TRABAJO FIN DE GRADO

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GRADO EN FÍSICA

Modelos cosmológicos alternativos al modelo concordante de materia y energía oscuras

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ABSTRACT

Throughout the history of physics different conceptions of the Universe of which we are part have existed and coexisted. The development the extension of the human senses provided by the modern instrumentation has allowed us to capture an increasingly realistic and humble vision of the cosmos. The so-called "concordance model" corresponds to the ΛCDM cosmological model supported by General Relativity in the macroscopic world and the standard model of particle physics in the microscopic world. The problem of unification between these two great theories is not the subject of this final degree project.

This cosmological model, as indicated by its acronym, denotes a Universe composed of two additional components to the baryonic matter: dark matter and dark energy. However, its nature is unknown and its existence cannot be firmly established. In this context, numerous reinterpretations of the observations that were used to postulate the dark matter and energy hypothesis arise, with the desire to be validated experimentally in the future. In this work, some of the most important models that have arisen as alternatives are considered: the negative mass model, the MOND theories, the $f(R)$ theories, the Chaplygin gas model or the entropic gravity model. Post-Newtonian parametrization or angular redshift fluctuations are also mentioned as observational constraints of new models that could help discarding some theories and supporting others.

RESUMEN

A lo largo de la historia de la física han existido y coexistido diferentes concepciones del Universo del que formamos parte. La extensión de los sentidos humanos que proporciona la instrumentación moderna ha permitido plasmar en nosotros una visión cada vez más realista y humilde del cosmos. El "modelo concordante" actual corresponde al modelo cosmológico ΛCDM sustentado por la Relatividad General en el mundo macroscópico y al modelo estándar de la física de partículas en el mundo microscópico. El problema de unificación existente entre estos dos grandes teorías no es objeto de este trabajo de fin de grado.

El modelo cosmológico, tal y como indican sus siglas, denota un Universo compuesto por dos componentes adicionales a la materia bariónica: la materia oscura y la energía oscura. Sin embargo se desconoce su naturaleza y no se puede asegurar con firmeza su existencia. En este contexto surgen numerosas reinterpretaciones de las observaciones que postulaban la materia y energía oscuras, con el afán de poder ser validadas experimentalmente en un futuro. En este trabajo se consideran algunas de los modelos más importantes que han surgido como alternativas: el modelo de masa negativa, las teorías MOND, las teorías $f(R)$, el modelo de gas de Chaplygin o el modelo de gravedad entrópica. También se mencionan la parametrización postnewtoniana o las fluctuaciones angulares de redshift como restricciones observacionales a los nuevos modelos que podrían ayudar a descartar algunas teorías y respaldar otras.

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1. Introduction

1.1. Dark Matter hypothesis and drawbacks

For a long time, invisible masses have already been detected due to their gravitational effects on objects that are visible. An example can be the detection of Neptune due to its gravitational effects on the orbit of Uranus. However, when observing irregularities in the orbit of Mercury, these were not due to any object that disturbed them. In this case it was Newton's theory that did not explain planetary orbits well enough. We see that with the development of General Relativity the peculiarities of the orbit of Mercury have been explained. Now for galaxy clusters their mass-luminosity ratios have been measured, which in some cases have given values around of 60. These values summed to other observations such as the rotation curves of galaxies may imply that there is a mass that is not visible but if we want to be rigorous we cannot discard that it can be again a problem in the theoretical formulation of the physics.

The rotation curves of the galaxies have contributed to the definition of the concept of dark matter (Figure 1). Since the rotation curves are asymptotically flat beyond the optical radius and do not decay with the keplerian factor $1/\sqrt{r}$, it becomes necessary to define much more mass than what is observed, the non-luminous mass. A halo of dark matter whose density decays roughly as r^{-2} for each galaxy is currently predicted (López-Corredoira, 2018).

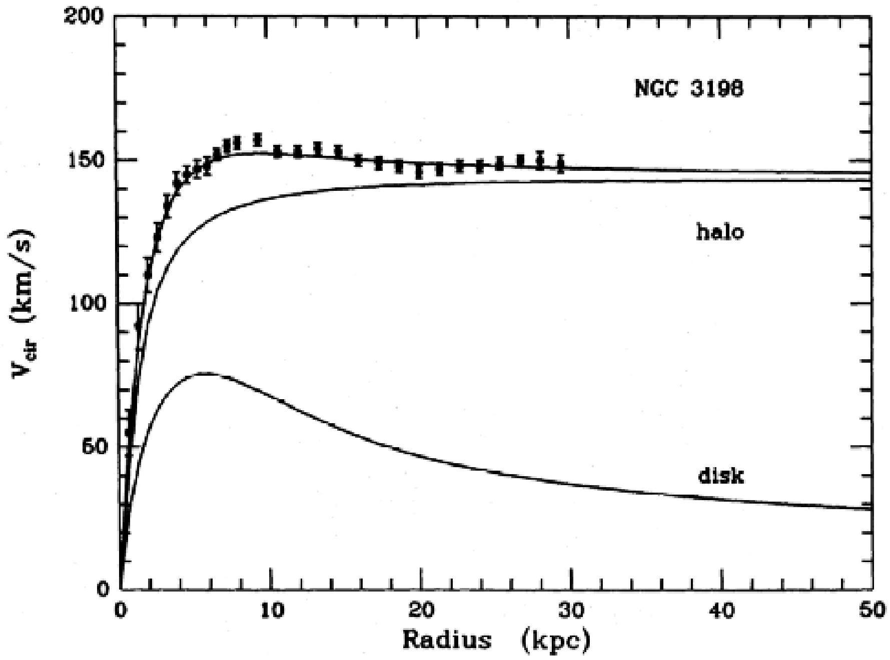

Figure 1: Rotation curve from the galaxy NGC 3196. The "disk" contribution will be the observable matter: stars and gas. The "halo" is the contribution of dark matter. If we put this together we obtain a curve that fits the observations (van Albada et al. , 1985).

As a clarification of figure 1, the curve corresponding to the halo corresponds to the following equation (van Albada et al., 1985):

$$\rho_{halo}(r) \propto \left[\left(\frac{a}{R_0}\right)^{\gamma} + \left(\frac{r}{R_0}\right)^{\gamma}\right]^{-1}$$

Where R_0 is a fiducial radius, a the radius of the core of the Galaxy and γ a free parameter. The following values are chosen: $R_0 = 8\,kpc, a = 8.5\,kpc,\ \gamma = 2.1$. In this equation a spherical halo is assumed.

Indeed, the beginnings of a serious discrepancy in this context date back to the 1930s, when Zwicky measured the mass of galaxy clusters such as Coma or Virgo in two independent ways. On the one hand he used an estimation for the sum of the luminosities of each galaxy and a mean mass to luminosity ratio as a proxy for the total mass, and on the other hand he considered the speeds of several galaxies for estimating the Virial mass of the cluster. The second mass estimator was around 400 times higher than the first (Zwicky, 1933). Despite this, no more research was done on the problem of the so-called "missing mass" until the 70s.

In reference to the nature of dark matter in the 70s, the possibility of non-baryonic dark matter was considered. In the realization of simulations of structures on a large scale, the model fitted well enough with the observations. In the 1980s, cold dark matter[1] prevailed over hot dark matter because it reproduced better the large scale structure of galaxies.

Some astrophysical objects were considered as possible dark (or low luminous) matter, such as brown dwarfs, small stars or black holes but none of them are good candidates to cover the missing material (López-Corredoira, 2018). Another form of dark matter would be constituted by non-baryonic particles that are not grouped into larger structures. Some examples are electrically neutral supersymmetric particles that do not interact in the strong force (neutrinos, photons, gravitons etc.), axions of very low mass and WIMPs (Weakly Interacting Massive Particles) such as the neutralino.

However, there is a discussion about whether this material really exists because the observations show some aspects that are far from those predicted by the standard cosmological model. For example, a central halo density lower than predicted is observed. The predicted angular momentum is also much smaller than what is observed. In barred galaxies, the bars go much faster than expected. You would expect them to rotate slower due to angular momentum transfer between the bar and the dark matter halo. The mass-luminosity ratios increase as the luminosity of the galaxy decreases, which is not predicted either (López-Corredoira, 2018). Apart from the above, it should be noted that dark matter has not yet been detected, although on the other hand this does not mean that it does not exist.

Some alternatives to dark matter are magnetic fields, theories of modified gravity, baryonic matter in the outskirts of the disk or non-circular orbits. The most popular alternative theory is the Modified Newtonian Dynamics (MOND), which modifies gravity for low accelerations but initially is not compatible with more general gravitation theories. To do so, the Tensor-Vector-Scalar (TeVeS) or the AQUAdratic Lagrangian theory (AQUAL) arise, which satisfy the principle of equivalence as well as the conservation of energy and momentum. Their successes are on a galactic scale but not in large structures.

[1] Cold dark matter is a type of dark matter that moves much slowly than the velocity of light. On the other hand, hot dark matter travels at ultrarelativistic velocities.

1.2. Dark energy hypothesis and drawbacks

Till the nineties of the last century it was already known that the Universe was expanding. Due to the attractive force of gravity, an expanding universe was expected to slow down. At this time, the groups *Supernova Cosmology Project* and *The high-z SN Search* wanted to quantify exactly this deceleration caused by gravity. The results that both research groups obtained were not as expected. Until that moment the cosmological model predicted an open universe with $\Omega_m = 0.2$. According to their conclusions, now there is a point of inflection about seven billion years ago in which the Universe converts its decelerated expansion into an accelerated expansion. This new result was admitted because the luminosity of the supernovae was lower than expected (Perlmutter, 2011).

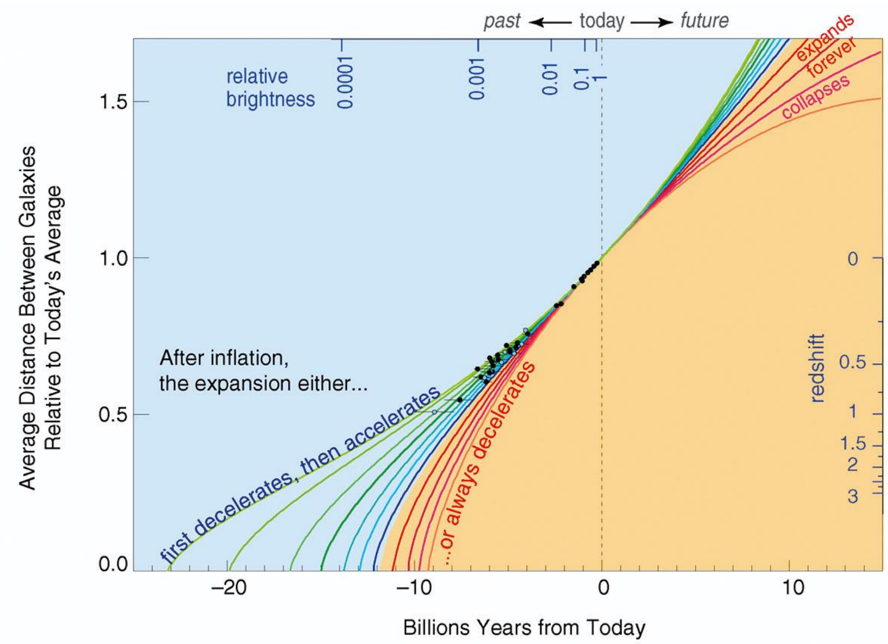

Figure 2: We have some curves in the graphic indicating the type of Universe according to its evolution in time. The black points are the supernova data that matches with an accelerated expansion of the Universe (Perlmutter, 2011).

These observations therefore needed a new explanation, since matter and radiation were both attractive types of energy. Therefore it was necessary at least a third constituent to contrast the effects of matter and radiation and cause an accelerated expansion by exerting a negative pressure. This ad hoc constituent was called dark energy and should represent approximately 70% of the matter observed in the Universe to fit the observations. From then on, a nature had to be established for this new component of the Universe. It was decided to recover the cosmological constant Λ that Einstein had already introduced to obtain a static universe. This, including the concept of dark matter, is how the current cosmological model ΛCDM was born.

