At the first level we consider all possible values for m_1, and at the second, we analyze the values for n_2. We verify that all solutions generated by the branch and bound are worse than the initial upper bound. Hence, the optimal solution corresponds to the values ($m_1 = 3$, $n_2 = 1$). From (5.35), we have that $t_0 = 2.3692$, and, $t_2 = t_0/n_2 = 2.3692$ and $t_1 = t_2/m_1 = 0.7897$. Remark that the cost incurred by this policy is 911.6841 $\$/time\ unit$.

Case 2: $t_0 = n_i t_i$, $t_0 = n_j t_j$ and $t_j = m_i t_i$, with $i = 2$ and $j = 1$.

Under these assumptions problem P1 does not change with respect to the previous case. Therefore, the replenishment intervals which solve problem P1 continue to be $t_0 = 0.9009$, $t_1 = 0.8490$ and $t_2 = 2.8302$.

Now, we have to determine the integer values n_1 and m_2. Observe that the value n_2 can be derived from $n_2 = n_1 m_2$. The optimal real values m_2^* and n_1^* are computed as follows: $m_2^* = t_1/t_2 = 0.8490/2.8302 = 0.2999$ and $n_1^* = t_0/t_1 = 0.9009/0.8490 = 1.0611$, and the two admissible combinations ($m_2 = 1$, $n_1 = 1$) and ($m_2 = 1$, $n_1 = 2$) are feasible. In addition, an initial upper bound for problem P given in (5.3)-(5.8), is 984.0812 $\$/time\ unit$ which corresponds to the values ($m_2 = 1$, $n_1 = 1$). This upper bound allow us to discard branches in the enumeration tree. It can be easily verified that all other branches of the tree have an associated cost greater than the upper bound. Thus, for this case, the optimal solution corresponds to the values ($m_2 = 1$, $n_1 = 1$). Now, we can determine t_0 from (5.35) to give $t_0 = 2.1542$, and hence, $t_1 = t_0/n_1 = 2.1542$ and $t_2 = t_1/m_2 = 2.1542$. The cost incurred by this policy is 984.0812 $\$/time\ unit$.

Case 3: $t_0 = n_i t_i$, $t_j = n_j t_0$ and $t_j = m_i t_i$, with $i = 1$ and $j = 2$.

In this case we must recalculate values H_0 and H_2. In particular,

$$\begin{aligned} H_0 &= h_0(1 - \frac{d_1}{P})d_1 = 80 \\ H_2 &= d_2 h_2 + \frac{h_0 d_2 d_2}{P} = 268.8 \end{aligned}$$

The value H_1 does not change, that is, $H_1 = 108$.

Initially, we should solve problem P1 using the KKT conditions, and therefore, we face the following possibilities for the Lagrange multipliers.

Case 1: $\lambda_1 = \lambda_2 = 0$.

We have that $t_1 = \sqrt{2k_1/H_1} = 0.4303$ and $t_2 = \sqrt{2k_2/H_2} = 2.7277$. However, these replenishment intervals violate conditions (5.18) and (5.20). Therefore, we should analyze Case 2.

Case 2: $\lambda_i = 0$ and $\lambda_j > 0$.

If $i = 1$ and $j = 2$, (5.27) implies $\lambda_2 < 0$. Hence this case cannot occur.

On the contrary, if $i = 2$ and $j = 1$, applying (5.27), we obtain $\lambda_1 = 0.9451$, and using (5.25) and (5.26), we have that $t_1 = 0.7857$ and $t_2 = 2.6194$. Moreover, the replenishment interval t_0 can be computed from (5.22) to give $t_0 = 1.1180$. Since these replenishment intervals satisfy the KKT conditions it is not needed to consider Case 3.

The next step consists of computing the integer values n_1 and n_2. Note that value m_1 can be obtained from n_1 and n_2 as $m_1 = n_1 n_2$.

Now, we compute the optimal real values $n_1^* = t_0/t_1 = 1.1180/0.7857 = 1.4229$ and $n_2^* = t_2/t_0 = 2.6194/1.1180 = 2.3429$. Therefore, in order to find an initial feasible solution we analyze the following combinations: ($n_1 = 1$, $n_2 = 2$), ($n_1 = 1$, $n_2 = 3$), ($n_1 = 2$, $n_2 = 2$) and ($n_1 = 2$, $n_2 = 3$). Only solutions ($n_1 = 1$, $n_2 = 2$) and ($n_1 = 1$, $n_2 = 3$) are feasible. Moreover, the initial upper bound for problem P given in (5.3)-(5.8) is 884.4553 $\$/time\ unit$, which corresponds to combination ($n_1 = 1$, $n_2 = 3$). In addition, this solution cannot be improved since all solutions analyzed by the branch and bound yield worse solutions. Hence, for this case, the optimal solution corresponds to the values ($n_1 = 1$, $n_2 = 3$). The replenishment interval t_0 associated with these values is calculated from (5.43) to give $t_0 = 0.8894$. Hence, $t_2 = n_2 t_0 = 2.6682$ and $t_1 = t_0/n_1 = 0.8894$. The cost incurred by this policy is 884.4553 $\$/time\ unit$.

Case 4: $t_0 = n_i t_i$, $t_j = n_j t_0$ and $t_j = m_i t_i$, with $i = 2$ and $j = 1$.

In this case, values H_0, H_1 and H_2 are given by

$$\begin{aligned} H_0 &= h_0(1 - \frac{d_2}{P})d_2 = 91.2 \\ H_1 &= d_1 h_1 + \frac{h_0 d_1 d_1}{P} = 140 \\ H_2 &= d_2(h_2 - h_0) + \frac{2h_0 d_2}{P}(d_1 + d_2) = 225.6 \end{aligned}$$

In order to solve problem P1, we must analyze the different possibilities for the Lagrange multipliers.

Case 1: $\lambda_1 = \lambda_2 = 0$.

If both Lagrange multipliers are equal to zero, then, $t_1 = \sqrt{2k_1/H_1} = 0.3779$ and $t_2 = \sqrt{2k_2/H_2} = 2.9774$. Since these replenishment intervals do not satisfy conditions (5.18) and (5.20), we have to verify the other cases for λ_1 and λ_2.

Case 2: $\lambda_i = 0$ and $\lambda_j > 0$.

If $i = 1$ and $j = 2$, applying (5.27) we obtain that $\lambda_2 < 0$, which violates the initial assumptions.

If $i = 2$ and $j = 1$, from (5.27) we have that $\lambda_1 = 1.3903$, and from (5.25) and (5.26) we obtain that $t_1 = 0.8336$ and $t_2 = 2.7790$. The value t_0 is computed from (5.22) to give $t_0 = 1.0471$. Since these replenishment intervals satisfy the KKT conditions we do not need to consider Case 3.

At this point the relaxed problem P1 is already solved, and hence we apply the branch and bound scheme to determine the optimal integer values n_1 and n_2. Note that m_2 can be derived from n_1 and n_2 using the relation $m_2 = n_1 n_2$.

First, we compute the optimal real values $n_1^* = t_1/t_0 = 0.8336/1.0471 = 0.7961$ and $n_2^* = t_0/t_2 = 1.0471/2.7790 = 0.3767$. Therefore, only the combination ($n_1 = 1$, $n_2 = 1$) should be analyzed. Moreover, this solution verifies the feasibility constraint (5.45), and its cost is 984.0812 $\$/time\ unit$. Thus, this cost represents an initial upper bound for problem P given in (5.3)-(5.8). Since the costs of all other solutions analyzed by the branch and bound are greater than the current upper bound, the optimal solution corresponds to the values ($n_1 = 1$, $n_2 = 1$). Applying (5.43) we have that $t_0 = 2.1542$, and hence, $t_1 = n_1 t_0 = 2.1542$ and $t_2 = t_0/n_2 = 2.1542$. The cost incurred by this policy is 984.0812 $\$/time\ unit$.

We summarize in Table 5.2 the solutions obtained for this numerical example.

Table 5.2: Summary of policies obtained for the numerical example

Case	Optimal solution	Optimal cost
$t_0 = n_1 t_1$, $t_0 = n_2 t_2$ and $t_2 = m_1 t_1$	$(t_0 = 2.3692, t_1 = 0.7897, t_2 = 2.3692)$	911.6841
$t_0 = n_1 t_1$, $t_0 = n_2 t_2$ and $t_1 = m_2 t_2$	$(t_0 = 2.1542, t_1 = 2.1542, t_2 = 2.1542)$	984.0812
$t_0 = n_1 t_1$, $t_2 = n_2 t_0$ and $t_2 = m_1 t_1$	$(t_0 = 0.8894, t_1 = 0.8894, t_2 = 2.6682)$	884.4553
$t_1 = n_1 t_0$, $t_0 = n_2 t_2$ and $t_1 = m_2 t_2$	$(t_0 = 2.1542, t_1 = 2.1542, t_2 = 2.1542)$	984.0812

Therefore, the optimal solution for problem P is given by $t_0 = 0.8894$, $t_1 = 0.8894$ and $t_2 = 2.6682$, and the cost incurred by this policy is 884.4553 $\$/time\ unit$.

It is worth noting that in the optimal integer-ratio solution the replenishment interval at buyer 2 is greater than t_0. This is due to the fact that the ordering cost at buyer 2 is much higher than the ordering cost at buyer 1 and than the setup cost at the vendor. Under this situation, it is not surprising that it is preferable that buyer 2 places order less often.

Many previous works force the replenishment intervals at the buyers to be smaller than the time between setups at the vendor. However, in practice it can be profitable to allow some buyers to have a replenishment interval greater than the time between setups at the vendor.

To this point we have studied the single-vendor two-buyer system. In the next section, we extend the analysis to the multi-buyer case.

5.4 The single-vendor multi-buyer problem

Now, we are extending the results obtained in the previous section in order to compute integer-ratio policies for the general case with multiple buyers. Finally, we also solve the problem considering that the decision system is decentralized, that is, each buyers make decisions independently.

5.4.1 Integer-ratio policies

First, we focus on the formulation of the single-vendor multi-buyer problem in terms of integer-ratio policies where the quotients t_j/t_{j-1}, $j = 2, ..., N$, are positive integers. These constraints ensure that $t_1 \leq t_2 \leq ... \leq t_N$ and that each time a buyer with replenishment interval t_j orders, the remaining buyers with replenishment interval $t_i \leq t_j$ also order. Thus, similar to the two-buyer case, there will be points in time where the vendor should supply all buyers simultaneously, and others where it only supplies some of the buyers. Therefore, in the latter case, the vendor should start the production later than when the vendor supplies all buyers. Then, in this case the time interval between two consecutive setups may be also non-constant. However, as we explained for the two-buyer case, in order to formulate the problem we can also use t_0 which remains constant. Recall that t_0 represents the time interval between two consecutive setups when it is only considered the buyers with replenishment interval $t_j \leq t_v$. Moreover, if we take into account all buyers, including those with replenishment interval $t_j > t_v$, the time interval between two consecutive setups can be easily obtained from t_0. See Figure 5.5.

