An educated review of "Quantum work statistics, Loschmidt echo and information scrambling"[1]

Eduardo González Padrón

Supervised by
Dr. Daniel Alonso Ramírez

at
Sección de Física
Facultad de Ciencias
UNIVERSIDAD DE LA LAGUNA

Julio 2020

Acknoledgements

Gracias a todas las personas que me han acompañado en estos cinco años de carrera. A Famara y a Aitor por estar en los tiempos más convulsos. A mis compañeros de carrera. A mis amigas y amigues. A mis imprescindibles Miguel, Rebeca y Timón. A Raquel, mi fiel y querida compañera.

Gracias también a mi tutor Daniel Alonso, por la ciencia transmitida en las discusiones y por su confianza y paciencia conmigo.

A mis hermanos y a mis padres, pero especialmente a mi madre, a mi abuela y a mi hermana.

Dedicado a mi abuelo Eliseo, nuestro gran desconocido.

Abstract

The formulation of quantum work statistics as a dynamical problem through the Loschmidt echo is at the heart of this work. An introduction to each of these concepts is presented together with the notion of information scrambling, which extends the scope of this work to areas such as quantum chaos or even black hole physics. Using the paper of *A. Chenu et al.* [1] as the guidelines, we first show that the work statistics associated with an arbitrary driving protocol of an isolated quantum system in a generic initial state is equivalent to the Loschmidt echo dynamics of a purified density matrix in an enlarged Hilbert space. When the initial state is thermal, the purification leads to a thermofield double state, which is used to describe eternal black holes through the AdS/CFT correspondence, often argued to be the fastest information scramblers. The field of quantum chaotic systems is shown to emerge naturally from the previous content, and a full description of it in terms of Random Matrix Theory is also presented. Numerical and analytical results are finally obtained for the quantities introduced after imposing time-reversal symmetry in our problem, hence selecting the Gaussian Orthogonal Ensemble as the framework within we shall take our averages.

Contents

List of Figures

List of Tables

1. Introduction

Abstract

El artículo que se analiza en este proyecto [1] es altamente interdisciplinar: conecta áreas de la física desde la termodinámica cuántica hasta el física de agujeros negros. Esta introducción sirve como manual de *cómo leer el trabajo*, explicando el contenido de las secciones, objetivos y estructura interna del mismo. La contextualización del trabajo se realiza de manera continua en los siguientes capítulos para facilitar la lectura.

The attempt to unify the different theories that currently describe our world has lead to a very interdisciplinary scenario in modern physics. The extension of standard thermodynamics and non-equilibrium statistical physics to ensembles of sizes well below the thermodynamic limit, in non-equilibrium situations and with the full inclusion of quantum effects [2] resulted in the emerging research field of *quantum thermodynamics*. The increasing interest in time reversibility, a classical thermodynamic problem, and the description of chaotic systems in the quantum domain has motivated the development of *quantum chaos*, giving rise to new tools to assess the sensibility of quantum evolution to perturbations as the *Loschmidt echo* or the out-of-time order correlators (OTOCs). The latter are widely used as a way to measure the *scrambling of information* in quantum systems, a concept that originates within the quantum information framework but whose late interest makes it a necessary background for quantum chaos analysis and, most surprisingly, black hole physics. In somewhat the same direction, the AdS/CFT correspondence is possibly the latest and most ambitiously interdisciplinary proposal, which theorises that a direct connection exists between quantum gravity theories (that uses Anti-de-Sitter spaces) and quantum field theories (or its extension, Conformal Field Theories). All the previous areas of physics are condensed in the reviewed paper [1], which establishes, in particular, firm connections between quantum work statistics, Loschmidt echo and information scrambling.

Quantum work statistics arises from the definition of *quantum work* as a stochastic variable, making it necessary a description in terms of a work probability density function. In section §2.1 it is applied to the case of an isolated quantum system that is driven out of equilibrium, allowing a discussion on the possibility of finding quantum analogs to the well-established fluctuation relations by the end of the section. The main result concerning quantum work statistics is equation (2.11), which expresses the work characteristic function as a two-time quantum correlation function. It is also the starting point to establish the first important relation of the work. Note that we take the Boltzmann and reduced Planck constants to be equal to unity ($\hbar = 1$, $k_B = 1$) throughout the text.

Time reversibility is closely tied to thermodynamics, and when studied in the quantum domain, the Loschmidt echo arises as a natural tool to assess it. More specifically, it measures the extent to which quantum evolution can be reversed upon an imperfect time-reversal operation [1]. Section §2.2 introduces both historically and mathematically this quantity, which can be identified after some calculations with the work characteristic function of the dynamics of the previous part as expressed in equation (2.23). It is the major result of the section.

Information scrambling is a concept that appears in the context of thermalisation of quantum many body systems, and accounts for the process of hiding and spreading the initial local information of these systems. As stated in section §2.3, a rigorous mathematical formalism of this quantity is deprecated in favour of a more intuitive, heuristic presentation. Still, its relation with the Loschmidt echo is summarised in equation (2.27). The end of the section introduces the thermofield double state, which connects the work to black hole physics through the AdS/CFT correspondence. Relation (2.30) can be seen as the main connection between quantum work statistics, Loschmidt echo and information scrambling.

Quantum chaos is finally put into play in chapter §3 after being implicitly around throughout the whole text. In section §3.1 it is first generally defined from a quantum system whose underlying equations of motion are so complicated that it is meaningless to treat it in any other way than statistically. A more precise definition in terms of random matrix theory is then given, presenting the analytical framework for the specific case of a system with time-reversal symmetry: the Gaussian Orthogonal Ensemble. Bringing quantum chaotic systems is in fact a way of particularising the system that is employed in the first chapter: Hamiltonians can be extracted from the previous ensemble, and results can be finally obtained in section §3.2. The main results of this section are summarised in the figures and the discussions that surround them. Particularly of relevance is the behaviour of the

Loschmidt echo evolution, that exhibits features common to scrambling dynamics and hence ratifies its ability to diagnose chaos.

1.1 Motivation and objectives

Approaching chaos was the original motivation of this work, which combined with a profound interest in the quantum world lead us to the reading and revision of the work of *A. Chenu et al.* [1]. The level of the concepts that are employed in it are, however, quite high for an undergraduate student to understand it in a first read. The main objective of this work is then to present an *educated review* of the reference paper [1], to make its reading possible for a student in our situation. This necessarily requires the following steps:

- To introduce and contextualise the main concepts of the paper, quantum work statistics, Loschmidt echo and information scrambling, together with the notion of chaos and random matrix theory.
- Derive the relations that allow us to connect the previous concepts, writing out the full mathematical derivations to bring transparency in the results.
- Reproduce the numerical results that are found for chaotic systems in the paper. As a way of contributing to the referenced work and to broaden up the analysis, another objective is to include an analytical framework to the discussion. This is in fact *innovative*: analytical results using Gaussian Unitary Ensemble have been obtained for comparison with numerical results, but no reference on using the Gaussian Orthogonal Ensemble has been found.

It should be noted that the procedure we use for assessing the paper is to read and analyse it in equal measure and in a very progressive way, breaking it apart and focusing on the material that appears along the way. A deep read is then carried out so that every concept or even *word* that is read sounds meaningful, precise and well-defined and can be put in context for a complete understanding. This in fact defines the inner structure of our work: an extensive introduction that condenses all the theoretical material is substituted by an effort in continuously presenting historical and critical contextualisation on each chapter, defining new tools on the fly and including brief explanatory sections and proofs when needed. This way we try to prioritise a light, self-consistent read over an excessively fragmented presentation of the work.

2. Paper review and analysis

Abstract

Comenzamos presentando el sistema hamiltoniano que se va a tratar y el proceso que se quiere estudiar en él. A continuación, se introduce la definición de trabajo cuántico y se describe el proceso de dos medidas proyectivas de energía en el que se apoya, sustituyendo seguidamente la descripción del mismo en términos de la función de densidad de probabilidad por la de su función característica. Las expresiones que se derivan a partir de aquí son empleadas en las siguientes secciones para establecer las relaciones buscadas con el *Loschmidt echo* y el *information scrambling*, conceptos que son presentados igualmente.

We start out by giving a description of the problem, characterising the system that is studied and the process it is subjected to. Some of the formalism that is used throughout the work is also summarised; note that we mostly use the notation of the reference paper [1], although external concepts and definitions are brought in order to explain the derivations that are omitted in it.

Let us consider a quantum system in a Hilbert space $\mathcal{H}$ whose dynamics is governed by a time-dependent Hamiltonian $\hat{H}_s$. Otherwise isolated, assume that we are able to evolve it or *drive* it according to a prescribed protocol in the time s by switching a parameter of the Hamiltonian at a finite rate from $s = 0$ to $s = \tau$, leading to the evolution $\hat{H}_0 \to \hat{H}_\tau$. It is convenient for our purposes to decompose the Hamiltonian using its eigenstates and eigenvalues —that we assume are non-degenerate— as $\hat{H}_s = \sum_n E_n^s |n_s\rangle\langle n_s|$, so that

- $H_0 = \sum_n E_n^0 |n_0\rangle\langle n_0|$, where $(E_n^0, |n_0\rangle)$ is the nth eigenvalue-eigenstate pair of the initial Hamiltonian
- $H_\tau = \sum_m E_m^\tau |m_\tau\rangle\langle m_\tau|$, where $(E_m^\tau, |m_\tau\rangle)$ is the mth eigenvalue-eigenstate pair of the final Hamiltonian

Consider that the system is prepared in the initial state[1] given by the density matrix $\hat{\rho}$ (right before the parameter is switched at s=0), and that the operator that dictates its time-evolution from $s = 0$ to $s = \tau$ obeys the Schrödinger equation and can thus be written via a Dyson series expansion [3] [4] by

$$\hat{U}(\tau) = \mathcal{T} \exp\left\{ -i \int_0^\tau \hat{H}_s ds \right\} \quad , \tag{2.1}$$

where $\mathcal{T}$ is the time-ordering operator, which protects the exponential function from the possible non-commutativity between the final and initial Hamiltonians. This expression also includes the case of instantaneous processes or *quenches* [2], for which $\tau \to 0^+$ and hence $\hat{U}(\tau) = \mathbf{1}$; even in this case, the Hamiltonian changes and so we will maintain the notation H_τ for the final Hamiltonian in order to distinguish it from the initial one H_0.

A very natural observation to do within the context of quantum thermodynamics (as noticed in [7]) is that the prescribed protocol resembles a standard thermodynamic transformation: a classical external force that acts on the parameters or variables of an isolated system and makes it evolve from an initial configuration to a final, different one. It is clear that we are not dealing with quasistatic processes and our protocol takes the system away from equilibrium, being the sudden quench the extreme case, but some questions arise: how can we define a *quantum work* for this process? How does it relate to its classical counterpart? Do the fluctuation relations (to be introduced) hold in this quantum regime? These questions are solved in the following sections.

2.1 Quantum work statistics

Since the system under study is isolated (there is no "heat" transfer during the process, even though the initial $\hat{\rho}$ may be previously prepared in a thermal state by using a heat bath), it is natural for us to define the difference in energy between the final and the initial states as the *work performed on the system during the*

[1]We may be using interchangeably the word *state* for a vector of the form $|\Psi\rangle \in \mathcal{H}$ and for the density operator/matrix that describes the system itself, as widely used in the bibliography.

[2]The concept of quench already refers to a unitary evolution in time following "the sudden change of the parameters" [5], although it is sometimes written as *sudden* quench or *instantaneous* quench as there are authors that prefer to extend its definition to account for *slow changes in the system parameters* [6]. Quench dynamics is indeed a very active and broad field.

prescribed protocol. Quantum mechanically, two projective energy measurements need to be carried out, one at the initial time $s = 0^-$ using H_0, before the external driving starts, and another one at the final time $s = \tau$ with H_τ, so that the energy outcomes give the work done as $W = E_m^\tau - E_n^0$.

This measuring protocol is typically known as the two-point measurement scheme (TPM) [8] [9] and inherently defines work as a stochastic quantity due to the probabilistic nature of quantum measures (pointing out an important difference with respect to work and fluctuations in the classical regime, whose randomness comes only from the statistical description of the initial ensemble). This statement forces us to express work as a probability density function (PDF); to show the construction of this distribution via this scheme we essentially follow the arguments given by Kurchan [9] and Tasaki [10]:

1. Perform an energy measurement at time $s = 0^-$, obtaining the outcome E_n^0 with the probability $p_n^0 = \left\langle \hat{P}_n^0 \right\rangle = \mathrm{Tr}\left\{ \hat{P}_n^0 \hat{\rho} \right\} = \langle n_0 | \hat{\rho} | n_0 \rangle$, where $\langle ... \rangle$ denotes the expected value in the correspondent state and $\mathrm{Tr}\{...\}$ the trace over the Hilbert space $\mathcal{H}$. The state after the measurement is then the eigenstate $|n_0\rangle$ of the initial Hamiltonian H_0.

2. Evolve the post-measurement state using the time-evolution operator of the equation (2.1) so that at time $s = \tau$ the state becomes $\hat{U}(\tau) |n_0\rangle$.