From this moment, numerous theories have been proposed for replacing dark energy such as $f(R)$ theories, taking a variation in the Einstein-Hilbert action, fact that we will comment in chapter six. Some cons that may have dark matter can be the following: if we take into account the extinction of the light of a supernova in its route to the observer or the possible metallicity dependence in the supernovae Ia, the necessity of a cosmological constant can be questioned. Other models, without even taking into account the above, attribute the cosmic acceleration to a cosmic variation of the speed of light or the gravitational constant, or that the universe experiences phases of acceleration-deceleration, among others.

It should be mentioned at this point that one of the biggest problems we have today in physics is the problem of the cosmological constant. The cosmological constant can be interpreted as energy of vacuum. It can be interpreted as the energy of virtual particles that can exist due to the limitations of the uncertainty principle. Thus, the virtual particles, depending also on its mass, have a short enough lifetimes that the uncertainty principle enables. Also the particles can be created in pairs to not violate the charge principle conservation. This vacuum energy has been experimentally checked with some effects such as the Casimir effect or the Hawking radiation of black holes. Quantifying these fluctuations and interpreting them as the cosmological constant, its value would be 123 orders of magnitude higher than is observed (Cepa, 2007). This last value anyway cannot contribute to the expansion of the Universe because it would be a huge accelerated expansion that is not observed. Therefore, this is considered the worst theoretical prediction in the history of physics.

2. Objectives

Dark matter and dark energy are the most abundant constituents of the Universe today but their nature is unknown. That two elements added to ordinary matter constitutes the ΛCDM, the actual model in cosmology. Obviously, these new characteristics are postulated by observational facts. However, a lot of alternative theories arose with the aim of substituting these two ad hoc elements, in part due to the fact that they have not been detected in many years and in the other hand because one problem usually has more than one possible solution.

The present project aims to compile and analyse different theories proposed to explain the observations in a different perspective, contributing in ideas that may not initially be considered enough. For this, an attempt has been made to search for the most up-to-date bibliography as well as on some occasions also the original bibliography of the subject in question. Books, papers and internet are used.

After an introduction to General Relativity, we present first the recent model of negative masses that substitute dark matter fitting rotation curves and dark energy (substituting the cosmological constant). MOND theories are also presented in order to replace dark matter and also $f(R)$ theories explaining among other things the accelerating expansion of the Universe or its structure. We introduce the Chaplygin gas model as an explanation of both dark matter and energy. We consider the post-Newtonian formalism as a way to check new theories, the angular redshift fluctuations as an observational tool that provides information about dark energy and also for testing new theories. All of the subjects are presented, developed and also discussed briefly.

3. Introduction to General Relativity

The General Relativity was developed by Albert Einstein in the twentieth century. It constitutes the theoretical framework more accepted for the description of the gravitational field in the Universe.

Before explaining the subject we will define two very important concepts in this context: a SRI and a SnRI. An inertial reference system (SRI) is a system in which you can choose Cartesian coordinates corresponding to a Euclidean space whose axioms express that space is homogeneous and isotropic. It is also assumed that time is homogeneous and that the law of inertia is fulfilled. In these systems the theory of special relativity can be applied.

Therefore, a non-inertial reference system (SRnI) will not fulfill some of the above properties. Generally, a non-inertial system will have acceleration with respect to an inertial system and therefore the law of inertia cannot be fulfilled.

By means of the well-known experiment of the elevator, Einstein concludes that the gravitational mass and the inertial mass of a body are equal, which constitutes the principle of weak equivalence.

Within a gravitational field all massive objects will experience an acceleration and therefore a priori we can't say that we are in an inertial reference system. However, a reference system in free fall in this gravitational field will possess the same acceleration as the acceleration of gravity, and therefore its effect on the objects that are in such system will be suppressed and it can be considered as a local inertial system (SLI). This constitutes the principle of strong equivalence. In a general gravitational field, changing in space and time, our SLI will be valid in a moment of time and in region of space small enough to consider that the gravitational field is homogeneous and static.

The formula of General Relativity, using a Newtonian description, relates the sources of the field with the components of the stress-energy tensor in the famous Einstein equation:

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R + \Lambda g_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta} \equiv G_{\alpha\beta} \tag{1}$$

Where $G_{\alpha\beta}$ is defined as the Einstein tensor, $g_{\alpha\beta}$ are the components of the metric tensor, which are obtained from gravitational field sources (mass densities). $R_{\alpha\beta}$ are the components of the Ricci tensor that are obtained by a contraction in the first and third indices of the Riemann Curvature Tensor, which gives an idea of the curvature of space time. R is the Ricci scalar and is obtained by a contraction of the Ricci tensor. $R_{\alpha\beta}$ and R can be obtained from the metric since the Riemann curvature tensor depends on the Christoffel symbols which in turn depend on the first derivatives of the metric.

$\Lambda > 0$ is the cosmological constant that would represent a base level of positive energy and is often interpreted as a vacuum energy. This term would generate repulsion in great scales, and was introduced by Einstein to compensate the gravitational attraction between the objects in order to obtain a static universe. Finally, $T_{\alpha\beta}$ are the components of the stress-energy tensor.

A solution of Einstein's equations would describe the evolution of the Universe on large scales, but for this purpose the components of the metric and the components of the stress-energy tensor must be known. There are many ways to represent each of the tensors and each one arises from some initial assumptions.

The most general metric in spherical coordinates that takes into account that space is homogeneous and isotropic and has an arbitrary curvature which it's constant, can be shown to be the Robertson-Walker metric:

$$ds^2 = -c^2 dt^2 + a^2(t) \cdot \left[\frac{dr^2}{1-kr^2} + r^2 d\Theta^2 + r^2 \sin^2\Theta \, d\varphi^2\right] \tag{2}$$

Where $a(t)$ would be the scale factor of the Universe with units of length and k would represent the curvature. The coordinate r is dimensionless because it is scaled with the radius of curvature. According to the value of k we can speak of a plane ($k = 0$), closed ($k = +1$), or open ($k = -1$) universe.

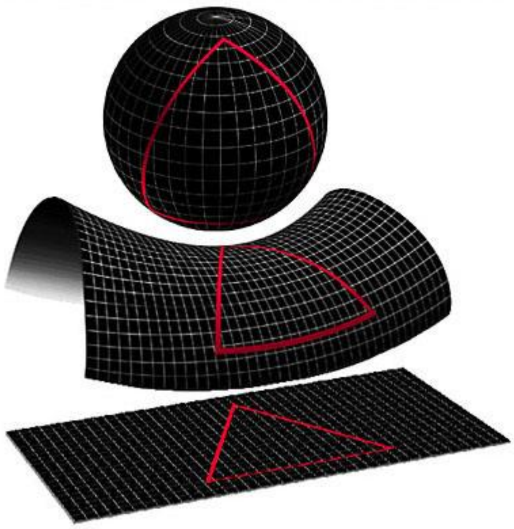

Figure 3: Three types of Universes. Above it's represented a closed Universe, in the middle an open Universe and below a flat Universe.

On the other hand it can also be shown that for an ideal fluid the strerss-energy tensor will adopt the following form:

$$T_{\alpha\beta} = \left(\rho + \frac{P}{c^2}\right) u_\alpha u_\beta + P g_{\alpha\beta} \tag{3}$$

Where $\rho(t)$ would be the energy density of the fluid, $P(t)$ its pressure, u_α the components of the quadrivelocity: assuming the static fluid would only have temporal component not null and equal to 1, and finally $g_{\alpha\beta}$ would be the components of the metric.

Assuming the metric and the previous energy-impulse tensor, the components of the Ricci tensor and the Ricci scalar are calculated and, by applying the Einstein equation, we obtain the Friedmann equations, which describe the evolution of the scale factor.

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda c^2}{3} - \frac{kc^2}{a^2} \tag{4}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) + \frac{\Lambda c^2}{3} \tag{5}$$

To provide a solution for the scale factor, the energy density and the pressure that are shown in the previous equations we need another equation, the state equation. The state equation relates the pressure with the energy density:

$$P = \omega\rho \qquad (6)$$

There are some types of fluids that satisfy that equation. As an example, for $\omega = 0$ we obtain an equation for non-relativistic matter. On the other hand $\omega = -1$ represents the cosmology constant. The density and pressure shown in the Friedmann equations are the sum of densities and pressures of the different state equations present in the universe, if considered independent (e.g.: a relativistic particle remain always relativistic).

Cassini-Huygens Mission

The Cassini-Huygens mission was launched with the collaboration of NASA, ESA and ASI to study the planet Saturn, with its rings and satellites. It has been active since 1997 until 2017, of which the first years it has traveled the planets Venus, the Earth, an asteroid and Jupiter by means of the flyby effect and it has been the last 13 years orbiting Saturn.

One of the experiments carried out was a test of General Relativity. Specifically, the objective was to measure the Shapiro time delay, an effect predicted by General Relativity in which light travels a greater distance when passing through a massive object such as the Sun because the space it crosses is curved. Therefore the light would take to travel the distance Saturn-Sun-Saturn a little more than usual. To check it, radio waves have been used. What has really been measured is the change in frequency between the one emitted and the one received once they have passed very close to the Sun.

Once the experiment was done, the validity of General Relativity has been shown measuring the γ parameter which in General Relativity is 1. The result was $\gamma = 1 + (2.1 \pm 2.3) \cdot 10^{-5}$ (Bertotti, Iess & Tortora, 2003). This leads us to think that that the effect of the new parameters that the alternatives theories introduce is reduced.

4. Model of negative masses

4.1. Motivation

In this model (Farnes, 2018), negative masses are proposed as an alternative to both dark matter and dark energy. It is therefore suggested a negative mass fluid that Einstein had already raised years ago, thus modifying the standard model of current cosmology. This mass would replace the cosmological constant and therefore the dark energy, and would explain the flat rotation curves at large radii. Therefore it would intend to not depend on dark matter either. It will be seen that this theory will give rise to a cyclic universe.

The negative masses are a candidate to replace the cold dark matter because as we will see below, they repel each other and therefore cannot coalesce into larger structures as well as the positive masses do, and consequently they cannot emit light as the stars do. On the other hand, since negative masses repel each other, they lead to the expansion of the universe. Since the negative masses are attracted by the positive masses they would apply pressure to the positive masses making the rotation curves asymptotically flat with a specific distribution of the negative matter.

The model explains the distribution of dark mass and dark energy from first principles, it makes predictions and has the potential to be consistent with observational evidences such as distant supernovae or microwave background radiation. Physical laws such as conservation of energy or momentum are not violated and are consistent with General Relativity.

4.2. Theoretical framework

Positive and negative masses are considered, as well as by analogy there are also positive and negative electric charges or magnetic poles north and south. It is considered that the negative masses also fulfil the principle of equivalence and therefore their inertial mass is equal to their gravitational mass. So, if we equate the gravitational force with the inertial force, we have:

$$F_g = -\frac{GM_1M_2}{r} = M_1 a \tag{7}$$

From this expression we can obtain that two positive masses are attracted, two negative masses are repelled and finally, as the most curious case, for a positive and a negative mass the positive mass is repelled by the negative but the negative is attracted to the positive. When $M_1 = -M_2$ the mass of the pair of particles is 0 and can accelerate to $v = c$ taking place the so-called runaway motion.

At this point we start with Einstein's equations in a Universe governed by the Robertson-Walker metric (Equation 2) with the Universe considered as a perfect fluid. From there, the same Friedmann's equations can be obtained (Equations 4 and 5). Finally we consider an equation of state (Equation 6).