The key idea to formulate the problem consists of classifying the buyers into three sets, denoted by G, L and E. In Roundy (1985), those buyers with $t_j > t_v$ are allocated to set G. In set E are those buyers with $t_j = t_v$, and finally, those buyers with $t_j < t_v$ belong to set L. As we have commented before, when the production rate is finite, we should let the time interval between two consecutive setups to be non-constant. Hence, sets G, L and E cannot be defined as in Roundy (1985). However, once t_0 has been defined, we can use it to define sets G, L and E similar to those in Roundy (1985). In particular, we set $G = \{j|t_j > t_0\}$, $E = \{j|t_j = t_0\}$ and $L = \{j|t_j < t_0\}$.

In the next subsection we focus on computing the average inventory at the vendor and at the buyers. The idea is similar to that introduced for the single-vendor two-buyer problem.

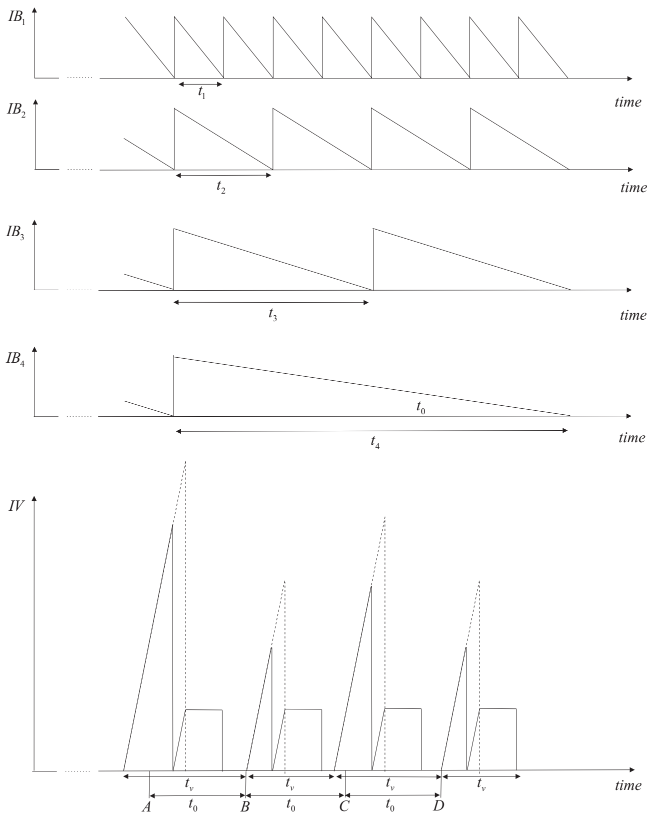

Figure 5.5: Inventory fluctuations at the vendor and at the buyers considering that t_v is non-constant. Buyer 1 and buyer 2 belongs to set $E \cup L$, and buyer 3 and buyer 4 are in set G. The dotted line represents the production rate. At instants A, B, C and D the vendor begins the production of the items which will be shipped to buyer 1 and buyer 2

Stock holding cost

First we state the production-inventory patterns. The cycle length is $T = \max_{j=1,\ldots,N}\{t_j\}$, and it starts with the first production setup.

For the previous example, Figure 5.6 illustrates the inventory pattern within a typical production cycle for the vendor, for the buyers and for the total system. Remark that buyer 1 and buyer 2 belong to set $E \cup L$ and buyer 3 and buyer 4 are in set G. Since the buyers follow the classical EOQ pattern, the holding cost per unit time at the buyers are directly obtained. On the other hand, the average inventory at the vendor cannot be computed so easily. For the single-vendor single-buyer problem, Hill (1999) showed that the average total inventory is directly derived. Thus, he computed the average inventory at the vendor as the average total inventory less the average inventory at the buyer.

However, we can see in Figure 5.6 that the pattern of total inventory, depicted with a dotted line, differs from the model with a single buyer, and it is more difficult to compute. It is important to note that the differences between both patterns are caused by the buyers in set G.

Recall that the vendor only holds inventory for the buyers in set G during the production time, but not after the shipment. Accordingly, in this case, we can also distinguish two types of inventory at the vendor: the inventory which will be used to satisfy the demand at buyers in set $E \cup L$, $I_{0,E\cup L}$, and the inventory which will be shipped to buyer $j \in G$, $I_{0,j}$. Thus, now $I_0^t = I_{0,E\cup L}^t + \sum_{j\in G} I_{0,j}^t$. Similar to the two-buyer case, we denote by $IT_{E\cup L}^t$ and $\overline{IT_{E\cup L}}$ the total inventory in the system for buyers in set $E \cup L$ at instant t, and the average total inventory for buyers in set $E \cup L$ during a cycle, respectively. In addition, $\overline{I_{0,E\cup L}}$ and $\overline{I_{0,j}}$, $j \in G$, represent the average inventory at the vendor during a cycle for buyers in set $E \cup L$ and for buyer $j \in G$, respectively. Then, the average vendor inventory, $\overline{I_0}$, can be computed as $\overline{I_0} = \overline{I_{0,E\cup L}} + \sum_{j\in G} \overline{I_{0,j}}$. Moreover, in Figure 5.7, we plot $I_{0,E\cup L}^t + \sum_{j\in E\cup L} I_j^t$, $t \in [0, T)$, which yields a pattern equal to that obtained for the single-buyer case. Thus, $\overline{I_{0,E\cup L}}$ can be determined as $\overline{IT_{E\cup L}} - \sum_{j\in E\cup L} \overline{I_j}$.

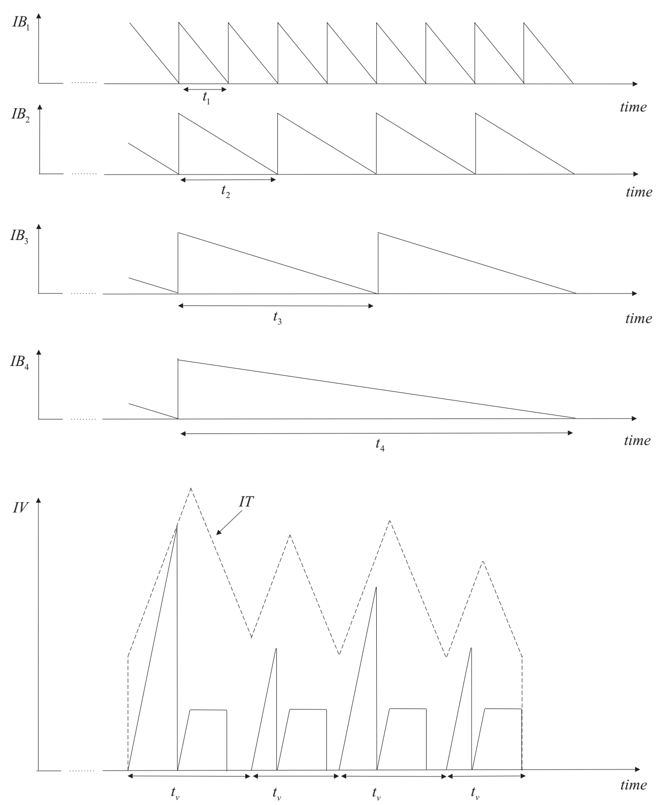

Figure 5.6: Inventory fluctuations at the vendor and at the buyers when t_v is non-constant. The dotted line represents the inventory fluctuations for the total system, IT.

We can determine $\overline{IT_{E\cup L}}$ using Figure 5.7. When production starts, the vendor has not inventory and the value of $IT^0_{E\cup L}$ is minimum. Moreover, the inventories at buyers in set $E \cup L$ are just enough to satisfy their demands until the next deliveries arrive. Taking into account that buyer j always orders $d_j t_j$, it is easy to see that I^0_j, $j \in E \cup L$, is $d_j \sum_{i\in E\cup L} d_i t_i/P$. Hence, $IT^0_{E\cup L}$ is given by $\sum_{j\in E\cup L} d_j \sum_{i\in E\cup L} d_i t_i/P$. Then, from the latter value $IT^0_{E\cup L}$, $IT^t_{E\cup L}$ keeps on increasing at a rate of $P - \sum_{j\in E\cup L} d_j$ during the time needed to manufacture the sum of the quantities demanded by buyers in set $E \cup L$ during a cycle t_0, that is, during $t_0 \sum_{j\in E\cup L} d_j$. Besides, in Figure 5.7 we can see that the instant where $IT_{E\cup L}$ reaches its maximum is $t' = t_0 \sum_{j\in E\cup L} d_j/P$ which coincides with the moment when the production finishes. Thus, the maximum value for $IT_{E\cup L}$ is

$$\frac{\sum_{j\in E\cup L} d_j \sum_{i\in E\cup L} d_i t_i}{P} + (1 - \frac{\sum_{j\in E\cup L} d_j}{P}) t_0 \sum_{j\in E\cup L} d_j$$

Accordingly, $\overline{IT_{E\cup L}}$ can be written as follows

$$\begin{aligned} \overline{IT_{E\cup L}} &= \frac{1}{t_0}\left[\frac{\sum_{j\in E\cup L} d_j \sum_{i\in E\cup L} d_i t_i}{P} t_0 + t_0(1 - \frac{\sum_{j\in E\cup L} d_j}{P}) \frac{t_0 \sum_{j\in E\cup L} d_j}{2}\right] = \\ &= \frac{\sum_{j\in E\cup L} d_j \sum_{i\in E\cup L} d_i t_i}{P} + (1 - \frac{\sum_{j\in E\cup L} d_j}{P}) \frac{t_0 \sum_{j\in E\cup L} d_j}{2} \end{aligned}$$

Observe that this expression is an extension of that obtained for the two-buyer case.

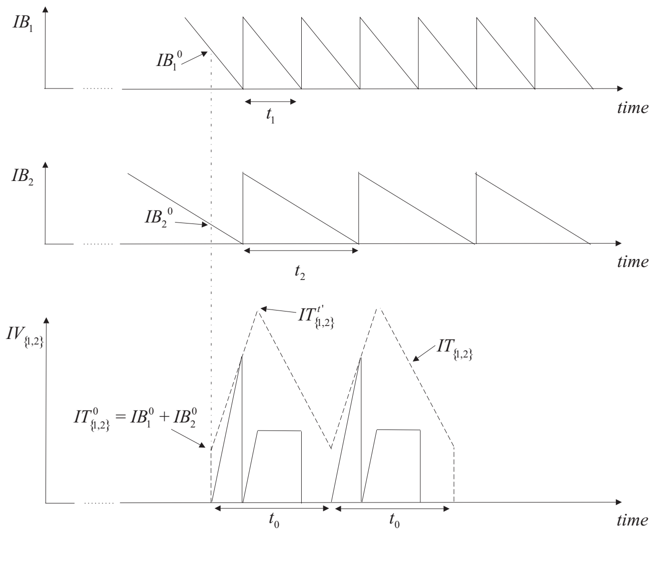

Figure 5.7: Inventory fluctuations at buyer 1 and buyer 2, and inventory located at the vendor which will be used to satisfy the demand at buyer 1 and buyer 2, namely, $IV_{E\cup L}$. The dotted line shows the sum of both inventories, that is, $IT_{E\cup L}$

Now, $\overline{I_{0,E\cup L}}$ can be computed from the following expression

$$\begin{aligned}\overline{I_{0,E\cup L}} &= \overline{IT_{E\cup L}} - \sum_{j\in E\cup L} \overline{I_j} = \\ &= \frac{\sum\limits_{j\in E\cup L} d_j \sum\limits_{i\in E\cup L} d_i t_i}{P} + (1 - \frac{\sum\limits_{j\in E\cup L} d_j}{P})\frac{t_0 \sum\limits_{j\in E\cup L} d_j}{2} - \sum_{j\in E\cup L} \frac{d_j t_j}{2}\end{aligned}$$

Next, we focus on determining $\overline{I_{0,j}}$, for each buyer $j \in G$.

