UNIVERSIDAD DE LA LAGUNA

«El efecto de variar los coeficientes en problemas de valores de contorno elipticos semilineales. Desde soluciones clasicas a metasoluciones.»

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CERTIFICA:

Que la presente memoria, titulada **"The effect of varying coefficients in semilinear elliptic boundary value problems. From classical solutions to metasolutions"**, ha sido realizada bajo mi dirección por la licenciada en Ciencias Matemáticas Doña Rosa M. Gómez Reñasco, y que constituye su tesis para optar al grado de Doctora en Ciencias Matemáticas.

Y para que conste a los efectos oportunos, firmo la presente en Madrid, a veinte de Enero de 1999.

Fdo. Don Julián López Gómez

D. José Sabina de Lis, Prof. Titular del Depto. de Análisis Matemático de la Universidad de La Laguna, certifica que es tutor, en el Programa de Doctorado de dicho departamento, de D^a. Rosa M. Gómez Reñasco, y que ha elaborado la memoria con título "The effect of varying coefficients in semilinear elliptic boundary value problems. From classical solutions to metasolutions", bajo la dirección de D. Julián López Gómez, Prof. Titular de Universidad en la Un. Complutense de Madrid, lo que se certifica en escrito adjunto. Por tanto, a los efectos de que la interesada presente dicho trabajo para optar al grado de doctor, quien suscribe *ratifica* la mencionada certificación.

En La Laguna, a 11 de Febrero de 1999.

Fdo. José Sabina de Lis.

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ÍNDICE

Introducción

Introducción

En esta memoria analizamos la estructura del conjunto de soluciones positivas del problema

$$-\Delta u = \lambda u - a(x)|u|^p u \quad \text{en} \;\; \Omega\,, \qquad u|_{\partial\Omega} = 0\,, \tag{0.1}$$

donde Ω es un dominio acotado de $\mathbb{R}^N$, $N \geq 1$, con frontera $\partial\Omega$ de clase C^2, $\lambda \in \mathbb{R}$ será considerado como un parámetro de continuación y $p \in \mathbb{R}\,,\; p > 0$. Suponemos que $a \neq 0$ es una función medible acotada en $\overline{\Omega}$ y escribimos

$$\Omega_\pm := \{\, x \in \Omega \,:\, a^\pm(x) > 0 \,\}\,,$$

siendo $a^+ := \max\{a, 0\}$ la parte positiva y $a^- := a^+ - a$ la parte negativa de a. Además supondremos que Ω_+ y Ω_- son conjuntos abiertos de clase C^2 y que $a^\pm$ están acotadas lejos de cero en subconjuntos compactos de $\Omega_\pm$. Nótese que bajo estas suposiciones Ω_+ y Ω_- tienen un número finito de componentes.

El problema (0.1) nos proporciona los estados de equilibrio del modelo parabólico asociado

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = \lambda u - a(x)|u|^p u\,, & (x,t) \in \Omega \times (0,\infty)\,, \\ u(\cdot,t)|_{\partial\Omega} = 0\,, & t > 0\,, \\ u(\cdot,0) = u_0 \geq 0\,, & \text{in} \;\; \Omega\,, \end{cases} \tag{0.2}$$

que es muy conocido en dinámica de poblaciones; cuando $a > 0$ está acotada lejos de cero se trata de la ecuación logística parabólica ([Ok80], [Mu93]). Aunque en biología matemática se suele considerar el coeficiente $a(x)$ no negativo, la ecuación (0.2) admite un significado biológico aún en el caso general cuando $a(x)$ cambia de signo. En este caso $u(\cdot, t)$ es la densidad de población en el tiempo t de una especie que habita la región Ω, λ es el índice de crecimiento neto de la especie, u_0 es la densidad de población inicial, y el coeficiente $a(x)$ mide el efecto de saturación en respuesta al estrés de la población en Ω_+, mientras que en Ω_- mide el efecto de simbiosis debido a la cooperación intraespecífica entre los individuos de la especie. En la región

$$\Omega_0 := \Omega \setminus (\overline{\Omega}_+ \cup \overline{\Omega}_-)\,,$$

los individuos de la especie están libres de cualquiera otros efectos mas que la difusión y, por lo tanto, las componentes de Ω_0 (tantas como haya en un número finito) pueden considerarse como *refugios*.

Durante las últimas décadas los problemas de valores de contorno elípticos semilineales del mismo tipo que el problema (0.1) han despertado un gran interés, sin embargo la mayoría de los estudios realizados tratan del modelo sublineal puro ($\Omega_+ = \Omega$), o del modelo superlineal puro ($\Omega_- = \Omega$). Nuestro objetivo en la presente memoria es analizar no sólo estos casos especiales, sino también el caso general cuando (0.1) es del tipo superlineal indefinido ($\Omega_+ \neq \emptyset$, $\Omega_- \neq \emptyset$).

Hemos dividido la memoria en tres capítulos. En el Capítulo I analizamos el problema sublineal, en el Capítulo II tratamos el problema superlineal indefinido y, finalmente, en el Capítulo III analizamos el problema superlineal puro, aunque permitiendo que se anule el coeficiente de la no linealidad. A continuación resumimos los contenidos y resultados de cada uno de estos tres capítulos.

En el Capítulo I analizamos el problema sublineal general, es decir $\Omega_- = \emptyset$. Sorprende que la situación general cuando la especie u está libre de efectos de saturación en algún subdominio de Ω (cuando Ω_+ es un subdominio propio de Ω y por lo tanto $\Omega_0 \neq \emptyset$), no se haya estudiado hasta muy recientemente (ver [BO86], [Ou92], [FKLM96], [GGLS98]).

En [BO86] se estableció, por medio de métodos variacionales, la existencia y la unicidad de las soluciones positivas de (0.1). El correspondiente problema de Neumann se estudió en [Ou92] usando métodos de continuación y se obtuvieron los mismos resultados que en [BO86]. En [FKLM96], y por medio de técnicas de comparación, se extendieron estos resultados a problemas no necesariamente autoadjuntos con coeficientes suaves y bajo condiciones de contorno más generales. Por último, en [GGLS98] se probó que en el caso en que Ω_0 es conexo y $N \geq 2$, las soluciones positivas de (0.1) crecen a infinito en la región donde el coeficiente $a(x)$ se anula, esto es en Ω_0, mientras que en su soporte, Ω_+, las soluciones positivas se estabilizan a la solución positiva minimal de la ecuación original sujeta a condiciones de Dirichlet infinitas en la frontera de Ω_+.

En el análisis llevado a cabo en el Capítulo I no es necesaria la imposición de que Ω_0 sea conexo y, junto al estudio de la existencia, comportamiento puntual y cálculo numérico de las soluciones clásicas de (0.1), también estudiamos una familia de soluciones no clásicas de (0.1) que no pertenecen a $\cup_{p=1}^{\infty} L^p_{loc}(\Omega)$. Nuestras soluciones no clásicas, a las que llamamos ***metasoluciones***, pueden valer infinito en algunas de las regiones del dominio soporte. Podemos pensar en las metasoluciones como una clase de extensión por infinito a todo el dominio de una solución clásica ***grande*** (ver [MV97]), pero contrariamente a lo que ocurre con estas soluciones grandes extendidas, las metasoluciones pueden anularse en algunas de las componentes de la frontera del dominio mientras crecen a infinito en el resto.

Aunque el análisis de soluciones clásicas grandes ha atraído la atención de investigadores durante los últimos años ([BM91], [Ve92], [MV97]), parece que este es el primer trabajo donde se introduce el concepto de metasolución, se analiza su existencia y se lleva a cabo su cálculo numericamente.

Desde el punto de vista biológico, la importancia de las metasoluciones viene del hecho de que nos proporcionan todos los posibles perfiles límite de la población en función del índice de crecimiento λ, cuando el tiempo crece a infinito.

En lo que concierne al tratamiento numérico del problema, debemos apuntar que nos hemos encontrado con serias dificultades para calcular las metasoluciones para algunos rangos de los parámetros envueltos en el problema; principalmente estos problemas han surgido del hecho de tener que trabajar con números muy grandes. En el Capítulo I se detallan todos estos aspectos.

El Capítulo II está dedicado al estudio del problema superlineal indefinido, es decir, el problema general cuando $\Omega_- \neq \emptyset$ and $\Omega_+ \neq \emptyset$. Hemos enfocado nuestra atención en analizar cómo cambia la dinámica de las soluciones del problema (0.1) cuando varía el comportamiento nodal del coeficiente $a(x)$. También hemos analizado el comportamiento *capa-pico* de las soluciones positivas de una versión uno-dimensional de (0.1) cuando el parámetro λ decrece a $-\infty$.

Entre las referencias clásicas que tratan del problema superlineal puro ($\Omega_- = \Omega$), podemos citar [AR73], [TU74], [BT77], [GS281], [FLN82], [Ou91]. Para este problema los métodos variacionales han sido muy útiles para probar la existencia de soluciones *capa-pico* de (0.1), pero desafortunadamente para el problema superlineal indefinido general no existen resultados de esta naturaleza. La dificultad principal, como ya se apuntó en [BCN95], viene del hecho de que los métodos variacionales no se pueden aplicar para tratar el caso cuando $a(x)$ cambia de signo.

Relativamente pocas han sido las características que se han establecido para el problema superlineal indefinido general ([BCN95], [AT96], [Lo97], [AL98]), y la mayoría de ellas se han encontrado muy recientemente. Entre todos los resultados disponibles, en [AL98] se probó, por medio de las teorías de bifurcación global y de índice de punto fijo, que si la bifurcación a soluciones positivas desde la solución trivial en $\lambda = \sigma_1^\Omega$ es supercrítica y las soluciones de (0.1) poseen cotas a priori uniformes en L_∞ para λ variando en subintervalos compactos de $\mathbb{R}$, entonces existe $\sigma_1^\Omega < \lambda^* < \sigma_1^{\Omega\setminus\overline{\Omega}_+}$ tal que (0.1) posee una solución positiva para cada $\lambda \in (-\infty, \lambda^*]$, y posee como mínimo dos soluciones positivas para cada $\lambda \in (\sigma_1^\Omega, \lambda^*)$. De aquí en adelante $\sigma_1^{\Omega_1}$ denota el autovalor principal de $-\Delta$ en Ω_1 bajo condiciones de contorno de Dirichlet homogéneas.

El teorema principal del Capítulo II (teorema II.3.7) completa este resultado probando que (0.1) posee una única solución positiva linealmente estable si, y sólo si, $\lambda \in (\sigma_1^\Omega, \lambda^*)$. Esta solución nos proporciona un atractor local para el modelo parabólico asociado. La unicidad del estado estable es bastante sorprendente ya que (0.1) posee, como mínimo, dos soluciones positivas para cada $\lambda \in (\sigma_1^\Omega, \lambda^*)$ y, además, nuestro estudio numérico sugiere que existen ejemplos con tantas soluciones como queramos, aún en espacios de una dimensión, si elegimos un apropiado coeficiente $a(x)$.