With all this we now consider the content of the Universe as the due to various contributions, including that of the negative mass.

$$\Omega_{M+} + \Omega_{M-} + \Omega_r + \Omega_\Lambda = \Omega \tag{8}$$

Where each Ω represent each density scaled to the critical density (Eq. 11) that is obtained for a null curvature in the Friedmann equation (Eq. 4).

For the standard cosmological model it is considered:

$$\Omega_{M-} = 0\ ;\ \Omega_{M+} = \Omega_b + \Omega_{CDM}\ ;\ \Omega_\Lambda > 0 \tag{9}$$

For our concrete case in which a negative mass is included, we assume a Universe dominated by matter, therefore:

$$\Omega_r = \Omega_\Lambda = 0\ ;\ \Omega = \Omega_{M+} + \Omega_{M-} \tag{10}$$

Now we can consider the following three situations:

1) The positive mass in the Universe is dominant

First of all we consider the critical density of the Universe, that is, the density in which we consider that the geometry of the Universe is flat and the first derivative of the scale factor with time tends asymptotically to 0, with the difference that now it will be due to two contributions:

$$\rho_c = \frac{3H^2}{8\pi G} \tag{11}$$

Taking into account both contributions of mass, the density parameter is:

$$\Omega = \frac{\rho_+ + \rho_-}{\rho_c} \;;\; \rho_+ > \rho_- \tag{12}$$

Taking into account a Universe with null curvature and dominated by matter and part of it is negative, the first Friedmann equation result in:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}(\rho_+ + \rho_-) \tag{13}$$

2) Massless Universe

The Universe on a large scale has zero mass. Therefore, on a large scale the impulse-energy tensor is null and the total density considered in the Friedmann equations due to both contributions is null

$$T_{\mu\nu} = 0 \;;\; \rho_+ + \rho_- = \rho = 0 \;;\; \Omega = 0 \tag{14}$$

The Big Bang is considered as an energy conservation event and can be created from nothing considering that it would be a highly unlikely event. This case corresponds to a Dirac-Milne Universe: governed by FRLW metric, in which is considered the spatial curvature parameter $k = -1$ and the scale factor evolving linearly with time (Benoit-Lévy & Chardin, 2009).

$$a(t) = ct \tag{15}$$

3) Negative mass Universe

In this case, taking into account that we have a universe dominated by matter and we have subtracted the cosmological constant of Friedmann's equations, we can only have physical solutions if $k = -1$, which denotes an expansion.

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}(\rho_+ + \rho_-) + \frac{c^2}{a^2} \,;\, |\,\rho_+| < |\rho_-| \tag{16}$$

Another possibility is the consideration of a positive cosmological constant and therefore obtain several possible values for k. In any case:

$$\Omega_{M+} + (\Omega_{M-} + \Omega_\Lambda) = \Omega_{M+} + \Omega_{degen} = \Omega \tag{17}$$

Where Ω_{degen} is a degenerate parameter in which we can vary the values of Ω_Λ and Ω_{M-} to obtain a given value. We can break the degeneracy if we take into account the parameter ω of the state equation. In that case, observations show that for Ω_{degen} the parameter is close to -1, and this is interpreted as the existence of the cosmological constant and thus, dark energy.

Solutions to the Friedmann equation:

We take the first Friedmann equation and consider negligible the positive mass matter, and in consequence as we said before for that case: $= -1$. Then:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_- + \frac{c^2}{a^2} = \frac{\Lambda c^2}{3} + \frac{c^2}{a^2} \,;\; \Lambda = \frac{8\pi G\rho_-}{c^2} < 0 \tag{18}$$

We also assumed that Λ is a natural explanation of the cosmologic constant taking negative masses. Taking these considerations the solution will be:

$$a(t) = \sqrt{\frac{-3}{\Lambda c^2}}\sin\sqrt{\frac{-\Lambda c^2}{3}}\,t \tag{19}$$

The Hubble parameter is defined as $H(t) = \frac{\dot{a}}{a}$, then:

$$H(t) = \sqrt{\frac{-\Lambda c^2}{3}}\cot\sqrt{\frac{-\Lambda c^2}{3}}\,t \tag{20}$$

As we said before there is a cycle of expansion and contraction. Considering $\sqrt{\frac{-\Lambda c^2}{3}}t = 0$ and $\sqrt{\frac{-\Lambda c^2}{3}}t = \pi$ we find the timescale $t = \sqrt{\frac{-3\pi^2}{\Lambda c^2}}$. For cosmological observations an approximation for the cosmological constant is obtained and from there it is obtained that the universe will recollapse in about 105 Gyr from its beginnings. Even adding positive mass the universe with negative cosmological constant would recollapse due to the extra attractive force. Taking into account the Hubble parameter observed at present and clearing the time of the previous equation it can be calculated also the lifetime of the current Universe in this cosmology, which is about 13.8 Gyr, consistent with the age of the concordance cosmological model ΛCDM.

As previously mentioned, this model is able to explain the rotation curves observed in galaxies. First, we consider a star within a galaxy at distance r from the center and having a stable circular orbit. In this case the gravitational force is equal to the centripetal:

$$\frac{GM_+M_{galaxia}}{r^2} = \frac{M_+v^2}{r} \tag{21}$$

From there we obtain the standard Kepler rotation: $v \propto \frac{1}{\sqrt{r}}$ that we know is not observed.

In the proposed model, the positive mass (the luminous mass) is surrounded by a fluid with a negative mass of density ρ_- and a total mass M_-. In this context we define the cosmological constant as we had previously done:

$$\Lambda = \frac{8\pi G\rho_-}{c^2} \tag{22}$$

In the weak field approximation and for small velocities a model is developed that reproduces Newton mechanics (Farnes, 2018). In the sources of the gravitational field, apart from the positive masses we also consider:

$$\rho_{vac} = \frac{\Lambda c^2}{4\pi G} \tag{23}$$

Using the Poisson equation we obtain an expression for the potential and later an expression for the force that we will equal to the centripetal force. Finally we get the speed. For very small and negative values of the cosmological constant, an explanation of the observed rotation curves is obtained, including a growth of the velocity at already very large radii.

$$v = \sqrt{\frac{GM_+}{r} - \frac{8\pi G\rho_-}{3}r^2} \quad (24)$$

The results do not agree at all with all the observations but it must be remembered that we start from some approximate assumptions: the mass of a galaxy is not only the mass of the central bulb, the galaxy has an halo etc. Solutions as a negative cosmological constant had already been proposed but were dismissed as incompatible with supernova observations, yet this argument can be reinterpreted (Farnes, 2018). However, N-body simulations (see next subsection) are consistent with observations.

4.3. N-body simulations

One of the most used methods to test models like the one developed in this chapter is via N-body simulations. In short, in a simulation of this type, a three-dimensional representation of N bodies is performed in which positions and velocities of the particles are evaluated. For this particular case, 5.000 particles of positive mass and 45.000 of negative mass are considered and mass creation is not taken into account.

Considering the above requirements, the result is the formation of a halo of negative matter with a radius several times larger than that of the positive mass. The halo is not cuspy which could give a solution to the cuspy problem[2] that the N-body simulations of CDM normally have. An asymptotically flat rotation curve is observed because the negative mass "pushes" the positive mass. Starting from a uniform distribution of positive and negative mass, the positive component in turn is surrounded by the negative. Voids and filaments similar to those observed are found in the formation of structures.

[2] The cuspy problem is caused by a discrepancy between the N-body simulations of dark matter and the density profiles of the low-mass galaxies. Thus, N-body simulations predict a higher density of dark matter for low radii than is observed from the rotation curves of disk galaxies.

5. MOND theories

5.1. Motivation of MOND

MOND (Modified Newtonian Dynamics), postulated by Mordehai Milgrom, was born in 1983 with the aim of explaining the incompatibilities observed in the rotation curves of galaxies.

If we take into account the approximation that the speed of the stars around the nucleus of the galaxy is circular and constant, this will mean that the forces on them will be nullified. Each star suffers mainly two forces: the centripetal force and the gravitational force:

$$G\frac{Mm}{r^2} = F_g = F_C = ma_c = m\frac{v^2}{r} \tag{25}$$

$$v = \sqrt{\frac{GM}{r}} \tag{26}$$

Where G is the universal gravitational constant, M is the mass of the galaxy contained within the orbit of the star, which can be approximated by the total mass of the galaxy, m is the mass of a star, r is the distance from the star to the nucleus of the galaxy and v is the speed of rotation of the star around the nucleus of the galaxy.

This expression of the velocity foresees a decay if we increase the distance of the star to the nucleus. Innumerable observations have shown that this is not the case, and that in fact the tendency for large radii is a stabilization of the rotation speed of the star. Tully and Fisher established the following expression that did contrast the observations (Bugg, 2015):

$$v_\infty = \sqrt[4]{GMa_0} \tag{27}$$

Where a_0 is an empirical constant.

In the previous case, the Keplerian velocity was based on Newton's gravitation scheme. However, what physical foundation would explain this last relation?

5.2. Theoretical Framework

The MOND theory is a modification of Newton's theory of gravitation for low accelerations, which correspond to the accelerations of stars that are far from the nucleus of the galaxy, all in the framework of weak field.

$$a = \frac{g_N}{\mu(x)} \tag{28}$$

g_N is the Newtonian gravity, a is the real centripetal acceleration of the star. $x = \frac{a}{a_0}$ is a parameter with a_0 as a postulated universal constant.

If $a \gg a_0 : (x) = 1$. In any other case: $\mu(x) \approx x$

For accelerations much greater than a_0 we would recover Newton's laws. a_0 would act as a parameter analogous to others already known as the Planck constant h or the speed of light c, which in certain limits separate classical physics from the current physics.
If the accelerations are not much greater than a_0 :

$$a = \frac{g_N}{\mu(x)} \approx \frac{g_N}{x} = \frac{g_N a_0}{a} \tag{29}$$

$$a = \sqrt{g_N a_0} \tag{30}$$

If we take into account this information we can put equation 28 as follows:

$$\frac{v^2}{r} = \sqrt{\frac{GMa_0}{r^2}} \tag{31}$$

$$v = \sqrt[4]{GMa_0} \tag{32}$$

We see that Tully-Fisher relation is obtained. Expressing this equation with respect to mass/luminosity ratio, a constant a_0[3] is obtained from numerous observations of luminosity and speed at different radii of galaxies (Aversa).

[3] An estimate value for this constant is $1.2 \cdot 10^{10}\ m\, s^{-2}$

$$\log L = 4 \cdot \log V - \log\left(Ga_0 \frac{M}{L}\right) \quad (33)$$

However, we see that it is necessary in each case to adjust the $\frac{M}{L}$ parameter of the galaxy.

We show now a typical rotation curve of a barred spiral galaxy.

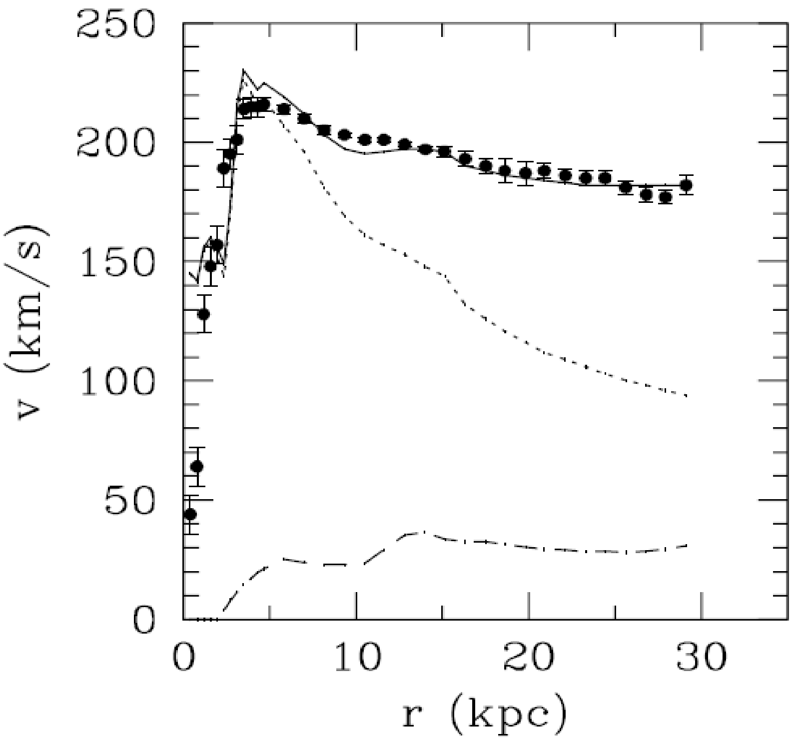

Figure 4: Galaxy rotation curve obtained from the galaxy NGC 2903. The Newtonian rotation curves due to the visible components of the galaxy are the dotted lines, and the dot-dashed lines are the Newtonian rotation curves due to the neutral hydrogen gaseous component, as measured in the radio at 21 cm.
The observations don't agree with the Newtonian components. It's proposed a new line, the solid line, from MOND theory that will fit much better the observations.