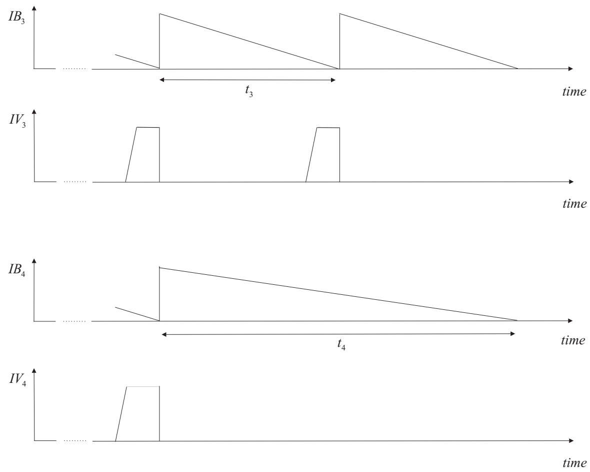

Figure 5.8: Inventory at buyer 3 and buyer 4, and inventory located at the vendor which will be shipped to buyer 3, IV_3, and to buyer 4, IV_4

As it can be seen in Figure 5.8, for buyers in set G the vendor only holds inventory during the production time, but not after the shipment. In addition, the vendor requires $d_j t_j/P$ units of time to produce the quantity ordered by buyer $j \in G$ every t_j. Moreover, since we are assuming that t_j/t_{j-1} is a positive integer for $j = 2, ..., N$, it holds that $t_1 \leq t_2 \leq ... \leq t_N$ and, each time a buyer with replenishment interval t_j orders, the remaining buyers with replenishment interval $t_i \leq t_j$ also order. Therefore, when a buyer $j \in G$ places an order all buyers $i < j$ also order. As it is shown in Figure 5.8, in this case, once the vendor has produced the units which will be sent to buyer $j \in G$ they are holding during the time that the vendor needs to produce the units for all buyers i, $i < j$, that is, during $\sum_{i=1}^{j-1} d_i t_i / P$ units of time. Accordingly, we can compute the average inventory at the vendor which will be shipped to buyer $j \in G$ as follows

$$\overline{I_{0,j}} = \frac{1}{t_j}\left[\frac{d_j t_j}{P}\frac{d_j t_j}{2} + \frac{d_j t_j \sum_{i=1}^{j-1} d_i t_i}{P}\right] = \frac{d_j t_j d_j}{2P} + \frac{d_j \sum_{i=1}^{j-1} d_i t_i}{P}$$

Thus, the average total inventory at the vendor is given by

$$\begin{aligned}
&\overline{I_0} = \overline{I_{0,E\cup L}} + \sum_{j\in G}\overline{I_{0,j}} = \\
= \;& \frac{\sum_{j\in E\cup L} d_j \sum_{i\in E\cup L} d_i t_i}{P} + (1 - \frac{\sum_{j\in E\cup L} d_j}{P})\frac{t_0 \sum_{j\in E\cup L} d_j}{2} - \frac{\sum_{j\in E\cup L} d_j t_j}{2} + \\
&+\sum_{j\in G}(\frac{d_j t_j d_j}{2P} + \frac{d_j \sum_{i=1}^{j-1} d_i t_i}{P}) \\
= \;& \frac{\sum_{j\in E\cup L} d_j \sum_{i\in E\cup L} d_i t_i}{P} + \sum_{j\in G}\frac{d_j t_j d_j}{2P} + \sum_{j\in G}\frac{d_j \sum_{i\in E\cup L} d_i t_i}{P} + \\
&+\sum_{j\in G}\frac{d_j \sum_{i\in G\cap\{i/i<j\}} d_i t_i}{P} + (1 - \frac{\sum_{j\in E\cup L} d_j}{P})\frac{t_0 \sum_{j\in E\cup L} d_j}{2} - \frac{\sum_{j\in E\cup L} d_j t_j}{2}
\end{aligned}$$

which can be rewritten as follow

$$\begin{aligned}
\overline{I_0} \;=\; & \sum_{i\in E\cup L}\frac{d_i t_i}{P}\sum_{j=1}^{N} d_j + \sum_{j\in G}\frac{d_j t_j}{P}(\frac{d_j}{2} + \sum_{i=j+1}^{N} d_i) + && (5.46)\\
& +(1 - \frac{\sum_{j\in E\cup L} d_j}{P})\frac{t_0 \sum_{j\in E\cup L} d_j}{2} - \frac{\sum_{j\in E\cup L} d_j t_j}{2} && (5.47)
\end{aligned}$$

Once we have the average total inventory at the vendor, the average total cost is simple to obtain. We address this task in the next subsection.

Average total cost

Since the buyers follow an EOQ pattern, the average total cost for a buyer can be easily determined from the following expression

$$C_j = \frac{k_j}{t_j} + \frac{h_j d_j t_j}{2} \tag{5.48}$$

On the other hand, the average total cost for the vendor is the sum of the average holding cost, that is, $h_0\overline{I_0}$, where $\overline{I_0}$ is given by (5.46), plus the average setup cost, that is k_0/t_0. Accordingly, C_0 can be expressed as follows

$$\begin{aligned} C_0 &= \frac{k_0}{t_0} + h_0\left[\sum_{i\in E\cup L}\frac{d_i t_i}{P}\sum_{j=1}^{N} d_j + \sum_{j\in G}\frac{d_j t_j}{P}\left(\frac{d_j}{2} + \sum_{i=j+1}^{N} d_i\right)\right] + \\ &\quad + h_0\left[\left(1 - \frac{\sum_{j\in E\cup L} d_j}{P}\right)\frac{t_0 \sum_{j\in E\cup L} d_j}{2} - \frac{\sum_{j\in E\cup L} d_j t_j}{2}\right] \end{aligned} \tag{5.49}$$

Then, the total cost per unit time is

$$\begin{aligned} C_T &= C_0 + \sum_{j=1}^{N} C_j = \\ &= \frac{k_0}{t_0} + h_0\left[\sum_{i\in E\cup L}\frac{d_i t_i}{P}\sum_{j=1}^{N} d_j + \sum_{j\in G}\frac{d_j t_j}{P}\left(\frac{d_j}{2} + \sum_{i=j+1}^{N} d_i\right)\right] + \\ &\quad + h_0\left[\left(1 - \frac{\sum_{j\in E\cup L} d_j}{P}\right)\frac{t_0 \sum_{j\in E\cup L} d_j}{2} - \frac{\sum_{j\in E\cup L} d_j t_j}{2}\right] \\ &\quad + \sum_{j=1}^{N}\left(\frac{k_j}{t_j} + \frac{h_j d_j t_j}{2}\right) \end{aligned} \tag{5.50}$$

Since $t_j = t_0$, $j \in E$, we can rearrange (5.50) to give

$$C_T = \frac{K_0}{t_0} + \frac{t_0 H_0}{2} + \sum_{j\in L\cup G}\left[\frac{k_j}{t_j} + \frac{t_j H_j}{2}\right] \tag{5.51}$$

where

$$
\begin{aligned}
K_0 &= k_0 + \sum_{j \in E} k_j \\
H_0 &= h_0\Big(1 - \frac{\sum_{j \in E \cup L} d_j}{P}\Big) \sum_{j \in E \cup L} d_j + h_0 \frac{2 \sum_{j \in E} d_j}{P} \sum_{j=1}^{N} d_j + \sum_{j \in E} d_j (h_j - h_0) \\
H_j &= \begin{cases} d_j(h_j - h_0) + \frac{2h_0 d_j}{P} \sum_{i=1}^{N} d_i & \text{if } j \in L \\ d_j h_j + \frac{2h_0 d_j}{P}\big(\frac{d_j}{2} + \sum_{i=j+1}^{N} d_i\big) & \text{if } j \in G \end{cases}
\end{aligned}
$$

It is worth noting that these expressions extend those obtained for the two-buyer case. It is also important to remark that the vendor holds inventory not only during the time until the buyers place an order but also while the whole lot is produced. Hence the expression H_j consists of two parts. Specifically, the first term $d_j(h_j - h_0)$, if $j \in L$, or $d_j h_j$, if $j \in G$, represents the echelon holding cost and the classical holding cost at buyer j, respectively. The second term corresponds to the holding cost incurred by the vendor during the time needed to produce the units required by the buyers. Moreover, notice that the expressions for H_0 and H_j are consistent with those obtained by Roundy (1985) for the case where the production is instantaneous, that is, $P \to \infty$.

The constraints

The cost formulation in (5.51) is based on the integer-ratio constraints, i.e., both t_j/t_0, $j \in G$, and t_0/t_j, $j \in L$, should be a positive integer. Furthermore, we also assume that each time that a buyer places an order, those buyers with smaller replenishment intervals also order, that is, the quotients t_j/t_{j-1}, $j = 2, ..., N$, are positive integers. However, it is easy to see that some of the constraints can be dropped because they can be obtained from two others. Accordingly, if we assume that $L = \{1, ..., l\}$, $E = \{l+1, ..., e\}$ and $G = \{e+1, ..., N\}$, then, it suffices to consider the following constraints

$$
\begin{aligned}
t_j &= r_{j-1} t_{j-1},\ r_{j-1} \text{ a positive integer, } j \in \{2, .., l\} \\
t_0 &= r_l t_l,\ r_l \text{ a positive integer} \\
t_{e+1} &= r_{e+1} t_0,\ r_{e+1} \text{ a positive integer} \\
t_j &= r_j t_{j-1},\ r_j \text{ a positive integer, } j \in \{e+2, .., N\}
\end{aligned}
$$

Moreover, to ensure the feasibility of a solution we require that the vendor delivers the orders on time. Remark that the vendor supplies all buyers only at the beginning of each cycle. Then, we should guarantee that the time needed to produce $\sum_{j=1}^{N} d_j t_j$ units of item is smaller than the replenishment intervals at the buyers. That is,

$$\frac{\sum\limits_{j=1}^{N} d_j t_j}{P} \leq t_i, \quad i = 1, ..., N$$

Therefore, the single-vendor multi-buyer problem can be formulated as follows

$$\min C_T = \frac{K_0}{t_0} + \frac{t_0 H_0}{2} + \sum_{j \in L \cup G} \left[\frac{k_j}{t_j} + \frac{t_j H_j}{2} \right] \tag{5.52}$$

s.t.

$$\begin{aligned}
t_j &= r_{j-1} t_{j-1},\ r_{j-1} \text{ a positive integer, } j \in \{2, .., l\} && (5.53)\\
t_0 &= r_l t_l,\ r_l \text{ a positive integer} && (5.54)\\
t_{e+1} &= r_{e+1} t_0,\ r_{e+1} \text{ a positive integer} && (5.55)\\
t_j &= r_j t_{j-1},\ r_j \text{ a positive integer, } j \in \{e+2, .., N\} && (5.56)\\
\frac{\sum\limits_{j=1}^{N} d_j t_j}{P} &\leq t_i, \quad i \in \{1, ..., N\} && (5.57)
\end{aligned}$$

Algorithm for computing sets G, L and E

The procedure that computes sets G, L and E is based on Algorithm 2.2 proposed by Roundy (1985) and it is given in detail in Algorithm 5.1.