En efecto, el análisis numérico llevado a cabo sugiere que si $N = 1$, Ω_- tiene n componentes y $a(x)$ tiene un único mínimo local en cada una de estas componentes, entonces (0.1) posee $2^n - 1$ soluciones positivas para cada $\lambda < 0$ acotado lejos de cero; n de estas soluciones tendrán un pico localizado en cada uno de los mínimos de $a(x)$, $\frac{n(n-1)}{2}$ soluciones exhibirán dos picos, y en general, $\binom{n}{j}$, $1 \le j \le n$, soluciones exhibirán j picos.

Desde el punto de vista biológico, si $\lambda \le \sigma_1^\Omega$ entonces $u = 0$ es la única solución no negativa estable de (0.1), pero como (0.1) posee una solución positiva necesariamente inestable, la especie puede evitar la extinción si la población inicial cae en la variedad estable de alguno de estos estados de equilibrio positivos. En este caso, para evitar la extinción, la especie debería concentrarse en las regiones donde la cooperación intraespecífica tiene lugar, es decir en Ω_-, y, más especificamente, tal concentración debería de ser mayor alrededor de los valores donde la simbiosis es más alta, es decir alrededor de cada uno de los mínimos negativos de $a(x)$.

Nuestros cálculos numéricos sugieren también que la estructura del conjunto de soluciones positivas de (0.1) se basa en gran medida en las propiedades de simetría del coeficiente $a(x)$. De hecho mostramos que romper la simetría de $a(x)$ se traduce en una pérdida de simetría del diagrama de bifurcación de las soluciones positivas del modelo. El diagrama global pasa de exhibir un tenedor global a exhibir un par de componentes globales. Una de ellas es la rama que emana del estado trivial, y la otra es una componente global acotada lejos del estado cero. También hemos analizado como cambian las soluciones en esta última componente global cuando $a(x)$ cambia, y mostramos como en muchos ejemplos estas soluciones se aproximan a algunas metasoluciones del problema para algunos valores críticos del parámetro. Por lo tanto, las metasoluciones no son una característica específica del modelo sublineal, sino que también surgen en problemas superlineales indefinidos. En el Capítulo II se completan los detalles.

El hecho de que el número de soluciones positivas de un problema subcrítico superlineal puede sufrir cambios drásticos cuando varía la forma de Ω está bien documentado en la literarura (ver [HV84], [Da88], [Da90], [Ce95]). De hecho, romper la convexidad de Ω puede producir un número arbitrariamente grande de soluciones. Nuestro análisis muestra que el mismo efecto surge cuando se varía el coeficiente del problema en lugar de la forma del dominio, aún en los modelos más simples de una dimensión. Este es el porqué de la importancia de la unicidad del estado estable. La unicidad no depende ni de la geometría del dominio ni del comportamiento nodal de $a(x)$; la unicidad de la solución positiva estable es una propiedad universal de (0.1).

En el Capítulo III abordamos el problema de la existencia de cotas a priori para las soluciones positivas radialmente simétricas de (0.1) cuando $N \geq 3$, Ω es una bola de $\mathbb{R}^N$ y $\Omega_+ = \emptyset$. En este caso la bifurcación desde la solución trivial es subcrítica. De hecho (0.1) no admite una solución positiva cuando $u = 0$ es inestable ($\lambda > \sigma_1^\Omega$). En [AL98] se probó la existencia de cotas a priori uniformes para las soluciones positivas radialmente simétricas de (0.1) bajo la única suposición de que $p+1 < \frac{N+2}{N-2}$, si $N \geq 3$, y supuesto que Ω_- es una bola. En este capítulo analizamos numericamente si el exponente crítico es o no optimal en todas las situaciones. Nuestros cálculos numéricos sugieren que, en el caso en que el coeficiente $a(x)$ se anule en alguna región de Ω, el exponente crítico para la existencia de cotas a priori depende de la estructura nodal de la función peso $a(x)$ más que del crecimiento a infinito de la no linealidad, es decir del tamaño de p. De hecho, si $a(x)$ se anula en alguna bola y vale una constante negativa en el resto, los resultados numéricos que hemos obtenido indican que el modelo posee una solución positiva radialmente simétrica para $\lambda < 0$ acotado lejos de cero, aún cuando $p+1$ sea igual al exponente crítico, sugiriendo fuertemente que en este caso las soluciones positivas radialmente simétricas del modelo deberían tener cotas a priori; mientras que si $a(x)$ permanece negativo en alguna bola y se anula en el resto, entonces se pierden las cotas a priori de las soluciones positivas radialmente simétricas, y éstas crecen a infinito en algún $\lambda^* > 0$. Debemos apuntar que el análisis incluído en este Capítulo está actualmente en progreso; en este Capítulo III hemos querido mostrar la dirección en que se encamina nuestra investigación futura.

Por último queremos hacer notar el hecho de que en esta memoria hemos utilizado herramientas teóricas y numéricas. El análisis teórico ayuda al estudio numérico, y el numérico confirma, completa e ilustra el análisis. El numérico, además, nos ha proporcionado algunos resultados para los cuales no hay conclusiones teóricas en la actualidad y que, por tanto, son problemas que quedan abiertos.

Introduction

In this memoir we analyze the structure of the set of positive solutions of

$$-\Delta u = \lambda u - a(x)|u|^p u \quad \text{in } \ \Omega\,, \qquad u|_{\partial\Omega} = 0\,, \tag{0.1}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 1$, whose boundary $\partial\Omega$ is of class C^2, $\lambda \in \mathbb{R}$ will be regarded as a continuation parameter and $p \in \mathbb{R}$, $p > 0$. We suppose that $a \neq 0$ is a bounded measurable function on $\overline{\Omega}$ and put

$$\Omega_{\pm} := \{\, x \in \Omega \,:\, a^{\pm}(x) > 0 \,\}\,,$$

where $a^+ := \max\{a, 0\}$ is the positive and $a^- := a^+ - a$ is the negative part of a. In addition we assume that Ω_+ and Ω_- are open sets of class C^2 and that $a^{\pm}$ is bounded away from zero on compact subsets of $\Omega_{\pm}$. Note that under these assumptions Ω_+ and Ω_- have only finitely many components. Problem (0.1) provides us with the steady states of the associated parabolic model

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = \lambda u - a(x)|u|^p u\,, & (x,t) \in \Omega \times (0,\infty)\,, \\ u(\cdot,t)|_{\partial\Omega} = 0\,, & t > 0\,, \\ u(\cdot,0) = u_0 \geq 0\,, & \text{in } \ \Omega\,, \end{cases} \tag{0.2}$$

which is very well known in population dynamics; the parabolic logistic equation when $a > 0$ is bounded away from zero, [Ok80], [Mu93]. Although in mathematical biology the coefficient $a(x)$ is usually taken to be non-negative, (0.2) admits a biological meaning as well even in the general case when $a(x)$ changes of sign. Typically, $u(\cdot,t)$ is the density at time t of a single species inhabiting Ω, λ is the net growth rate of the species, u_0 is the initial population density, and the coefficient $a(x)$ measures the saturation effect responses to the population stress in Ω_+, while in Ω_- it measures the symbiosis effects due to the intraspecific cooperation among the individuals of the species. In the region

$$\Omega_0 := \Omega \setminus (\overline{\Omega}_+ \cup \overline{\Omega}_-)\,,$$

these individuals are free from other effects than diffusion and hence the components of Ω_0 (at most finitely many) can be though as *refuges* for the species.

Semilinear elliptic boundary value problems of the same type as (0.1) have attracted a great deal of interest during the last few decades, although most of the published papers dealt, however, either with the pure sublinear model ($\Omega_+ = \Omega$) or with the pure superlinear one ($\Omega_- = \Omega$), being our attempt in this memoir to analyze not only these special cases but also the general case when (0.1) is of superlinear indefinite type. The memoir is divided in three chapters. In Chapter I we analyze the pure sublinear problem, in Chapter II we treat the superlinear indefinite problem and finally in Chapter III we shall be concerned with the pure superlinear problem where in addition the damping coefficient in front of the nonlinearity is allowed to vanish. In the sequel we shall shortly summaryze the containts and results in each of the three chapters.

In Chapter I we analyze the general sublinear problem, $\Omega_- = \emptyset$. Quite surprisingly the general situation when the species u is free from crowding effects on some subdomain of Ω (Ω_+ is a proper subdomain of Ω and hence $\Omega_0 \neq \emptyset$) has not been tackled until very recently (cf. [BO86], [Ou92], [FKLM96], [GGLS98], and the references there in).

In [BO86] it was established the existence and the uniqueness of the positive solution of (0.1) by means of variational methods. In [Ou92] the Neumann problem was dealt with and the same results as in [BO86] were found; this time using continuation methods. All these results were further extended in [FKLM96] to not necessarily self-adjoint problems with smooth coefficients under rather general boundary conditions by means of comparison techniques. In [GGLS98] it was shown that in the case when Ω_0 is connected and $N \geq 2$ the positive solutions of (0.1) grow to infinity in the region where the coefficient $a(x)$ vanishes, Ω_0, while on its support, Ω_+, the positive solutions stabilize to the minimal positive solution of the original equation subject to infinity Dirichlet conditions on the boundary of Ω_+.

For the analysis carried out in Chapter I we do not need impossing that Ω_0 is connected and beside the existence, the point-wise behavior and the numerical computation of the classical positive solutions of (0.1), we study a family of non-classical solutions of (0.1) which do not belong to $\cup_{p=1}^{\infty} L^p_{loc}(\Omega)$. Our non-classical solutions, called metasolutions here in, are allowed to be infinity on some regions of the support domain. The metasolutions can be thought as a sort of prolongation by infinity to the whole domain of a large classical solution (cf. [MV97] and the references there in), but contrarily to what happens with these extended large solutions, the metasolutions can vanish on some of the components of the boundary of the domain while can grow to infinity on the remaining ones.

Although the analysis of large classical solutions has attracted the attention of many researchers during the last few years ([BM91], [Ve92], [MV97] and the references there in), it seems to be that this is the first work where the concept of metasolution has been introduced and the problem of their existence and numerical computation has been addressed.

From the biological point of view, the relevance of the metasolutions comes from the fact that they provide us with all the possible limiting profiles of the population as time grows to infinity accordingly to the size of the birth rate λ.

As far to the numerics concerns we should point out that we have found serious difficulties to compute the metasolutions for some ranges of the parameters involved in the setting of the problem, those difficulties mainly coming from the fact that one has to deal with very high numbers in the computations. We refer to Chapter I for further details.