3. Perform an energy measurement with H_τ at time $s = \tau$ that yields the result E_m^τ. The probability of measuring this eigenvalue for the final Hamiltonian is the transition probability $p_{m|n}^\tau = | \langle m_\tau | \hat{U}(\tau) |n_0\rangle |^2$. It is in fact a *conditional probability* and can be read as *the probability of measuring E_m^τ at time $s = \tau$ given that we measured E_n^0 at time $s = 0$.*

4. The energy difference or work $W = E_m^\tau - E_n^0$ for this process is then obtained with probability $p_{m,n} = p_n^0 p_{m|n}^\tau$. If we average over all the possible initial conditions and outcomes, we can write the *discrete* work probability distribution by means of the Dirac delta function as

$$p(W) := \sum_{n,m} p_n^0 p_{m|n}^\tau \delta[W - (E_m^\tau - E_n^0)] \quad . \tag{2.2}$$

2.1.1 Work characteristic function

The first connection that is pursued in the reference paper between quantum work statistics and Loschmidt echo is established through the characteristic function of work. It is defined as

$$\chi(t,\tau) \equiv \left\langle e^{itW} \right\rangle \tag{2.3}$$

$$= \int_{-\infty}^{\infty} dW p(W) e^{itW} \tag{2.4}$$

which can be seen both as the classical average in probability theory or as a Fourier transform of the distribution $p(W)$. Note that it contains full information regarding the statistics of the random variable W, since the work probability density can be easily recovered by undoing the transformation

$$p(W) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \chi(t,\tau) e^{-itW} \quad . \tag{2.5}$$

Before introducing the Loschmidt echo in §2.2, we need to do some algebraic manipulation to the characteristic function: by expressing it as a quantum correlation function, some fundamental properties of quantum work emerge and allow us to connect it with other thermodynamic quantities, giving rise to the (quantum) fluctuation theorems (although we do not go into the details of this relations). We start by applying the definition of the δ-Dirac from $p(W)$ to the equation (2.4), followed by using the explicit form of the probabilities p_n^0 and $p_{m|n}^\tau$:

$$\begin{aligned}
\chi(t,\tau) &= \sum_{n,m} p_n^0 p_{m|n}^\tau e^{it(E_m^\tau - E_n^0)} \\
&= \sum_{n,m} p_n^0 \langle m_\tau | \hat{U}(\tau) | n_0 \rangle\langle n_0 | \hat{U}^\dagger(\tau) | m_\tau \rangle \, e^{it(E_m^\tau - E_n^0)} \\
&= \sum_m \langle m_\tau | \hat{U}(\tau) e^{-it\hat{H}_0} \left(\sum_n p_n^0 | n_0 \rangle\langle n_0 | \right) \hat{U}^\dagger(\tau) e^{it\hat{H}_\tau} | m_\tau \rangle \\
&= \mathrm{Tr}\left\{ \hat{U}(\tau) e^{-it\hat{H}_0} \hat{\rho}_{mix} \hat{U}^\dagger(\tau) e^{it\hat{H}_\tau} \right\} \\
&= \mathrm{Tr}\left\{ \hat{U}^\dagger(\tau) e^{it\hat{H}_\tau} \hat{U}(\tau) e^{-it\hat{H}_0} \hat{\rho}_{mix} \right\}
\end{aligned}$$

where the cyclic invariance of the trace has been used in the last equality. In the derivation, we have also defined the quantity $\hat{\rho}_{mix} = \sum_n p_n^0 | n_0 \rangle\langle n_0 | = \sum_n \hat{P}_n^0 \hat{\rho} \hat{P}_n^0$, as in the reference paper[3]; it represents the diagonal part of $\hat{\rho}$ in the eigenbasis of the initial Hamiltonian $\hat{H}_0$. This result allows us to express the characteristic function as an average taken with respect to this new mixed state $\hat{\rho}_{mix}$

$$\chi(t,\tau) = \mathrm{Tr}\Big\{\hat{U}^\dagger(\tau)e^{it\hat{H}_\tau}\hat{U}(\tau)e^{-it\hat{H}_0}\hat{\rho}_{mix}\Big\} \tag{2.6}$$

$$= \Big\langle \hat{U}^\dagger(\tau)e^{it\hat{H}_\tau}\hat{U}(\tau)e^{-it\hat{H}_0}\Big\rangle_{mix} \quad . \tag{2.7}$$

Further manipulation can be carried out with the intent of simplifying the expression and putting together both Hamiltonians under the same exponential function, so that there exists the possibility of expressing work as a Hermitian operator; the latter will prove an impossible task under the definition of work given (see the next subchapter for a full discussion on this topic). We proceed by defining an *effective* Hamiltonian $\hat{H}^{\mathrm{eff}}_\tau = \hat{U}^\dagger(\tau)\hat{H}_\tau\hat{U}(\tau)$, which is the Hamiltonian $\hat{H}_\tau$ in the Heisenberg picture; since $\hat{H}_\tau$ and $\hat{H}^{\mathrm{eff}}_\tau$ are related by a unitary transformation, we can easily show that their spectrum is the same:

$$\hat{H}_\tau |m_\tau\rangle = E^\tau_m |m_\tau\rangle \quad \Rightarrow \quad \hat{U}^\dagger(\tau)\hat{H}_\tau\hat{U}(\tau)\hat{U}^\dagger(\tau)|m_\tau\rangle = \hat{U}^\dagger(\tau)E^\tau_m|m_\tau\rangle \tag{2.8}$$

$$\Rightarrow \quad \hat{H}^{\mathrm{eff}}_\tau\Big(\hat{U}^\dagger(\tau)|m_\tau\rangle\Big) = E^\tau_m\Big(\hat{U}^\dagger(\tau)|m_\tau\rangle\Big) \quad . \tag{2.9}$$

Not only their spectrum is the same, but the time backward-evolved eigenstates $\Big\{\hat{U}^\dagger(\tau)|m_\tau\rangle\Big\}$ form a complete basis for $\mathcal{H}$, so that we can use them for tracing since the completeness relation $\mathbf{1} = \sum_m \hat{U}^\dagger(\tau)|m_\tau\rangle\langle m_\tau|\hat{U}(\tau)$ holds. Applying it to the equation (2.6), we have

$$\begin{aligned}\chi(t,\tau) &= \sum_m \langle m_\tau|\hat{U}(\tau)\hat{U}^\dagger(\tau)e^{it\hat{H}_\tau}\hat{U}(\tau)e^{-it\hat{H}_0}\hat{\rho}_{mix}\hat{U}^\dagger(\tau)|m_\tau\rangle \\ &= \sum_m \langle m_\tau|e^{itE^\tau_m}\hat{U}(\tau)e^{-it\hat{H}_0}\hat{\rho}_{mix}\hat{U}^\dagger(\tau)|m_\tau\rangle \\ &= \sum_m \langle m_\tau|\hat{U}(\tau)e^{it\hat{H}^{\mathrm{eff}}_\tau}e^{-it\hat{H}_0}\hat{\rho}_{mix}\hat{U}^\dagger(\tau)|m_\tau\rangle \quad .\end{aligned}$$

[3]It is said [1] that *the first projective energy measurement in the initial eigenbasis generally leads to the (post-measurement) mixed state* $\hat{\rho}_{mix}$, although this cannot be seen in a single realisation of the process (where the projector $\hat{P}^0_n$ yields the intermediate *pure* state $\hat{\rho}_n = |n_0\rangle\langle n_0|$). If we wanted to view it as a post-measurement mixed state, it would be *if we perform a measurement but we do not record the results, the post-measurement state is given by* $\hat{\rho}_{mix}$ [11], but it does not correspond to the measuring scheme of the process; $\hat{\rho}_{mix}$ does appear though due to the *invasive* nature of the first measurement, that actually alters the measured work itself (see next section for further discussion)

Analogously to the expressions (2.6) and (2.7), we can write the prior expression in the simple form

$$\chi(t,\tau) = \mathrm{Tr}\Big\{ e^{it\hat{H}_\tau^{\mathrm{eff}}} e^{-it\hat{H}_0} \hat{\rho}_{mix} \Big\} \tag{2.10}$$

$$= \Big\langle e^{it\hat{H}_\tau^{\mathrm{eff}}} e^{-it\hat{H}_0} \Big\rangle_{mix} \quad . \tag{2.11}$$

The fact that the expressions that we have found so far for the characteristic function are given in terms of two-time quantum correlation functions and not by the expectation value of a (single time) operator gives us hints to start the following section.

2.1.2 Why not a work operator? A critical review

The apparent freedom of choice at the time of defining a quantum analog for work has motivated controversy around which requirements it should meet and which basic features it should be built upon. Some authors as Allahverdyan [12] [13] have tried to define work as the average of a Hermitian operator, while most of the latest research has advanced by using the TPM scheme, achieving results that serve as strong arguments for this definition of work. The late debate seems to be guided by the possibility of finding quantum analogs to the classical (work) fluctuation theorems, although sometimes the lack of a clear differentiation between the *inclusive* and *exclusive* perspectives in the definition of work has hindered the progress in this sense. Revising these concepts seems hence a good starting point for giving a consistent answer to the proposed question.

Let us consider that the time-dependent Hamiltonian of our process has the explicit form $\hat{H}_s = \hat{H}_0 + \hat{X}_s$, where $\hat{X}_s$ is the external time-dependent perturbation that we switch in order to perform the evolution $\hat{H}_0 \to \hat{H}_\tau$. We define the *exclusive* viewpoint as that which does not count the coupling to the external work-source $\hat{X}_s$ as internal energy; in the measuring scheme described, this means that the second measurement is made with $\hat{H}_0$, so that the *exclusive work* is defined as[4]

$$W_0 = E_m^0 - E_n^0 \quad . \tag{2.12}$$

[4]Notice that the time evolution of the system, $\hat{U}(\tau)$, remains unchanged since the process has not been altered, only our definition of work.

It is not hard to prove that the same relations found for W hold for W_0 if we change $\hat{H}_\tau$ by $\hat{H}_0$ where needed. In the contrary, the *inclusive* viewpoint is the one that does count the interaction term $\hat{X}(t)$ as part of the work. This is the perspective adopted throughout this text and the most used in the bibliography.

These concepts do not only apply to the quantum regime: the classical definitions of work are also formulated taking this into account [14]. In fact, it allows us to split the main (classical) fluctuation theorems in two: the Bochkov-Kuzovlev (BK) equality and BK work fluctuation relation, that use an *exclusive* approach to work; and the Jarzynski equality and Crooks fluctuation theorem, that adopt the *inclusive* viewpoint. With this in mind, we claim that the only definition of quantum work that is able to reproduce so far the fluctuation theorems in the quantum regime is the one that employs the two projective energy measurement (TPM) scheme (following the work of Campisi [15], who carries out a complete revision of the evolution of the concept and definition of quantum work, and also in-depth derivations of the formulae that are shown in the table 2.1, which are beyond the scope of this work).

Exclusive perspective (W_0)	Inclusive perspective (W)
$\left\langle e^{-\beta W_0} \right\rangle = 1$	$\left\langle e^{-\beta W} \right\rangle = e^{-\beta \Delta F}$
$\frac{p_F^0(W_0)}{p_R^0(-W_0)} = e^{\beta W_0}$	$\frac{p_F(W)}{p_R(-W)} = e^{\beta(W - \Delta F)}$

Table 2.1: Work fluctuation theorems presented symmetrically from the exclusive and inclusive perspectives. The exclusive perspective column shows the BK equality and the BK fluctuation relation in comparison to the inclusive pespective one, showing the Jarzkynski equality and Crooks fluctuation theorem. These formulae can be understood both in classical and quantum regimes, with the correct definition and interpretation of each term. The subscripts F and R and superscript 0 on the probability densities p_F^0, p_R^0 denote the forward and reverse processes and the exclusive perspective, respectively; ΔF is the free-energy difference and $\beta = 1/k_B T$. Further conditions are required for proving them as *microreversibility* and assuming an initial canonical distribution [15].

With this strong back-up for our definition of work, we can now bring back the discussion over whether work can be expressed as a Hermitian operator or not. The alternative definition that is usually proposed [12][13] is (we write here its analogous definition to the problem described in our case, maintaining hence the notation)

$$\widetilde{W} = \mathrm{Tr}\left\{ (\hat{H}_\tau^{\mathrm{eff}} - \hat{H}_0)\hat{\rho} \right\} = \left\langle \hat{H}_\tau^{\mathrm{eff}} - \hat{H}_0 \right\rangle \tag{2.13}$$

which seems consistent with the classical definition of work as the average energy acquired by the system from its interaction with the work source. Can we relate this definition to the one we have? Are they equivalent? First thing to notice is that $W = E_m^\tau - E_n^0$ gives the work for a single realisation (repetition) of the process; if we average in the classical sense[5], we find that

$$\begin{aligned}
\langle W \rangle &= \sum_{n,m} p_n^0 p_{m|n}^\tau (E_m^\tau - E_n^0) \\
&= \sum_{n,m} p_n^0 \langle m_\tau | \hat{U}(\tau) | n_0 \rangle\langle n_0 | \hat{U}^\dagger(\tau) | m_\tau \rangle (E_m^\tau - E_n^0) \\
&= \sum_{m} \langle m_\tau | \hat{H}_\tau \hat{U}(\tau) \hat{\rho}_{mix} \hat{U}^\dagger(\tau) - \hat{U}(\tau) \hat{\rho}_{mix} \hat{H}_0 \hat{U}^\dagger(\tau) | m_\tau \rangle \\
&= \mathrm{Tr}\Big\{ \hat{H}_\tau \hat{U}(\tau) \hat{\rho}_{mix} \hat{U}^\dagger(\tau) \Big\} - \mathrm{Tr}\Big\{ \hat{\rho}_{mix} \hat{H}_0 \Big\} \\
&= \mathrm{Tr}\Big\{ (\hat{H}_\tau^{\text{eff}} - \hat{H}_0) \hat{\rho}_{mix} \Big\} = \Big\langle \hat{H}_\tau^{\text{eff}} - \hat{H}_0 \Big\rangle_{mix}
\end{aligned} \tag{2.14}$$

where similar arguments employed in the previous derivations have been used. As we can see, Eqs. (2.13) and (2.14) differ from one another only by the state with respect to which they are averaged. The latter only reduces to the former if the condition $\left[\hat{\rho}, \hat{H}_0\right] = 0$ holds, so that $\hat{\rho}_{mix} = \hat{\rho}$; from a physical perspective, this condition implies that the first measurement is not invasive in the sense that it does not alter the initial state, i.e., it leaves the system and work *untouched*. In classical mechanics, work is always untouched (the interaction between the system and the measurement device can be made arbitrarily weak without limiting the precision of the outcome), which could be seen as supporting eq. (2.13). Nevertheless, this definition of work as an already averaged quantity of a Hermitian operator does not allow one to determine fluctuations in a meaningful way and disregards the inherent process dependence of work [16]; indeed, it fails to reproduce the classic fluctuation theorems, leading to the erroneous conclusion that *there is no direct analog of the classical BK (and Jarzynski) equality in the quantum domain* [12]. We can conclude hence that:

1. Work is not an observable [17]. If we had a Hermitian operator $\hat{\Omega}$ representing work, then its eigenvalue spectrum should be composed of the possible values that work can take, with orthogonal eigenstates that spawn the Hilbert space $\mathcal{H}$; from the definition of work as the random variable $W = E_m^\tau - E_n^0$, it follows that the number of possible values is typically

[5]Special caution must be taken at the time of viewing the first and the last angular brackets $\langle ... \rangle$ that appear in the derivation: the first is to be understood as the classical average of the *random variable* W, whereas the last one is a quantum expectation value. In any way W can be viewed as a work operator.

larger than the dimension of $\mathcal{H}$ and hence $\hat{\Omega}$ cannot exist. Consequently, work cannot be determined in a single projective measurement (although it can be understood in terms of a generalised measurement scheme using POVMs [18] at a single time) except for the odd case of commuting Hamiltonians $\left[\hat{H}_s, \hat{H}_{s'}\right] = 0$ for all $s, s' \in [0, \tau]$ [16]. This does not imply in any way that work is not measurable; it does implies that:

2. Work characterises a process, not a state. The fact that the measurement scheme alters its average (see discussion on eq. (2.14)) should not be taken as a negative critic, but rather point us that work, as in thermodynamics (where it is not a state function), is not an instantaneous state of the system but rather a quantity characterising a process. Indeed, the TPM changes the configuration of the system, so when we first described the protocol (prior to §2.1) and asked ourselves about the *work performed on the system*, we maybe should have implemented the measuring scheme in the description itself. Even though work and free-energy differences are process dependent and are altered by the measurements [19], a strong result shows that the (quantum) fluctuation theorems are not affected by them [20].

2.2 Loschmidt echo

As it was briefly stated in the previous section, together with the initial canonical distribution, *reversibility* of the microscopic equations of motion is one of the fundamental principles from which the fluctuation theorems can be derived (see reference [15]). Nevertheless, these relations reveal a clear asymmetry between the forward and reverse processes, showing that time direction chooses energy consuming processes over energy releasing ones with an exponential ratio; this time-reversal symmetry breaking or *arrow of time*, which is ultimately a consequence of the *principle of causality* [21], is the responsible of the existence of irreversible macroscopic behaviour. How to reconcile these two opposite premises has actually been one of the central issues in statistical mechanics for more than a century, when Josef Loschmidt questioned Boltzmann's approach to his *H theorem*, apparently just based on collisions and (time-reversal) dynamics: "if one were to reverse all the velocities of the particles, then it should be equally likely to see a process in which entropy decreases"[6]. Although the arguments of Boltzmann based on the probabilistic interpretation of the second law and the importance of the initial conditions were undeniably valid, the controversy acquired a historical transcendence with the name of *Loschmidt paradox* and still motivates works and discussions nowadays.

Indeed, the profound objection posed by Loschmidt still reverbs any time we speak about the possibility of reversing a process. In a classical system it is for all practical purposes impossible to accomplish such a motion reversal experiment due to the large amount of particles it contains and its high sensitivity to possible variations in the initial conditions. However, the few degrees of freedom and manifest linearity in quantum systems make it meaningful to address time-reversal experiments and study how they are affected by a perturbation or *echo* in the reversed process. The quantity that measures this sensitivity to an imperfect time-reversal procedure is known as the *Loschmidt echo*, and the derivation of its relation to the quantum work previously defined is the objective of this section.

In order to properly define this quantity, consider a quantum system described by the state $|\psi_0\rangle$ that evolves during a time t under the time-independent Hamiltonian $\hat{H}_1$ and, after a sudden quench, a second time-independent Hamiltonian $-\hat{H}_2$ is

[6] The story goes that Boltzmann's reply to Loschmidt's question regarding velocity reversal of the particles was: "Then try to do it!" [22]

applied to the system for the same time t with the aim of recovering the initial state $|\psi_0\rangle$. The Loschmidt echo is then defined as [23]

$$\mathcal{L}(t) = |A(t)|^2 = \left| \langle\psi_0|e^{i\hat{H}_2 t}e^{-i\hat{H}_1 t}|\psi_0\rangle \right|^2 \tag{2.15}$$

where $A(t)$ is then the Loschmidt amplitude. A perfect recovery of the initial state requires a sudden negation of the initial Hamiltonian $\hat{H}_2 = \hat{H}_1$, which leads to $\mathcal{L}(t) = 1$, but this is an impossible task in realistic problems and $\mathcal{L}(t)$ is usually a decreasing function in t. Besides, a time-evolution according to $-\hat{H}_2$ is equivalent to a reversed evolution (in $-t$) with H_2, which connects with the concept of time-reversibility. Note that this quantity can be seen from two perspectives:

(a) As a survival probability $|\langle\psi_0|\psi_t\rangle|^2$ in which $|\psi_t\rangle = e^{i\hat{H}_2 t}e^{-i\hat{H}_1 t}\Psi_0$, which coincides with the setup described above. From this viewpoint, Loschmidt echo measures the degree of irreversibility of the process.

(b) As the overlap between two copies of the initial state after evolving separately under $\hat{H}_1$ and $\hat{H}_2$ during a time t, a perspective that quantifies the sensitivity of the evolution to perturbations. In this case, Loschmidt echo is usually called *fidelity*.

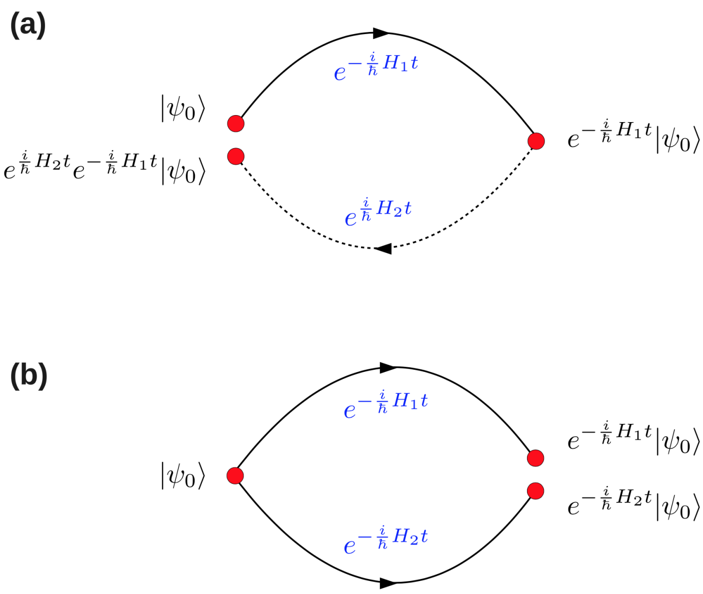


Figure 2.1: Time evolution in Loschmidt echo seen as (**a**) a survival amplitude and (**b**) the fidelity. Figure taken from reference [23]

2.2.1 Connection to work statistics

In what follows we look back to our original system (§2.1) and the general expression derived for its work characteristic function (2.11), particularising it for different setups in order to establish a relation with the Loschmidt echo just introduced. We will call s the *physical time* from now on in order to distinguish it from the second time of evolution t that appears in this context (and will be simply called t throughout). Besides, we will describe the echo protocols by making use of the Heaviside function $\Theta(t)$ in the following fashion:

$$\hat{H}(t) = \hat{H}_1\Theta(-t) - \hat{H}_2\Theta(t) = \begin{cases} \hat{H}_1 & t \leqslant 0 \\ -\hat{H}_2 & t > 0 \end{cases}$$

which consider that the system is initialised at negative t with $\hat{H}_1$, is subjected to the instantaneous quench at $t = 0$ and then evolves with $-\hat{H}_2$ up to the time t. This is the notation also employed in the reference paper [1].

First case

Consider that the initial state is the eigenstate $|j_0\rangle$ of the initial Hamiltonian $\hat{H}_0$, so that $\hat{\rho}_{mix} = \hat{\rho} = |j_0\rangle\langle j_0|$, i.e., we have an initial pure state. Besides, assume that we have a sudden quench $\tau \to 0^+$ so that the time evolution operator becomes $U(0^+) = \mathbf{1}$. The characteristic function then simplifies to

$$\chi(t, 0^+) = \mathrm{Tr}\left\{e^{it\hat{H}_\tau}e^{-it\hat{H}_0}\,|j_0\rangle\langle j_0|\right\} \tag{2.16}$$

$$= \langle j_0|e^{it\hat{H}_\tau}e^{-it\hat{H}_0}|j_0\rangle \tag{2.17}$$

which is easily recognisable as a Loschmidt amplitude $A(t)$. Hence, the variable t that emerged in §2.1.1 as the transform variable in the Fourier transform of $p(W)$ can be identified with a second time of evolution, different from the physical time s; the Hamiltonian that drives this (hypothetical) echo process is

$$\hat{H}(t) = \hat{H}_0\Theta(-t) - \hat{H}_\tau\Theta(t) \quad . \tag{2.18}$$

We stress the fact that the system under study is not experiencing a Loschmidt echo protocol: remember that the physical evolution (in s) of the system under this quench dynamics is $\hat{H}_0 \to \hat{H}_\tau$, not $\hat{H}_0 \to -\hat{H}_\tau$. What the equivalence established does imply is that work statistics for the previous process can be in principle measured by making use of the Loschmidt echo in a system evolved under the Hamiltonian in (2.18). In this case, both systems are initialised in the same state $|j_0\rangle$, but this will not happen in the following case.

General case

In this case, no conditions over the initial state nor over the protocol are imposed: $\hat{\rho}$ can be any mixed state and $\hat{U}(\tau)$ can refer to any generic (unitary) evolution. The procedure that is used in order to show that such a *universal* relation between quantum work statistics and Loschmidt echo exists requires the introduction of the concepts of *partial tracing* and *purification*.

The partial trace is an operation that is used in the context of composite systems; for simplicity, let us restrict ourselves to a bipartite system $\mathcal{H}_L \otimes \mathcal{H}_R$, where L and R refer to "left" and "right". If a measurement of an observable whose projector is $\hat{P}_x$ wants to be performed solely on —say— the left system, the postulates of quantum mechanics force us to *upgrade* the operator to $\hat{P}_x \otimes \mathbf{1}_R$ and calculate the probability of the outcome p_x through *Born's rule*: $p_x = \mathrm{Tr}\left\{\hat{\rho}\hat{P}_x \otimes \mathbf{1}_R\right\}$, where $\hat{\rho}$ is the density operator of the composite system and $\mathrm{Tr}\{...\}$ denotes the usual trace —over $\mathcal{H}_L \otimes \mathcal{H}_R$ here—. The partial trace is the operation that allows us to extract a density operator $\hat{\rho}_L$ from the total $\hat{\rho}$ such that Born's rule can be computed in $\mathcal{H}_L$ alone via $p_x = \mathrm{Tr}\left\{\hat{\rho}_L \hat{P}_x\right\}$, individuating the subsystem L and ignoring the rest. In fact, it is the *unique mapping* that guarantees this correspondence [11] and is defined such that, for any $|a_1\rangle, |a_2\rangle \in \mathcal{H}_L$ and any $|b_1\rangle, |b_2\rangle \in \mathcal{H}_R$,

$$\mathrm{Tr}_R\left\{|a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2|\right\} = |a_1\rangle\langle a_2|\,\mathrm{Tr}\{|b_1\rangle\langle b_2|\} \tag{2.19}$$

up to the linearity condition. The definition of the reduced density operator $\hat{\rho}_L$ in $\mathcal{H}_L$ follows directly as $\hat{\rho}_L = \mathrm{Tr}_R\{\hat{\rho}\}$.