For notation convenience we again relabel the buyers, so that, $L = \{1, ..., l\}$, $E = \{l + 1, ..., e\}$ and $G = \{e + 1, ..., N\}$.

Notice that, in general, we cannot guarantee that this approach for sorting the buyers yields an optimal classification. This is due to the fact that in the expression of the replenishment interval at a buyer $i \in G$ are involved the parameters of other buyers $j \in G$.

Once we have shown how to determine sets G, L and E, we next introduce a heuristic procedure for computing near-optimal integer-ratio policies.

Algorithm 5.1 Algorithm for computing sets G, L and E

Step 1

Set $E = G = \emptyset$, $L = \{1, ..., N\}$. Then C_T can be written as

$$C_T = \frac{K_0}{t_0} + \frac{t_0 H_0}{2} + \sum_{j \in L \cup G} \left[\frac{k_j}{t_j} + \frac{t_j H_j}{2} \right]$$

where $H_0 = h_0(1 - \sum_{j \in L} d_j/P) \sum_{j \in L} d_j$ and $H_j = d_j(h_j - h_0) + 2h_0 d_j \sum_{i=1}^{N} d_i/P$. Differentiating C_T with respect to t_0 and t_j's we obtain the following replenishment intervals

$$t_0 = \left[\frac{2k_0}{H_0} \right]^{1/2} \tag{5.58}$$

$$t_j = \left[\frac{2k_j}{H_j} \right]^{1/2}, \; j \in \{1, ..., N\} \tag{5.59}$$

Now, we should sort the values t_j's to give a nondecreasing sequence. Without loss of generality we can assume that $t_1 \leq t_2 \leq ... \leq t_N$. Set $i = N$.

Step 2

If $t_i \geq t_0$, update E and L as follows: $E \longleftarrow E \cup \{i\}$, and $L \longleftarrow L \backslash \{i\}$

Since sets E and L have changed, H_0 and K_0 should be recalculated.

Assuming that $i \in G$, we compute the replenishment interval at the vendor and at buyer i, and we denote them by t_0' and t_i', respectively. Afterward, if $t_i' > t_0'$ sets E and G have to be updated as follows: $E \longleftarrow E \backslash \{i\}$, and $L \longleftarrow L \cup \{i\}$

Consequently, H_0, K_0 and H_i have to be updated and t_0 is set to t_0'.

Set $i = i - 1$. If $i > 0$ go to Step 2. Otherwise, go to Step 3.

Step 3

Using sets G, L and E we can compute the final values of H_0, K_0 and H_j, $j = 1, ..., N$. Then, set

$$t_0 = \left[\frac{2K_0}{H_0} \right]^{1/2} \tag{5.60}$$

$$t_j = t_0, \; j \in E \tag{5.61}$$

$$t_j = \left[\frac{2k_j}{H_j} \right]^{1/2}, \; j \in \{1, ..., N\} \tag{5.62}$$

Heuristic approach

In general, obtaining the total average cost for the single-vendor multi-buyer problem with finite production rate is an arduous task. However, the use of the integer-ratio policies facilitates the determination of the total average cost. Accordingly, the problem can be formulated as in (5.52)-(5.57). Nevertheless, this problem is a nonlinear mixed integer programming problem and computing its optimal solution could be computationally inefficient for a large number of buyers. In this section, we develop a heuristic method based on an iterative approach for solving problem (5.52)-(5.57) with minimum computational effort.

First, we compute sets $L = \{1, ..., l\}$, $E = \{l+1, ..., e\}$ and $G = \{e+1, ..., N\}$ and then, we determine the replenishment intervals t_i's by using (5.60)-(5.62). Obviously, these replenishment intervals only solve the relaxed problem, that is, dropping the constraints (5.53)-(5.57). If these replenishment intervals do not satisfy the constraint set in (5.57), then we add such a set into the objective function using Lagrange multipliers and we solve the dual problem. Thus, we have an initial solution which minimizes (5.52) subject to (5.57). A scheme of the Lagrangian relaxation is described at the end of this section.

Next, we compute the optimal values r_i^*, $i = 1, ...l, e+1, ..., N$, substituting the replenishment intervals corresponding to the initial solution into constraints given in (5.53)-(5.56). However, in most cases, these values are not integers. Hence, we will use an iterative approach to determine near-optimal integer values r_i, $i = 1, ...l, e+1, ..., N$. This approach can be represented by a decision tree with $N-|E|$ levels, and at each level we will analyze two possible integer values for r_i, $i = 1, ...l, e+1, ..., N$. Remark that values r_i, $i = l+1, ..., e$, are not considered in the formulation of problem (5.52)-(5.57) because they correspond to buyers in set E.

At each iteration we use one of the constraints in (5.53)-(5.56) to isolate the corresponding replenishment interval as a function of the value r_i. Thus, we can express the cost function in terms of the value r_i. In order to decide which branch of the decision tree should be explored in the next iteration, we compute the total cost for the two values that we analyze for r_i. Accordingly, the node with greatest cost associated will be discarded.

Since there are four kind of constraints and at each iteration we use one of them to compute a value r_i, we should distinguish the following cases:

Case 1: If $i \in \{1, ..., l-1\}$.

In this iteration, the values r_1,..., r_{i-1} have been already calculated and we want to compute the value r_i. Moreover, we know the values t_j's and it holds that $t_j = r_{j-1}t_{j-1}$, $j \in \{2, ..., i\}$. Now, we introduce the constraint $t_{i+1} = r_i t_i$ which can be applied to obtain the real value r'_i as

$$r'_i = \frac{t_{i+1}}{t_i} \tag{5.63}$$

Then, depending on the value of the cost function, we will choose between $r_i = \lceil r'_i \rceil$ or $r_i = \lfloor r'_i \rfloor$. If $r'_i < 1$, we set $r_i = 1$.

Now, taking into account that $t_j = r_{j-1}t_{j-1}$, $j \in \{2, ..., i+1\}$, the total cost C_T given in (5.52) can be reformulated in terms of t_0, t_j, with $j \in \{i+1, ..., l\} \cup \{e+1, ..., N\}$ and r_i. Notice that values r_j's, $j \in \{1, ..., i-1\}$, have been already determined and therefore they are fixed values. Thus, the total cost (5.52) can be stated as follows

$$C_T = \frac{K_0}{t_0} + \frac{t_0 H_0}{2} + \frac{K'_{i+1}}{t_{i+1}} + \frac{t_{i+1}H'_{i+1}}{2} + \sum_{j\in\{i+2,...,l\}\cup G} \left[\frac{k_j}{t_j} + \frac{t_j H_j}{2}\right] \tag{5.64}$$

where

$$\begin{aligned} K'_{i+1} &= k_{i+1} + \sum_{j=1}^{i} (\prod_{u=j}^{i-1} r_u) r_i k_j \\ H'_{i+1} &= H_{i+1} + \sum_{j=1}^{i} (\prod_{u=j}^{i-1} \frac{1}{r_u}) \frac{1}{r_i} H_j \end{aligned}$$

Then, taking the derivative of (5.64) with respect to t_{i+1} and setting it equal to zero, we obtain the expression of the new replenishment interval at buyer $i+1$, for known values r_j's , $j \in \{1, ..., i-1\}$. That is,

$$t_{i+1} = \left[\frac{2K'_{i+1}}{H'_{i+1}}\right]^{1/2} \tag{5.65}$$

It is worth noting that the replenishment intervals t_0 and t_j's, $j \in \{i+2, ..., l\} \cup G$, do not change. On the other hand, for each buyer $j \in \{1, ..., i\}$, the replenishment interval can be recalculated using constraints in (5.53).

Thus, the cost function in (5.64) can be rearranged using (5.65) to give

$$C_T(r_i) = \sqrt{2K_0H_0} + \sqrt{2K'_{i+1}H'_{i+1}} + \sum_{j\in\{i+2,...,l\}\cup G} \sqrt{2k_jH_j} \tag{5.66}$$

At this point, taking into account (5.63), we compute $C_T(r_i = \lfloor r'_i \rfloor)$ and $C_T(r_i = \lceil r'_i \rceil)$. We set $r_i = \lfloor r'_i \rfloor$ if $C_T(r_i = \lfloor r'_i \rfloor) < C_T(r_i = \lceil r'_i \rceil)$. Otherwise, we set $r_i = \lceil r'_i \rceil$. If $r'_i < 1$, we set $r_i = 1$.

Once the integer value r_i is determined, we calculate the new replenishment interval t_{i+1} from (5.65). The replenishment intervals t_j's, $j = 1, ..., i$, are also updated using constraints in (5.53), that is, $t_j = t_{j+1}/r_j$, $j = 1, ..., i$.

Case 2: If $i = l$.

In this case, the values r_1,..., r_{l-1} are already known and we use the constraint $t_0 = r_lt_l$ to determine the real value r'_l. That is,

$$r'_l = \frac{t_0}{t_l} \tag{5.67}$$

Moreover, since $t_j = r_{j-1}t_{j-1}$, for those buyers $j \in \{2, ..., l\}$ and $t_0 = r_lt_l$, it is easy to see that the cost function given in (5.52) can be reformulated in terms of t_0, t_j, $j \in \{e+1, ..., N\}$ and r_l to give

$$C_T = \frac{K'_0}{t_0} + \frac{t_0H'_0}{2} + \sum_{j=e+1}^{N} \left[\frac{k_j}{t_j} + \frac{t_jH_j}{2}\right] \tag{5.68}$$

where

$$\begin{aligned} K'_0 &= K_0 + \sum_{j=1}^{l} (\prod_{u=j}^{l-1} r_u) r_l k_j \\ H'_0 &= H_0 + \sum_{j=1}^{l} (\prod_{u=j}^{l-1} \frac{1}{r_u}) \frac{1}{r_l} H_j \end{aligned}$$

Taking the derivative of (5.68) with respect to t_0 and setting it equal to zero, we have that the expression for the new replenishment interval t_0 is given by

$$t_0 = \left[\frac{2K'_0}{H'_0}\right]^{1/2} \tag{5.69}$$

Substituting (5.69) into (5.68), the total cost C_T in (5.68) can be formulated as follows

$$C_T(r_l) = \sqrt{2K_0'H_0'} + \sum_{j\in G}\sqrt{2k_jH_j} \tag{5.70}$$

Now, if $C_T(r_l = \lfloor r_l' \rfloor) < C_T(r_l = \lceil r_l' \rceil)$ we set $r_l = \lfloor r_l' \rfloor$. On the contrary, we set $r_l = \lceil r_l' \rceil$. Again, if $r_l' < 1$, we set $r_l = 1$.