Chapter II is devoted to the analysis of the superlinear indefinite problem, i.e. the general problem when $\Omega_- \neq \emptyset$ and $\Omega_+ \neq \emptyset$. We mainly focus our attention into the problem of analyzing how changes the dynamics of the solutions of problem (0.1) as the nodal behavior of the coefficient $a(x)$ varies. Also, we analyze the spike layer behavior of all the positive solutions of some one-dimensional versions of (0.1) as $\lambda \downarrow -\infty$.

Classical papers dealing with the pure superlinear problem, $\Omega_- = \Omega$, are [AR73], [TU74], [BT77], [GS281], [FLN82], [Ou91], and the references there in. For the pure superlinear problem, variational methods have proven to be very useful to show the existence of some spike layer solutions of (0.1), but unfortunately no result of this nature is available for the general superlinear indefinite problem. The main difficulty coming from the fact that the standard variational methods do not apply to treat the case when $a(x)$ changes of sign, as already pointed out in [BCN95].

Relatively few features has been established for the general superlinear indefinite problem (cf. [BCN95], [AT96], [Lo97], [AL98] and the references there in), and most of them were found very recently. Among all the results available, in [AL98] it was shown, by means of global bifurcation and fixed point index theory, that if Ω_+ and Ω_- are not empty, the bifurcation to positive solutions from the trivial solution at $\lambda = \sigma_1^\Omega$ is supercritical and in addition the positive solutions of (0.1) possess uniform L_∞ a priori bounds for λ varying in compact subintervals of $\mathbb{R}$, then there exists $\sigma_1^\Omega < \lambda^* < \sigma_1^{\Omega\setminus\overline{\Omega}_+}$

such that (0.1) possesses a positive solution for each $\lambda \in (-\infty, \lambda^*]$, and it possesses at least two positive solutions for each $\lambda \in (\sigma_1^{\Omega}, \lambda^*)$. Hereafter $\sigma_1^{\Omega_1}$ stands for the principal eigenvalue of $-\Delta$ in Ω_1 under homogeneous Dirichlet boundary conditions.

Our main analytical result in Chapter II completes that picture by showing that (0.1) possesses a linearly stable positive solution if, and only if, $\lambda \in (\sigma_1^{\Omega}, \lambda^*)$, and that in this case the stable solution is unique and it provides us with a local attractor for the associated parabolic model. The uniqueness of the stable state is quite striking, since (0.1) possesses at least two positive solutions for each $\lambda \in (\sigma_1^{\Omega}, \lambda^*)$ and it looks like that there are examples having as many solutions as we wish by chosing an appropiate $a(x)$, even in one-dimensional spaces.

Our numerical analysis in Chapter II suggest that if $N = 1$, Ω_- has n components, and $a(x)$ has an unique local minimum on each of these components, then (0.1) possesses $2^n - 1$ positive solutions for each $\lambda < 0$ bounded away from zero; n solutions among them will have one single peak located on each of the minima of $a(x)$, $\frac{n(n-1)}{2}$ solutions will exhibit two peaks, and in general $\binom{n}{j}$, $1 \leq j \leq n$, solutions will exhibit j peaks.

From the biological point of view, if $\lambda \leq \sigma_1^{\Omega}$ then $u = 0$ is the unique stable non-negative solution of (0.1), but since (0.1) possesses a positive solution neccesarily unstable, the species can avoid extintion if the initial population lies on the stable manifold of some of these positive steady states. In this case, in order to avoid extintion the species should concentrate within the regions where intraspecific cooperation takes place, i.e. in Ω_-, and more specifically such concentration should be more emphasized around the values where the symbiosis rate is higher, i.e. around each of the negative minima of $a(x)$.

Our numerical computations in Chapter II also suggest that the structure of the set of positive solutions of (0.1) is strongly based upon the symmetry properties of the damping coefficient $a(x)$. In fact we have shown that breaking down the symmetry of $a(x)$ results into a lost of the symmetry of the bifurcation diagram of positive solutions of the model. The global diagram passes from exhibiting a global pitchfork to exhibiting a couple of global components. One of them, the branch emanating from the trivial state, and the other a global folding bounded away from the zero state. We have also analyzed how change the solutions on the global folding as $a(x)$ changes and we have shown how in many instances these solutions approach to some metasolutions of the problem at some critical value of the parameter. So metasolutions are not an specific feature of the sublinear model but also arise in superlinear indefinite problems. The reader should go to Chapter II for further details and a more complete discussion.

The fact that the number of positive solutions of a superlinear subcritical problem can suffer drastical changes as the shape of Ω varies is well documented in the literature (cf. [HV84], [Da88], [Da90], [Ce95], and the references there in). In fact, breaking down the convexity of Ω can result into an arbitrarily large number of solutions. Our analysis shows that the same effect arises by varying the coefficients of the model, instead of the shape of the domain, even in the simplest one-dimensional models. This is why the uniqueness of the stable state is so relevant, since it does not depend neither on the geometry of the domain nor on the nodal behavior of $a(x)$; the uniqueness of the stable positive solution is an universal property of (0.1).

In Chapter III we address the problem of the existence of a priori bounds for the radially symmetric positive solutions of (0.1) when $N \geq 3$, Ω is a ball of $\mathbb{R}^N$, and $\Omega_+ = \emptyset$. Now, the bifurcation from the trivial solution is subcritical. In fact, (0.1) does not admit a positive solution when $u = 0$ is unstable ($\lambda > \sigma_1^{\Omega}$). In [AL98] it was shown the existence of uniform a priori bounds for the positive radially symmetric solutions of (0.1) under the sole assumption that $p + 1 < \frac{N+2}{N-2}$, if $N \geq 3$, provided that Ω_- is a ball, and in Chapter III we have analyzed numerically if the critical exponent is or not optimal in all situations. Our numerical computations suggest that in the case when the coefficient $a(x)$ vanishes in some region of Ω, the critical exponent for the existence of a priori bounds is strongly dependent on the nodal structure of the weight function $a(x)$ rather than on the growth at infinity of the nonlinearity, i.e. the size of p. In fact, if $a(x)$ vanishes on some ball while it is kept as a negative constant on its complement then it seems that the model possesses a radially symmetric positive solution for $\lambda < 0$ bounded away from zero, even when $p+1$ equals critical exponent, so strongly suggesting that in this case the radially symmetric positive solutions of the model should have a priori bounds; while if $a(x)$ remains negative on any ball, then the a priori bounds of the positive radially symmetric solutions are lost and the solutions grow to infinity at some $\lambda^* > 0$. We should point out that the research included in Chapter III is in progress at present and it will completed in the near future. In some sense, in Chapter III we want to show some of the future directions of our research.

Finally, we would like to emphasize that in this memoir we have used both theoretical and numerical tools. The analysis aids the numerical study, and the numerics confirm, complete and illustrate the analysis. The numerics in addition provide us with some further results for which at first glance analytical tools are not available at present.

Chapter I
The Sublinear Problem

Chapter I

The Sublinear Problem

I.1. Introduction

In this Chapter we analyze the existence and point-wise behavior of the positive solutions of

$$-\Delta u = \lambda u - a(x)|u|^p u \qquad \text{in } \ \Omega\,, \qquad u|_{\partial\Omega} = 0\,, \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 1$, of class C^2, $\lambda \in \mathbb{R}$, $p > 0$, and $a \neq 0$ is a bounded measurable non-negative function in $\overline{\Omega}$ satisfying the following assumptions:

(H1) The set

$$\Omega_+ := \{\, x \in \Omega \;:\; a(x) > 0 \,\}$$

is open and it possesses a finite number of components

$$\Omega_{+,j}\,, \quad 1 \leq j \leq h\,,$$

such that $\overline{\Omega}_{+,i} \cap \overline{\Omega}_{+,j} = \emptyset$ if $i \neq j$. Moreover, each of these components is of class C^3.

(H2) $a(x)$ is bounded away from zero on any compact subset of Ω_+, and if Γ_+ and Γ are components of $\partial\Omega_+$ and $\partial\Omega$, respectively, such that $\Gamma_+ \cap \Gamma \neq \emptyset$, then $\Gamma_+ = \Gamma$ and $a(x)$ is bounded away from zero on Γ_+.

(H3) If Γ_+ is a component of $\partial\Omega_+$ such that $\Gamma_+ \cap \partial\Omega = \emptyset$, then

$$a(x) = o(\text{dist }(x, \Gamma_+)) \qquad \text{as} \quad \text{dist }(x, \Gamma_+) \downarrow 0\,. \tag{1.2}$$

(H4) The open set

$$\Omega_0 := \Omega \setminus \overline{\Omega}_+$$

possesses a finite number of components,

$$\Omega^i_{0,j}\,, \quad 1 \leq i \leq m\,,\ 1 \leq j \leq n_i\,,$$

such that $\overline{\Omega}^i_{0,j} \cap \overline{\Omega}^{\hat{i}}_{0,\hat{j}} = \emptyset$ if $(i,j) \neq (\hat{i},\hat{j})$, and

$$\sigma_1^{\Omega^i_{0,j}} = \sigma_1^{\Omega^i_{0,j+1}}\,, \qquad 1 \leq i \leq m\,,\ \ 1 \leq j \leq n_i - 1\,, \tag{1.3a}$$

$$\sigma_1^{\Omega^i_{0,1}} < \sigma_1^{\Omega^{i+1}_{0,1}}\,, \qquad 1 \leq i \leq m-1\,. \tag{1.3b}$$

Hereafter, given a regular subdomain D of Ω and $V \in L^\infty(D)$, we denote by $\sigma_1^D[-\Delta + V]$ the principal eigenvalue of $-\Delta + V$ in D subject to homogeneous Dirichlet boundary conditions, and set $\sigma_1^D := \sigma_1^D[-\Delta]$.

Nevertheless, in Section 4 it will be shown how condition (H3) is not necessary if either $N = 1$, or if $N \geq 2$ and in addition (1.1) possesses radial symmetry.

Problem (1.1) provides us with the steady states of a well known model in population dynamics, the parabolic logistic equation, [Mu93], [Ok80]. Typically, $u(\cdot)$ is the density of a single species inhabiting Ω, λ is the net growth rate of the species, and the coefficient $a(x)$ measures the saturation effect responses to the population stress in Ω_+. In the region Ω_0 the individuals of the population are free from other effects than diffusion and this is why the components of Ω_0 will be referred to as ***refuges***. Given two arbitrary refuges R_1 and R_2, it will be said that R_1 ***is larger than*** R_2 if $\sigma_1^{R_1} < \sigma_1^{R_2}$, independently of the relation between their Lebesgue measures.