In fact, the converse operation is also possible, and it is what we are interested in: if we start from a general density operator $\hat{\rho}_L$ acting on $\mathcal{H}_L$, and expand it in its diagonal basis $\hat{\rho}_L = \sum_k \lambda_k |\varphi_k\rangle\langle\varphi_k|$, it is possible to embed it in a bipartite system $\mathcal{H}_L \otimes \mathcal{H}_R$[7] and express it as the partial trace of a density operator built from the

[7]The right system does not necessarily have to be a copy of the left system, just to have a dimension at least equal to the number of nonzero eigenvalues of $\hat{\rho}$.

state vector $|\psi\rangle \in \mathcal{H}_L \otimes \mathcal{H}_R$ given by $|\psi\rangle = \sum_k \sqrt{\lambda_k} |\varphi_k\rangle \otimes |\varphi_k\rangle$. Indeed, if we perform partial trace over $\mathcal{H}_R$ we recover $\hat{\rho}_L$:

$$\begin{aligned}
\mathrm{Tr}_R \{|\psi\rangle\langle\psi|\} &= \mathrm{Tr}_R \left\{ \sum_{k,k'} \sqrt{\lambda_k \lambda_{k'}} \, |\varphi_k\rangle\langle\varphi_{k'}| \otimes |\varphi_k\rangle\langle\varphi_{k'}| \right\} \\
&= \sum_{k,k'} \sqrt{\lambda_k \lambda_{k'}} \, \mathrm{Tr}_R \{|\varphi_k\rangle\langle\varphi_{k'}| \otimes |\varphi_k\rangle\langle\varphi_{k'}|\} \\
&= \sum_{k,k'} \sqrt{\lambda_k \lambda_{k'}} \, |\varphi_k\rangle\langle\varphi_{k'}| \, \mathrm{Tr}\{|\varphi_k\rangle\langle\varphi_{k'}|\} \\
&= \sum_{k} \lambda_k \, |\varphi_k\rangle\langle\varphi_k| = \hat{\rho}_L
\end{aligned} \tag{2.20}$$

where linearity has been used in going from the first line to the second, and the fact that $\mathrm{Tr}\{|\varphi_k\rangle\langle\varphi_{k'}|\} = \delta_{kk'}$ in the last two lines. This process is called *purification* for it allows us to write a general mixed-state (with a *purity* $\mathrm{Tr}\{\hat{\rho}^2\} < 1$) in a pure form (with maximum purity $\mathrm{Tr}\{|\psi\rangle\langle\psi|^2\} = 1$, since $\langle\psi|\psi\rangle = 1$). The auxiliary system that is created in this process ($\mathcal{H}_R$) is usually called the *reference system* and it is a fictitious one, without an a priori direct physical significance [24].

Going back to the problem of interest, we perform a purification of the state $\hat{\rho}_{mix}$, which is already diagonal in the basis of H_0, by embedding it in the composite system $\mathcal{H}_L \otimes \mathcal{H}_R$, with $\mathcal{H}_L = \mathcal{H}_R = \mathcal{H}$: the resulting double-copy, purified state is

$$|\Psi_0\rangle = \sum_n \sqrt{p_n^0} \, |n_0\rangle_L \otimes |n_0\rangle_R \tag{2.21}$$

which reduces to $\hat{\rho}_{mix}$ by taking the partial trace over $\mathcal{H}_R$ in an analogous derivation as that performed in equation (2.20). Let us now introduce the *echo matrix* $\hat{M}(t,\tau) = e^{it\hat{H}_\tau^{\mathrm{eff}}} e^{-it\hat{H}_0}$, which contains the time evolution described for an echo protocol, and upgrade it to the operator $\hat{M}(t,\tau) \otimes \mathbf{1}_R$ so that it can be used in the composite system $\mathcal{H}_L \otimes \mathcal{H}_R$. If we now apply it to the "initial" $|\Psi_0\rangle$, we obtain the state $|\Psi_t\rangle = \hat{M}(t,\tau) \otimes \mathbf{1}_R \, |\Psi_0\rangle$ in which the left copy is evolved in time following a quench of the form

$$\hat{H}(t) = \hat{H}_0 \Theta(-t) - \hat{H}_\tau^{\mathrm{eff}} \Theta(t) \tag{2.22}$$

while the right copy is left unchanged. As already stated in the theoretical explanation, the left copy is the one that represents our system while the right one is just a mathematical tool: purification allows us to express this Loschmidt echo operation on $\hat{\rho}_{mix}$, but no actual change is made to the system. By performing the overlap between the two states we end up with the survival amplitude

$$\langle\Psi_0|\Psi_t\rangle = \langle\Psi_0|\hat{M}(t,\tau)\otimes\mathbf{1}_R|\Psi_0\rangle$$

which is recognisable as the Loschmidt amplitude $A(t,\tau)$. Its connection with the characteristic function is established using the Born's rule correspondence between the composite system $\mathcal{H}_L\otimes\mathcal{H}_R$ and the original one $\mathcal{H}_L$:

$$\begin{aligned} A(t,\tau) &= \mathrm{Tr}\Big\{\hat{M}(t,\tau)\otimes\mathbf{1}_R\,|\Psi_0\rangle\langle\Psi_0|\Big\} \\ &= \mathrm{Tr}\Big\{\hat{M}(t,\tau)\hat{\rho}_{mix}\Big\} \\ &= \Big\langle e^{it\hat{H}_\tau^{\mathrm{eff}}}e^{-it\hat{H}_0}\Big\rangle_{mix} \\ &= \chi(t,\tau) \end{aligned} \tag{2.23}$$

where the partial trace $\hat{\rho}_{mix} = \mathrm{Tr}_R\{|\Psi_0\rangle\langle\Psi_0|\}$ is the operation that allows this equivalence (two first lines), and the last identification follows from equation (2.11). We have hence a proof that the relation between Loschmidt echo and work statistics $\mathcal{L}(t) = |A(t,\tau)|^2 = |\chi(t,\tau)|^2$ holds for a universal setting in the original system, extending the relation proved for the first case (which was first found by Silva [7]). Note aside, we remark that the protocols from which this equality is built are not symmetrical with respect to the initial state in this case: while the TPM scheme is initialised at a general mixed state $\hat{\rho}$, the echo protocol starts from its diagonal part in the $\hat{H}_0$ eigenbasis, $\hat{\rho}_{mix}$.

Some observations can be made at this point:

- The work characteristic function verifies $\chi(t,\tau)^* = \chi(-t,\tau)$, which directly follows from its definition (and note that $p(W)$ is a real-valued distribution). This property entails that the Loschmidt echo $\mathcal{L}(t)$ is an even function: $\mathcal{L}(-t) = \chi(-t,\tau)^*\chi(-t,\tau) = \chi(t,\tau)\chi(-t,\tau) = \chi(t,\tau)\chi(t,\tau)^* = \mathcal{L}(t)$.
- The link between work statistics and Loschmidt echo can be also seen through the quantum work fluctuations ΔW. If we assume that the cumulant generating function of the work probability density $K(t) = \ln\chi(t,\tau) = \ln\langle e^{itW}\rangle$ is an analytic function, then we can expand it as $K(t) = \sum_{n=1}^{\infty}(it)^n\kappa_n/n!$

and use it as follows

$$
\begin{aligned}
\ln \mathcal{L}(t) &= \ln \chi(t,\tau) + \ln \chi(-t,\tau) \\
&= \sum_{n=1}^{\infty} \frac{(it)^n}{n!}\kappa_n + \sum_{n=1}^{\infty} \frac{(-1)^n (it)^n}{n!}\kappa_n \\
&= 2\sum_{n=1}^{\infty} \frac{(it)^{2n}}{2n!}\kappa_{2n} = 2\sum_{n=1}^{\infty} \frac{(-1)^n (t)^{2n}}{2n!}\kappa_{2n} \\
&= -\kappa_2 t^2 + \mathcal{O}(t^4)
\end{aligned}
\tag{2.24}
$$

where the second order cumulant is $\kappa_2 = \langle W^2 \rangle - \langle W \rangle^2 = \Delta W^2$, i.e., the variance of the work. Hence, the short time dynamics ($t \ll 1$) of the Loschmidt echo (in the system following the echo protocol) is of a Gaussian form where the width is given by the work fluctuations due to the original evolution. What is the physical interpretation of this (sensitivity of the system to perturbations vs work fluctuations)?

2.3 Information scrambling

The concept that is about to be put into play is at the heart of recent investigation on the fields of quantum information, quantum chaotic systems, condensed matter physics and even black hole physics. A complete understanding of the scope of information scrambling and its relations would require us to deepen into intricate concepts such as quantum channels or holography and use a formalism that is well beyond the intention of this work; mathematical formalism will then be reduced to the least possible and replaced by a heuristic, intuitive presentation of the physics behind it. In fact, research on this matter do not seem to handle a universal definition for this concept but rather work on the basis of a general understanding of it while proposing different theoretical and experimental tools to study it. This approach seems then reasonable both in the context of this work and because of the nature of the concept itself.

Information scrambling is a recent terminology used to describe the propagation of quantum information in the process of *thermalisation*. The latter arises in the context of quantum many body systems and (quantum) statistical mechanics and encodes two simple questions: will an isolated quantum system reach thermal equilibrium after a long-time evolution? Can the unitarity of quantum mechanics evolution account for this behaviour? In order to explain this, let us consider a generic Hamiltonian quantum system with a macroscopic number of degrees of freedom —say, for example, we have a large number of spins localised in a lattice, with different positions in space—, that is initialised in the state $\hat{\rho}(0)$. We say that the system *thermalises* if in its long-time evolution, any subsystem S_A can be described by a density operator $\hat{\rho}_A(t) = \mathrm{Tr}_B\{\hat{\rho}(t)\}$ with the form of the canonical distribution, where $\hat{\rho}(t) = \hat{U}(t)\hat{\rho}(0)\hat{U}^\dagger(t)$ is the evolved density operator of the whole system and the complementary subsystem S_B that we are tracing out can be understood as a heat bath for S_A. In such process, it is natural to assume that the system has lost its identity or *information*: if in the beginning the system had any localised excitation that could be accessed via local measurements, at late times such a measurement would not yield any valuable information about the initial state; the system is then effectively described by macroscopic, averaged quantities such as the temperature [25]. Nevertheless, under unitary dynamics it is clear that a system cannot *erase* or *forget* its initial state, since the evolution can be in principle reversed. A sharpened perspective on this lose of memory issue is that the information is hidden, spread, *scrambled* along the system degrees of freedom, not lost; the initial information becomes inaccessible without carrying out a large amount of local measurements.

The special cases of systems that do not thermalise (see Anderson-localized systems and the process of Many-Body Localisation —MBL—[25]) and can locally remember information about their local initial conditions are of great interest in the field of quantum information and computation; however, in this work we focus on those systems that do scramble information. There are many recent tools that have proven useful in order to give account of this process and even quantify and parametrise it in many different systems and setups. The one that we find the most consistent with the concept introduced and from which a direct correspondence can be established with the previous content of this work is the decay of out-of-time ordered correlators (OTOCs). These are defined as four-point correlators of the form

$$F_\beta(t) = \left\langle \hat{A}^\dagger(t)\hat{B}^\dagger\hat{A}(t)\hat{B}\right\rangle_\beta \tag{2.25}$$

where $\hat{A}$ and $\hat{B}$ are two local, unitary and commuting operators that are supported on the subsystems S_A and S_B of the complete system under study, $\langle ... \rangle_\beta$ is the thermal ensemble average at inverse temperature $\beta = 1/T$ (note that $k_B = 1$, as already stated in §1) and $\hat{A}(t) = \hat{U}^\dagger(t)\hat{A}\hat{U}(t)$ is the time-evolved operator $\hat{A}$ in the Heisenberg picture. As time advances, the operator $\hat{A}(t)$ will become non-local and overlap with $\hat{B}$, an effect that can be diagnosed by the growth of the commutator $\left[\hat{A}(t), \hat{B}\right]$. This is a clear sign that the initial localised information is spreading throughout the entire system; its relation with the introduced OTOC can be derived in the following way

$$\begin{aligned}
\left\langle \left[\hat{A}(t), \hat{B}\right]^\dagger \left[\hat{A}(t), \hat{B}\right] \right\rangle &= (\hat{B}^\dagger\hat{A}^\dagger(t) - \hat{A}^\dagger(t)\hat{B}^\dagger)(\hat{A}(t)\hat{B} - \hat{B}\hat{A}(t)) \\
&= 2 - \left\langle \hat{A}^\dagger(t)\hat{B}^\dagger\hat{A}(t)\hat{B}\right\rangle_\beta - \left\langle (\hat{A}^\dagger(t)\hat{B}^\dagger\hat{A}(t)\hat{B})^\dagger\right\rangle_\beta \\
&= 2 - 2\,\mathrm{Re}\left\{\left\langle \hat{A}^\dagger(t)\hat{B}^\dagger\hat{A}(t)\hat{B}\right\rangle_\beta\right\}
\end{aligned}$$

and hence

$$\mathrm{Re}\{F_\beta(t)\} = 1 - \frac{1}{2}\left\langle \left[\hat{A}(t), \hat{B}\right]^\dagger \left[\hat{A}(t), \hat{B}\right] \right\rangle \quad . \tag{2.26}$$

This equation implies that information scrambling can be indeed measured by the decay of the real part of the OTOC $F_\beta(t)$. The next step we take is towards establishing a relation between this tool to characterise information scrambling and Loschmidt echo: such task has already been accomplished in reference [26]. We are not going into the details, but the derivation basically consists on averaging $F_\beta(t)$ over all unitary operators $\hat{A}$ and $\hat{B}$ supported on the subsystems S_A and S_B (denoted by an overbar to point out a specific *Haar* average) to account for a general delocalisation of the information. This quantity can then be coarse-grained equalled to a Loschmidt echo of the form

$$\overline{F_{\beta=0}(t)} \approx \left| \left\langle e^{i\hat{H}_{B1}t} e^{-i\hat{H}_{B2}t} \right\rangle_{\beta=0} \right|^2 \tag{2.27}$$

where $\hat{H}_{B1}$ and $\hat{H}_{B2}$ contain both the part of the total system Hamiltonian acting on the subsystem S_B and the averaged effect of possible noises in the time-evolution of the system. Note that this result can also be generalised to finite temperatures [26]. However, restricting ourselves to the intention of this section, the importance of this formula for us is not the quantitative relation between these two quantities, but rather the assertion that information scrambling in a system can be assessed by Loschmidt echo measurements. Tracing back this concept in our work, we have also established a relation between the information scrambling and quantum work statistics.