Afterward, the new replenishment interval t_0 is computed using (5.69). The replenishment interval t_l is updated taking into account constraint (5.54), i.e., $t_l = t_0/r_l$, and the values t_j's, $j = 1, ..., l-1$, are recalculated using constraints in (5.53).

Case 3: If $i = e+1$.

Recall that values r_i, $i = l+1, ..., e$, are not considered because they correspond to buyers belonging to set E.

Now, we have already computed the values r_1,..., r_l, and the objective is to determine the value r_{e+1}. In this case, the real value r_{e+1}' is obtained from the constraint $t_{e+1} = r_{e+1}t_0$. Thus, we have

$$r_{e+1}' = \frac{t_{e+1}}{t_0} \tag{5.71}$$

Similar to the previous cases, we can reformulate the cost function given in (5.52) to depend only on the values t_j's with $j \in \{e+1, ..., N\}$ and r_{e+1}. That is, C_T can be written as follows

$$C_T = \frac{{K_{e+1}}'}{t_{e+1}} + \frac{t_{e+1}H_{e+1}'}{2} + \sum_{j=e+2}^{N}\left[\frac{k_j}{t_j} + \frac{t_jH_j}{2}\right] \tag{5.72}$$

where

$$\begin{aligned}
K_{e+1}' &= k_{e+1} + r_{e+1}K_0 + \sum_{j=1}^{l}(\prod_{u=j}^{l} r_u)r_{e+1}k_j \\
H_{e+1}' &= H_{e+1} + \frac{H_0}{r_{e+1}} + \sum_{j=1}^{l}(\prod_{u=j}^{l}\frac{1}{r_u})\frac{1}{r_{e+1}}H_j
\end{aligned}$$

Hence, from (5.72) we obtain that the new replenishment interval at buyer $e+1$ is given by the following expression

$$t_{e+1} = \left[\frac{2K'_{e+1}}{H'_{e+1}}\right]^{1/2} \tag{5.73}$$

Next, similar to Case 1 and Case 2, (5.72) can be reformulated using (5.73) to give

$$C_T(r_{e+1}) = \sqrt{2K'_{e+1}H'_{e+1}} + \sum_{j\in G}\sqrt{2k_jH_j} \tag{5.74}$$

Now, we set $r_{e+1} = \lfloor r'_{e+1}\rfloor$ if $C_T(r_{e+1} = \lfloor r'_{e+1}\rfloor) < C_T(r_{e+1} = \lceil r'_{e+1}\rceil)$. Otherwise, we set $r_{e+1} = \lfloor r'_{e+1}\rfloor$. If $r'_{e+1} < 1$, we set $r_{e+1} = 1$.

Finally, substituting the value r_{e+1} into (5.73) we have the new replenishment interval t_{e+1}. Moreover, from (5.55) we determine the new replenishment interval t_0 as $t_0 = t_{e+1}/r_{e+1}$, and the replenishment intervals t_l and t_j's, $j = 1, ..., l-1$, are also updated using the constraints (5.54) and (5.53), respectively.

Case 4: If $i \in \{e+2, ..., N\}$.

We must compute the value r_i taking into account that the values r_1,..., r_l and r_{e+1},..., r_{i-1} have been already calculated. Furthermore, the constraints (5.53)-(5.55) hold, and also $t_j = r_jt_{j-1}$, $j \in \{e+2, ..., i-1\}$. In this iteration we introduce the constraint $t_i = r_it_{i-1}$, which is used to determine the real value r'_i as follows

$$r'_i = \frac{t_i}{t_{i-1}}$$

Thus, the total cost given in (5.52) can be expressed in the following way

$$C_T = \frac{K'_i}{t_i} + \frac{t_iH'_i}{2} + \sum_{j=i+1}^{N}\left[\frac{k_j}{t_j} + \frac{t_jH_j}{2}\right] \tag{5.75}$$

where

$$\begin{aligned} K'_i &= k_i + (\prod_{u=e+1}^{i-1} r_u)r_iK_0 + \sum_{j=e+1}^{i-1}(\prod_{u=j+1}^{i-1} r_u)r_ik_j + \sum_{j=1}^{l}(\prod_{u=j}^{l} r_u)(\prod_{u=e+1}^{i-1} r_u)r_ik_j \\ H'_i &= H_i + (\prod_{u=e+1}^{i-1}\frac{1}{r_u})\frac{1}{r_i}H_0 + \sum_{j=e+1}^{i-1}(\prod_{u=j+1}^{i-1}\frac{1}{r_u})\frac{1}{r_i}H_j + \sum_{j=1}^{l}(\prod_{u=j}^{l}\frac{1}{r_u})(\prod_{u=e+1}^{i-1}\frac{1}{r_u})\frac{1}{r_i}H_j \end{aligned}$$

In this case, the above expressions seems to be more complex than those obtained in the previous iterations. This is due to the fact that in this case the four kind of constraints are involved in the formulation of the cost.

Then, taking the derivative of (5.75) with respect to t_i and setting it equal to zero, we obtain the expression of the new replenishment interval t_i

$$t_i = \left[\frac{2K'_i}{H'_i}\right]^{1/2} \tag{5.76}$$

and, then, using (5.76) the total cost (5.75) can be written as follows

$$C_T(r_i) = \sqrt{2K'_i H'_i} + \sum_{j=i+1}^{N} \sqrt{2k_j H_j} \tag{5.77}$$

Now we compute $C_T(r_i = \lfloor r'_i \rfloor)$ and $C_T(r_i = \lceil r'_i \rceil)$. If $C_T(r_i = \lfloor r'_i \rfloor) < C_T(r_i = \lceil r'_i \rceil)$, then, $r_i = \lfloor r'_i \rfloor$; otherwise $r_i = \lceil r'_i \rceil$. Again, if $r'_i < 1$, we set $r_i = 1$.

Once the integer value r_i is determined, we compute the new replenishment interval t_i from (5.76). Similar to the above cases, the other replenishment intervals are recalculated using the constraints in (5.53)-(5.56).

After computing the corresponding replenishment intervals we should check if they satisfy the constraint set in (5.57). If at any level of the decision tree this set of constraints does not hold, then we add such a set into the objective function using Lagrange multipliers and we solve the dual problem. A sketch of the heuristic procedure is given in Algorithm 5.2.

Algorithm 5.2 Heuristic for computing integer-ratio policies

Step 1
Compute the values t_j's using (5.60)-(5.62) and check if the constraints in (5.57) hold. If these constraints are not satisfied, then add them into the objective function using the Lagrange multipliers and solve the dual problem.
Set $s = 1$ and go to Step 2.
Step 2
if $s \in \{1, ..., l-1\}$ (Case 1), **then**
  recalculate the replenishment intervals using (5.53)
end if
if $s = l$ (Case 2), **then**
  compute the new replenishment intervals from (5.53) and (5.54)
end if
if $s \in \{l+1, ..., e\}$ **then**
  go to Step 4
end if
if $s = e + 1$ (Case 3), **then**
  update the replenishment intervals using (5.53)-(5.55)
end if
if $s \in \{e+2, ..., N\}$ (Case 4), **then**
  detemine the new replenishment intervals from (5.53)-(5.56)
end if
Then prove if the constraints in (5.57) hold. If these constraints are satisfied go to Step 4. Otherwise, go to Step 3.
Step 3
Add such constraints into the objective function using the Lagrangian relaxation and solve the dual problem. Go to Step 4.
Step 4
Set $s = s + 1$. If $s \leq N$ go to Step 2. Otherwise all the replenishment intervals have been already computed. Stop.

Lagrangian relaxation Here we present a sketch of the Lagrangian relaxation which is used to minimize the total cost (5.52) subject to (5.57). Let $\mu_1, \mu_2, ..., \mu_N$ be the Lagrange multipliers associated with the constraints in (5.57). Then the Lagrangian function can be written as

$$L(t_0, ..., t_N, \overline{\mu}) = C_T(t_0, ..., t_N) + \overline{\mu} A \overline{t}^t$$

where

$$A = \begin{pmatrix} d_1 - P & d_2 & .. & d_N \\ d_1 & d_2 - P & .. & d_N \\ . & . & . & . \\ . & . & . & . \\ d_1 & d_2 & .. & d_N - P \end{pmatrix}, \overline{\mu} = (\mu_1, \mu_2, ..., \mu_N) \text{ and } \overline{t} = (t_1, t_2, ..., t_N).$$

It is important to note that at each iteration of the heuristic we isolate one value t_i, $i = 0, 1, ...N-1$, in terms of other value t_j. Taking this into account, the set of constraints in (5.57) changes at each iteration, and so, A has to be updated.

Now, the dual problem can be expressed as follows

$$\max_{\overline{\mu}} \theta(\overline{\mu})$$

where $\theta(\overline{\mu}) = \min L(t_0, ..., t_N, \overline{\mu})$.

In order to solve it, we will apply a standard subgradient method. Accordingly, we need to compute an upper bound UB. In this case UB is obtained by solving the problem assuming that all buyers use the same replenishment interval, that is, $t_j = t$, $j \in \{1, ..., N\}$, and also that $t_0 = nt$, with n a positive integer.

Under this situation, the cost function is reduced to

$$C_T = \frac{k_0}{t_0} + \frac{t_0 h_0 D}{2}(1 - \frac{D}{P}) + \frac{1}{t}\sum_{j=1}^{N} k_j + \frac{t}{2}\sum_{j=1}^{N} d_j(h_j - h_0) + \frac{2h_0}{P}D^2$$

where $D = \sum_{j=1}^{N} d_j$.

In addition, it is easy to see that $\sum_{j=1}^{N} d_j t_j / P \leq t_i$, $i \in \{1, .., N\}$ is equivalent to considering $D \leq P$. Therefore, the optimal t_0 is given by

$$t_0 = \left[\frac{2k_0}{h_0 D(1 - \frac{D}{P})}\right]^{1/2}$$

and the optimal t is computed as follows

$$t = \left[\frac{2\sum_{j=1}^{N} k_j}{\sum_{j=1}^{N} d_j(h_j - h_0) + \frac{2h_0}{P}D^2}\right]^{1/2}$$

Now, we can determine the optimal real value n as $n = t_0/t$. Then, taking into account that $t = t_0/n$ the cost function can be reformulated to give

$$C_T = \frac{k_0 + n\sum_{j=1}^{N} k_j}{t_0} + \frac{t_0}{2}\left[h_0 D(1-\frac{D}{P}) + \frac{1}{n}(\sum_{j=1}^{N} d_j(h_j - h_0) + \frac{2h_0}{P}D^2)\right]$$

Taking the derivative of C_T with respect to t_0 and setting it equal to zero we obtain that

$$t_0 = \left[\frac{2(k_0 + n\sum_{j=1}^{N} k_j)}{h_0 D(1-\frac{D}{P}) + \frac{1}{n}(\sum_{j=1}^{N} d_j(h_j - h_0) + \frac{2h_0}{P}D^2)}\right]^{1/2}$$

and then, the cost function can be expressed in the following way

$$C_T(n) = \sqrt{2(k_0 + n\sum_{j=1}^{N} k_j)\left[h_0 D(1-\frac{D}{P}) + \frac{1}{n}(\sum_{j=1}^{N} d_j(h_j - h_0) + \frac{2h_0}{P}D^2)\right]}$$

At this point, we compute $C_T(n = \lfloor n \rfloor)$ and $C_T(n = \lceil n \rceil)$. If $C_T(n = \lfloor n \rfloor) < C_T(n = \lceil n \rceil)$ we set $n = \lfloor n \rfloor$ and $UB_0 = C_T(n = \lfloor n \rfloor)$. Otherwise, we set $n = \lceil n \rceil$ and $UB_0 = C_T(n = \lceil n \rceil)$. If $n < 1$, we set $n = 1$.