In this work, beside the existence, the point-wise behavior and the numerical computation of the positive classical solutions of (1.1), we study a family of non-classical solutions of (1.1) that do not belong to $\cup_{p=1}^{\infty} L^p_{loc}(\Omega)$. These solutions will be referred to as ***metasolutions***. Setting

$$\Omega_k := \Omega \setminus \bigcup_{i=1}^{k} \bigcup_{j=1}^{n_i} \overline{\Omega}^i_{0,j}\,, \qquad 1 \leq k \leq m\,, \tag{1.4}$$

a function $u : \Omega \to [0, \infty]$ is said to be a ***metasolution of order*** k of (1.1) supported in Ω_k if the following conditions are satisfied:

(M1) $u|_{\Omega_k}$ is a ***large classical solution*** of

$$\begin{cases} -\Delta u = \lambda u - a(x) u^{p+1} & \text{in} \quad \Omega_k\,, \\ u|_{\partial\Omega \cap \partial\Omega_k} = 0\,, \quad u|_{\partial\Omega_k \setminus \partial\Omega} = \infty\,. \end{cases} \tag{1.5}$$

(M2) $u = \infty$ in $(\Omega \setminus \Omega_k) \cup (\partial\Omega_k \setminus \partial\Omega)$.

Although the analysis of large solutions has attracted the attention of many researchers during the past few years (cf. [BM91], [Ve92], [MV97] and the references there in), this seems to be the first work where metasolutions have been introduced and the problem of their existence and numerical computation has been addressed. It should be pointed out that if u is a metasolution of order k supported in Ω_k and in addition $\partial\Omega \cap \partial\Omega_k = \emptyset$, then $u|_{\Omega_k}$ provides us with a large solution of (1.1) in Ω_k, in the sense of [MV97]. However, from the definition itself it is rather clear that not all metasolutions provide us with large solutions, since a metasolution can vanish on some of the components of the boundary while grows to infinity on the remaining. Nevertheless, it should be pointed out that in the setting of [MV97] and the references there in talking about metasolutions does not make sense, since this concept actually relies on the existence of some refuge and the models dealt with there in do not possess any.

As far as to the existence of classical solutions concerns, it is already known that (1.1) possesses a strong solution if, and only if,

$$\sigma_1^{\Omega} < \lambda < \sigma_1^{\Omega^1_{0,1}}\,. \tag{1.6}$$

Moreover, it is unique, if it exists (cf. [BO86], [Ou92], [FKLM96], [GGLS98], [AL98] and the references there in). Throughout this chapter it will be denoted by $\theta^{\Omega}_{[\lambda,a]}$. In this chapter, setting

$$\lambda_1 := \sigma_1^{\Omega^1_{0,1}}\,.$$

we show that the mapping $\lambda \to \theta^{\Omega}_{[\lambda,a]}$ is increasing and that

$$\lim_{\lambda\uparrow\lambda_1} \theta^{\Omega}_{[\lambda,a]}(x) = \infty \qquad \text{for all} \quad x \in \bigcup_{j=1}^{n_1} \overline{\Omega}^1_{0,j}\,, \tag{1.7}$$

while

$$\lim_{\lambda\uparrow\sigma_1^{\Omega^1_{0,1}}} \theta^{\Omega}_{[\lambda,a]}(x) := \Xi^{\Omega_1}_{[\lambda_1,a]}(x) \in \mathbb{R} \qquad \text{for all} \quad x \in \Omega_1 := \Omega \setminus \bigcup_{j=1}^{n_1} \overline{\Omega}^1_{0,j}\,. \tag{1.8}$$

In fact, $\Xi^{\Omega_1}_{[\lambda_1,a]}$ provides us with the minimal positive classical solution of

$$\begin{cases} -\Delta u = \lambda_1 u - a(x) u^{p+1} & \text{in} \quad \Omega_1\,, \\ u|_{\partial\Omega\cap\partial\Omega_1} = 0\,, \quad u|_{\partial\Omega_1\setminus\partial\Omega} = \infty\,. \end{cases} \tag{1.9}$$

This result was found in [GGLS98] for the case when Ω_0 is connected but it is new under our general framework. Using the concept of metasolution the previous result shows that the $[0,\infty]$-valued function $\mathcal{M}^{\Omega}_{[\lambda_1,a]}$ defined by

$$\mathcal{M}^{\Omega}_{[\lambda_1,a]} := \begin{cases} \Xi^{\Omega_1}_{[\lambda_1,a]} & \text{in} \quad \Omega_1 \cup (\partial\Omega_1 \cap \partial\Omega)\,, \\ \infty & \text{in} \quad (\Omega\setminus\Omega_1)\cup(\partial\Omega_1\setminus\partial\Omega)\,, \end{cases} \tag{1.10}$$

is a metasolution of order one of (1.1) supported in Ω_1 for $\lambda=\lambda_1$. More generally, in this chapter the following result is obtained.

Theorem 1.1. *Assume $1\le k\le m$ and*

$$\sigma_1^{\Omega^k_{0,1}} \le \lambda < \sigma_1^{\Omega^{k+1}_{0,1}}\,, \tag{1.11}$$

where we take $\sigma_1^{\Omega^{m+1}_{0,1}} := \infty$. Then, (1.1) possesses a metasolution of order k supported in Ω_k. Moreover, if $1\le k\le m-1$ and (λ_n,u_n), $n\ge 1$, is a sequence of metasolutions of order k supported in Ω_k with

$$\sigma_1^{\Omega^k_{0,1}} \le \lambda_n < \sigma_1^{\Omega^{k+1}_{0,1}}\,, \qquad \lim_{n\to\infty}\lambda_n = \sigma_1^{\Omega^{k+1}_{0,1}}\,,$$

then there exists a subsequence $(\lambda_{n_\ell},u_{n_\ell})$, $\ell\ge 1$, such that the point-wise limit

$$\lim_{\ell\to\infty} u_{n_\ell}$$

provides us with a metasolution of order $k+1$ supported in Ω_{k+1}.

From the biological point of view, Theorem 1.1 is a crucial result, since the metasolutions provide us with all the possible limiting profiles of the population as time grows to infinity according to the size of the birth rate λ. Adopting this point of view, the analytical results of this chapter read as follows.

- If $\lambda \le \sigma_1^{\Omega}$, then Ω is unable to support the species u and the population is driven to extinction.

• If $\sigma_1^{\Omega} < \lambda < \sigma_1^{\Omega_{0,1}^1}$, then the population stabilizes to the fixed profile $\theta_{[\lambda,a]}^{\Omega}$ as time grows to infinity.

• If $k \in \{1, ..., m-1\}$ and $\sigma_1^{\Omega_{0,1}^k} \leq \lambda < \sigma_1^{\Omega_{0,1}^{k+1}}$, then the population stabilizes to a metasolution of order k supported in Ω_k. In particular, it grows to infinity in all the refuges $\Omega_{0,j}^i$, $1 \leq i \leq k$, $1 \leq j \leq n_i$, while it stabilizes to a bounded profile in Ω_k.

• If $\lambda \geq \sigma_1^{\Omega_{0,1}^m}$, then the population stabilizes to a metasolution of order m supported in Ω_m. In particular, it grows to infinity in any refuge, while it stabilizes to a bounded profile in Ω_m.

As far as to the numerical computation of the positive classical solutions concerns, path following coupled with pseudo-spectral methods have shown to be very efficient and robust to compute global compact arcs of solutions in many reaction diffusion equations and systems of interest in applications (cf. [Ei86], [CHQZ88], [LDEM92], [GGLS98], and the references there in). Nevertheless, however looking so simple as (1.1) does, computing its classical positive solutions for $\lambda < \lambda_1$, $\lambda \simeq \lambda_1$, is far from being an easy task. The main difficulty coming from the fact that the curve of positive solutions bifurcates from infinity at $\lambda = \lambda_1$ with respect to any L^p-norm, $1 \leq p \leq \infty$, and therefore, in order to approximate that curve, a huge number of modes should be considered in the discretizations of (1.1), which makes increasing the computational cost. In fact, not only the positive solution $\theta_{[\Omega,a]}^{\Omega}$ but also its gradient diverges to infinity on $\cup_{j=1}^{n_1} \partial\Omega_{0,j}^1$ as $\lambda \uparrow \lambda_1$ and therefore, the number of modes should indeed grow to infinity (cf. [CHQZ88]). All the practical difficulties that we have found to compute the classical solutions are extremely magnified when we compute the metasolutions, since in this case we should deal with the additional difficulty coming from the fact that the large associated solutions take the value infinity on some of the components of the boundary. In Sections 5, 6 we attack all these computational problems for a simple radially symmetric prototype of (1.1).

This chapter is distributed as follows. In Section 2 we characterize the point-wise growth of the classical positive solutions of (1.1) as $\lambda \uparrow \lambda_1$. The corresponding results are substantial extensions of those previously found in [FKLM96] and [GGLS98]. Then, we obtain the existence of positive solutions for a general class of non-homogeneous Dirichlet boundary conditions. These results will be used in Section 3 to prove the existence of metasolutions of any order. In Section 3 we introduce the concept of metasolution and prove Theorem 1.1 above. In Section 4 we show that in order to get the previous results condition (H3) is not necessary if either $N = 1$, or $N \geq 2$ and in addition (1.1) possesses radial symmetry. This result complements to those found in [LS98]. In Section 5 we use path-following coupled with pseudo-spectral methods to compute the classical solutions of a radially symmetric problem. Finally, in Section 6 we attack the problem of computing the metasolutions of the model analyzed in Section 5.

I.2. Classical solutions

In this section we characterize the existence and analyze the point-wise growth of the classical positive solutions of

$$-\Delta u = \lambda u - a(x)|u|^p u \qquad \text{in } \ D\,, \qquad u|_{\partial D} = 0\,, \tag{2.1}$$

where D is a bounded domain of $\mathbb{R}^N$, $N \geq 1$, of class C^2, $\lambda \in \mathbb{R}$, $p > 0$, and $a \neq 0$ is a bounded measurable non-negative function in $\overline{D}$ satisfying (H1-4). Throughout this section it will be convenient using the notations $D_+ = \Omega_+$, $D_{+,j} = \Omega_{+,j}$, $D_0 = \Omega_0$, and $D^i_{0,j} = \Omega^i_{0,j}$.

Definition 2.1. *A function $u : \overline{D} \to [0,\infty)$ is said to be a classical positive solution of* (2.1) *if $u \in L^\infty(D) \cap W^{1,p_0}(D)$ for some $p_0 > 1$ and* (2.1) *is satisfied in the weak sense.*

Lemma 2.2. *If u is a classical positive solution of* (2.1)*, then $u \in W^{2,p}(D) \cap W_0^{1,p}(D)$ for all $p > 1$ and hence, $u \in C^{1,\nu}(\overline{D})$ for all $0 < \nu < 1$. Moreover, u is a.e. in D twice classically differentiable. In other words, u is a strong solution of* (2.1).