To formalize the theoretical model of the theory, Beckenstein and Milgrom (1984) proposed a non-relativistic Lagrangian. Despite the fact that we are in a region of speeds much lower than velocity of light $\left(100 - 250\,\frac{\text{km}}{\text{s}}\right)$, we anticipate that the proposed Lagrangian will also have a limited validity since it does not consider relativistic effects. Next we present the Newtonian Lagrangian and the modified Lagrangian proposed.

$$L_N = -\int d^3r\{\rho\varphi_N + (8\pi G)^{-1}(\nabla\varphi_N)^2\} \quad (34)$$

$$L = -\int d^3r \left\{\rho\varphi_N + (8\pi G)^{-1} a_0^2 F\left[\frac{(\nabla\varphi_N)^2}{a_0^2}\right]\right\} \tag{35}$$

ρ is the mass density and φ_N the Newtonian potential.

After a series of operations, for the Newtonian case we obtain the Poisson equation from which the Newtonian potential can be obtained, which relates the sources of the field (the mass densities) with the gravitational potential. In the non-Newtonian case we obtain an analogous equation that also gives an expression for the potential. The expressions are the following:

$$\varphi_N \sim -\frac{GM}{r} \tag{36}$$

$$\varphi \rightarrow -\sqrt{GMa_0}\ln\left(\frac{r}{r_0}\right) \tag{37}$$

Where r_0 is the radius in the galaxy corresponding to the acceleration a_0. Taking into account equation 37 and $a = -\nabla\varphi$, the relation of Tully-Fisher is reached:

$$\frac{v^2}{r} = a = -\nabla\varphi(r) = -\frac{d\varphi(r)}{dr} = \frac{\sqrt{GMa_0}}{r} \tag{38}$$

$$v^4 = GMa_0 \tag{39}$$

5.3. Cosmological implications

In MOND theory, it is proposed that the constant a_0 is also likely to have cosmological implications. Taking the current value of the Hubble constant the following ad hoc expression is postulated (Milgrom, 2015):

$$2\pi a_0 \approx cH_0 \tag{40}$$

This already indicates that a_0 varies in time as the Hubble constant does.

The following two parameters are also defined:

$$l_M = \frac{c^2}{a_0} \approx 7.5 \cdot 10^{28} cm \qquad (41)$$

$$M_M = \frac{c^4}{G a_0} \approx 10^{57}\, g \qquad (42)$$

Both parameters would be of the order of the maximum observable distance as well as the maximum observable mass of the visible Universe, which if true would be quite surprising starting from a theory that only intended to explain some velocity discrepancies at the galactic level.

MOND also establishes that there are not objects within the observable Universe that are in the MOND regime and at the same time are relativistic.

An object that is in the MOND region will imply the following expression:

$$\frac{MG}{r^2} < a_0 \qquad (43)$$

A relativistic object satisfies:

$$\frac{MG}{r} \sim c^2 \qquad (44)$$

If we divide:

$$r = \frac{\frac{MG}{r}}{\frac{MG}{r^2}} > \frac{c^2}{a_0} = l_M \qquad (45)$$

Therefore, an object with these characteristics would be outside the observable Universe.

In spite of all this, MOND, in the cosmological framework, is not capable of making predictions as it does in galactic systems with the different $\frac{M}{L}$ bands.

5.3.1. Formation of structures

Although MOND theory is not a definitive theory because as we said earlier it does not include General Relativity, it can be treated in non-relativistic situations. Thus, many of the results that are dealt with by Newton's laws (for example cosmological simulations) will be treated in the same way in the MOND framework.

The first implication of MOND theory is the change it produces in the epoch of matter-radiation equality in the Universe, compared with the standard cosmological model. Because all the matter in a MOND theory is considered baryonic, it implies that the epoch of equality matter-radiation occurs approximately at $z_{eq} \approx 400$, while for the current cosmological model $z_{eq} \approx 3300$. This gives rise to the fact that in the MOND theory the equality of matter-radiation occurs later and not before, that the epoch of recombination[4]. In conclusion, structures cannot be formed until a period after the recombination (Mcgaugh, 2015).

Another characteristic in the formation of structures is that they coincide quite well with those of ΛCDM (performed with numerical simulations of N-bodies), however in a MOND Universe they are developed before due to the non-linear force law existing in the low acceleration regime.

On the other hand, one of the best observational disagreements of MOND theory is the observational evidence provided by the Bullet Cluster, a clear example in favour of dark matter. Analysing X-ray gas and the lensing effects from the Cluster is seen that MOND gravity does not explain the observations.

5.4. Several MOND theories

There are different theories of modified gravity that meet the essential requirements of a MOND theory. There are non-relativist theories and relativistic theories.

[4] In the period of recombination protons and electrons come together to form neutral atoms approximately when the Universe has 380000 years old.

Among the non-relativistic formulations of MOND we can name the Modified Poisson gravity, that makes a change in the Lagrangian in such a way that the respective Poisson equation results in:

$$\nabla \cdot \left[\mu\left(\frac{|\nabla\varphi|}{a_0}\right)\nabla\varphi\right] = 4\pi G\rho \tag{46}$$

In the Deep MOND limit we obtain $\mu\left(\frac{|\nabla\varphi|}{a_0}\right) \approx \frac{|\nabla\varphi|}{a_0}$, and if we define $\eta(\mathbf{r}, t) = G a_0 \rho(\boldsymbol{r}, t)$ we obtain two field equations:

$$\nabla \cdot (|\nabla\varphi|\nabla\varphi) = 4\pi\eta \tag{47}$$

$$\dot{v} = -\nabla\phi \tag{48}$$

As for relativistic theories, the first that was formalized was the so-called Tensor-Vector-Scalar (TeVeS), formulated by Beckenstein. The main idea of this theory that gives rise to its name is the consideration that gravity can not only be represented as a metric tensor but in fact it would also be formed by a vector field U_α and a scalar field φ:

$$\tilde{g}_{\alpha\beta} = e^{-2\varphi}\left(g_{\alpha\beta} + U_\alpha U_\beta\right) - e^{2\varphi}U_\alpha U_\beta \tag{49}$$

6. $f(R)$ Theories

6.1. Motivation of $f(R)$ theories

The theory of General Relativity has had the opportunity to be immensely tested and proven, and therefore is very well established in the current scientific field. However, apart from the problems discussed about the accelerated expansion of the Universe and the rotation curves of the galaxies, it must be added that there is still no observational evidence of how the gravitational field behaves in extreme high-curvature regimes such as neutron stars or black holes.

Due in part to the aspects that have been commented, a set of theories have been born, the $f(R)$ theories which generalises Einstein's General Relativity. Actually is a family of theories, each one defined by a different function $f(R)$. R is the Ricci scalar. The simplest case that corresponds to the Einstein's General Relativity is when $f(R) = R$. This theory $f(R)$ was first proposed in 1970 by Hans Adolph Buchdahl. Then, it was used ϕ instead of *f*.

With this arbitrary function, there may be freedom to explain the accelerated expansion and structure formation of the Universe without adding unknown forms of dark matter and dark energy. Some of these functional forms are inspired by corrections arising by a quantum theory of gravity.

Is an active field of research since the work of Starobinsky on cosmic inflation. There's a wide range of possibilities from this theory if we adopt different functions. However, we can rule out some of the functions because of observational grounds or pathological theoretical problems.

6.2. Theoretical framework

6.2.1. Metric version

In a $f(R)$ theory when the action of Einstein-Hilbert is generalized making the change $R \to f(R)$ we obtain the following expression:

$$S[g] = \int \frac{1}{2\kappa} f(R)\sqrt{-g}d^4x + \int d^4x L_M\left(g_{\mu\nu}, \psi\right) \quad (50)$$

Where $\int d^4x L_M\left(g_{\mu\nu}, \psi\right) = S_M$ is the action corresponding to the fields of mass. Also $\kappa = \frac{8\pi G}{c^4}$ and $g = \det\left(g_{\mu\nu}\right)$.

We now take a variation in the action and focus only on the first term. We also define $F(R) = \frac{df(R)}{dR}$ and we take into account the result obtained in the first appendix (Davie, 2017):

$$\int \frac{1}{2\kappa}\left(\delta f(R)\sqrt{-g} + f(R)\delta\sqrt{-g}\right)d^4x = \int \frac{1}{2\kappa}\left(F(R)\delta R\sqrt{-g} - f(R)\frac{1}{2}\sqrt{-g}\delta g^{\mu\nu} g_{\mu\nu}\right)d^4x \quad (51)$$

We make an incision to treat the differential of the Ricci scalar. R is defined as follows:

$$R = g^{\mu\nu}R_{\mu\nu} \quad (52)$$

Where $R_{\mu\nu}$ is the Ricci tensor.

Its variation with respect to the inverse of the metric is as follows:

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu} \quad (53)$$

From appendices 2 and 3 we obtain that (Trilleras, 2012):

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\left(\delta\Gamma^{\rho}_{\nu\mu\,;\,\rho} - \delta\Gamma^{\rho}_{\rho\mu\,;\,\nu}\right) = R_{\mu\nu}\delta g^{\mu\nu} - \nabla_\mu\nabla_\nu\delta g^{\mu\nu} + g_{\mu\nu}\Box\delta g^{\mu\nu} \quad (54)$$

$\Box$ is the D'Alembert operator.