This result should be of no surprise: an a priori natural reasoning would be to relate the sensitivity to perturbations in the system to *quantum chaos*, and through it an intuitive approach to the mixture of information in the system would emerge. We have now arrived at a stronger argument supporting this naive perspective.

2.3.1 Thermofield double state

In what follows, we go back to the system described in the beginning of the chapter. So far, we have not specified the form of the initial state $\hat{\rho}$. The connection just established with information scrambling makes it appropriate to particularise it to an initial thermal state

$$\hat{\rho} = \hat{\rho}_{th} = \frac{e^{-\beta \hat{H}_0}}{\mathrm{Tr}\left\{ e^{-\beta \hat{H}_0} \right\}} \tag{2.28}$$

with $\beta = 1/T$ the inverse of temperature. In this case, $\hat{\rho}$ is diagonal in the eigenbasis of $\hat{H}_0$, so $\hat{\rho} = \hat{\rho}_{mix}$. The proposed way of treating this mixed state according to the tools described in §2.2 is by introducing the purified state

$$|\Psi_0\rangle = \frac{1}{\sqrt{Z(\beta)}} \sum_n e^{-\frac{\beta}{2}E_n^0} |n_0\rangle_L \otimes |n_0\rangle_R \tag{2.29}$$

where $Z(\beta) = \mathrm{Tr}\left\{e^{-\beta \hat{H}_0}\right\}$ denotes the partition function at inverse temperature β. This entangled state is called the *thermofield double state* (TFD) and connects us directly with black hole physics: through the AdS/CFT correspondence[8], the TFD state represents both an entangled state of two conformal field theories (CFT) and an eternal black hole from the AdS perspective [27]. In order words, conjectures about black holes can be in principle rephrased in terms of unitary quantum dynamics (studying black holes in the laboratory!). In particular, it has been claimed that black holes are the physical systems that scramble information at the highest rate in the universe, also calling them *fast scramblers* [28], with several recent works proposing a bound on this exponential information scrambling rate [29].

Leaving aside such pioneering considerations, we can use the expression (2.23) with the initial state $\hat{\rho}_{th}$ to arrive at the formula

$$\begin{aligned}\mathcal{L}(t) &= |\chi(t,\tau)|^2 = |\langle\Psi_0|\Psi_t\rangle|^2 \\ &= \frac{1}{|Z(\beta)|^2}\left|\mathrm{Tr}\left\{e^{it\hat{H}_\tau^{\mathrm{eff}}} e^{-(\beta+it)\hat{H}_0}\right\}\right|^2\end{aligned} \tag{2.30}$$

which is the main result with respect to the connection between quantum work statistics, Loschmidt echo and information scrambling. Since the quantity that is more accessible is $p(W)$, it is reasonable to invert $\chi(t,\tau)$ through its Fourier transform as in equation (2.5) to express the Loschmidt echo as

$$\mathcal{L}(t) = \iint dW dW' p(W) p(W') \cos[(W-W')t] \tag{2.31}$$

where we have used the fact that $\mathcal{L}(t)$ is real to cancel out the $i\sin[(W-W')t]$ contribution from the complex exponential.

[8]It is well beyond the scope of this work, but it is a recent approach to connect a quantum gravitational theories (which employ Anti-de-Sitter spaces) and quantum field theories (extended to conformal field theories or CFT).

3. Reproduction of the results

Abstract

Reunimos todas las relaciones que hemos obtenido en la sección anterior y las analizamos en profundidad particularizando el sistema inicial. Introducimos para ello la noción de caos y los sistemas cuánticos caóticos, presentando con ellos la teoría de matrices aleatorias en la que se apoya su definición. Asumiendo una simetría de inversión temporal de nuestro sistema, explotamos las propiedades del Gaussian Orthogonal Ensemble para obtener resultados analíticos en el caso de un protocolo con una inversión temporal perfecta ($\hat{H}_\tau = -\hat{H}_0$), resultados que comparamos con las medias numéricas obtenidas de promediar con 10000 matrices aleatorias de esta misma distribución. Para el caso de una operación de inversión temporal imperfecta, los resultados son sólo numéricos. Ambos casos son acompañados de gráficas y discusiones.

There is something we have not been worrying about yet: what is the form of the work probability density $p(W)$, its characteristic function $\chi(t,\tau)$ or the Loschmidt echo $\mathcal{L}(t)$? How does the latter evolve in time and what does this evolution say about information scrambling? What is their dependence on the temperature or on the spectra of the initial and the final Hamiltonians? The closest we have been to these points is when we proved at the end of section §2.2 that the short time behaviour of $\mathcal{L}(t)$ has the form of a Gaussian decay, which does not seem to be affected by the chosen Hamiltonians further than through the width $\kappa = \Delta W^2$. In this section, we want to fully characterise the system under study and the protocol we subject it to in order to obtain results for the expressions derived and give them some physical interpretation. Hence, this chapter is arranged in the following fashion: in section §3.1 we introduce chaotic systems and explain why it is a natural choice to make for the system of this work. The Hamiltonians of our problem are then extracted from matrix ensembles of the general Random Matrix Theory (RMT), a usual way of defining quantum chaotic systems; therefore, a brief description of this field and its achievements is presented in subsection §3.1.1. By stating the time-reversal symmetry of our system, the Gaussian Orthogonal Ensemble (GOE) is selected and we explain in subsection §3.1.2 how we will use it for obtaining expectation values both analytically —exploiting its RMT properties through the correlation functions— and numerically in the next section. Having specified where the Hamiltonians come from, section §3.2 contains the two different protocols we study and the results we obtain for them. On the one hand, we treat the perfect time-reversal operation with $\hat{H}_\tau = -\hat{H}_0$; analytical expressions are derived and then compared with the numerical results in different figures. On the other hand, numerical results for an imperfect time-reversal operation —with $\hat{H}_\tau$ obtained independently from $\hat{H}_0$— are also obtained and analysed. Note that we depart from the specifications made to the system until now: we maintain the initial thermal state $\hat{\rho}_{th}$, and add the condition of instantaneous quench $\hat{U}(0^+) = \mathbf{1}$.

3.1 Quantum chaotic systems

The notion of *chaos* has already been introduced in this work: in the hypothetical motion-reversal procedure proposed by Loschmidt, we claimed that such operation was impossible for all practical purposes due to the high sensitivity of a many-particle system to possible variations in the initial conditions. That is in fact a direct link to the most widely used definition of chaos [30]: when the forces and interactions are so complicated that either we cannot write the corresponding differential equation, or when we can[1], the whole situation is unstable in the sense that a small change in the initial conditions produces a large difference in the final outcome, we call the system chaotic. The systems that have differential equations simple enough to be solved and are stable in the previous sense are called *integrable*; although they are the most ubiquitous in contemporary textbooks on classical or quantum mechanics, systems in nature are not integrable, so chaos is indeed of great interest for us.

Nevertheless, the linearity and unitarity of quantum mechanics prevents the definition of a (quantum) chaotic system in terms of sensitivity to initial conditions: two slightly different states evolving with an identical Hamiltonian will continue to be at equal distance (measured by the fidelity or Loschmidt echo) at any time of their evolution. The concept of quantum chaos is then somewhat diffuse and can

[1]The concept of *deterministic chaos* is used to account for this phenomenon of knowing the equations of motion with no random elements involved and still observing chaotic behaviour: the entire past and future of such systems may be deduced from a knowledge of their present state, "but the approximate present does not approximately determine the future", as stated by Edward Lorenz.

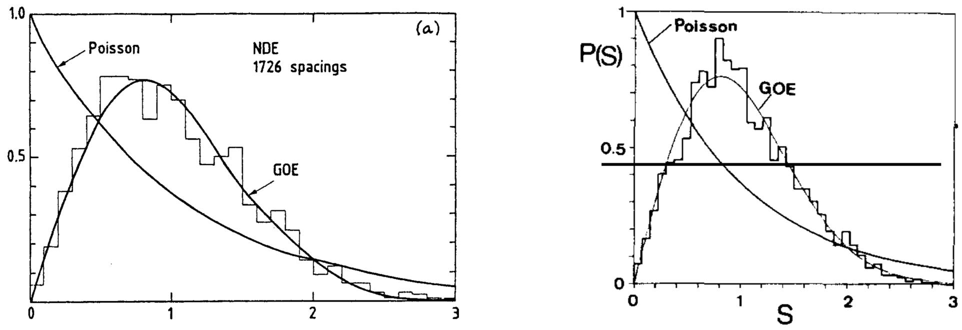
accommodate both the study of how classical chaotic systems can be described in the quantum regime and also the analysis of quantum systems with very complicated underlying equations of motion (or even unknown!) that, despite this, show some distinctive behaviour that can be described with the correct mathematical construction. The latter is to some extent closer to the perspective we adopt in this work and it is a direct link to the next subject: Random Matrix Theory.

Note that the choice of this type of system for our work is then clearly not arbitrary, as we have already used natural tools to assess quantum chaos as the Loschmidt echo and information scrambling, and we expect to observe some characteristic features of chaos in these quantities.

3.1.1 Random Matrix Theory

The theory of random matrices was first developed in mathematical statistics in the 1930s, but an intensive study of their properties in connection with nuclear physics came with the work of Wigner in the 1950s. He proposed that *the characteristic energies of chaotic systems behave locally as if they were the eigenvalues of a matrix with randomly distributed elements* [30]; the vast amounts of experimental data on the excitation spectra of various nuclei helped at the time of proving the validity of this hypothesis. This is quite a remarkable result: instead of trying to calculate the exact solutions of a system with a high complexity, one should rather try to determine the statistical properties of such systems.

For us, the approach to study a nucleus would naturally follow the standard quantum mechanics procedure: find the Hamiltonian that best describes its interactions and solve its eigenvalue equation to obtain its eigenvalue spectrum and eigenstates. Any physical information should be then deduced from this knowledge. However, the nucleus adjusts to a quite good extent to the definition we have given of a quantum chaotic system: we do not usually know the Hamiltonian and, even if we did, it would be too complicated to attempt to solve the corresponding equation. Therefore, we follow Wigner's spirit: focus on what is known about the system and then simply roll a dice. First, we reduce the problem to the discrete part of the spectrum so that we can work on a finite dimensional Hilbert space. Second, we represent the Hamiltonian operators $\hat{H}$ as matrices *compatible with the general symmetry properties of our original system*: each element is taken to be random —say, from a Gaussian distribution—, but the overall matrix symmetry must meet



(a) Case of nuclear (energy) levels: 1407 resonance levels belonging to 30 sequences (no missing levels and same J^P) of 27 different nuclei. Taken from ref. [30].

(b) Case of a hydrogen atom in a strong magnetic field, with energy levels sampled near the (scaled) binding energy. Taken from ref. [31].

Figure 3.1: Probability density for the nearest neighbor spacings $P(s)$ versus the scaled level spacing $s = S/D$, where D is the mean level spacing, in two complex experimental settings. The histograms represent the experimental data in both cases, while the solid lines indicate: the Poisson distribution (no correlation between eigenvalues) and that for the eigenvalues of a matrix extracted from the Gaussian Ortogonal Ensemble (GOE).

the one of the system under study. Finally, the quantities that are to be calculated are obtained as averages over an ensemble of these matrices, where each particular matrix can be thought of as a different nucleus with similar properties. “The statistical theory will not predict the detailed level sequence of any one nucleus, but it will describe the general appearance and the degree of irregularity of the level structure that is expected to occur in any nucleus which is too complicated to be understood in detail” [32]. In figure 3.1 we can see how this statement becomes real: the (averaged) level spacing of the eigenvalues of the random matrix reproduces quite reliably the level spacing from the experimental nuclear data.

From a group theoretical analysis, Dyson found that the number of matrix ensembles to characterise the different symmetries that a system can have were only three, naming them orthogonal, unitary and symplectic ensembles. In this work we assume that the system under study has time-reversal symmetry[2], an interesting assumption with all the background given in the previous sections. In the even-spin case (also rotationally invariant) [30], it can be proved without difficulty that the ensemble from which we should extract the matrices for our Hamiltonians is the Gaussian Orthogonal Ensemble.

[2]The time-reversal operator $\hat{T}$ transforms the states of a system as $|\psi^R\rangle = \hat{T}\,|\psi(t)\rangle = |\psi(-t)\rangle$ and its observables as $\hat{A}^R = \hat{T}\hat{A}\hat{T}^{-1}$. A physical system is invariant under time reversal if its Hamiltonian verifies that $\hat{H}^R = \hat{T}\hat{H}\hat{T}^{-1} = \hat{H}$.

3.1.2 Gaussian Orthogonal Ensemble

The Gaussian Orthogonal Ensemble is defined as those real symmetric matrices H that meet the following two conditions [30]:

1. The ensemble is invariant under every transformation
$$H \to W^T H W \tag{3.1}$$
where W is any real orthogonal matrix.

2. The elements of the matrix H_{mn}, $m \leq n$, are statistically independent.

There are two ways for working within the GOE and capture all its statistics in our results. On the one hand, we can propose a *numerical* method for sampling matrices of the ensemble and work with many of them so that the averaged quantities contain the statistics associated with the ensemble. On the other hand, although more mathematically involving, we can deepen into the theory of random matrices and work with its joint probability density function, correlation functions and heavy statistical material to calculate *analytical* averages. Both approaches are used in this work and are summarised below.