Proof

The L^p-estimates of Agmon, Douglis and Nirenberg (cf. [GT83, Chapter IX] show that $u \in W^{2,p}(D) \cap W_0^{1,p}(D)$ if $p > 1$. The remaining assertions follow from the embedding $W^{2,p}(D) \hookrightarrow C^{2-\frac{N}{p}}(\overline{D})$, $p > N$, and [St70, Theorem VIII.1]. $\square$

Remark 2.3. *If u is a classical positive solution of (2.1), then the maximum principle implies $u(x) > 0$ for all $x \in D$ and $\frac{\partial u}{\partial n}(x) < 0$ for all $x \in \partial D$, where n is the outward unit normal to D at x, i.e. u lies in the interior of the cone P of positive functions of $C_0^1(\overline{D})$.*

In the sequel, given a function $w \in L^\infty(D)$ we say that $w > 0$ if $w \geq 0$ and $w \neq 0$. If $w \in C_0^1(\overline{D})$, it will be said that $w \gg 0$ if $w \in \operatorname{int} P$.

Theorem 2.4. *Assume $a(x)$ satisfies $(H1-4)$. Then, the following assertions are true:*

(i) *The problem* (2.1) *possesses a classical positive solution if, and only if,*

$$\sigma_1^D < \lambda < \sigma_1^{D^1_{0,1}} \,. \tag{2.2}$$

Moreover, it is unique if it exists.

(ii) *Suppose* (2.2) *and let $\theta_{[\lambda,a]}^D$ denote the unique classical positive solution of* (2.1)*. Then,*

$$\lim_{\lambda \downarrow \sigma_1^D} \|\theta_{[\lambda,a]}^D\|_{L^\infty(D)} = 0 \,, \tag{2.3}$$

and

$$\lim_{\lambda \uparrow \sigma_1^{D_{0,1}^1}} \theta_{[\lambda,a]}^D = \infty \qquad \textit{uniformly in } \bigcup_{j=1}^{n_1} \overline{D}_{0,j}^1 \setminus \partial D\,. \tag{2.4}$$

(iii) *The mapping $\lambda \to \theta_{[\lambda,a]}^D$ is point-wise increasing and if we regard to it as a mapping from $(\sigma_1^D, \sigma_1^{D_{0,1}^1})$ into $C^{1,\nu}(\overline{D})$, $0<\nu<1$, then it is differentiable and $\frac{\partial \theta_{[\lambda,a]}^D}{\partial \lambda} \in W^{2,p}(D) \cap W_0^{1,p}(D)$ for all $p>1$.*

(iv) *The principal eigenvalue of the linearization at $\theta_{[\lambda,a]}^D$ converges to zero as $\lambda \uparrow \sigma_1^{D_{0,1}^1}$. In other words,*

$$\lim_{\lambda \uparrow \sigma_1^{D_{0,1}^1}} \sigma_1^D[-\Delta + (p+1)a(x)[\theta_{[\lambda,a]}^D]^p - \lambda] = 0\,.$$

Some preliminary versions of this result were given in [BO86] by means of variational methods, and in [Ou92] by means of global continuation arguments. In the special case when D_0 is connected, (ii) and (iii) were found in [GGLS98] and [FKLM96], this time using the method of sub and supersolutions. Our proof uses some ideas and techniques introduced in [FKLM96], [GGLS98] and [LS98].

Proof of Theorem 2.4

(i) Let u_0 be a classical positive solution of (2.1). Then, $u_0|_{\partial D} = 0$ and

$$(-\Delta + a(x)u_0^p)u_0 = \lambda u_0\,.$$

Hence, u_0 is a positive eigenfunction associated with the eigenvalue λ of the operator $-\Delta + au_0^p$ under homogeneous Dirichlet boundary conditions. Thus, by the uniqueness of the principal eigenvalue,

$$\lambda = \sigma_1^D[-\Delta + a(x)u_0^p]\,. \tag{2.5}$$

and hence, by the monotonicity of the principal eigenvalue with respect to the potential,

$$\lambda > \sigma_1^D ,$$

since $a > 0$ and $u_0 \gg 0$. Moreover, since $a = 0$ in $D_{0,1}^1$, we find from (2.5) and the monotonicity of the principal eigenvalue with respect to the domain that

$$\lambda = \sigma_1^D[-\Delta + a(x)u_0^p] < \sigma_1^{D_{0,1}^1} .$$

Thus, (2.2) is necessary for the existence of a classical positive solution of (2.1). To show the sufficiency of (2.2) we use the method of sub and supersolutions. Assume (2.2) and let φ denote the principal eigenfunction associated with σ_1^D. Then, it is easily seen that for $\varepsilon > 0$ small enough $\varepsilon\varphi$ is a positive subsolution of (2.1). To complete the proof of the existence it remains to construct a supersolution $\overline{u} \geq \varepsilon\varphi$. For $\delta > 0$ sufficiently small, consider the δ-neighborhood of D_0 in D

$$D_0^\delta := \{\, x \in D \,:\, \text{dist }(x, D_0) < \delta \,\} .$$

Then,

$$D_0^\delta = \bigcup_{i=1}^{m} \bigcup_{j=1}^{n_i} D_{0,j}^{i,\delta} , \tag{2.6}$$

where

$$D_{0,j}^{i,\delta} := \{\, x \in D \,:\, \text{dist }(x, D_{0,j}^i) < \delta \,\} .$$

Thanks to (H4), $\overline{D}_{0,j}^i \cap \overline{D}_{0,\hat{j}}^{\hat{i}} = \emptyset$ if $(i,j) \neq (\hat{i},\hat{j})$, and hence, $\delta > 0$ can be chosen sufficiently small so that

$$\overline{D}_{0,j}^{i,\delta} \cap \overline{D}_{0,\hat{j}}^{\hat{i},\delta} = \emptyset \qquad \text{if } \ (i,j) \neq (\hat{i},\hat{j}) . \tag{2.7}$$

For each $1 \leq i \leq m$ and $1 \leq j \leq n_i$ the component $D_{0,j}^i$ is a proper subdomain of D, since $D_+ \neq \emptyset$. Thus, $\partial D_{0,j}^i \cap D \neq \emptyset$ and hence, $D_{0,j}^i$ is a proper subdomain of $D_{0,j}^{i,\delta/2}$ and $D_{0,j}^{i,\delta/2}$ is a proper subdomain of $D_{0,j}^{i,\delta}$. Therefore, by the monotonicity of the principal eigenvalue with respect to the domain,

$$\sigma_1^{D_{0,j}^{i,\delta}} < \sigma_1^{D_{0,j}^{i,\delta/2}} < \sigma_1^{D_{0,j}^{i}}, \qquad 1 \le i \le m\,, \;\; 1 \le j \le n_i\,. \tag{2.8}$$

Thanks to (H1-2), if Γ_0 and Γ are components of ∂D_0 and ∂D, respectively, such that $\Gamma_0 \cap \Gamma \neq \emptyset$, then $\Gamma_0 = \Gamma$. Thus, it follows from (2.6) and (2.7) that

$$\partial D_0^\delta \setminus \partial D \subset D_+$$

and that each $D_{0,j}^{i,\delta}$ can be obtained from $D_{0,j}^{i}$ through by an holomorphic transformation, (cf. [LS98, Theorem 3.2]). Hence,

$$\lim_{\delta \downarrow 0} \sigma_1^{D_{0,j}^{i,\delta}} = \sigma_1^{D_{0,j}^{i}} \qquad 1 \le i \le m\,, \;\; 1 \le j \le n_i\,. \tag{2.9}$$

Therefore, by (1.3), (2.2), (2.8) and (2.9), for each $\delta > 0$ sufficiently small we have

$$\lambda < \sigma_1^{D_{0,j}^{1,\delta}} < \sigma_1^{D_{0,\hat{j}}^{i,\delta}}, \qquad 1 \le j \le n_1\,, \;\; 2 \le i \le m\,, \;\; 1 \le \hat{j} \le n_i\,. \tag{2.10}$$

Now, for each $1 \le i \le m$ and $1 \le j \le n_i$, we denote by $\varphi_{0,j}^{i,\delta} \gg 0$ the principal eigenfunction associated with $\sigma_1^{D_{0,j}^{i,\delta}}$, and consider the function

$$\Phi(x) := \begin{cases} \varphi_{0,j}^{i,\delta}(x) & \text{if } x \in D_{0,j}^{i,\delta/2} \text{ for some } 1 \le i \le m\,, \;\; 1 \le j \le n_i\,, \\[2ex] \psi(x) & \text{if } x \in \overline{D} \setminus D_0^{\delta/2}\,, \end{cases} \tag{2.11}$$

where $\psi(x)$ is any regular extension of

$$\bigcup_{i=1}^{m} \bigcup_{j=1}^{n_i} \varphi_{0,j}^{i,\delta}$$

outside $D_0^{\delta/2}$ which is positive and bounded away from zero in $\overline{D} \setminus D_0^{\delta/2}$. We claim that if $\kappa > 1$ is sufficiently large, then the function

$$\overline{u} := \kappa\Phi$$

provides us with a supersolution of (2.1) satisfying $\overline{u} \geq \varepsilon\varphi$. Indeed, in the set

$$\overline{D} \setminus D_0^{\delta/2} \subset \overline{D}_+$$

the function ψ is positive and bounded away from zero and, thanks to (H2), $a(x)$ is positive and bounded away from zero as well. Thus, if κ is sufficiently large

$$-\Delta\psi \geq \lambda\,\psi - a(x)\,\kappa^p\psi^{p+1} \qquad \text{in} \quad \overline{D} \setminus D_0^{\delta/2}\,.$$

Moreover, thanks to (2.10), in each of the components of $D_0^{\delta/2}$, e.g. $D_{0,j}^{i,\delta/2}$, we have

$$-\Delta(\kappa\Phi) = \kappa\,\sigma_1^{D_{0,j}^{i,\delta}}\varphi_{0,j}^{i,\delta} > \lambda\kappa\varphi_{0,j}^{i,\delta} - a(x)(\kappa\varphi_{0,j}^{i,\delta})^{p+1}\,,$$

and therefore, $k\Phi$ provides us with a positive supersolution of (2.1). By construction, on any component Γ of ∂D we have $\Phi = 0$ and $\frac{\partial\Phi}{\partial n}(x) < 0$ for all $x \in \Gamma$, where n is the outward unit normal, if $\Gamma \notin \partial D_+$, while $\Phi(x) > 0$ for all $x \in \Gamma$ if $\Gamma \in \partial D_+ \cap \partial D$. Therefore, κ can be chosen sufficiently large so that $\kappa\Phi > \varepsilon\varphi$. This completes the proof of the existence of a classical positive solution.