Note that now all the terms contain the factor $\delta g^{\mu\nu}$. Then, we continue with the previous development:

$$\int \frac{1}{2\kappa}\left(F(R)\delta R\sqrt{-g} - f(R)\frac{1}{2}\sqrt{-g}\delta g^{\mu\nu}g_{\mu\nu}\right)d^4x =$$

$$= \int \frac{1}{2\kappa}\sqrt{-g}\left(F(R)\left(R_{\mu\nu}\delta g^{\mu\nu} - \nabla_\mu\nabla_\nu\delta g^{\mu\nu} + g_{\mu\nu}\Box\delta g^{\mu\nu}\right) - f(R)\frac{1}{2}\delta g^{\mu\nu}g_{\mu\nu}\right)d^4x =$$

$$= \int \frac{1}{2\kappa}\sqrt{-g}\delta g^{\mu\nu}\left(F(R)\left(R_{\mu\nu} - \nabla_\mu\nabla_\nu + g_{\mu\nu}\Box\right) - f(R)\frac{1}{2}\delta g^{\mu\nu}g_{\mu\nu}\right)d^4x \qquad (55)$$

Now we have in mind that the action is invariant through variations of the metric $\frac{\delta S}{\delta g^{\mu\nu}} = 0$

Then:

$$\frac{1}{2\kappa}\sqrt{-g}\left(F(R)\left(R_{\mu\nu} - \nabla_\mu\nabla_\nu + g_{\mu\nu}\Box\right) - f(R)\frac{1}{2}g_{\mu\nu}\right) = -\frac{\delta L_M}{\delta g^{\mu\nu}} \qquad (56)$$

$$F(R)\left(R_{\mu\nu} - \nabla_\mu\nabla_\nu + g_{\mu\nu}\Box\right) - f(R)\frac{1}{2}g_{\mu\nu} = -\frac{2\kappa}{\sqrt{-g}}\frac{\delta L_M}{\delta g^{\mu\nu}} \qquad (57)$$

We define as the stress-energy tensor of mass sources as the quantity: ${T_{\mu\nu}}^{(M)} = -\frac{2}{\sqrt{-g}}\frac{\delta L_M}{\delta g^{\mu\nu}}$

$$F(R)\left(R_{\mu\nu} - \nabla_\mu\nabla_\nu + g_{\mu\nu}\Box\right) - f(R)\frac{1}{2}g_{\mu\nu} = \kappa {T_{\mu\nu}}^{(M)} \qquad (58)$$

${T_{\mu\nu}}^{(M)}$ satisfies the continuity equation as a consequence of the principle of action:

$$\nabla^\mu {T_{\mu\nu}}^{(M)} = 0 \qquad (59)$$

If we now take the trace in the equation 58:

$$g^{\mu\nu}F(R)\left(R_{\mu\nu} - \nabla_\mu\nabla_\nu + g_{\mu\nu}\Box\right) - f(R)\frac{1}{2}g^{\mu\nu}g_{\mu\nu} = \kappa g^{\mu\nu}{T_{\mu\nu}}^{(M)} \qquad (60)$$

$$F(R)\left(R - g^{\mu\nu}\nabla_\mu\nabla_\nu + \delta^\mu_\mu\Box\right) - f(R)\frac{1}{2}\delta^\mu_\mu = \kappa T \qquad (61)$$

Where $\delta^\mu_\mu = 4$, $g^{\mu\nu}\nabla_\mu\nabla_\nu = \Box$ and $g^{\mu\nu}{T_{\mu\nu}}^{(M)} = T$

$$F(R)(R + 3\Box) - 2f(R) = \kappa T \qquad (62)$$

These equations would generalize the General Relativity equation found by Einstein. In fact, taking $f(R) = R$; $F(R) = \frac{df(R)}{dR} = 1$ we obtain:

$$R = -\kappa T \tag{63}$$

At this point we show equations equivalent to the Friedmann equations in General Relativity. In appendix 4 we demonstrate the first of them. To do this we suppose that the Universe is dominated by the Robertson-Walker metric and that the stress-energy tensor of matter corresponds to a perfect fluid.

$$3FH^2 = \frac{FR-f}{2} - 3H\dot{F} + \kappa\rho \tag{64}$$

$$-2F\dot{H} = \ddot{F} - H\dot{F} + \kappa(\rho + P) \tag{65}$$

Finally, pressure and gravity satisfy the following continuity equation:

$$\rho + 3\dot{H}\rho = 0 \tag{66}$$

6.2.2. Equivalent formalism: Brans-Dicke theory

We start again from the $S[g]$ action formulated at the beginning. In this case we introduce an auxiliary field X, so that our action is transformed as follows:

$$S[g] = \int \frac{1}{2\kappa}\sqrt{-g} f(R) d^4x + S_M = \int \frac{1}{2\kappa}\sqrt{-g}[f'(X)(R - X) + f(X)] d^4x + S_M \tag{67}$$

That scalar field follows the equation:

$$f''(X)(R - X) = 0 \tag{68}$$

The regions of X where $f''(X) \neq 0$ will be called branches (Hindawi, Ovrut & Waldram, 1996). In these regions we have from the previous equation that $R = X$, that replaced in the new action we return to obtain the action formulated at the beginning (Eq. 50).

To write the action now in function of a scalar field ϕ that fulfills the condition $\phi = f'(X)$:

$$S[g] = \int \frac{1}{2\kappa}\sqrt{-g}\left[\phi\left(R - X(\phi)\right) + f\left(X(\phi)\right)\right]d^4x + S_M \quad (69)$$

You can rewrite the action as follows:

$$S[g] = \int \sqrt{-g}\left[\frac{\phi R}{2\kappa} - U(\phi)\right]d^4x + S_M \quad (70)$$

With:

$$U(\phi) = \frac{\phi R(\phi) - f\left(R(\phi)\right)}{2\kappa} \quad (71)$$

This action is a particular case of the action described in the Brans-Dicke theory, which contains an additional term proportional to ω_{BD} (the Brans-Dicke parameter). Therefore our case corresponds to $\omega_{BD} = 0$ (Pérez Bergliaffa, 2011). The complete action principle of the Brans-Dicke theory is the following expression (De Felice & Tsujikawa):

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2}\phi R - \frac{\omega_{BD}}{2\phi}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - U(\phi)\right) + S_M \quad (72)$$

To obtain the field equations from Eq. 72 we use the same procedure as in subsection 6.2.1. We take a variation of the terms with respect to the scalar field ϕ to obtain a first field equation and we take a variation with respect to the metric $g_{\mu\nu}$ to obtain a second field equation.

If we return to our particular case, the calculus of the resulting equation taking a variation of the action of Eq. 70 is specified at the appendix 5.

$$3\phi\square + 2V(\phi) - \phi\frac{\delta V(\phi)}{\delta\phi} = \kappa T \quad (73)$$

Where $V(\phi) = 2\kappa U(\phi)$

6.2.3. Palatini version

There are other methods to obtain the field equations, such as the one introduced by Palatini. In this method, the metric and the connection are considered independent. In a similar way to the previous method, we start from the following principle of action:

$$S_P = \int \frac{1}{2\kappa} f(R)\sqrt{-g}d^4x + S_M \tag{74}$$

Taking a variation of the action:

$$\delta S_P = \int \frac{1}{2\kappa} \delta \left(f(R)\sqrt{-g}\right) d^4x + \delta S_M \tag{75}$$

A similar development to the previous one (in the metric version) is taken, but in this case the terms calculated in Appendix 3 are eliminated due to the condition explained above. Then, we obtain the following equations:

$$F(R)R_{\mu\nu} - f(R)\frac{1}{2}g_{\mu\nu} = \kappa T_{\mu\nu}{}^{(M)} \tag{76}$$

Taking the trace we obtain:

$$g^{\mu\nu}F(R)R_{\mu\nu} - f(R)\frac{1}{2}g^{\mu\nu}g_{\mu\nu} = \kappa g^{\mu\nu}T_{\mu\nu}{}^{(M)} \tag{77}$$

$$F(R)R - 2f(R) = \kappa T \tag{78}$$

R y T have an algebraic relationship.

Now we consider a variation with respect to the connection. If we take into account the development in Appendix 2:

$$\delta S_P = \int \frac{1}{2\kappa} \sqrt{-g}\, g^{\mu\nu} F(R) \cdot \left(\delta\Gamma^{\rho}_{\nu\mu\,;\,\rho} - \delta\Gamma^{\rho}_{\rho\mu\,;\,\nu}\right) d^4x \tag{79}$$

$$\delta S_P = \int \frac{1}{2\kappa} F(R) \cdot \left(\nabla_\rho \sqrt{-g} g^{\mu\nu} \delta\Gamma^{\rho}_{\nu\mu} - \nabla_\nu \sqrt{-g} g^{\mu\nu} \delta\Gamma^{\rho}_{\rho\mu}\right) d^4x \tag{80}$$

$$\delta S_P = \int \frac{1}{2\kappa} \left(\nabla_\rho\left(F(R)\sqrt{-g}g^{\mu\nu}\right) - \nabla_\nu\left(F(R)\sqrt{-g}g^{\mu\nu}\right)\delta^{\nu}_{\rho}\right) \delta\Gamma^{\rho}_{\nu\mu} d^4x \tag{81}$$

Imposing that the variation of the action is invariant to the connection and calculating the trace, we obtain:

$$\nabla_\rho\left(F(R)\sqrt{-g}g^{\mu\nu}\right) = 0 \tag{82}$$

Now we can obtain that:

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{F(R)} g^{\alpha\lambda} \left(\partial_\mu (F(R) g_{\alpha\nu}) + \partial_\nu \big(F(R) g_{\alpha\mu}\big) - \partial_\alpha \big(F(R) g_{\mu\nu}\big)\right) \tag{83}$$

Since it has been proved that the relationship between R and T is algebraic, then the connection can be a function of the metric and a function of T, that is, on the mass fields.

6.3. Particular models

6.3.1. Starobinsky's model

Starobinsky proposed the first inflation model (for early stages of the Universe) in which the function $f(R)$ takes the following form:

$$f(R) = R + \frac{R^2}{6M^2} \tag{84}$$

Where M has mass dimensions.

For this model, it is considered the absence of mass fluids e.g.: $\rho = 0$ because in that epoch of the Universe the inflationary potential dominated over others. If we take into account the generalized Friedmann equations and Ricci's dependence on the parameter H and its derivative $\dot{H}$ we obtain the following equations:

$$\ddot{H} - \frac{\dot{H}^2}{2H} + \frac{1}{2} M^2 H = -3H\dot{H} \tag{85}$$

$$\ddot{R} + 3HR + M^{\dot{2}} R = 0 \tag{86}$$

We can consider the first two terms negligible in the first equation. This will be true while we are in the inflation period. This leads to: $\dot{H} \cong -\frac{M^2}{6}$. From there we can obtain a solution for the Hubble parameter H, and therefore also for the scale factor a:

$$H \cong H_i - \frac{M^2}{6}(t - t_i) \tag{87}$$

$$a \cong a_i exp\left(H_i(t-t_i) - \frac{M^2}{12}(t-t_i)^2\right) \tag{88}$$

We assume $t = t_i$ as the beginning of inflation, in which the Hubble parameter and the scale factor were H_i and a_i, respectively (De Felice & Tsujikawa, 2010).

6.3.2. Hu-Sawicki model

In this model, the function $f(R)$ takes the following form:

$$f(R) = R - \frac{c_1 R_{HS}\left(\frac{R}{R_{HS}}\right)^p}{c_2\left(\frac{R}{R_{HS}}\right)^p + 1} \tag{89}$$

c_1, c_2, R_{HS} are parameters and $p > 0$ is a positive constant.

Not all the parameters are independent.

6.3.3. Tsujikawa model

In that case, the form of $f(R)$ is:

$$f(R) = R - \mu R_T \tanh\left(\frac{R}{R_T}\right) \tag{90}$$

Where μ and R_T are two positive constants.

6.3.4. Exponential gravity model

The $f(R)$ form is:

$$f(R) = R - \beta R_E\left(1 - e^{-R/R_E}\right) \tag{91}$$

Where β and R_E are the parameters.

6.4. Observational tests

One of the parameters that are determined at large scales is the Hubble parameter H. Different types of techniques are used to obtain these measurements. One of them is the Cosmic Chronometers technique. This method, based on the evolution of the passively evolving early-type galaxies, determines the Hubble parameter.

For a Universe that follows the Robertson-Walker metric, we show in the appendix 6 that the Hubble function can be expressed as follows depending on the redshift (Nunes, 2016):

$$H = -\frac{1}{1+z}\frac{dz}{dt} \tag{92}$$

So, measuring $\frac{dz}{dt}$ we obtain directly the value of H.

We can constrain $f(R)$ using also data of supernovae Ia. One of the latest attempts start from a certain form of the function: $f(R) = R - \beta R^{-n}$ The parameters are adjusted with the experimental measurements and some of the restrains are the following: $\beta \in [2.3, 7.1]$, $n \in [-0.25, 0.35]$ (Santos, 2018).

Baryonic acoustic oscillations[5] are also used to constrain the expansion history of the Universe and some measurements are made in that direction (Nunes, 2016).

Those observational data aim to constrain the free values of the $f(R)$ parameters for models as the mentioned above in section 6.3.

[5] The visible baryonic matter suffers fluctuations in the density due to acoustic density waves in the early plasma of the Universe.