Numerical

Let the dimension of our system be N. We can sample a matrix from the GOE by taking an $N \times N$ matrix A with all its elements being standard normals ($\langle A_{ij} \rangle = 0$, $\langle A_{ij}^2 \rangle = \sigma^2 = 1$) and constructing the matrix $H = (A + A^T)/2$. The matrix H is real symmetric and meets the requirements of the above definition. We can then use a large number of matrices constructed as H to condense GOE statistics in our results: we calculate quantities using Hamiltonians sampled in this manner, and we perform a simple, standard average by adding them up and dividing the sum by the number of matrices employed.

Analytical

The analytical frame is somewhat mathematically involving and we restrict ourselves to the strictly necessary material. It uses the typical machinery of statistical mechanics: consider the probability density function $P(H)$ which is defined such that the quantity $P(H)dH$ is the probability that a randomly chosen matrix of the GOE belongs to the volume element $dH = \Pi_{i \leq j} dH_{ij}$. We would then be able to calculate analytical *GOE averages* of a quantity Q with any dependence on the Hamiltonian by computing the integral

$$\langle Q \rangle_{GOE} = \int QP(H)dH \tag{3.2}$$

for which an explicit form of $P(H)$ would be necessary. This task would be facilitated by expressing the various components of H in terms of its N eigenvalues and other mutually independent variables p_μ, ending up with the joint probability density function of the eigenvalues $P_N(E_1, ..., E_N)$ after integrating over p_μ. However, we will not work with this formalism so we do not need any explicit form for them; instead, we will make use of the *correlation functions* of the ensemble. The k-point correlation function is defined as the probability density of finding a level (regardless of labelling) around each of the eigenvalues $E_1, E_2, ..., E_k$ with the positions of the remaining levels being unobserved [30]. It is given by the expression

$$R_k(E_1, ..., E_k) = \frac{N!}{(N-k)!} \int_{\infty}^{\infty} ... \int_{\infty}^{\infty} P_N(E_1, ..., E_N) dE_{k+1} ... dE_N \tag{3.3}$$

where one can trivially see, in particular, that the first order correlation function $R_1(E)$ corresponds to the overall level density. In order to connect the ensemble averages of the quantities of interest to this theoretical tools we will make use of the density of states $\rho(E) = \sum_n \delta(E - E_n^0)$, which verifies $\langle \rho(E) \rangle_{GOE} = R_1(E)$. In this work we only require $R_1(E)$ and $R_2(E_1, E_2)$; its explicit forms can be derived following [30] and assuming $N = 2m$ (even) for simplicity:

$$R_1(E) = S_N(E, E) \tag{3.4}$$

$$= K_N(E, E) + \frac{1}{2}\sqrt{\frac{N}{2}} \varphi_{N-1}(E) \int_{\infty}^{\infty} dz \; sgn(E - z) \varphi_N(z) \tag{3.5}$$

$$= \langle \rho(E) \rangle_{GUE} + \sqrt{\frac{N}{2}} \varphi_{N-1}(E) \int_0^E dz \varphi_N(z) \tag{3.6}$$

where $K_N(E_p, E_q) = \sum_{n=0}^{N-1} \varphi_n(E_p) \varphi_n(E_q)$ is called the *kernel* of the Gaussian Unitary Ensemble (GUE), $\langle \rho(E) \rangle_{GUE}$ is the overall level density in the GUE, $sgn(z)$ is the signuum function and the $\varphi_j(E)$'s are the "oscillator wave functions" defined in terms of the Hermite polynomials $H_j(x)$ as

$$\varphi_j(x) = (2^j j! \sqrt{\pi})^{-1/2} \exp\bigl(-x^2/2\bigr) H_j(x) \quad . \tag{3.7}$$

Instead of computing the two-point correlation function, we employ the definition of the two-level cluster function $T_2(E_1, E_2) = -R_2(E_1, E_2) + R_1(E_1)R_1(E_2)$ [30], which can be obtained as $T_2(E_1, E_2) = \mathrm{Tr}\{\sigma_N(E_1, E_2)\sigma_N(E_2, E_1)\}/2$ [33], with

$$\sigma_N(E_1, E_2) = \begin{pmatrix} S_N(E_1, E_2) & DS_N(E_1, E_2) \\ JS_N(E_1, E_2) & S_N(E_2, E_1) \end{pmatrix} \tag{3.8}$$

where $S_N(x, y)$ is usually called the *kernel* of the GOE as it generates the rest of the functions [30]:

$$S_N(x, y) = K_N(x, y) + \sqrt{\frac{N}{2}} \varphi_{N-1}(x) \int_0^y dz \varphi_N(z) \tag{3.9}$$

$$DS_N(x, y) = -\frac{d}{dy} S_N(x, y) = \sum_{i=0}^{m-1} -\varphi_{2i}(x)\varphi'_{2i}(y) + \varphi'_{2i}(x)\varphi_{2i}(y) \tag{3.10}$$

$$IS_N(x, y) = \frac{1}{2} \int_{-\infty}^{\infty} dt \; sgn(x - t) S_N(t, y) \tag{3.11}$$

$$= \sum_{i=0}^{m-1} \int_0^x dt \varphi_{2i}(t) \varphi_{2i}(y) - \varphi_{2i}(x) \int_0^y dt \varphi_{2i}(t) \tag{3.12}$$

$$JS_N(x, y) = IS_N(x, y) - \frac{1}{2} sgn(x - y) \quad . \tag{3.13}$$

A simple calculation of the previous trace gives us

$$T_2(E_1, E_2) = S_N(E_1, E_2) S_N(E_2, E_1) + DS_N(E_1, E_2) JS_N(E_2, E_1) \quad . \tag{3.14}$$

Note that we do not have simpler expressions for the correlation functions in GOE (except for the integral of $\langle \rho(E) \rangle_{GUE}$, as will be stated in the next section), so even for the analytical case we will use raw "numerical" computation. We are now in any case in the right position to go back to our system and obtain results.

3.2 Results and discussion

In what follows we return to the system we had in section §2.3.1, where we specified an initial thermal density operator $\hat{\rho}_{th}$ in order to arrive to equation (2.30). To this condition we now add the *instantaneous quench* protocol, $\hat{U}(0^+) = \mathbf{1}$, in order to facilitate the computation of results, which are mainly condensed in the figures of the chapter and its discussions. The derivations and physical remarks that precede them in the first subsection are still necessary for the analytical description that follows. Note that we do not restrict ourselves to a simple reproduction of the results of the reference paper (which can be seen in fig. 3.5), but we bring analytical derivations and discussions so that a more in-depth analysis can be carried out in the same fashion as in reference [34].

3.2.1 First case: perfect time-reversal operation

When the system Hamiltonian is symmetrical under time-reversal, the implementation of the time-reversal operation is equivalent to the sudden negation of the Hamiltonian in the original protocol [1] such that $\hat{H}_\tau = -\hat{H}_0$. If we do so, the characteristic function becomes

$$\chi(t, 0^+) = \frac{1}{Z(\beta)} \operatorname{Tr}\left\{e^{-(\beta+2it)\hat{H}_0}\right\} = \frac{Z(\beta + 2it)}{Z(\beta)} \tag{3.15}$$

and the Loschmidt echo in equation (2.30) transforms into

$$\begin{aligned} \mathcal{L}(t) = \left|\chi(t, 0^+)\right|^2 &= \left|\frac{1}{Z(\beta)} \operatorname{Tr}\left\{e^{-(\beta+2it)\hat{H}_0}\right\}\right|^2 \\ &= \left|\frac{Z(\beta + 2it)}{Z(\beta)}\right|^2 \end{aligned} \tag{3.16}$$

where $Z(\beta + 2it)$ is to be identified as the analytic continuation of the partition function. Bringing back momentarily the discussion on the connection between the Loschmidt echo and the characteristic function for the purified state $|\Psi_0\rangle$ of section §2.2, we can look at this result as

$$\begin{aligned} \left| \langle\Psi_0|e^{it(-\hat{H}_0\otimes\mathbf{1}_R)}e^{-it(\hat{H}_0\otimes\mathbf{1}_R)}|\Psi_0\rangle\right|^2 &= \left|\frac{Z(\beta + 2it)}{Z(\beta)}\right|^2 \\ &= \mathcal{L}(t) \end{aligned}$$

the LHS being a very clear version of Loschmidt echo for the left copy of the purified state since the full evolution is now under the exponential sign. It is also easy to see that any independent *unquenched* protocol for the right copy would leave the result untouched.

Finally and before proceeding to the analytical derivations and main results, we can trivially calculate the mean work done in order to reverse the dynamics of the system by particularising the expression 2.14 for the present conditions:

$$\langle W \rangle = -2 \left\langle \hat{H}_0 \right\rangle_\beta \tag{3.17}$$

which seems consistent with the time-reversal protocol studied.

Characteristic function

From the very beginning we will be making an approximation called *annealing* [34] at the time of calculating analytical GOE averages of the quantities of interest: we will take the expectation value of a quotient as the quotient of expectation values. In the case of the characteristic function of equation 3.15, it reads

$$\langle \chi(t, 0^+) \rangle_{GOE} = \left\langle \frac{Z(\beta + 2it)}{Z(\beta)} \right\rangle \approx \frac{\langle Z(\beta + 2it) \rangle_{GOE}}{\langle Z(\beta) \rangle_{GOE}} \quad . \tag{3.18}$$

Hence, we will take the numerical averages as the more physically accurate of our results, and explain the differences between curves as consequences of deviations in the analytical part.

As a means to calculate the GOE average of the (analytically continued) partition function, we perform the following manipulation [35]:

$$Z(\sigma) = \sum_n e^{-\sigma E_n^0} = \int_{-\infty}^{\infty} dE e^{-\sigma E} \sum_n \delta(E - E_n^0) \tag{3.19}$$

$$= \int_{-\infty}^{\infty} dE \rho(E) e^{-\sigma E} \tag{3.20}$$

where σ can be any complex number. Its GOE average can then be expressed in terms of a (known, see equation (3.4)) correlation function,

$$\langle Z(\sigma) \rangle_{GOE} = \int_{-\infty}^{\infty} dE \, \langle \rho(E) \rangle_{GOE} \, e^{-\sigma E} = \int_{-\infty}^{\infty} dE R_1(E) e^{-\sigma E} \quad . \tag{3.21}$$

It can be further simplified by doing

$$\langle Z(\sigma)\rangle_{GOE} = \int_{-\infty}^{\infty} dE e^{-\sigma E} \langle \rho(E)\rangle_{GUE} + \int_{-\infty}^{\infty} dE e^{-\sigma E} f(E) \tag{3.22}$$

$$= e^{\sigma^2/4} L^1_{N-1}\left(-\frac{\sigma^2}{2}\right) + \int_{-\infty}^{\infty} dE e^{-\sigma E} f(E) \tag{3.23}$$

where the first integral has been expressed in terms of the generalised Laguerre polynomial $L^1_{N-1}(x)$, a result extracted from a GUE analysis in reference [34]. Using this expression we can find the GOE average by computing the division in equation (3.18).

The numerical and analytical results are in great accordance as can be seen in the left of figure 3.2. The oscillations observed in the numerical solutions for the two temperature cases considered are not a problem: they converge very slowly and tend to zero for large times. The short time coincidences of the curves, both in the multiple peaks of the $\beta = 0$ case and in the smooth decay for $\beta = 1$, are good indicators of the validity of our results: the next quantities will indeed reproduce correctly the short time dynamics.

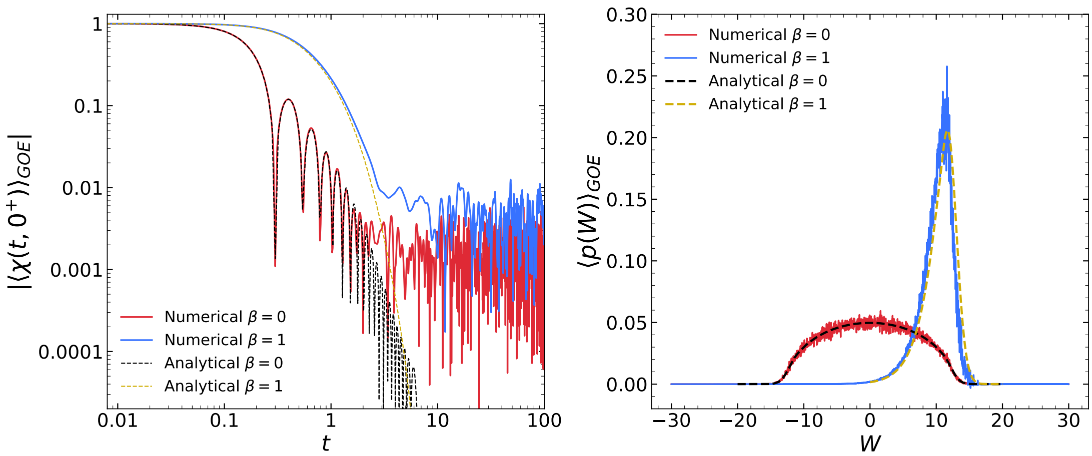


Figure 3.2: Analytical and numerical results for the time evolution of the norm of the work characteristic function (left) and the work probability density function (right) for two different temperatures. The case of infinite temperature ($\beta = 0$) is coloured in solid red lines for the numerical average and in dashed black for the analytical case. In the finite temperature regime ($\beta = 1$), the solid blue lines represents the numerical average and the dashed yellow the analytical result. Numerical calculations use 10000 matrices extracted from the GOE.

Probability density function

The fastest computation of the work probability density function after calculating the work characteristic function is to perform its Fourier transform. In fact, that is what we do for obtaining the numerical solution, which we simplify as

$$\langle p(W)\rangle_{GOE} = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt \left\langle \chi(t,0^+)\right\rangle_{GOE} e^{-itW} \tag{3.24}$$

$$= \frac{1}{\pi}\,\mathrm{Re}\left\{\int_{0}^{\infty} dt \left\langle \chi(t,0^+)\right\rangle_{GOE} e^{-itW}\right\} \tag{3.25}$$

where we have used the property $\chi(t,0^+)^* = \chi(-t,0^+)$ for splitting the integral and reassuring that $p(W)$ is real.