To show the uniqueness assume (2.2) and let u_0, u_1 be two positive solutions of (2.1). Then,

$$(-\Delta + a(x)\frac{u_0^{p+1} - u_1^{p+1}}{u_0 - u_1} - \lambda)(u_0 - u_1) = 0\,. \tag{2.12}$$

Moreover, since $u_0 \neq u_1$ we have

$$\frac{u_0^{p+1} - u_1^{p+1}}{u_0 - u_1} > u_0^p$$

and hence, we find from (2.5) that

$$\sigma_1^D[-\Delta + a(x)\frac{u_0^{p+1} - u_1^{p+1}}{u_0 - u_1} - \lambda] > \sigma_1^D[-\Delta + a(x)u_0^p - \lambda] = 0\,.$$

Therefore, the strong maximum principle applied to (2.12) implies $u_0 = u_1$, which is a contradiction. This completes the proof of Part (i).

Relation (2.3) in Part (ii) follows from the fact that $\lambda = \sigma_1^D$ is a bifurcation value to positive solutions of (2.1) from $u = 0$, [CR71], and the proof of Part (iii) can be easily accomplished by using the strong maximum principle and the implicit function theorem, as in [FKLM96] and [GGLS98]. So, we omit the details.

We now show (2.4). Differentiating (2.1) with respect to λ gives

$$(-\Delta + (p+1)a(x)[\theta_{[\lambda,a]}^D]^p - \lambda)\frac{d\theta_{[\lambda,a]}^D}{d\lambda} = \theta_{[\lambda,a]}^D \qquad \text{in } \; D\,,$$

and $\frac{d\theta_{[\lambda,a]}^D}{d\lambda} = 0$ on ∂D. Moreover, $a = 0$ in the open set

$$D_0^1 := \bigcup_{j=1}^{n_1} D_{0,j}^1$$

and hence,

$$(-\Delta - \lambda)\frac{d\theta_{[\lambda,a]}^D}{d\lambda} = \theta_{[\lambda,a]}^D \qquad \text{in } \; D_0^1\,.$$

Now, pick

$$\hat\lambda \in (\sigma_1^D, \sigma_1^{D_{0,1}^1})$$

and consider $c > 0$ such that for each $1 \le j \le n_1$,

$$\theta_{[\hat\lambda,a]}^D > c\varphi_{0,j}^1 \qquad \text{in } \;\; D_{0,j}^1\,.$$

Recall that $\varphi_{0,j}^1$ is the principal eigenfunction associated with $\sigma_1^{D_{0,j}^1}$. Then, thanks to Part (iii), for each $\lambda \in (\hat{\lambda}, \sigma_1^{D_{0,1}^1})$ we have

$$\theta_{[\lambda,a]}^D > \theta_{[\hat{\lambda},a]}^D > c\varphi_{0,j}^1 \qquad \text{in} \quad D_{0,j}^1\,, \quad 1 \leq j \leq n_1\,.$$

Moreover, for each $\lambda \in (\hat{\lambda}, \sigma_1^{D_{0,1}^1})$ and $1 \leq j \leq n_1$ the operator $-\Delta - \lambda$ satisfies the strong maximum principle in $D_{0,j}^1$ and hence,

$$\frac{d\theta_{[\lambda,a]}^D}{d\lambda} > c\,(-\Delta - \lambda)^{-1}\varphi_{0,j}^1 = \frac{c}{\sigma_1^{D_{0,1}^1} - \lambda}\varphi_{0,j}^1 \qquad \text{in} \;\; D_{0,j}^1\,,$$

since by assumption (H4) we have $\sigma_1^{D_{0,j}^1} = \sigma_1^{D_{0,1}^1}$ for each $1 \leq j \leq n_1$. Moreover, for each $1 \leq j \leq n_1$, the function $\varphi_{0,j}^1$ is bounded away from zero on any compact subset of $D_{0,j}^1$ and hence,

$$\lim_{\lambda \uparrow \sigma_1^{D_{0,1}^1}} \frac{d\theta_{[\lambda,a]}^D}{d\lambda} = \infty \qquad \text{uniformly in compact subsets of} \;\; D_0^1\,.$$

Therefore,

$$\lim_{\lambda \uparrow \sigma_1^{D_{0,1}^1}} \theta_{[\lambda,a]}^D = \infty \qquad \text{uniformly in compact subsets of} \;\; D_0^1\,.$$

To show that

$$\lim_{\lambda \uparrow \sigma_1^{D_{0,1}^1}} \theta_{[\lambda,a]}^D(x) = \infty \qquad \text{for all} \quad x \in \partial D_0^1 \setminus \partial D\,, \tag{2.13}$$

we consider $\delta > 0$ sufficiently small, pick up λ satisfying

$$\sigma_1^{D_{0,j}^{1,\delta}} < \sigma_1^{D_{0,j}^{1,\delta/2}} < \lambda < \sigma_1^{D_{0,j}^1}\,, \qquad 1 \leq j \leq n_1\,,$$

and introduce the function $u_\delta \in C(\overline{D})$ defined by

$$u_\delta(x) = \begin{cases} C\,\varphi_{0,j}^{1,\delta}(x) & \text{if } x \in \overline{D}_{0,j}^{1,\delta/2} \text{ for some } 1 \le j \le n_1\,, \\ 0 & \text{if } x \in \overline{D} \setminus \cup_{j=1}^{n_1} D_{0,j}^{1,\delta/2}\,, \end{cases} \tag{2.14}$$

where $C > 0$ is a positive constant. Then, the argument of the proof of Theorem 4.3 in [LS98] can be easily adapted to show that under condition (H3) there exists $C = C(\delta) > 0$ such that u_δ is a subsolution of (2.1) satisfying $\lim_{\delta \downarrow 0} u_\delta(x) = \infty$ for each $x \in \partial D_0^1 \setminus \partial D$. From the strong maximum principle it is easily seen that $u_\delta \le \theta_{[\lambda,a]}^D$ and hence, (2.13) holds. The uniform divergence in ∂D_0^1 follows from the point-wise monotonicity in λ as an immediate consequence from Dini's theorem.
Part (iv) follows readily from the following estimate

$$0 = \sigma_1^D[-\Delta + a(x)[\theta_{[\lambda,a]}^D]^p - \lambda] < \sigma_1^D[-\Delta + (p+1)a(x)[\theta_{[\lambda,a]}^D]^p - \lambda] < \sigma_1^{D_{0,1}^1} - \lambda\,.$$

This completes the proof. □

The following result will be crucial in the next section to show the stabilization in $D \setminus \overline{D}_0^1$ of the positive solutions, as $\lambda \uparrow \sigma_1^{D_{0,1}^1}$.

Theorem 2.5. *Assume that $a(x)$ satisfies $(H1-4)$ and that there are $q \ge 1$ components $\Gamma_{+,j}$, $1 \le j \le q$, of ∂D such that $a(x)$ is positive and bounded away from zero on each $\Gamma_{+,j}$. Given q arbitrary positive constants $\alpha_j > 0$, $1 \le j \le q$, consider the boundary value problem*

$$\begin{cases} -\Delta u = \lambda u - a(x)|u|^p u \quad \text{in } \; D\,, \\ u|_{\Gamma_{+,j}} = \alpha_j > 0\,, \qquad 1 \le j \le q\,, \\ u|_{\partial D \setminus \cup_{j=1}^q \Gamma_{+,j}} = 0\,. \end{cases} \tag{2.15}$$

Then, the following assertions are true:

(i) *If (2.15) possesses a classical positive solution, then $\lambda < \sigma_1^{D_{0,1}^1}$. Moreover, under condition (2.2) it possesses a unique classical pos-*

itive solution which will be denoted by $\Theta^D_{[\lambda,a,\Gamma_+,\alpha]}$, *where* $\Gamma_+ = (\Gamma_{+,1},...,\Gamma_{+,q})$ *and* $\alpha = (\alpha_1,...,\alpha_q)$.

(ii) *The map* $(\sigma_1^D, \sigma_1^{D^1_{0,1}}) \to C(\overline{D})$, $\lambda \to \Theta^D_{[\lambda,a,\Gamma_+,\alpha]}$ *is point-wise increasing, as well as the map* $(0,\infty)^q \to C(\overline{D})$, $\alpha = (\alpha_1,...,\alpha_q) \to \Theta^D_{[\lambda,a,\Gamma_+,\alpha]}$. *Moreover,*

$$\lim_{\lambda\uparrow\sigma_1^{D^1_{0,1}}} \Theta^D_{[\lambda,a,\Gamma_+,\alpha]} = \infty \qquad \textit{uniformly in } \bigcup_{j=1}^{n_1} \overline{D}^1_{0,j} \setminus \partial D\,. \tag{2.16}$$

Proof

(i) Assume that (2.15) possesses a positive solution, say u_0. Then,

$$(-\Delta + a(x)u_0^p - \lambda)u_0 = 0 \quad \text{in} \;\; D\,,$$

and $u|_{\partial D} > 0$. Thus, u_0 a is positive strict supersolution of $-\Delta + a(x)u_0^p - \lambda$ in D under homogeneous Dirichlet boundary conditions and hence, it follows from the characterization of the strong maximum principle found in [LM94] (cf. [Lo96] and [AL98] as well) that

$$\sigma_1^D[-\Delta + a(x)u_0^p - \lambda] > 0\,. \tag{2.17}$$

Therefore,

$$\lambda < \sigma_1^D[-\Delta + a(x)u_0^p] < \sigma_1^{D^1_{0,1}}\,,$$

since $a = 0$ in $D^1_{0,1}$. To show that under condition (2.2) the problem (2.15) possesses a unique positive solution we use the method of sub and supersolutions as in the proof of Theorem 2.4. Assume (2.2). As in the proof of Theorem 2.4, $\varepsilon\varphi > 0$ provides us with a positive subsolution if $\varepsilon > 0$ is sufficiently small. We now show that the function $\overline{u} = \kappa\Phi$, where Φ is given by (2.11) and κ is sufficiently large, provides us with a positive supersolution such that $\overline{u} > \varepsilon\varphi$. It suffices to check that if κ is sufficiently large, then

$\kappa\psi|_{\Gamma_{+,j}} > \alpha_j$ for each $1 \le j \le q$. This is true, since $\Gamma_{+,j} \subset \overline{D} \setminus D_0^{\delta/2}$ and by construction ψ is positive and bounded away from zero in $\overline{D} \setminus D_0^{\delta/2}$. The uniqueness of the positive solution follows from (2.17) adapting the corresponding argument of the proof of Theorem 2.4. This completes the proof of Part (i).