7. Chaplygin gas

As we already know, observations of Supernovae type Ia indicates that the Universe expands in an accelerated way. ΛCDM explains this inconsistency with dark energy of which its nature is not known. Chaplygin gas is one of the models of dark energy and also can explain dark matter at small scales. It consists as an exotic substance that unifies both dark phenomena that satisfies the equation of a perfect fluid (Eq. 3) and the following equation of state:

$$P = -\frac{A}{\rho} \tag{93}$$

Where P is the pressure, ρ the density and A is a positive constant.

Also we can work with a generalization of the previous model (The modified Chapygin gas):

$$P = B\rho - \frac{A}{\rho^{\alpha}} \tag{94}$$

Where $0 < \alpha \leq 1$ and B is constant.

We consider now the speed of sound in the context of this model.

$$v_s^2 = \frac{\partial P}{\partial \rho} = B + \alpha \frac{A}{\rho^{\alpha+1}} \tag{95}$$

The speed of sound squared is positive and bounded if ≥ 0 , $B \geq 0$ and $0 \leq \alpha \leq 1$. In this limit Modified Chapygin gas behaves as dark matter (Avelino & Ferreira, 2015).

For the generalized model, the dependence of the energy density ρ on the cosmological radius a assuming a Universe governed by the Robertson-Walker metric is as follows:

$$\rho = \left(A + \frac{B}{a^{3(1+\alpha)}}\right)^{\frac{1}{1+\alpha}} \tag{96}$$

A y B are constants of integration. $\sqrt{A}$ corresponds to the cosmologic constant.

The application of this Chaplygin gas method is not only limited to cosmological studies but is also related to string theory. The equation of state of this gas can be obtained from the

action of Nambu-Goto within the framework of string theory. A theoretical basis can also be developed in the cosmological framework. In both cases we can get the same state equation.
The cosmological predictions of this gas model can be contrasted with different types of investigation lines such as the cosmic microwave background radiation, type Ia supernovae, gravitational lenses, inhomogeneities in the large-scale structure of the Universe etc. The last mentioned point is crucial to test the proposed model of Chaplygin gas with the large-scale structure of the universe (Gorini, 2004).

In short, Chaplygin Gas is able to explain the accelerated expansion of the Universe, give a description of both dark matter and dark energy and describes a transition between a deceleration of the Universe to acceleration, fact that is observed with supernovae Ia (Perlmutter, 2011). However, new observational data is needed to definitely prove its existence.

8. Constraints

8.1. Post-Newtonian formalism

In the post-Newtonian formalism a series of corrections are introduced to the Newtonian theory, which depend on a parameter. This parameter is the velocity of the object that creates the gravitational field divided by the speed of light. One of the correction models is the theory of General Relativity.

The parametrized post-Newtonian formalism (PPN) is a generalization that details the parameters in which Newtonian gravity differs from a particular theory of gravity. This formalism does not at first presuppose any theory as correct, which constitutes an advantage for testing alternative theories. In fact, it has been a very useful tool for the testing of the theory of General Relativity, in which the parameters introduced by this theory have to have specific values. Post-Newtonian formalism was already used by Eddington in 1922 and different versions have emerged from there. Each version has included its own parameters with its own notation. In the case of Wil, Ni, and Misner in the 70s, 10 parameters were described, within the notation called beta-delta, which completely characterized the theory in a weak gravitational field. These parameters were intended to measure effects such as changes in gravity produced by unit pressure, per unit of internal energy, per unit of kinetic energy etc. Below we show the coefficients and their description (Will, 1971):

γ	Curvature created in space that is produced by a mass at rest
β	Non-linearity that could exist in the superposition of gravity
β_1	Gravity produced by kinetic energy
β_2	Gravity produced by gravitational potential energy unit
β_3	Gravity produced by internal energy
β_4	Gravity produced by unit pressure
ζ	Difference between transversal and kinetic energy about gravity
η	Difference between the radial and transversal tension to the gravity
Δ_1	Drag of inertial reference systems per moment unit
Δ_2	Difference between the radial and transverse moment in the drag of the inertial reference systems.

Figure 5: Table. 10 parameters defined in the framework of parametrized post-Newtonian formalism.

For General Relativity all the coefficients are worth 1 except $\eta = \zeta = 0$.

There is a more recent notation of Will & Nortvedt and Will of the 80s in which 10 parameters are also introduced which are linear combinations of the parameters of the previous formalism, in order to describe with the new parameters preferred frame effects or failures in the conservation of energy, momentum or angular momentum. γ and β are the only parameters that retain their identity (Will & Nordtvedt, 1972).

As already mentioned, the PPN formalism is useful for testing different theories of gravity, and there is already a process more or less established for it. Thus, a large ammount of theories has been compared using such parameters, such as Nordström's flat theory of gravity, quasi-linear theories such as Whitehead[6] generalized relativity, bimetric theories etc. (Alemañ, 2016). All of them have not been included in this work because they have been experimentally discarded.

8.2. Angular redshift fluctuations

We know that redshift can be expressed as follows:

$$z = \frac{\lambda_2 - \lambda_1}{\lambda_1} \cong \frac{H_0}{c} D \tag{97}$$

λ_2 is the observed wavelengh and λ_1 is the emitted wavelengh. H_0 is the current value of the Hubble constant and D is the distance to the object with redshift z. The speed at which we move away from the object in question due to the expansion of the Universe is:

$$v = H_0 D \cong zc \tag{98}$$

The galaxies also have peculiar speeds apart from their recession due to the increase of the scale factor of the Universe. We will denote with the letter V the radial component (with

[6] Whitehead's theory for a long time faced the theory of General Relativity. The discrepancy between Whitehead and Einstein was his way of seeing in each case the relationship that matter and space-time possessed. However, after an experimental test carried out in 1965, Whitehead's theory had serious problems in moving forward because this theory forced an anisotropy in the Earth's gravity field that was disproportionately high in relation to the observations and also violated the weak equivalence principle (Alemañ, 2016).

respect to the line of sight to the observer) of these peculiar speeds. Then the previous expression taking into account this effect will take the following form:

$$v = H_0 D + V \tag{99}$$

We see then:

$$z \cong \frac{H_0 D}{c} + \frac{V}{c} = z_{Hubble} + z_{pec} \tag{100}$$

We consider now a set of galaxies $j = 1, \dots, N$ in which we observe the redshift z_j. We define the following window function.

$$W_j = exp\left(-\frac{(z_{obs} - z_j)^2}{2\sigma_z^2}\right) \tag{101}$$

The values z_{obs} and σ_z are respectively the central redshift of the distribution of redshifts assumed Gaussian and the width of that Gaussian.

The fluctuation of the redshift is defined as follows:

$$\delta z(n_i) = \frac{\sum_{j \in i}(z_j - \bar{z})W_j}{\sum_{j \in i} W_j} \tag{102}$$

n_i is the i-th pixel of the image obtained from the sky. In each one of them there would be a determined number of galaxies, each one with its redshift. $\bar{z}$ is the average of the redshift taking into account the window function.

$$\bar{z} = \frac{\sum_j z_j W_j}{\sum_j W_j} \tag{103}$$

The angular fluctuations of the redshift provide additional information on both velocity fields and densities obtained with a different systematics. It also provides information about dark energy, adding restrictions and testing other models proposed for the same purpose (Monteagudo, 2017). As we can see from the previous equations, the fluctuations depend on that parameter that we have added regarding the movement between the galaxies that does not depend on the expansion of the universe, that is, on the peculiar velocities.

9.Conclusions

First, the observation. Then the hypothesis. These are some of the steps of the methodology used by the scientific method to obtain new knowledge, to advance our vision and understanding of the Universe. A specific fact of the proposed models is that many of them make predictions, that is, they need more observations to be able to verify their full scope. Therefore, the scientific method becomes cyclical on many occasions.

The observation. Also generally called problem, when it questions hypotheses and models previously elaborated. The observations that are exposed in this work are on the one hand the problem of the lost mass as for example in the curves of rotation of the galaxies and on the other hand the discovery of the accelerated expansion of the Universe.

The hypothesis and the model. Dark matter and dark energy. These are the explanation accepted today in the scientific community, which have given rise to the current cosmological model ΛCDM. However, their nature is currently unknown. Nor they have been detected after many years of experimentation designed for this purpose. But the most important fact that allows the flowering of alternative theories lies in the premise that the present observations that we have allow the existence of several possibilities to explain the reality of our environment, each from a different point of view, breaking preconceived schemes and providing new and creative ideas.

In this work we have exposed some of the alternative current models, such as the negative mass model, MOND theories, $f(R)$ theories and the Chaplygin gas, as well as some theoretical restrictions such as post-Newtonian parametrization or experimental such as angular fluctuations of redshift. Each proposed model defends its ideology with the artillery at its disposal against the healthy criticism that is generated from the outside. . In this way the models evolve and become stronger or are dismissed. One of the objectives of this work has been to present these models as impartially as possible, without giving any of them as valid or less probable, and for the reader to judge what he considers appropriate.

As previously mentioned, many models need new observations for their testing, and this is the case of all the models presented in this paper. In this way the cycle dies and the future will determine a new birth.

Finally we present a summary table of the theories that have been proposed.

	Is an alternative of	Tries to explain	Characterized by	Pros	Cons
Model of negative masses	Dark matter and dark energy	The cosmological constant and the rotation curves of galaxies	Introduces a new parameter ρ_- in the Friedmann equations that represents the negative masses	Considering N-body simulations with positive and negative masses it gets asymptotically flat rotation curves and also gives a solution to the cuspy problem. In large scale we observe voids and filaments.	There are discussions of the incompatibility of a negative cosmologic constant with observations of supernovae.
MOND theories	Dark matter	Rotation curves of galaxies	Generalization of the second Newton law at very small accelerations introducing a new parameter: a_0	Fits very well the rotation curves of a lot of galaxies and dwarf galaxies.	Is an ad hoc model. The Bullet cluster is a point against due observed lensing effects and it does not explain the loss mass in galaxy clusters
$f(R)$ theories	Dark matter and dark energy	Structure formation of the Universe, regions of extreme curvature and the accelerated expansion of the Universe	Generalization of the Einstein-Hilbert action substituting in the integrand the Ricci scalar by a function of it	Post-Newtonian formalism or the Cassini Huygens in their observational limits confirm General Relativity and $f(R)$ gravity	Is an ad hoc model. More precise observations are needed to constrain some parameters
Chaplygin gas	Dark matter and dark energy	The accelerated expansion of the Universe and the loss mass problem introducing a square bounded sound speed	Its proposed an exotic substance with negative pressure that causes the accelerated expansion	It has been contrasted with gravitational lenses, supernovae type Ia or inhomogeneities in the large-scale structure of the Universe. It needs more tests.	The gas is hypothetic and its nature is unknown.

Figure 6: Table. Overview of the theories presented in this work.