Nevertheless, we also want to exploit the tools of random matrix theory and be able to find an expression of $p(W)$ *independently* from the characteristic function. If we use the definition of $\chi(t,0^+)$ combined with equation 3.20 we have that

$$\langle p(W)\rangle_{GOE} = \frac{1}{2\pi\left\langle Z(\beta)\right\rangle_{GOE}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dt dE e^{-\beta E}\left\langle \rho(E)\right\rangle_{GOE} e^{-it(W+2E)} \tag{3.26}$$

$$= \frac{1}{\left\langle Z(\beta)\right\rangle_{GOE}}\int_{-\infty}^{\infty} dE e^{-\beta E}\left\langle \rho(E)\right\rangle_{GOE}\delta(W+2E) \tag{3.27}$$

$$= \frac{1}{2\left\langle Z(\beta)\right\rangle_{GOE}} e^{\beta W/2}\left\langle \rho\left(-\frac{W}{2}\right)\right\rangle_{GOE} \tag{3.28}$$

$$= \frac{e^{\beta W/2}}{2\left\langle Z(\beta)\right\rangle_{GOE}}\left[\left\langle \rho\left(-\frac{W}{2}\right)\right\rangle_{GUE} + f\left(-\frac{W}{2}\right)\right] \tag{3.29}$$

$$= \frac{e^{\beta W/2}}{2\left\langle Z(\beta)\right\rangle_{GOE}}\left[\sum_{j=0}^{N-1}\varphi_j^2\left(\frac{W}{2}\right) + f\left(\frac{W}{2}\right)\right] \tag{3.30}$$

where we have used, in order, the definition of the Dirac delta function in terms of the Fourier transform, the property $\delta(cx) = \delta(x)/|c|$, the definition of $\langle\rho(E)\rangle_{GOE}$ as given by equation (3.4), and we have denoted by $f(E)$ the second term of the same equation. In going to the last line, we have also made use of the even property of both $\varphi_j(x)$ and $f(E)$ (remember that N is even). In fact, note that the only term that prevents $\langle p(W)\rangle_{GOE}$ from being an even function is the exponential $e^{\beta W/2}$; in figure 3.2 we can corroborate that for the infinite temperature case the distribution is symmetrical with respect to the $W = 0$ axis for both numerical and analytical results, which perfectly overlap. This is also expected from the fact that the initial and final Hamiltonians are drawn from the same ensemble: the mean work after many repetitions should be zero. On the contrary, in the case of finite temperature ($\beta = 1$), the first measurement is more likely to fall on the low-energy spectrum due to the initial thermal state, increasing the probability of obtaining a positive work after the second measurement as manifested in the figure.

Loschmidt echo

Taking into consideration the annealing approximation, we want to calculate

$$\langle \mathcal{L}(t) \rangle_{GOE} \approx \frac{\langle |Z(\beta + 2it)| \rangle_{GOE}^2}{\left\langle |Z(\beta)|^2 \right\rangle_{GOE}} = \frac{\langle Z(\beta + 2it) Z(\beta - 2it) \rangle_{GOE}}{\langle Z(\beta)^2 \rangle_{GOE}} \tag{3.31}$$

where we note that the denominator is just the numerator with $t = 0$, which acts as a normalisation factor. We can hence focus on the GOE average of the numerator alone, that we will call *spectral form factor* and denote $g(\beta, t)$:

$$\left\langle |Z(\beta + 2it)|^2 \right\rangle_{GOE} = \tag{3.32}$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dE_1 dE_2 \, \langle \rho(E_1) \rho(E_2) \rangle_{GOE} \, e^{-(\beta+2it)E_1} e^{-(\beta-2it)E_2} \tag{3.33}$$

$$= \langle Z(2\beta) \rangle_{GOE} + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dE_1 dE_2 R_2(E_1, E_2) e^{-(\beta+2it)E_1} e^{-(\beta-2it)E_2} \tag{3.34}$$

$$= \langle Z(2\beta) \rangle_{GOE} + |\langle Z(\beta + 2it) \rangle_{GOE}|^2 + g_c(\beta, t) = g(\beta, t) \tag{3.35}$$

where the following steps have been made:

- In going from the second line to the third one, we have separated the diagonal part $E_1 = E_2$ from the off-diagonal terms $E_1 \neq E_2$; the discussion on the different definitions of the k-point correlation function $R_k(E_1, ..., E_k)$ is well explained on reference [33]; we will take

  $$\langle \rho(E_1) \rho(E_2) \rangle_{GOE} = \delta(E_1 - E_2) \, \langle \rho(E_1) \rangle_{GOE} + R_2(E_1, E_2) \tag{3.36}$$

  which is consistent with the two-point correlation function R_2 given in [30].

- We have split $R_2(E_1, E_2)$ using the definition of the two-level cluster function $T_2(E_1, E_2)$ given in section §3.1.2, and we have defined the *connected spectral* *form factor* $g_c(\beta, t)$ as

$$g_c(\beta, t) = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dE_1 dE_2 T_2(E_1, E_2) e^{-(\beta+2it)E_1} e^{-(\beta-2it)E_2} \quad . \tag{3.37}$$

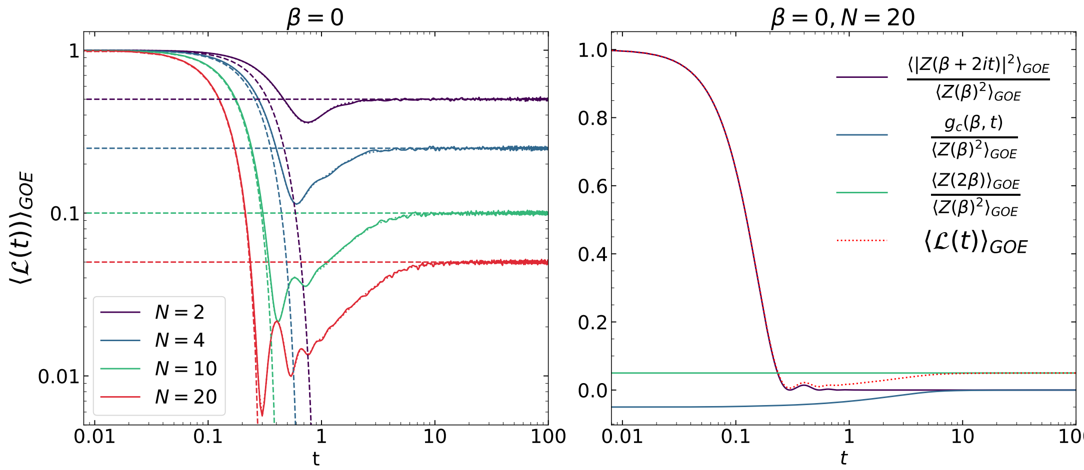


Figure 3.3: Time evolution of Loschmidt echo for $\beta = 0$ and different matrix dimensions N (left) and different contributions to the Loschmidt echo for $\beta = 0$ and $N = 20$ (right). The left plot shows numerical solutions in solid lines and analytical results in dotted lines (overlapped). The dashed horizontal lines are the contribution from the first term $\langle Z(2\beta) \rangle_{GOE}$, while the dashed Gaussian curves come from the second term $|\langle Z(\beta + 2it) \rangle_{GOE}|^2$. A clearest perspective on the contribution from each term is seen in the figure on the right. Note that the y-scale is not logarithmic to represent the negative values of $g_c(\beta, t)$.

The relevance of this separation into three terms is that it serves us to analyse the features of the Loschmidt echo evolution studying each term on its own [35]. We can distinguish three stages (common to scrambling dynamics [1]):

- **Gaussian decay**. The short time dynamics is dominated by the disconnected contribution $|\langle Z(\beta + 2it) \rangle|^2$, which behaves as a Gaussian whose width is approximately the second moment of the work PDF or work fluctuations ΔW^2. From figure 3.3 on the right, we can see that the other two terms compensate to give a zero net contribution.
- **Power law and dip**. The disconnected part dips below the plateau value (dashed horizontal lines in figure 3.3), reaching its first minimum followed by some oscillations near zero from the Laguerre polynomial. The increasing contribution from the connected term $g_c(\beta, t)$ starts to be noticed. The behaviour of the curve below the asymptotic value is usually called *correlation*

hole [36] as it is a consequence of the level of correlation between eigenvalues and, hence, of chaos.

- **Ramp and plateau**. The only nonzero terms that remain after the previous stage are the constant plateau value $\langle Z(2\beta)\rangle_{GOE}$ and the connected contribution, which approaches zero from below in an almost linear growth and vanishes completely at large times. The fact that the Loschmidt echo does not vanish for infinite times is a sign of the finiteness of the system [36], which does not allow for an absolute *scrambling* of the information in the system as could be expected for systems with a continuum spectrum and infinite dimension.

Note that the numerical and analytical results shown in figure 3.3 are almost perfectly overlapped. This is not the case for the finite temperature case (see 3.4), where the *annealing* approximation shows higher deviations the lower the temperature. That is why we have ommited an analogous plot to that of figure 3.3 for $\beta = 1$, although its general features can be seen in figures 3.4 and 3.5.

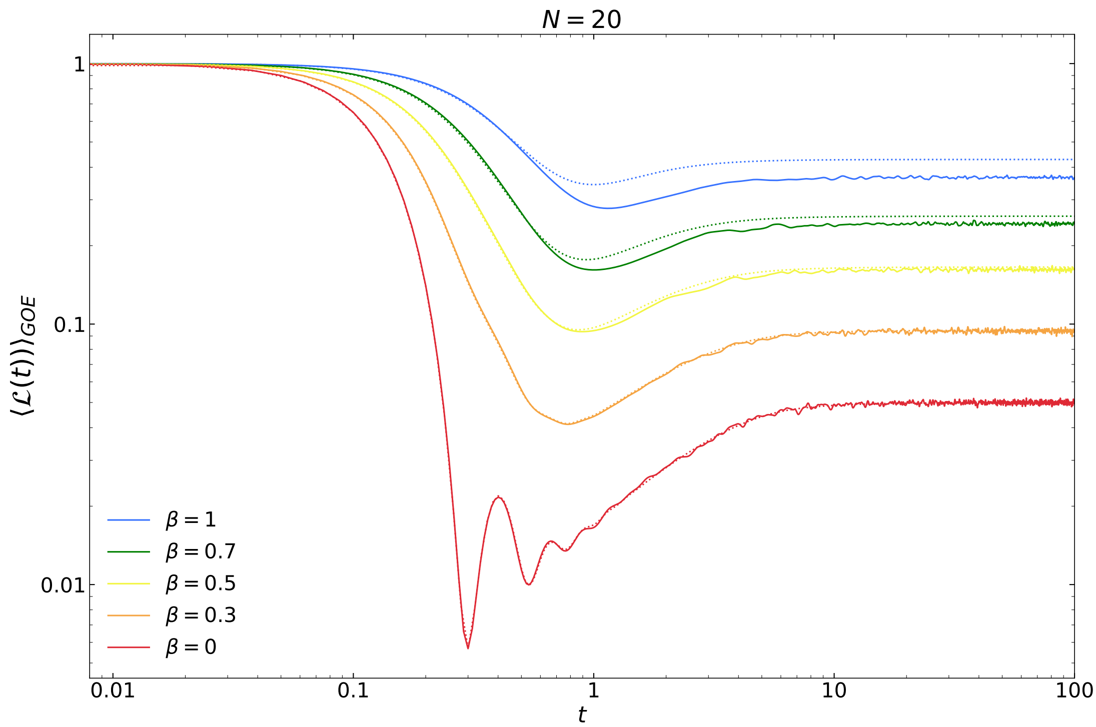


Figure 3.4: Time evolution of Loschmidt echo for different temperature scenarios. The numerical solutions are drawn in solid lines, and the analytical ones use dotted lines. The figure is shown to compare the dependence on β of the deviation caused by the annealing approximation.

3.2.2 Second case: imperfect time-reversal operation

By choosing the second Hamiltonian $\hat{H}_\tau$ from an independent Gaussian Orthogonal Ensemble, we can give account of an imperfect time-reversal operation. If we wanted to obtain analytical results for this case in order to compare them with the numerical averages, we would need to perform a double GOE average, which is a very involving task in which we are not deepening in this work. Instead, we restrict ourselves to the numerical calculations as in the reference paper. The employed expression for the GOE average of the work characteristic function is

$$\left\langle \chi(t, 0^+) \right\rangle_{GOE} = \left\langle \frac{1}{Z(\beta)} \operatorname{Tr}\left\{ e^{it\hat{H}_\tau} e^{-(\beta+it)\hat{H}_0} \right\} \right\rangle_{GOE} \tag{3.38}$$

whose Fourier transformation for numerical computation of $p(W)$ is already given in equation (3.25). The GOE average of the Loschmidt echo is simply

$$\langle \mathcal{L}(t) \rangle_{GOE} = \left\langle \left| \chi(t, 0^+) \right|^2 \right\rangle_{GOE} \quad . \tag{3.39}$$

The results of quantum work statistics and Loschmidt echo dynamics are condensed in figure 3.5, which contains a reproduction of the results of the reference paper and is presented in an analogous way. The left column contains the treated case of a perfect time reversal operation and repeats some of the curves displayed previously, while the right column show the pertinent solutions for this section. With this disposition of the results, it is easier to point some remarks:

1. The work characteristic function and work probability density functions for the sudden quench ((*b*) case) exhibit the same features as those commented in the previous case: $p(W)$ is symmetrical for $\beta = 0$ and shifted to the positive values for $\beta = 1$, and $\chi(t, 0^+)$ presents rapid oscillations at large t.