(ii) Let λ_1, $\lambda_2 \in (\sigma_1^D, \sigma_1^{D_{0,1}^1})$ such that $\lambda_1 < \lambda_2$. Then, setting

$$\Theta_i := \Theta^D_{[\lambda_i, a, \Gamma_+, \alpha]}\,, \qquad i = 1\,,\, 2\,,$$

we find from their definition that

$$(-\Delta + a(x)\frac{\Theta_2^{p+1} - \Theta_1^{p+1}}{\Theta_2 - \Theta_1} - \lambda_1)(\Theta_2 - \Theta_1) > 0 \qquad \text{in} \quad D \tag{2.18}$$

and that $\Theta_2 - \Theta_1 = 0$ on ∂D. Moreover,

$$\frac{\Theta_2^{p+1} - \Theta_1^{p+1}}{\Theta_2 - \Theta_1} \ge \Theta_1^p$$

and hence, using (2.17) gives

$$\sigma_1^D[-\Delta + a(x)\frac{\Theta_2^{p+1} - \Theta_1^{p+1}}{\Theta_2 - \Theta_1} - \lambda_1] \ge \sigma_1^D[-\Delta + a(x)\Theta_1^p - \lambda_1] > 0\,.$$

Thus, the operator

$$-\Delta + a(x)\frac{\Theta_2^{p+1} - \Theta_1^{p+1}}{\Theta_2 - \Theta_1} - \lambda_1$$

under homogeneous Dirichlet boundary conditions in D satisfies the strong maximum principle and therefore, it follows from (2.18) that

$$\Theta_2 - \Theta_1 \gg 0\,.$$

This completes the proof of the monotonicity in λ. This argument can be easily adapted to get the monotonicity in α.

Relation (2.16) follows easily from Theorem 2.4 taking into account that

$$\Theta^D_{[\lambda,a,\Gamma_+,\alpha]} \gg \theta^D_{[\lambda,a]}\,.$$

This completes the proof. $\square$

I.3. Stabilization and metasolutions

Thanks to Theorem 2.4, the problem (1.1) possesses a classical positive solution if, and only if,

$$\sigma_1^{\Omega} < \lambda < \lambda_1 := \sigma_1^{\Omega^1_{0,1}}\,. \tag{3.1}$$

Moreover, it is unique if it exists, and if we denote it by $\theta^{\Omega}_{[\lambda,a]}$, then

$$\lim_{\lambda\uparrow\lambda_1} \theta^{\Omega}_{[\lambda,a]} = \infty \qquad \text{uniformly in} \quad \overline{\Omega}{}^1_0 \setminus \partial\Omega\,, \tag{3.2}$$

where

$$\Omega^1_0 := \bigcup_{j=1}^{n_1} \Omega^1_{0,j}\,. \tag{3.3}$$

The following result shows the stabilization of $\theta^{\Omega}_{[\lambda,a]}$ in $\Omega\setminus\overline{\Omega}{}^1_0$ as $\lambda\uparrow\lambda_1$ providing us with a substantial generalization of Theorem 6.1 in [GGLS98], where the special case when $\Omega\setminus\overline{\Omega}_+$ is connected ($n_1 = m = 1$) was dealt with.

Theorem 3.1. *Assume that $a(x)$ satisfies $(H1-4)$. Then, the following assertions are true:*

(i) *Set*

$$\Omega_1 := \Omega \setminus \overline{\Omega}_0^1 . \tag{3.4}$$

Then, for each $x \in \Omega_1$ the point-wise limit

$$\Xi_{[\lambda_1,a]}^{\Omega_1}(x) := \lim_{\lambda \uparrow \lambda_1} \theta_{[\lambda,a]}^{\Omega}(x) < \infty \tag{3.5}$$

is well defined.

(ii) *For all $p > 1$ and $\nu \in (0,1)$ the function $\Xi_{[\lambda_1,a]}^{\Omega_1}$ defined by (3.5) satisfies*

$$\Xi_{[\lambda_1,a]}^{\Omega_1} \in W_{\ell oc}^{2,p}(\Omega_1) \cap C^{1,\nu}(\Omega_1)$$

and it provides us with a weak positive solution of

$$-\Delta u = \lambda_1 u - a(x) u^{p+1} \qquad in \quad \Omega_1 .$$

Moreover,

$$\lim_{\lambda \uparrow \lambda_1} \| \theta_{[\lambda,a]}^{\Omega} - \Xi_{[\lambda_1,a]}^{\Omega_1} \|_{C^{1,\nu}(\Omega_1)} = 0 . \tag{3.6}$$

(iii) *The following relations hold*

$$\Xi_{[\lambda_1,a]}^{\Omega_1}|_{\partial\Omega\cap\partial\Omega_1} = 0 , \qquad \lim_{dist\ (x,\partial\Omega_1\setminus\partial\Omega)\downarrow 0} \Xi_{[\lambda_1,a]}^{\Omega_1}(x) = \infty , \tag{3.7}$$

and therefore, $\Xi_{[\lambda_1,a]}^{\Omega_1}$ defines a weak positive solution of

$$\begin{cases} -\Delta u = \lambda_1 u - a(x) u^{p+1} \qquad in \quad \Omega_1 , \\ u|_{\partial\Omega\cap\partial\Omega_1} = 0 , \quad u|_{\partial\Omega_1\setminus\partial\Omega} = \infty . \end{cases} \tag{3.8}$$

In fact, $\Xi_{[\lambda_1,a]}^{\Omega_1}$ provides us with the minimal positive solution of (3.8).

Proof

(i) For each $\delta > 0$ sufficiently small we consider the δ-neighborhoods

$$\Omega_0^{1,\delta} := \bigcup_{j=1}^{n_1} \Omega_{0,j}^{1,\delta}, \quad \Omega_{0,j}^{1,\delta} := \{ x \in \Omega \,:\, d(x, \Omega_{0,j}^1) < \delta \}, \quad 1 \leq j \leq n_1 .$$

As in the proof of Theorem 2.4 we have that $\overline{\Omega}_{0,j}^{1,\delta} \cap \overline{\Omega}_{0,\hat{j}}^{1,\delta} = \emptyset$ if $j \neq \hat{j}$ and that $\Omega_{0,j}^1$ is a proper subdomain of $\Omega_{0,j}^{1,\delta}$ for each $1 \leq j \leq n_1$. Moreover, if Γ^δ is any component of $\partial\Omega_{0,j}^{1,\delta}$ with $\Gamma^\delta \cap \Omega \neq \emptyset$, then $\Gamma^\delta \subset \Omega_+$. By our assumptions on $a(x)$, the remaining components of $\Omega_{0,j}^{1,\delta}$ must be components of $\partial\Omega$. The proof of Part (i) will follow from Theorem 2.5 applied in the domain D defined by

$$D := \Omega \setminus \overline{\Omega}_0^{1,\delta} \subset \Omega_1 = \Omega \setminus \overline{\Omega}_0^1 .$$

It is straightforward to see that D satisfies all the requirements of Section 2. By construction, the components of D where $a(x)$ vanishes which provide us with the lowest principal eigenvalue are $\Omega_{0,j}^2$, $1 \leq j \leq n_2$. Hence,

$$D_{0,1}^1 = \Omega_{0,1}^2 .$$

Let $\Gamma_{+,j}$, $1 \leq j \leq q$, denote the components of ∂D in Ω_+. Fix

$$x_0 \in \bigcup_{j=1}^{q} \Gamma_{+,j}$$

and consider a ball B centered at x_0 such that $\overline{B} \subset \Omega_+$. Pick up $\lambda \in (\sigma_1^\Omega, \lambda_1)$. Then,

$$-\Delta\theta_{[\lambda,a]}^\Omega = \lambda\theta_{[\lambda,a]}^\Omega - a(\theta_{[\lambda,a]}^\Omega)^{p+1} < \lambda_1\theta_{[\lambda,a]}^\Omega - (\inf_{\overline{B}} a)(\theta_{[\lambda,a]}^\Omega)^{p+1} ,$$

and therefore, a rather standard comparison together with the strong maximum principle imply

$$\theta_{[\lambda,a]}^\Omega < \Psi_{[\lambda_1, \inf_{\overline{B}} a]}^B \qquad \text{in } B , \tag{3.9}$$

where $\Psi^B_{[\lambda_1,\inf_{\overline{B}} a]}$ is the unique positive solution of

$$\begin{cases} -\Delta\psi = \lambda_1\psi - (\inf_{\overline{B}} a)\psi^{p+1} \qquad \text{in } \ B\,, \\ \\ \psi|_{\partial B} = \infty\,, \end{cases} \tag{3.10}$$

whose existence and uniqueness is guaranteed by [MV97, Corollary 2.3]. Note that $\inf_{\overline{B}} a > 0$, since $\overline{B} \subset \Omega_+$. This shows that the family of positive solutions $\theta^{\Omega}_{[\lambda,a]}$, $\sigma^{\Omega}_1 < \lambda < \lambda_1$, is uniformly bounded above in a neighborhood of each point $x_0 \in \cup_{j=1}^q \Gamma_{+,j}$. Therefore, by a rather standard compactness argument, there exists $\alpha > 0$ such that

$$\theta^{\Omega}_{[\lambda,a]} \leq \alpha \qquad \text{in } \ \bigcup_{j=1}^{q} \Gamma_{+,j} \qquad \text{for all } \ \lambda \in (\sigma^{\Omega}_1, \lambda_1)\,. \tag{3.11}$$

Now, we set

$$\lambda_2 := \sigma_1^{\Omega^2_{0,1}} > \lambda_1 = \sigma_1^{\Omega^1_{0,1}}$$

and for each $\lambda \in (\sigma^{\Omega}_1, \lambda_2)$ consider the boundary value problem

$$\begin{cases} -\Delta u = \lambda u - a u^{p+1} \qquad \text{in } \ D\,, \\ \\ u|_{\partial D\cap\Omega_+} = \alpha\,, \quad u|_{\partial D\cap\partial\Omega} = 0\,, \end{cases} \tag{3.12}$$

where $\alpha > 0$ is any bound satisfying (3.11). Thanks to Theorem 2.5, (3.12) possesses a unique positive solution, denoted by $\Theta^D_{[\lambda,a,\partial D\setminus\partial\Omega,\alpha]}$. Moreover, by (3.11) for each $\lambda \in (\sigma^{\Omega}_1, \lambda_1)$ the function $\theta^{\Omega}_{[\lambda,a]}$ is a positive subsolution of (3.12) in D. Thus, the maximum principle implies

$$\theta^{\Omega}_{[\lambda,a]} \leq \Theta^D_{[\lambda,a,\partial D\setminus\partial\Omega,\alpha]} \qquad \text{in } \ D\,, \qquad \forall\, \lambda \in (\sigma^{\Omega}_1, \lambda_1)\,. \tag{3.13}$$

Therefore,

$$\lim_{\lambda\uparrow\lambda_1} \theta^{\Omega}_{[\lambda,a]} \leq \Theta^D_{[\lambda_1,a,\partial D\setminus\partial\Omega,\alpha]} \qquad \text{in } \ D\,.$$

Since this estimate is valid for all $\delta > 0$ sufficiently small, the proof of Part (i) is completed.