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APPENDIXES

Appendix 1

In this appendix we will demonstrate the following expression:

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\delta g^{\alpha\beta}g_{\alpha\beta} \tag{1}$$

We start representing the following matrix:

$$A = \begin{pmatrix} a_0 & 0 \\ 0 & a_1 \end{pmatrix} \tag{2}$$

The trace of that matrix is:

$$tr(A) = a_0 + a_1 \tag{3}$$

Now we define another matrix that has a relationship with the previous one:

$$B = e^A = \begin{pmatrix} e^{a_0} & 0 \\ 0 & e^{a_1} \end{pmatrix} \tag{4}$$

We take the determinant of that matrix:

$$\det(B) = e^{a_0}e^{a_1} = e^{a_0+a_1} = e^{tr(A)} = e^{tr(\ln B)} \tag{5}$$

$$\ln\det(B) = \ln e^{tr(\ln B)} = tr(\ln B) \tag{6}$$

If we take the derivative in both sides:

$$\frac{\partial \det(B)}{\det(B)} = tr\left(\frac{\partial B}{B}\right) \tag{7}$$

Taking $B = g_{\alpha\beta}$ y $\det(B) = g$

$$\delta g = gg^{\alpha\beta}\delta g_{\alpha\beta} \tag{8}$$

Once we obtained this we take a variation of the previous amount:

$$\delta\sqrt{-g} = -\frac{1}{2\sqrt{-g}}\delta g = -\frac{1}{2\sqrt{-g}}gg^{\alpha\beta}\delta g_{\alpha\beta} = \frac{1}{2\sqrt{-g}}(-g)g^{\alpha\beta}\delta g_{\alpha\beta} =$$

$$= \frac{1}{2\sqrt{-g}}\left(\sqrt{-g}\right)^2 g^{\alpha\beta}\delta g_{\alpha\beta} = \frac{1}{2}\sqrt{-g}g^{\alpha\beta}\delta g_{\alpha\beta} = -\frac{1}{2}\sqrt{-g}\delta g^{\alpha\beta}g_{\alpha\beta} \quad (9)$$

The last step takes into account that:

$$\delta\left(g^{\alpha\beta}g_{\alpha\beta}\right) = \delta(\delta^{\alpha}_{\alpha}) = 0 \quad (10)$$

Then:

$$\delta g^{\alpha\beta}g_{\alpha\beta} + g^{\alpha\beta}\delta g_{\alpha\beta} = 0 \quad (11)$$

$$\delta g^{\alpha\beta}g_{\alpha\beta} = -g^{\alpha\beta}\delta g_{\alpha\beta} \quad (12)$$

Therefore the equality is demonstrated (Davie, 2017).

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\delta g^{\alpha\beta}g_{\alpha\beta} \quad (13)$$

Appendix 2

In this appendix we will demonstrate the following expression:

$$\delta R_{\mu\nu} = \nabla_\rho \delta\Gamma^{\rho}_{\nu\mu} - \nabla_\nu \delta\Gamma^{\rho}_{\rho\mu} \qquad (14)$$

The curvature tensor can be shown to have the following dependency with respect to Levi-Civita connections:

$$R^{\rho}{}_{\sigma\mu\nu} = \Gamma^{\rho}_{\nu\sigma,\mu} - \Gamma^{\rho}_{\mu\sigma,\nu} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma} \qquad (15)$$

Taking a variation:

$$\delta R^{\rho}{}_{\sigma\mu\nu} = \delta\Gamma^{\rho}_{\nu\sigma,\mu} - \delta\Gamma^{\rho}_{\mu\sigma,\nu} + \delta\Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} + \Gamma^{\rho}_{\mu\lambda}\delta\Gamma^{\lambda}_{\nu\sigma} - \delta\Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma} - \Gamma^{\rho}_{\nu\lambda}\delta\Gamma^{\lambda}_{\mu\sigma} \qquad (16)$$

$\delta\Gamma^{\rho}_{\nu\sigma}$ is the difference between two connections and therefore it is a tensor. Being a tensor we can calculate its covariant derivative, which for a tensor once contravariant and twice covariant is as follows:

$$\delta\Gamma^{\rho}_{\nu\sigma\,;\,\mu} = \delta\Gamma^{\rho}_{\nu\sigma\,,\mu} + \Gamma^{\rho}_{\mu\lambda}\delta\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\lambda}_{\mu\nu}\delta\Gamma^{\rho}_{\lambda\sigma} - \Gamma^{\lambda}_{\mu\sigma}\delta\Gamma^{\rho}_{\nu\lambda} \qquad (17)$$

$$-\delta\Gamma^{\rho}_{\mu\sigma\,;\,\nu} = -\delta\Gamma^{\rho}_{\mu\sigma\,,\nu} - \Gamma^{\rho}_{\nu\lambda}\delta\Gamma^{\lambda}_{\mu\sigma} + \Gamma^{\lambda}_{\nu\mu}\delta\Gamma^{\rho}_{\lambda\sigma} + \Gamma^{\lambda}_{\nu\sigma}\delta\Gamma^{\rho}_{\mu\lambda} \qquad (18)$$

Then:

$$\delta R^{\rho}{}_{\sigma\mu\nu} = \delta\Gamma^{\rho}_{\nu\sigma\,;\,\mu} - \delta\Gamma^{\rho}_{\mu\sigma\,;\,\nu} + \Gamma^{\lambda}_{\mu\nu}\delta\Gamma^{\rho}_{\lambda\sigma} + \Gamma^{\lambda}_{\mu\sigma}\delta\Gamma^{\rho}_{\nu\lambda} - \Gamma^{\lambda}_{\nu\mu}\delta\Gamma^{\rho}_{\lambda\sigma} - \Gamma^{\lambda}_{\nu\sigma}\delta\Gamma^{\rho}_{\mu\lambda} + \delta\Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \delta\Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}$$

(19)

Using that the Christoffel symbols are symmetric with respect to the second and third index and also that being components of tensors (scalars) the commutative property is fulfilled:

$$\delta R^{\rho}{}_{\sigma\mu\nu} = \delta\Gamma^{\rho}_{\nu\sigma\,;\,\mu} - \delta\Gamma^{\rho}_{\mu\sigma\,;\,\nu} \tag{20}$$

We now find the variation of the Ricci tensor by a contraction in the first and third indices:

$$\delta R_{\sigma\nu} = \delta R^{\rho}{}_{\sigma\rho\nu} = \delta\Gamma^{\rho}_{\nu\sigma\,;\,\rho} - \delta\Gamma^{\rho}_{\rho\sigma\,;\,\nu} = \nabla_{\rho}\delta\Gamma^{\rho}_{\nu\sigma} - \nabla_{\nu}\delta\Gamma^{\rho}_{\rho\sigma} \tag{21}$$

Finally we make the following change $\sigma \to \mu$:

$$\delta R_{\mu\nu} = \nabla_{\rho}\delta\Gamma^{\rho}_{\nu\mu} - \nabla_{\nu}\delta\Gamma^{\rho}_{\rho\mu} \tag{22}$$

Appendix 3

In this appendix we will demonstrate the following expression:

$$g^{\mu\nu}\delta R_{\mu\nu} = -\nabla_\mu\nabla_\nu\delta g^{\mu\nu} + g_{\mu\nu}\square\delta g^{\mu\nu} \tag{23}$$

We will take the result in the appendix 2.

$$g^{\mu\nu}\delta R_{\mu\nu} = g^{\mu\nu}\left(\nabla_\rho\delta\Gamma^{\rho}_{\ \nu\mu} - \nabla_\nu\delta\Gamma^{\rho}_{\ \rho\mu}\right) = \nabla_\rho g^{\mu\nu}\delta\Gamma^{\rho}_{\ \nu\mu} - \nabla_\nu g^{\mu\nu}\delta\Gamma^{\rho}_{\ \rho\mu} \tag{24}$$

For the second term we make the following index change: $\nu \to \rho, \rho \to \gamma$

$$\nabla_\rho g^{\mu\nu}\delta\Gamma^{\rho}_{\ \nu\mu} - \nabla_\nu g^{\mu\nu}\delta\Gamma^{\rho}_{\ \rho\mu} = \nabla_\rho\left(g^{\mu\nu}\delta\Gamma^{\rho}_{\ \nu\mu} - g^{\mu\rho}\delta\Gamma^{\gamma}_{\ \gamma\mu}\right) \tag{25}$$

The Christoffel symbols are expressed in terms of the metric as follows:

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\alpha\lambda}\left(g_{\alpha\nu,\mu} + g_{\alpha\mu,\nu} - g_{\mu\nu,\alpha}\right) \tag{26}$$

We know from appendix 2 that this amount is a tensor, so it can be shown that:

$$\delta\Gamma^{\rho}_{\ \nu\mu} = \frac{1}{2}g^{\rho\alpha}\left(\delta g_{\alpha\mu;\nu} + \delta g_{\alpha\nu;\mu} - \delta g_{\nu\mu;\alpha}\right) \tag{27}$$

For the other quantity we have:

$$\delta\Gamma^{\gamma}_{\ \gamma\mu} = \frac{1}{2}g^{\alpha\gamma}\left(\delta g_{\alpha\mu;\gamma} + \delta g_{\alpha\gamma;\mu} - \delta g_{\gamma\mu;\alpha}\right) =$$

$$= \frac{1}{2}\left(\delta g^{\alpha\gamma}g_{\alpha\mu;\gamma} + g^{\alpha\gamma}\delta g_{\alpha\gamma;\mu} - \delta g^{\alpha\gamma}g_{\gamma\mu;\alpha}\right) =$$

$$\frac{1}{2}\left(\delta g^{\gamma}{}_{\mu:\gamma} + g^{\alpha\gamma}\delta g_{\alpha\gamma;\mu} - \delta g^{\alpha}{}_{\mu:\alpha}\right) \tag{28}$$

Making an index change in the third term $\to \gamma$:

$$\delta\Gamma^{\gamma}_{\ \gamma\mu} = \frac{1}{2}g^{\alpha\gamma}\delta g_{\alpha\gamma;\mu} \tag{29}$$

Once the two quantities $\delta\Gamma^{\rho}_{\ \nu\mu}$ and $\delta\Gamma^{\gamma}_{\ \gamma\mu}$ have been obtained, we are interested in expressing each of them as a function of $\delta g^{\mu\nu}$. For this we use the following results:

$$\delta g_{\alpha\beta} = -g_{\alpha\nu} g_{\beta\mu} \delta g^{\mu\nu} \; ; \; g^{\alpha\beta} g_{\beta\gamma} = \delta^{\beta}_{\gamma} \; ; \; \delta^{\beta}_{\gamma} g_{\alpha\nu} \delta g^{\nu\gamma} = g_{\alpha\mu} \delta g^{\beta\mu} \; ; \; \nabla^{\sigma} = g^{\sigma\gamma} \nabla_{\gamma} \quad (30)$$

Obtaining:

1) $\delta \Gamma^{\rho}_{\nu\mu}$

$$\delta \Gamma^{\rho}_{\nu\mu} = \frac{1}{2} g^{\rho\alpha} \left(\delta g_{\alpha\mu;\nu} + \delta g_{\alpha\nu;\mu} - \delta g_{\nu\mu;\alpha} \right) = \frac{1}{2} g^{\rho\alpha} \left(\nabla_{\nu} \delta g_{\mu\alpha} + \nabla_{\mu} \delta g_{\nu\alpha} - \nabla_{\alpha} \delta g_{\nu\mu} \right) =$$

$$= -\tfrac{1}{2} \left(g_{\mu\alpha} \nabla_{\nu} \delta g^{\rho\alpha} + g_{\nu\alpha} \nabla_{\mu} \delta g^{\rho\alpha} - g_{\nu\lambda} g_{\mu\sigma} \nabla^{\rho} \delta g^{\lambda\sigma} \right) \quad (31)$$

2) $\delta \Gamma^{\gamma}_{\gamma\mu}$

$$\delta \Gamma^{\gamma}_{\gamma\mu} = \tfrac{1}{2} g^{\alpha\gamma} \delta g_{\alpha\gamma;\mu} = -\tfrac{1}{2} g_{\lambda\sigma} \nabla_{\mu} \delta g^{\lambda\sigma} \quad (32)$$