In addition to the upper bound, we also have to choose an initial multiplier vector, $\overline{\mu}_0$, and determine the step size, θ. In particular, we set $\overline{\mu}_0$ equal to the zero vector and the step size at iteration j is given by the following expression

$$\theta_j = \frac{\lambda_j UB_j - L(\overline{\mu}_j)}{\left\|\overline{\mu}_{j+1} - \overline{\mu}_j\right\|^2}$$

where λ_j is a scalar which initially is set to 2. Then, λ_j is reduced by a factor of 2 whenever the best Lagrangian objective function value has not increased in three iterations. A scheme of the subgradient method is given in Algorithm 5.3.

Algorithm 5.3 Subgradient method

Initialization
Consider an initial upper bound UB_0
Fixe an initial multiplier vector $\overline{\mu}_0 \geq 0$
Set an initial factor $\lambda_0 \leftarrow 2$
for $j = 0, 1, ...,$ **do**
  $\theta_j = \frac{\lambda_j UB_j - L(\overline{\mu}_j)}{\left\|\overline{\mu}_{j+1} - \overline{\mu}_j\right\|^2}$ **{step size}**
  $\overline{\mu}_{j+1} = \max\{0, \overline{\mu}_j + \theta_j A\overline{t}\}$
  if $\left\|\overline{\mu}_{j+1} - \overline{\mu}_j\right\| < \epsilon$ **then**
    Stop
  end if
  if No improvements in more than three iterations then
    $\lambda_{j+1} = \frac{\lambda_j}{2}$
  end if
  $j \leftarrow j + 1$
end for

5.4.2 Decentralized policies

In this subsection we show how the problem should be addressed in case of independence among the vendor and the buyers. Under this situation, we propose a two-level optimization approach consisting of computing first the order quantities at the buyers, and then, determining the shipment schedule at the vendor. Accordingly, since the buyers follow an EOQ pattern, the total cost at buyer j can be easily obtained as $C_j = k_j/t_j + h_j d_j t_j/2$. Moreover, the optimal replenishment intervals at the buyers are given by the following expression

$$t_j^* = \sqrt{\frac{2k_j}{h_j d_j}},\ j \in \{1, ..., N\} \tag{5.78}$$

Taking into account that there is no relationship among these replenishment intervals, it seems obvious that the vendor behaves as an inventory system with time-varying demand. When the demand rate varies with time, the most widely known procedure for deriving the optimal solution is that credited to Wagner and Whitin (1958), although other more efficient approaches have been developed by Wagelmans et al. (1992), Federgruen and Tzur (1991) and Aggarwal and Park (1993). However, all these approaches consider infinite production rate, hence we cannot directly apply them in our case. Fortunately, Hill (1997a) showed how a dynamic lot-sizing problem with finite production rate can be reformulated to take the same form as the corresponding infinite production rate problem. Therefore, the Wagner and Whitin algorithm or any of the other techniques currently available, can be applied to the new reformulated problem.

The main problem is to determine the demand vector at the vendor. It is worth noting that the optimal replenishment intervals at the buyers are real values. Therefore, we cannot assure that a point in time exists where all buyers order simultaneously. Hence, the number of periods of the demand vector at the vendor could not be finite. To overcome this problem we use the approach introduced in Chapter 3 which consists of either truncating or rounding up the real replenishment intervals to rational times. It is clear that the solution provided by this method is not the optimal plan but it is quite a good approximation.

Once the demand vector is obtained, we should apply Hill's approach to obtain the optimal shipment schedule at the vendor.

5.4.3 Numerical example

In order to illustrate the solution procedures developed in this section, let us consider a single-vendor three-buyer system with the data given in Table 5.3.

Table 5.3: Input data for an instance of the single-vendor three-buyer problem

	d_j	k_j	h_j	P
Buyer 1	5	36	79	
Buyer 2	5	25	84	
Buyer 3	34	62	55	
Vendor		48	46	2853

Below we solve the problem using both the approach for computing integer-ratio policies and the decentralized procedure.

Integer-ratio policy

First, we must compute sets G, L and E using Algorithm 5.1. The steps involved in computing such sets are given below.

Step 1.

Set $E = G = \emptyset$ and $L = \{1, ..., N\}$. Then, from (5.58) and (5.59) we have $t_0 = 0.2194$, $t_1 = 0.6468$, $t_2 = 0.5036$ and $t_3 = 0.5916$.

Relabel the buyers so that, $t_1 \leq t_2 \leq t_3$. Thus, buyer 1 = buyer 2, buyer 2 = buyer 3 and buyer 3 = buyer 1, and $t_1 = 0.5036, t_2 = 0.5916$ and $t_3 = 0.6468$.

Set $i = 3$. Go to Step 2.

Step 2.

Iteration 1.

Since $t_3 = 0.6468 > t_0 = 0.2194$, update sets E and L so that, $E = \{3\}$ and $L = \{1, 2\}$.

Moreover, since $t'_3 = 0.4267 > t'_0 = 0.2329$, we conclude that buyer $3 \in G$. Hence, set $E = \emptyset$, $L = \{1, 2\}$, $G = \{3\}$, $t_3 = 0.4267$, $t_0 = 0.2329$ and $i = 2$.

Iteration 2.

Since $t_2 = 0.5916 > t_0 = 0.2329$, sets E and L are updated to give $E = \{2\}$ and $L = \{1\}$. Then, $t_0 = t_2 = 0.3218$.

Now, $t'_2 = 0.2558 < t'_0 = 0.6466$, so we cannot move buyer 2 to set G. Set $i = 1$.

Iteration 3.

Since $t_1 = 0.5036 > t_0 = 0.3218$, we update sets E and L so that, $E = \{1, 2\}$ and $L = \emptyset$. Moreover, as $t'_1 = 0.3445 > t'_0 = 0.3403$, we conclude that buyer $1 \in G$. Then, set $E = \{2\}$, $G = \{3, 1\}$, $t_1 = 0.3445$, $t_0 = 0.3403$ and $i = 0$. Go to Step 3.

Step 3.

The finals sets are $L = \emptyset$, $E = \{2\}$ and $G = \{1, 3\}$.

For notation convenience, we again relabel the buyers, so that buyer 1 = buyer 2 and buyer 2 = buyer 1. Then, $L = \emptyset$, $E = \{1\}$ and $G = \{2, 3\}$, and $t_0 = t_1 = 0.3403$, $t_2 = 0.3445$ and $t_3 = 0.4267$.

Now, using sets G, L and E, the problem can be stated as follows

$$\min \frac{K_0}{t_0} + \frac{t_0 H_0}{2} + \frac{k_2}{t_2} + \frac{t_2 H_2}{2} + \frac{k_3}{t_3} + \frac{t_3 H_3}{2}$$

s.t.

$$t_2 = r_2 t_0$$
$$t_3 = r_3 t_2$$

$$\frac{d_1t_0 + d_2t_2 + d_3t_3}{P} \leq t_0 \tag{5.79}$$

$$\frac{d_1t_0 + d_2t_2 + d_3t_3}{P} \leq t_2 \tag{5.80}$$

$$\frac{d_1t_0 + d_2t_2 + d_3t_3}{P} \leq t_3 \tag{5.81}$$

where

$$\begin{aligned}
K_0 &= k_0 + k_1 = 110 \\
H_0 &= h_0(1 - \frac{d_1}{P})d_1 + h_0\frac{2d_1}{P}\sum_{j=1}^{3} d_j + d_1(h_1 - h_0) = 1899.6025 \\
H_2 &= d_2h_2 + \frac{2h_0d_2}{P}(\frac{d_2}{2} + d_3) = 421.2092 \\
H_3 &= d_3h_3 + \frac{h_0d_3d_3}{P} = 395.4030
\end{aligned}$$

The next steps of the heuristic can be summarized as follows.

Step 1.

The initial replenishment intervals satisfied the feasibility constraints (5.79)-(5.81). Therefore, set $s = 1$ and go to Step 2.

Step 2.

Since $s = e$ go to Step 4.

Step 4.

Set $s = 2$ and go to Step 2.

Step 2.

From (5.74) it follows

$$C_T(r_2) = \sqrt{2(k_2 + r_2K_0)(H_2 + \frac{H_0}{r_2})} + \sqrt{2k_3H_3}$$

Since $r_2' = t_2/t_0 = 0.3445/0.3403 = 1.0123$, we choose between $r_2 = 1$ or $r_2 = 2$. As $C_T(r_2 = 1) = 960.3203 < C_T(r_2 = 2) = 988.3586$, we set $r_2 = 1$. Now, using (5.73) we recalculate t_2 to give $t_2 = 0.3410$. Hence, $t_1 = t_0 = t_2/r_2 = 0.5978$. Moreover, it is easy to see that the feasibility constraints (5.79)-(5.81) hold, and hence we proceed to go to Step 4.

Step 4.

Set $s = 3$ and go to Step 2.

Step 2.

Now (5.77) yields

$$C_T(r_3) = \sqrt{2(k_3 + r_3 k_2 + r_3 r_2 K_0)(H_3 + \frac{H_2}{r_3} + \frac{H_0}{r_3 r_2})}$$

Since $r'_3 = t_3/t_2 = 0.4267/0.3410 = 1.2513$, we should choose between $r_3 = 1$ or $r_3 = 2$. Given that $C_T(r_3 = 1) = 963.8181 < C_T(r_3 = 2) = 975.7843$, we set $r_3 = 1$. Next, from (5.76) we recalculate t_3 to give $t_3 = 0.3548$. Hence, $t_2 = t_3/r_3 = 0.3548$ and $t_1 = t_0 = t_2/r_2 = 0.3548$. Note that these replenishment intervals are feasible, and hence the procedure goes to Step 4.

Step 4.

Set $s = 4$. Since $s > N$, stop.

The integer-ratio policy is given by the following replenishment intervals $t_0 = t_1 = t_2 = t_3 = 0.3548$, and the cost incurred by this policy is 963.8181 $\$/time\ unit$.