2. Since the initial Gaussian decay of the Loschmidt echo depends on the width of the work PDF through ΔW^2, looking at the graphs it is clear that the $\beta = 0$ case has to decrease faster than the finite temperature case.

3. The correlation hole is greatly smoothed out in the Loschmidt echo dynamics for the (*b*) case, pointing out the lack of correlation between the initial and final Hamiltonians. The decay is accentuated, leading to lower asymptotic values after a broadened dip region with an almost nonexistent ramp.

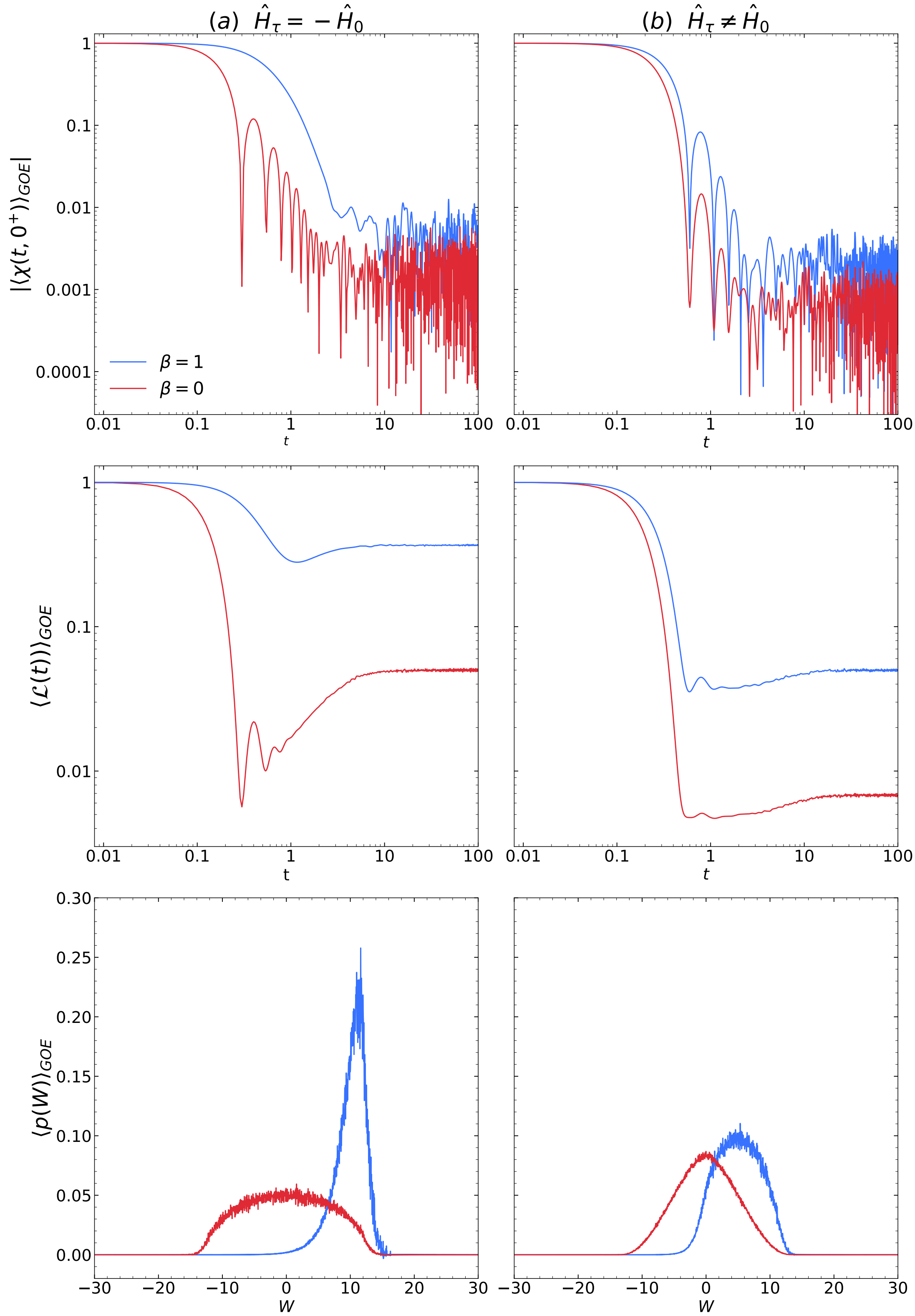


Figure 3.5: Quantum work statistics of time reversal and Loschmidt echo dynamics, reproducing the results of the reference paper. The left column shows the case of a perfect time reversal operation in the system, and the right column an arbitrary sudden quench with $\hat{H}_\tau$ being independent of the initial $\hat{H}_0$. Results are numerical averages computed with respect to a set of 10000 matrices extracted from the Gaussian Orthogonal Ensemble.

4. Conclusions

Through the text it has been necessary to progressively obtain many *minor* results before reaching the main findings of the reference paper [1]. Concerning quantum work statistics, we showed that quantum work is not an observable and that the TPM scheme is the one that accounts for its stochastic and process-dependence nature. By using a description of work in terms of its characteristic function, it was found that quantum thermodynamics can be recast into a dynamical problem. In fact, this can be seen as the major result [1]: a universal relation exists between quantum work statistics and Loschmidt echo, in particular, the characteristic function of an arbitrary protocol of an isolated system starting at any initial mixed state can be identified with the Loschmidt echo amplitude of a purified density operator in an enlarged Hilbert space, for a quench acting on one of the copies. The connection is then extended to information scrambling, which becomes clearest when specifying the initial mixed state to be thermal; the thermofield double state that appears describes an eternal black hole through the AdS/CFT correspondence [1], and black holes have been theorised to be the fastest information scramblers in nature. From here we conclude that black hole physics can be, in principle, studied in quantum mechanical terms. A direct link to the notion of chaos is also evident, and the results we obtain in the last section allows us to claim that random matrices are an appropriate scenario for studying quantum chaotic systems. Loschmidt echo also shows up to be a good quantum chaos diagnostic tool, as it is manifest in its behaviour shown in the figures.

This work could also be improved and extended in many ways. Black hole physics and information scrambling has been somewhat deprecated, so a finer study could be carried out for a better understanding and presentation. Also, we omitted the GOE analytical results for the second case to avoid excessive mathematics, but it would have allowed a broader analysis. And yet, this work should be understood as an introductory work to the field, and the minimum objective of reproducing and studying the paper results and derivations has been accomplished.

4. Bibliography

[1] A. Chenu, I. L. Egusquiza, J. Molina-Vilaplana, and A. del Campo, *Quantum work statistics, loschmidt echo and information scrambling*, Scientific Reports **8**, 10.1038/s41598-018-30982-w (2018).

[2] S. Vinjanampathy and J. Anders, *Quantum thermodynamics*, Contemporary Physics **57**, 545 (2016), https://doi.org/10.1080/00107514.2016.1201896 .

[3] J. Sakurai, *Advanced Quantum Mechanics*, Always learning (Pearson Education, Incorporated, 1967).

[4] H. Murayama, *Lecture notes on Time-Dependent Perturbation Theory* (Berkeley, 2006).

[5] A. Mitra, *Quantum quench dynamics*, Annual Review of Condensed Matter Physics **9**, 245 (2018).

[6] A. Polkovnikov, K. Sengupta, A. Silva, and M. Vengalattore, *Colloquium: Nonequilibrium dynamics of closed interacting quantum systems*, Reviews of Modern Physics **83**, 863 (2011), arXiv:1007.5331 [cond-mat.stat-mech] .

[7] A. Silva, *Statistics of the Work Done on a Quantum Critical System by Quenching a Control Parameter*, **101**, 120603 (2008), arXiv:0806.4301 [cond-mat.stat-mech] .

[8] I. García-Mata, A. J. Roncaglia, and D. A. Wisniacki, *Semiclassical approach to the work distribution*, EPL (Europhysics Letters) **120**, 30002 (2017), arXiv:1709.05115 [quant-ph] .

[9] J. Kurchan, *A Quantum Fluctuation Theorem*, arXiv e-prints , cond-mat/0007360 (2000), arXiv:cond-mat/0007360 [cond-mat.stat-mech] .

[10] H. Tasaki, *Jarzynski Relations for Quantum Systems and Some Applications*, arXiv e-prints , cond-mat/0009244 (2000), arXiv:cond-mat/0009244 [cond-mat.stat-mech] .

[11] M. G. A. Paris, *The modern tools of quantum mechanics. A tutorial on quantum states, measurements, and operations*, European Physical Journal Special Topics **203**, 61 (2012), arXiv:1110.6815 [quant-ph] .

[12] A. E. Allahverdyan and T. M. Nieuwenhuizen, *Fluctuations of work from quantum subensembles: The case against quantum work-fluctuation theorems*, Phys. Rev. E **71**, 066102 (2005).

[13] A. E. Allahverdyan, *Nonequilibrium quantum fluctuations of work*, Phys. Rev. E **90**, 032137 (2014).

[14] C. Jarzynski, *Comparison of far-from-equilibrium work relations*, Comptes Rendus Physique **8**, 495 (2007), arXiv:cond-mat/0612305 [cond-mat.stat-mech] .

[15] M. Campisi, P. Hänggi, and P. Talkner, *Colloquium: Quantum fluctuation relations: Foundations and applications*, Rev. Mod. Phys. **83**, 771 (2011).

[16] P. Talkner and P. Hänggi, *Aspects of quantum work*, Phys. Rev. E **93**, 022131 (2016).

[17] P. Talkner, E. Lutz, and P. Hänggi, *Fluctuation theorems: Work is not an observable*, Physical Review E **75**, 10.1103/physreve.75.050102 (2007).

[18] A. J. Roncaglia, F. Cerisola, and J. P. Paz, *Work measurement as a generalized quantum measurement*, Phys. Rev. Lett. **113**, 250601 (2014).

[19] M. Campisi, P. Talkner, and P. Hanggi, *Influence of measurements on the statistics of work performed on a quantum system*, Physical review. E, Statistical, nonlinear, and soft matter physics **83**, 041114 (2011).

[20] M. Campisi, P. Talkner, and P. Hänggi, *Fluctuation theorems for continuously monitored quantum fluxes*, Phys. Rev. Lett. **105**, 140601 (2010).

[21] D. J. Evans and D. J. Searles, *The fluctuation theorem*, Advances in Physics **51**, 1529 (2002), https://doi.org/10.1080/00018730210155133 .

[22] T. Gorin, T. Prosen, T. H. Seligman, and M. Žnidarič, *Dynamics of Loschmidt echoes and fidelity decay*, **435**, 33 (2006), arXiv:quant-ph/0607050 [quant-ph] .

[23] A. Goussev, R. A. Jalabert, H. M. Pastawski, and D. Wisniacki, *Loschmidt Echo*, arXiv e-prints , arXiv:1206.6348 (2012), arXiv:1206.6348 [nlin.CD] .

[24] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (2010).

[25] R. Nandkishore and D. A. Huse, *Many-body localization and thermalization in quantum statistical mechanics*, Annual Review of Condensed Matter Physics **6**, 15 (2015), https://doi.org/10.1146/annurev-conmatphys-031214-014726 .

[26] B. Yan, L. Cincio, and W. H. Zurek, *Information Scrambling and Loschmidt Echo*, **124**, 160603 (2020), arXiv:1903.02651 [quant-ph] .

[27] J. Maldacena, *Eternal black holes in anti-de Sitter*, Journal of High Energy Physics **2003**, 021 (2003), arXiv:hep-th/0106112 [hep-th] .

[28] Y. Sekino and L. Susskind, *Fast scramblers*, Journal of High Energy Physics **2008**, 065 (2008), arXiv:0808.2096 [hep-th] .

[29] J. Maldacena, S. H. Shenker, and D. Stanford, *A bound on chaos*, Journal of High Energy Physics **2016**, 106 (2016), arXiv:1503.01409 [hep-th] .

[30] M. L. Mehta, *Random matrices* (Elsevier, 2004).

[31] H. Friedrich and H. Wintgen, *The hydrogen atom in a uniform magnetic field — an example of chaos*, Physics Reports **183**, 37 (1989).

[32] F. J. Dyson, *Statistical theory of the energy levels of complex systems. i*, Journal of Mathematical Physics **3**, 140 (1962), https://doi.org/10.1063/1.1703773 .

[33] T. Guhr, A. Müller-Groeling, and H. A. Weidenmüller, *Random-matrix theories in quantum physics: common concepts*, **299**, 189 (1998), arXiv:cond-mat/9707301 [cond-mat] .

[34] A. Chenu, J. Molina-Vilaplana, and A. del Campo, *Work Statistics, Loschmidt Echo and Information Scrambling in Chaotic Quantum Systems*, arXiv e-prints , arXiv:1804.09188 (2018), arXiv:1804.09188 [quant-ph] .

[35] A. del Campo, J. Molina-Vilaplana, and J. Sonner, *Scrambling the spectral form factor: Unitarity constraints and exact results*, **95**, 126008 (2017), arXiv:1702.04350 [hep-th] .

[36] A. del Campo, J. Molina Vilaplana, L. F. Santos, and J. Sonner, *Decay of a Thermofield-Double State in Chaotic Quantum Systems*, arXiv e-prints , arXiv:1709.10105 (2017), arXiv:1709.10105 [cond-mat.str-el] .