Now we can evaluate the following term:

$$g^{\mu\nu} \delta \Gamma^{\rho}_{\nu\mu} - g^{\mu\rho} \delta \Gamma^{\gamma}_{\gamma\mu} =$$

$$= -\frac{1}{2} \left(g^{\mu\nu} g_{\mu\alpha} \nabla_{\nu} \delta g^{\rho\alpha} + g^{\mu\nu} g_{\nu\alpha} \nabla_{\mu} \delta g^{\rho\alpha} - g^{\mu\nu} g_{\nu\lambda} g_{\mu\sigma} \nabla^{\rho} \delta g^{\lambda\sigma} - g^{\mu\rho} g_{\lambda\sigma} \nabla_{\mu} \delta g^{\lambda\sigma} \right) =$$

$$= -\frac{1}{2} \left(\delta^{\nu}_{\alpha} \nabla_{\nu} \delta g^{\rho\alpha} + \delta^{\mu}_{\alpha} \nabla_{\mu} \delta g^{\rho\alpha} - \delta^{\mu}_{\lambda} g_{\mu\sigma} \nabla^{\rho} \delta g^{\lambda\sigma} - g_{\lambda\sigma} \nabla^{\rho} \delta g^{\lambda\sigma} \right) =$$

$$= -\frac{1}{2} \left(2 \nabla_{\alpha} \delta g^{\rho\alpha} - 2 g_{\lambda\sigma} \nabla^{\rho} \delta g^{\lambda\sigma} \right) = g_{\lambda\sigma} \nabla^{\rho} \delta g^{\lambda\sigma} - \nabla_{\alpha} \delta g^{\rho\alpha} =$$

$$= g_{\lambda\sigma} \nabla_{\rho} \nabla^{\rho} \left(\delta g^{\lambda\sigma} \right) - \nabla_{\rho} \nabla_{\alpha} (\delta g^{\rho\alpha}) = g_{\lambda\sigma} \nabla_{\rho} \nabla^{\rho} \left(\delta g^{\lambda\sigma} \right) - \nabla_{\lambda} \nabla_{\sigma} \left(\delta g^{\lambda\sigma} \right) =$$

$$= g_{\lambda\sigma} \nabla_{\lambda} \nabla^{\lambda} \left(\delta g^{\lambda\sigma} \right) - \nabla_{\lambda} \nabla_{\sigma} \left(\delta g^{\lambda\sigma} \right) = g_{\lambda\sigma} g^{\lambda\sigma} \nabla_{\lambda} \nabla_{\sigma} \left(\delta g^{\lambda\sigma} \right) - \nabla_{\lambda} \nabla_{\sigma} \left(\delta g^{\lambda\sigma} \right) =$$

$$= g_{\lambda\sigma} \Box \delta g^{\lambda\sigma} - \nabla_{\lambda} \nabla_{\sigma} \delta g^{\lambda\sigma} \quad (33)$$

If we make the following changes: $\rightarrow \mu$, $\sigma \rightarrow \nu$ we finally obtain (Trilleras, 2012):

$$g^{\mu\nu} \delta R_{\mu\nu} = -\nabla_{\mu} \nabla_{\nu} \delta g^{\mu\nu} + g_{\mu\nu} \Box \delta g^{\mu\nu} \quad (34$$

Appendix 4

The Robertson-Walker metric can be expressed as follows in Cartesian coordinates:

$$ds^2 = -dt^2 + a^2(t) \cdot [dx^2 + dy^2 + dz^2] \tag{35}$$

From this metric, the metric tensor, the Christoffel symbols and finally the corresponding Riemann curvature tensor are obtained. By means of the contraction of this tensor we obtain the Ricci tensor and by a second contraction the Ricci scalar.

Taking into account the following relations:

$$H = \frac{\dot{a}}{a} \tag{36}$$

$$\dot{H} = \frac{\ddot{a}a - \dot{a}\dot{a}}{a^2} = \frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 \tag{37}$$

The points denote temporal derivatives. We can express the Ricci scalar in the following way:

$$R = 6\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right) = 6\left(2H^2 + \dot{H}\right) \tag{38}$$

We also express the non-zero components of the Ricci tensor:

$$R_{00} = -3\frac{\ddot{a}}{a} = -3\left(\dot{H} + H^2\right) = -3\left(\dot{H} + 2H^2 - H^2\right) = -\frac{R}{2} + 3H^2 \tag{39}$$

$$R_{11} = R_{22} = R_{33} = a\ddot{a} + 2\dot{a}^2 = a^2\left(\dot{H} + 2H^2\right) \tag{40}$$

For a perfect fluid it can be shown that its stress-energy tensor is diagonal and takes the following form

:

$$T_{\mu\nu}{}^{(M)} = (\rho + P)u_\mu u_\nu + P g_{\mu\nu} \tag{41}$$

Reminding that the equations derived from the principle of action are the following:

$$F(R)R_{\mu\nu} + \left(-\nabla_\mu\nabla_\nu + g_{\mu\nu}\Box\right)F(R) - f(R)\frac{1}{2}g_{\mu\nu} = \kappa T_{\mu\nu}{}^{(M)} \tag{42}$$

We demonstrate the case: $\mu = \nu = 0$ taking into account the perfect fluid and the Robsertson-Walker metric.

$$F(R)R_{00} + (-\nabla_0\nabla_0 + g_{00}\Box)F(R) - f(R)\frac{1}{2}g_{00} = \kappa T_{00}{}^{(M)} \tag{43}$$

$$\left(-\frac{R}{2} + 3H^2\right)F(R) - (\nabla_0\nabla_0 + \Box)F(R) + f(R)\frac{1}{2} = \kappa\rho \tag{44}$$

We can use the following notation:

$$\nabla_0\nabla_0 F(R) = \partial_0^2 F(R) \tag{45}$$

It can be shown that:

$$\Box F(R) = -\partial_0^2 F(R) - 3\frac{\dot{a}}{a}\partial_0 F(R) \tag{46}$$

Applying the chain rule:

$$(\nabla_0\nabla_0 + \Box)F(R) = -3\frac{\dot{a}}{a}\partial_0 F(R) = -3\frac{\dot{a}}{a}\partial_R F(R)\partial_0 R = -3HF'(R)\dot{R} \tag{47}$$

Going back to the previous equation and substituting:

$$\left(-\frac{R}{2} + 3H^2\right)F(R) + 3HF'(R)\dot{R} + f(R)\frac{1}{2} = \kappa\rho \tag{48}$$

$$3FH^2 = \frac{FR-f}{2} - 3HF'(R)\dot{R} + \kappa\rho = \frac{FR-f}{2} - 3H\dot{F} + \kappa\rho \tag{49}$$

Appendix 5

In this appendix we will demonstrate the following expression:

$$3\phi\Box + 2U(\phi) - \phi\frac{\delta U(\phi)}{\delta\phi} = \kappa T \tag{50}$$

We start representing the Brans-Dicke action:

$$S[g] = \int \sqrt{-g}\left[\frac{\phi R}{2\kappa} - U(\phi)\right] d^4x + S_M \tag{51}$$

We take a variation of that action:

$$\delta S[g] = \int \left\{\delta\sqrt{-g}\left[\frac{\phi R}{2\kappa} - U(\phi)\right] + \sqrt{-g}\left[\frac{\delta\phi R}{2\kappa} + \frac{\phi\delta R}{2\kappa} - \delta U(\phi)\right]\right\} d^4x + \delta S_M \tag{52}$$

We take in mind some expressions:

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu} - \nabla_\mu\nabla_\nu\delta g^{\mu\nu} + g_{\mu\nu}\Box\delta g^{\mu\nu} \tag{53}$$

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\delta g^{\mu\nu}g_{\mu\nu} \tag{54}$$

We also need to consider this calculus:

$$\frac{\delta U(\phi)}{\delta\phi} = \frac{R(\phi)+R'(\phi)\phi-f'\left(R(\phi)\right)R'(\phi)}{2\kappa} = \frac{R(\phi)+R'(\phi)\phi-\phi R'(\phi)}{2\kappa} = \frac{R(\phi)}{2\kappa} \tag{55}$$

With all this, substituting δR ,$\delta\sqrt{-g}$, and bearing in mind the latest result, dividing by $\delta\phi$ and identifying common terms:

$$\frac{\delta S[g]}{\delta\phi} = \int \left\{\sqrt{-g}\left(-\frac{1}{2}g_{\mu\nu}\left[\phi\frac{\delta U(\phi)}{\delta\phi} - U(\phi)\right] + \left[\frac{\phi}{2\kappa}\left(R_{\mu\nu} - \nabla_\mu\nabla_\nu + g_{\mu\nu}\Box\right)\right]\right)\frac{\delta g^{\mu\nu}}{\delta\phi}\right\} d^4x + \frac{\delta S_M}{\delta\phi} \tag{56}$$

Taking the expression of S_M and taking $\frac{\delta S[g]}{\delta\phi} = 0$

$$\sqrt{-g}\left(-\frac{1}{2}g_{\mu\nu}\left[\phi\frac{\delta U(\phi)}{\delta\phi} - U(\phi)\right] + \left[\frac{\phi}{2\kappa}\left(R_{\mu\nu} - \nabla_\mu\nabla_\nu + g_{\mu\nu}\Box\right)\right]\right)\frac{\delta g^{\mu\nu}}{\delta\phi} = \frac{\delta L_M}{\delta\phi} \tag{57}$$

The stress-energy tensor is identified again $T_{\mu\nu}$:

$$-\frac{1}{2}g_{\mu\nu}\left[\phi\frac{\delta U(\phi)}{\delta\phi} - U(\phi)\right] + \left[\frac{\phi}{2\kappa}\left(R_{\mu\nu} - \nabla_\mu\nabla_\nu + g_{\mu\nu}\Box\right)\right] = \frac{1}{2}T_{\mu\nu} \tag{58}$$

Multiplying by $g^{\mu\nu}$:

$$-\frac{1}{2}g^{\mu\nu}g_{\mu\nu}\left[\phi\frac{\delta U(\phi)}{\delta\phi} - U(\phi)\right] + \left[\frac{\phi}{2\kappa}g^{\mu\nu}R_{\mu\nu} - \frac{\phi}{2\kappa}g^{\mu\nu}\nabla_\mu\nabla_\nu + \frac{\phi}{2\kappa}g^{\mu\nu}g_{\mu\nu}\Box\right] = \frac{1}{2}g^{\mu\nu}T_{\mu\nu} \tag{59}$$

Taking into account that:

$$g^{\mu\nu}g_{\mu\nu} = \delta_\mu^\mu = 4 \tag{60}$$

$$\frac{g^{\mu\nu}R_{\mu\nu}}{2\kappa} = \frac{R}{2\kappa} = \frac{\delta U(\phi)}{\delta\phi} \tag{61}$$

$$g^{\mu\nu}\nabla_\mu\nabla_\nu = \Box \tag{62}$$

$$g^{\mu\nu}T_{\mu\nu} = T \tag{63}$$

$$V(\phi) = 2\kappa \cdot U(\phi) \tag{64}$$

We put the latest expression like as follows:

$$-2\cdot\frac{1}{2\kappa}\left[\phi\frac{\delta V(\phi)}{\delta\phi} - V(\phi)\right] + \frac{1}{2\kappa}\left[\phi\frac{\delta V(\phi)}{\delta\phi} - \phi\Box + 4\phi\Box\right] = \frac{1}{2}T \tag{65}$$

$$3\phi\Box + 2V(\phi) - \phi\frac{\delta V(\phi)}{\delta\phi} = \kappa T \tag{66}$$

Appendix 6

In that appendix we will demonstrate the following expression:

$$H = -\frac{1}{1+z}\frac{dz}{dt} \tag{67}$$

We start from the definition of the redshift:

$$z = \frac{\lambda_{obs} - \lambda_{em}}{\lambda_{em}} \tag{68}$$

λ_{obs} is the wavelength of the observed object and λ_{em} is the real wavelength emitted by that object. Thus:

$$1 + z = \frac{\lambda_{obs}}{\lambda_{em}} \tag{69}$$

Now we take into account that we have a Robertson-Walker Universe and the expansion of the Universe is related to an expansion of the wavelength. Then:

$$1 + z = \frac{a_0}{a} \tag{70}$$

a_0 is the scale factor of the actual Universe and a is the scale factor when the object emitted this wave. If we derive the expression by the time in both sides:

$$\dot{z} = \frac{dz}{dt} = -\frac{a_0 \dot{a}}{a^2} = -\frac{a_0}{a}\frac{\dot{a}}{a} = -(1+z) \cdot H \tag{71}$$

$$H = -\frac{1}{(1+z)}\frac{dz}{dt} \tag{72}$$