Decentralized policy

We first should compute the optimal replenishment intervals at the buyers using (5.78). Thus, we obtain $t_1 = 0.4269$, $t_2 = 0.3450$ and $t_3 = 0.2575$.

Next, the above values t_i's are rounded off to obtain the following replenishment intervals, $t_1 = 0.4$, $t_2 = 0.3$ and $t_3 = 0.3$. Consequently, the time cycle for the vendor is 1.2.

Moreover, the order quantities at the buyers are $Q_1 = 2$, $Q_2 = 1.5$ and $Q_3 = 10.2$

Now, it can be easily determined that the instants of time where the vendor receives an order are given by the following time vector

0.0	0.3	0.4	0.6	0.8	0.9

In addition, the quantities which are ordered at each instant, that is, the demand vector at the vendor is

13.7	11.7	2	11.7	2	11.7

We apply now the procedure devised by Hill (1997a) to obtain the quantities that the vendor should order at each instant. For this example, the solution provided by Hill's approach is the following

13.7	13.7	0	13.7	0	11.7

The cost incurred by the vendor when this policy is applied is 186.9660 \$/*time unit*, and the overall cost including the costs at the buyers and at the vendor is 989.4660 \$/*time unit*.

As you can see, for this example it is better to apply the integer-ratio policy than the decentralized strategy. However, as it is shown in the computational results this is not always the case.

5.5 Computational results

In order to illustrate the performance of the procedures introduced in subsection 5.4.1 and 5.4.2 we have performed a detailed numerical study. The results of this computational experiment are reported in Tables 5.4-5.11.

First, we evaluate the effectiveness of the integer-ratio policies computed by the heuristic. Then, we compare these integer-ratio policies with the decentralized strategies. Finally, we analyze the effect of the different parameters of the problem on the total cost for both strategies.

5.5.1 Performance of the heuristic

We consider the following parameter values. The number of buyers N is set to 5, 10, 15 and 20. The values for h_0, k_0, k_j and d_j are taken from a uniform distribution $U[1, 100]$. The parameter h_j has been chosen from a uniform distribution $U[h_0, 100+h_0]$. Finally, the production rate at the vendor is randomly generated from three uniform distributions $U[100 + D, 500 + D] \equiv U_{P1}$, $U[1000 + D, 5000 + D] \equiv U_{P2}$, and $U[10000 + D, 20000 + D] \equiv U_{P3}$, where $D = \sum_{j=1}^{N} d_j$. Notice that the possible combinations of N and P give a total of 12 problem sets and for each one we carried out 100 instances. Therefore, 1200 instances have been solved and the results are summarized in Table 5.4.

Let C_{IR} denote the cost of the integer-ratio policy computed by the heuristic and let LB be the lower bound for the problem defined by (5.52)-(5.57). This lower bound is simply obtained by solving the relaxed problem, that is, by dropping the constraints (5.53)-(5.56).

In Table 5.4 we compare the cost of the integer-ratio policy provided by the heuristic with the lower bound. The first column contains the number of buyers. In columns two, five and eight we show the average percentage deviation of the cost of the integer-ratio solution from the lower bound for when $P \sim U_{P1}$, $P \sim U_{P2}$ and $P \sim U_{P3}$, respectively. Similarly, columns three, six and nine contain the maximum percentage deviation of the cost of the integer-ratio solution from the lower bound. Finally, in the rest of columns we show the average number of times that the heuristic needs to use the Lagrangian relaxation to compute a feasible solution (Av. Lag.).

Table 5.4: Average and maximum deviations of the integer-ratio policies from the lower bound, and average number of times that the heuristic needs to use the Lagrangian relaxation to compute a feasible solution

	$P \sim U_{P1}$			$P \sim U_{P2}$			$P \sim U_{P3}$		
N	Av. Dev. (%)	Max. Dev. (%)	Av. Lag.	Av. Dev. (%)	Max. Dev. (%)	Av. Lag.	Av. Dev. (%)	Max. Dev. (%)	Av. Lag.
5	0.90	2.59	0.14	1.05	4.39	0	1.05	4.40	0
10	0.64	5.21	5.71	0.97	2.63	0.51	1.56	3.49	0
15	0.35	1.67	12.4	0.50	3.69	3.12	0.63	3.68	1.09
20	0.20	0.81	20.1	0.28	1.56	5.32	0.32	2.09	3.16

From Table 5.4, we can see that the average percentage deviation of the integer-ratio solution from the lower bound for all problem sets is lower than 1.6 %. Moreover, the maximum percentage deviation obtained is lower than 5.5 %. From these results, we can conclude that the heuristic provides good integer-ratio policies in comparison with the lower bound. In addition, the average number of times that the heuristic needs to use the Lagrangian relaxation to compute a feasible solution is smaller than 6. It is worth noting that only when $P \sim U_{P1}$ and $N > 10$ the number of times that the Lagrangian relaxation is required increases.

5.5.2 Integer-ratio policies versus decentralized policies

Below we compare the integer-ratio policies with those obtained by using the decentralized approach. The cost of the decentralized policy is denoted by C_D.

In Table 5.5, we report for each problem set the percentage of instances where the integer-ratio heuristic provides both better and worse policies than the decentralized procedure, respectively.

Table 5.5: Comparison between integer-ratio and decentralized policies

	$P \sim U_{P1}$		$P \sim U_{P2}$		$P \sim U_{P3}$	
N	$C_{IR} < C_D$	$C_{IR} > C_D$	$C_{IR} < C_D$	$C_{IR} > C_D$	$C_{IR} < C_D$	$C_{IR} > C_D$
5	38	62	86	14	90	10
10	36.9	63.1	58	42	79	21
15	0	100	11.4	88.6	45.7	54.3
20	-	-	33.3	66.7	44.4	55.6

The results in Table 5.5 show that as the number of buyers increases so does the percentage of instances where the decentralized policy is better. On the other hand, another parameter which has a significant effect on which policy is better is the production rate at the vendor. In particular, when the production rate is selected from U_{P1}, in most cases, even when the number of buyers is small, it is preferable to follow the decentralized strategy instead of the integer-ratio policy. However, when the production rate is taken from either U_{P2} or U_{P3}, and the number of buyers is smaller than 10, the decentralized procedure provides in most cases worse policies than those given by the integer-ratio heuristic. In our opinion, this is due to the fact that as the production rate for the vendor increases, the coordination of the orders for all buyers becomes easier. Therefore, the integer-ratio policies can be more effective than the decentralized ones.

It is worth noting that for some problem sets there are instances for which one of the procedures cannot generate a feasible solution. Moreover, there are problem sets which yield infeasible solutions for all instances, mainly when $P \sim U_{P1}$. This is because the production rate is not enough to satisfy all buyer demands at the beginning of the cycle.

We also evaluate the difference between both costs, C_{IR} and C_D. Accordingly, in case of $C_{IR} > C_D$ we compute $Gap_{(C_{IR} - C_D)} = (C_{IR} - C_D/C_D) \times 100$. Otherwise, if $C_D \geq C_{IR}$, we calculate $Gap_{(C_D - C_{IR})} = (C_D - C_{IR}/C_{IR}) \times 100$. For each problem set the average percentages $\overline{Gap}_{(C_{IR} - C_D)}$ and $\overline{Gap}_{(C_D - C_{IR})}$ are presented in Table 5.6. From this table the following conclusions can be extracted.

When the production rate is selected from U_{P1} and $C_D \geq C_{IR}$, the average value of $\overline{Gap}_{(C_D - C_{IR})}$ is equal to 1.214. However, when $C_D < C_{IR}$, the average value of $\overline{Gap}_{(C_{IR} - C_D)}$ increases to 2.540. Therefore, under this situation when decentralized strategies are better than the corresponding integer-ratio policies, the difference between the costs is approximately twice as large as the opposite case.

Table 5.6: Percentage difference between the costs of the integer-ratio policies and the decentralized policies. $\overline{Gap}_1$ and $\overline{Gap}_2$ are $\overline{Gap}_{(C_D - C_{IR})}$ and $\overline{Gap}_{(C_{IR} - C_D)}$, respectively.

	$P \sim U_{P1}$		$P \sim U_{P2}$		$P \sim U_{P3}$	
N	$\overline{Gap}_1$	$\overline{Gap}_2$	$\overline{Gap}_1$	$\overline{Gap}_2$	$\overline{Gap}_1$	$\overline{Gap}_2$
5	1.472	2.627	3.180	0.863	4.035	0.666
10	0.956	2.454	1.767	1.374	2.153	0.848
15	-	-	2.034	2.450	1.657	1.550
20	-	-	0.985	0.986	1.539	0.067
Av.	1.214	2.540	1.991	1.418	2.346	0.782

In contrast, when the production rate is taken from either U_{P2} or U_{P3} the value of $\overline{Gap}_{(C_{IR} - C_D)}$ is generally tighter than the value of $\overline{Gap}_{(C_D - C_{IR})}$. This difference is even more remarkable when $N \leq 10$. This yields the integer-ratio policies to be a good compromise solution when $P \sim U_{P2}$ or $P \sim U_{P3}$.

5.5.3 Sensitivity analysis

We have carried out a sensitivity analysis involving parameters h_0, k_0, k_j, d_j, and h_j to assess their impact on our results. The number of buyers has been fixed to 10 and we have varied the ranges of the uniform distributions from which we select the above parameters. Specifically, we have analyzed 45 new problem sets. Some of the new problem sets are obtaining by varying only the uniform distributions for one of the following parameters: d_j, k_j, h_j and k_0, while the rest of parameters are selected from their initial distributions. In other problem sets we vary the uniform distributions for both h_0 and h_j. For each new problem set we generate 100 instances and the results are summarized in Tables 5.7-5.11.

Table 5.7: Sensitivity analysis with respect to d_j. The value d_j is selected from the uniform distributions: $U_1 \equiv U[100, 500]$, $U_2 \equiv U[1000, 5000]$ and $U_3 \equiv U[10000, 20000]$

	$P \sim U_{P1}$		$P \sim U_{P2}$		$P \sim U_{P3}$	
N	$C_{IR} < C_D$	$C_{IR} > C_D$	$C_{IR} < C_D$	$C_{IR} > C_D$	$C_{IR} < C_D$	$C_{IR} > C_D$
U_1	1.472	2.627	3.180	0.863	4.035	0.666
U_2	0.956	2.454	1.767	1.374	2.153	0.848
U_3	-	-	2.034	2.450	1.657	1.550

In Table 5.7, if $P \sim U_{P1}$ there are always instances where at least one of the procedures provides infeasible solutions, with independence of the value of the demand. When $P \sim U_{P2}$ and $P \sim U_{P3}$, we can conclude that as the quotient D/P decreases, namely, when the production rate is significantly greater than the total demand, the percentage of instances where the integer-ratio policies are better than the decentralized strategies increases.

Table 5.8: Sensitivity analysis with respect to k_j. The value k_j is selected from the uniform distributions: $U_1 \equiv U[100, 500]$, $U_2 \equiv U[1000, 5000]$ and $U_3 \equiv U[10000, 20000]$

	$P \sim U_{P1}$		$P \sim U_{P2}$		$P \sim U_{P3}$	
N	$C_{IR} < C_D$	$C_{IR} > C_D$	$C_{IR} < C_D$	$C_{IR} > C_D$	$C_{IR} < C_D$	$C_{IR} > C_D$
U_1	0	100	0	100	22	78
U_2	0	100	0	100	0	100
U_3	-	-	-	-	-	-

Table 5.9: Sensitivity analysis with respect to h_j. The value h_j is selected from the uniform distributions: $U_1 \equiv U[100, 500]$, $U_2 \equiv U[1000, 5000]$ and $U_3 \equiv U[10000, 20000]$

	$P \sim U_{P1}$		$P \sim U_{P2}$		$P \sim U_{P3}$	
N	$C_{IR} < C_D$	$C_{IR} > C_D$	$C_{IR} < C_D$	$C_{IR} > C_D$	$C_{IR} < C_D$	$C_{IR} > C_D$
U_1	0	100	44	56	77	23
U_2	5	95	25	75	28	72
U_3	79	21	39	61	39	61

Regarding parameters k_j and h_j we can conclude from Tables 5.8 and 5.9 that if the replenishment or holding costs at the buyers increase, then, in most instances, it is preferable that the buyers make decisions independently.

It is important to remark that the integrated models reduce the costs for the vendor, but increase the costs for the buyers. Thus, if the replenishment and holding costs at the buyers increase considerably, then the total costs at the buyers increase in an amount that is greater than that in which the cost at the vendor is reduced. Hence, under this situation it is better to follow a decentralized policy.

In contrast, the results in Table 5.10 show that if the setup cost at the vendor increases then the integer-ratio policies are better than those provided by the decentralized procedure, mainly when $P \sim U_{P2}$ or $P \sim U_{P3}$.

Table 5.10: Sensitivity analysis with respect to k_0. The value k_0 is selected from the uniform distributions: $U_1 \equiv U[100, 500]$, $U_2 \equiv U[1000, 5000]$ and $U_3 \equiv U[10000, 20000]$

	$P \sim U_{P1}$		$P \sim U_{P2}$		$P \sim U_{P3}$	
N	$C_{IR} < C_D$	$C_{IR} > C_D$	$C_{IR} < C_D$	$C_{IR} > C_D$	$C_{IR} < C_D$	$C_{IR} > C_D$
U_1	35	65	95	5	99	1
U_2	64	36	96	4	96	4
U_3	76	24	97	3	98	2

Table 5.11: Sensitivity analysis with respect to h_0 and h_j. The value h_0 is selected from the uniform distributions: $U_1 \equiv U[100, 500]$, $U_2 \equiv U[1000, 5000]$ and $U_3 \equiv U[10000, 20000]$, and the value h_j is selected from the uniform distributions: $U_4 \equiv U[100+h_0, 500+h_0]$, $U_5 \equiv U[1000+h_0, 5000+h_0]$and $U_6 \equiv U[10000+h_0, 20000+h_0]$

	$P \sim U_{P1}$		$P \sim U_{P2}$		$P \sim U_{P3}$	
N	$C_{IR} < C_D$	$C_{IR} > C_D$	$C_{IR} < C_D$	$C_{IR} > C_D$	$C_{IR} < C_D$	$C_{IR} > C_D$
U_1	0	100	58	42	85	15
U_2	14	86	60	40	87	13
U_3	0	100	62	38	91	9

Finally, as it can be seen in Table 5.11, if we simultaneously vary h_0 and h_j the results are similar to those showed in Table 5.5.

5.6 Conclusions

In this chapter, we consider a single-vendor multi-buyer system in which the vendor supplies an item to the buyers at a finite production rate. Previous works mostly focused on the single-vendor single-buyer problem or on the single-vendor multi-buyer system in which the vendor supplies a different item to each buyer. In this chapter, we assume that the buyers order the same item to the vendor and that the vendor can supply items to the buyers before the whole lot is produced. Furthermore, we allow the replenishment interval at any buyer to be greater than the time between setups at the vendor. This assumption gives more freedom in determining the replenishment intervals at the buyers, which are not forced to be smaller than the replenishment interval at the vendor. Moreover, there are many situations in practice where it is profitable to consider this assumption.

The single-vendor two-buyer problem is the simplest case within the single-vendor multi-buyer systems. Hence, we first have focused on this problem. Besides, the study of such model offers insights into the possible strategies that can be analyzed for the multi-buyer case. We formulate the problem in terms of integer-ratio policies and we develop a solution method which computes an optimal policy. Depending on how the replenishment intervals at the buyers and at the vendor are related, the expression of the average inventory at the vendor is different. Thus, we have analyzed two different cases in the statement of the problem, and we have obtained the optimal integer-ratio policy for both cases. We show that the single-vendor two-buyer system can be formulated as a nonlinear mixed integer programming problem. Moreover, we propose a solution method that combines the Karush-Kuhn-Tucker conditions with a branch and bound scheme.

The rest of the chapter is devoted to the general problem with multiple buyers. For this case, assuming that the decision system is centralized we formulate the problem in terms of integer-ratio policies and we propose a heuristic procedure. Additionally, we also show how to handle the problem if the vendor and the buyers are considered as independent installations. Under this situation, the vendor behaves as an inventory system with time-varying demand and we propose a two-level optimization approach for computing near-optimal solutions.

We have implemented both procedures and the computational results show that either the integer-ratio policies or the decentralized policies can be effective strategies. That is, depending on the parameter values one strategy outperforms the other.

We have carried out a sensitivity analysis to study the effect of the different parameters of the problem on the total cost for both strategies. The results suggest that as either the quotient D/P decreases or the setup cost at the vendor increases the integer-ratio policies are more effective than the decentralized policies. In contrast, as the replenishment or the holding costs at the buyers increase so does the percentage of instances where it is preferable to apply a decentralized policy.

In conclusion, depending on the parameter values it could be preferable to apply either the integer-ratio policies or the decentralized strategies. Nevertheless, in most cases we can use the integer-ratio policies since the gap between the costs of both class of policies in the instances where the decentralized strategies dominate the integer-ratio policies is tighter than the gap in the reverse case.

Conclusions and Future Research

Multi-echelon inventory systems are very common in practice. For example, consumers often do not purchase products directly from the producer. Instead, products are usually distributed, for example, through regional warehouses and local retailers to the consumer, that is, through a multi-echelon distribution system. Similarly, in the production context, stocks of raw materials, components, and finished products are similarly coupled to each other.

This thesis is concerned to the study of the multi-echelon inventory systems. However, before analyzing such systems, we have summarized in Chapter 1 the basic concepts and models in inventory control. It is worth noting that these basic models are fundamental for a good understanding of the multi-echelon inventory systems. Then, in Chapter 2 we have introduced the most important structures that we can find in multi-echelon inventory systems, namely, the serial, assembly and distribution systems. Besides, we have reviewed the most important models and algorithms for solving lot sizing problems for these systems with constant demand rates. For the two-level serial and assembly systems, optimal policies have to be stationary and nested. However, for distribution systems the form of the optimal policies can be very complex even when we restrict ourselves to the two-level distribution systems, that is, to the one-warehouse N-retailer problem. Hence, it is not surprising that for these models many authors have restricted themselves to compute an optimal policy within a simpler class of policies. In particular, in Chapter 3 we have analyzed the stationary and nested policies which are very easy to apply in practice. The main contribution of this chapter is the development of a new $\mathrm{O}(N\log N)$ heuristic which in most cases computes more effective single-cycle policies than those provided by the existing approaches. In addition, in this chapter we also have formulated the problem assuming that the decision system is decentralized, that is, assuming that each installation try to minimize its total costs. Finally, we have developed a computational experience in order to compare both strategies. From this experience, we can conclude that as the number of retailers increases so does the number of instances where the decentralized policies are better. In addition, given a number of retailers, we have carried out an analysis of sensitivity of the parameters. This analysis suggests that, under specific conditions of the replenishment and holding costs at the warehouse, the decentralized policies can provide better solutions. In particular, the results in this chapter are included in Abdul-Jalbar et al. (2003, 2006).

The single-cycle policies are very efficient in many situations and have clear managerial advantages. However, in some cases as for example when relatively high replenishment costs are combined with relatively low demand rates, the performance of these policies get worse. In order to achieve more effective strategies, in Chapter 4 we have dropped the assumptions of stationary and nested and we have analyzed a more general class of centralized strategies known as integer-ratio policies. For this case, we have developed an $\mathrm{O}(N \log N)$ heuristic which has been compared with the most referenced method in the literature, namely, the Roundy approach. Some of the contributions in this chapter have been already published in Abdul-Jalbar et al. (2005).

Finally, in Chapter 5 we have extended the study to the case where the warehouse produces the items at a finite rate. Most works in the literature analyzing this situation focus on the single-vendor single-buyer problem. Hence, an important contribution of this chapter is the formulation of the problem for the general case with multiple buyers. In particular, we first have focused on the single-vendor two-buyer systems and then, we have analyzed the problem with multiple buyers. We have formulated the problem in terms of integer-ratio policies and also assuming that the vendor and the buyers make decisions independently. We have implemented and compared both policies and the computational results show that depending on the parameter values one strategy outperforms the other. The results introduced in this chapter are compiled in Abdul-Jalbar et al. (2004a, 2004b).

The problem that we have analyzed in this thesis can be extended to consider shortages. Some mathematical models have been developed considering this restriction. See, for example, Mitchell (1987), Atkins and Sun (1995, 1997), Chen (1998, 1999, 2000).

Another important extension of the model consists of assuming time varying demand at the retailers. Examples of such models are given in Zangwill (1969), Blackburn and Millen (1982, 1985), Joneja (1990), Federgruen and Tzur (1994b, 1999), Simpson and Erenguc (1995), Herer and Tzur (2001), etc. Many authors have also analyzed multi-echelon inventory system with probabilistic demands. The primary early work on these systems was done by Clark and Scarf (1960). Relevant contributions are also given by Sherbrooke (1968), Federgruen and Zipkin (1984), De Bodt and Graves (1985), Graves (1985, 1988), Rosling (1989), Axsäter (1993), Axsäter and Rosling (1993), Chen and Zheng (1994a, 1994b), Forsberg (1995), Wang (1995), Axsäter and Juntti (1996), among others.

In this dissertation we have analyzed the problem considering that the warehouse produces the items at a finite production rate. Some researches have extended the model to also consider finite production rate at the buyers. See, for example, Kim (1999), Hill (2000) and Bogaschewsky et al. (2001).

Another interesting variation of the problem is to allow lateral transshipments among the retailers. When a demand at a retailer cannot be satisfied directly from stock on-hand, it could be taken from an adjacent retailer that has stock on-hand. Examples of such models are given in Karmarkar and Patel (1977), Cohen et al. (1986), Jönsson and Silver (1987), Lee (1987), Axsäter (1990, 2003), Pyke (1990), Robinson (1990), Sherbrooke (1992), Diks and de Kok (1996), Evers (1996), Alfredsson and Verrijdt (1999), Banerjee et al. (2003), among others.

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