(ii) Let $O_1 \subset O$ two open subsets of Ω_1 such that $\overline{O}_1 \subset O$ and $\overline{O} \subset \Omega_1$. If in the proof of Part (i) $\delta > 0$ is taken to be sufficiently small, then $\overline{O} \subset D$ and hence, (3.13) gives

$$\theta^{\Omega}_{[\lambda,a]} \leq \Theta^{D}_{[\lambda_1,a,\partial D \setminus \partial\Omega,\alpha]} \qquad \text{in} \quad \overline{O} \tag{3.14}$$

for all $\lambda \in (\sigma_1^\Omega, \lambda_1)$. By the L^p-estimates of Agmon, Douglis & Nirenberg, for each $p > 1$ there exists a constant $C_1 = C(p, O_1)$ such that

$$\|\theta^{\Omega}_{[\lambda,a]}\|_{W^{2,p}(O_1)} \leq C_1 \qquad \lambda \in (\sigma_1^\Omega, \lambda_1)\,.$$

Thus, thanks to Morrey's embedding theorem, for each $\nu \in (0,1)$ there exists a constant $C_2 = C(\nu, O_1)$ such that

$$\|\theta^{\Omega}_{[\lambda,a]}\|_{C^{1,\nu}(\overline{O}_1)} \leq C_2 \qquad \lambda \in (\sigma_1^\Omega, \lambda_1)\,.$$

Now, a rather standard compactness argument together with the uniqueness of the point-wise limit (3.5) shows that

$$\lim_{\lambda \uparrow \lambda_1} \|\theta^{\Omega}_{[\lambda,a]} - \Xi^{\Omega_1}_{[\lambda_1,a]}\|_{C^{1,\nu}(\overline{O}_1)} = 0\,,$$

and therefore,

$$\lim_{\lambda \uparrow \lambda_1} \|\theta^{\Omega}_{[\lambda,a]} - \Xi^{\Omega_1}_{[\lambda_1,a]}\|_{C^{1,\nu}(\Omega_1)} = 0\,.$$

In particular, $\Xi^{\Omega_1}_{[\lambda_1,a]}$ is a weak solution of

$$-\Delta u = \lambda_1 u - a(x) u^{p+1} \qquad \text{in} \quad \Omega_1\,.$$

This completes the proof of Part (ii). Part (iii) follows straight away from (3.2) and Parts (i), (ii). $\square$

Based on Theorem 3.1 we introduce the following concepts.

Definition 3.2. *Let Ω_1 be a subdomain of class C^2 of Ω such that if Γ is a component of $\partial\Omega_1$, then either $\Gamma \subset \Omega$ or Γ is a component of $\partial\Omega$. Assume that $u \in W^{2,p}(\Omega_1)$ for all $p > 1$. It is said that u is a* **large classical solution** *of*

$$\begin{cases} -\Delta u = \lambda u - a u^{p+1} \qquad \text{in } \ \Omega_1 \,, \\[2ex] u|_{\partial\Omega_1 \cap \partial\Omega} = 0 \,, \qquad u|_{\partial\Omega_1 \setminus \partial\Omega} = \infty \,, \end{cases} \tag{3.15}$$

if it solves the differential equation, it satisfies $u = 0$ on $\partial\Omega_1 \cap \partial\Omega$ and

$$\lim_{dist\,(x, \partial\Omega_1 \setminus \partial\Omega) \downarrow 0} u(x) = \infty \,.$$

Definition 3.3. *A function $u : \Omega \to [0, \infty]$ is said to be a* **metasolution of order k** *of* (1.1) *if the following conditions are satisfied:*

(i) *There exists a subdomain $\Omega_k \subset \Omega$ such that $u|_{\Omega_k}$ is a large classical solution of*

$$\begin{cases} -\Delta u = \lambda u - a(x) u^{p+1} \qquad \text{in } \ \ \Omega_k \,, \\[2ex] u|_{\partial\Omega \cap \partial\Omega_k} = 0 \,, \quad u|_{\partial\Omega_k \setminus \partial\Omega} = \infty \,. \end{cases} \tag{3.16}$$

(ii) *$u = \infty$ in $(\Omega \setminus \overline{\Omega}_k) \cup (\partial\Omega_k \setminus \partial\Omega)$.*

(iii) *$\Omega \setminus \overline{\Omega}_k$ has a finite number of components $\Omega^i_{0,j}$, $1 \le i \le k$, $1 \le j \le n_i$, such that*

$$\sigma_1^{\Omega^i_{0,1}} = \cdots = \sigma_1^{\Omega^i_{0,n_i}} \qquad 1 \le i \le k \,,$$

$$\sigma_1^{\Omega^1_{0,1}} < \sigma_1^{\Omega^2_{0,1}} < \cdots < \sigma_1^{\Omega^k_{0,1}} \,.$$

More precisely, under the previous conditions it will be said that u is a metasolution of order k supported in Ω_k.

Using these concepts, the following result is an immediate consequence from Theorem 3.1.

Corollary 3.4. *Assume that* $a(x)$ *satisfies* $(H1-4)$. *Then, the function* $\mathcal{M}^{\Omega}_{[\lambda_1,a]}$ *defined by*

$$\mathcal{M}^{\Omega}_{[\lambda_1,a]} := \begin{cases} \Xi^{\Omega_1}_{[\lambda_1,a]} & \text{in} \quad \Omega_1 \cup (\partial\Omega_1 \cap \partial\Omega)\,, \\ \infty & \text{in} \quad (\Omega \setminus \overline{\Omega}_1) \cup (\partial\Omega_1 \cap \Omega)\,, \end{cases} \tag{3.17}$$

is a metasolution of order one of (1.1) *for* $\lambda = \lambda_1 (= \sigma_1^{\Omega^1_{0,1}})$ *(recall that* Ω_1 *was defined in* (3.4)*).*

The following result provides us with a sufficient condition for the existence of metasolutions of arbitrary order $k \le m$, where m is the integer arising in the statement of (H4).

Theorem 3.5. *Assume that* $a(x)$ *satisfies* $(H1-4)$, *fix* $k \in \{1, ..., m\}$ *and set*

$$\Omega_k := \Omega \setminus \bigcup_{i=1}^{k} \bigcup_{j=1}^{n_i} \overline{\Omega}^i_{0,j}\,. \tag{3.18}$$

Then, for each λ *satisfying*

$$\sigma_1^{\Omega^k_{0,1}} \le \lambda < \sigma_1^{\Omega^{k+1}_{0,1}} \tag{3.19}$$

the problem (1.1) *possesses a metasolution of order* k *supported in* Ω_k. *If* $k = m$, *then* (3.19) *should read as*

$$\sigma_1^{\Omega^m_{0,1}} \le \lambda\,.$$

The proof of this result will be based upon Corollary 3.4 and the following lemma.

Lemma 3.6. *Let* D *be a domain of class* C^2, *consider* $\lambda > \sigma_1^D$ *and pick up a finite number of components* Γ_j, $1 \le j \le q$, *of* ∂D. *Then, there exists a subdomain* $\hat{D}$ *of* D *such that*

$$\lambda = \sigma_1^{\hat{D}} \tag{3.20}$$

and each Γ_j, $1 \leq j \leq q$, *is a component of* $\partial \hat{D}$.

Proof

We can make a hole in D in such a way that the remaining portion of D has an arbitrarily small Lebesgue measure and hence, thanks to Faber-Krahn inequality, an arbitrarily large principal eigenvalue, [Lo96]. Since the principal eigenvalue varies continuously with the support domain and we are assuming that $\lambda > \sigma_1^D$, the hole can be taken so that the remaining portion of D, say $\hat{D}$, satisfy (3.20). This completes the proof, since the components of $\partial \hat{D}$ are the components of ∂D plus the boundary of the hole. $\square$

Proof of Theorem 3.5

Assume

$$\sigma_1^{\Omega_{0,1}^k} < \lambda < \sigma_1^{\Omega_{0,1}^{k+1}} . \tag{3.21}$$

Then, thanks to Lemma 3.6 for each $1 \leq i \leq k$ and $1 \leq j \leq n_i$ there exists a subdomain $\hat{\Omega}_{0,j}^i$ of $\Omega_{0,j}^i$ such that

$$\lambda = \sigma_1^{\hat{\Omega}_{0,j}^i}$$

and any component of $\partial\Omega_k \cap \partial\Omega_{0,j}^i$ is a component of $\partial\hat{\Omega}_{0,j}^i$ as well. Now, consider the new domain

$$\hat{\Omega} := \Omega_k \cup \bigcup_{i=1}^{k} \bigcup_{j=1}^{n_i} [\hat{\Omega}_{0,j}^i \cup (\partial\Omega_k \cap \partial\Omega_{0,j}^i)] . \tag{3.22}$$

By construction, $a(x)$ satisfies the same requirements with respect to $\hat{\Omega}$ as it satisfies with respect to Ω, but now its corresponding $\Omega_{0,j}^1$ components are given by the $\hat{\Omega}_{0,j}^i$'s. Applying Theorem 3.1 and Corollary 3.4 to the problem in $\hat{\Omega}$ completes the proof if (3.21) holds. If, instead of (3.21), the following holds

$$\lambda = \sigma_1^{\Omega_{0,1}^k}$$

then, the same argument works out. Now we only have to enshort the $\Omega^i_{0,j}$'s with $i<k$. This completes the proof. $\square$

The following result ascertains the point-wise behavior of the metasolutions of order k supported in Ω_k as $\lambda \uparrow \sigma_1^{\Omega^{k+1}_{0,1}}$. They stabilize in Ω_+ while they grow to infinity in $\cup_{j=1}^{n_{k+1}} \overline{\Omega}^{k+1}_{0,j} \setminus \partial\Omega$. This shows that the metasolutions of order $m-1$ are point-wise convergent to some metasolution of order m as $\lambda \uparrow \sigma_1^{\Omega^m_{0,1}}$. If $k \le m-2$ we can not be sure of the validity of the corresponding result since we were not able to show that these metasolutions stabilize in $\cup_{i=k+2}^m \cup_{j=1}^{n_i} \overline{\Omega}^i_{0,j}$. The main difficulty coming from the fact that we do not know whether or not the metasolutions are unique. Anyway, we conjecture that this is the case.

Theorem 3.7. *Assume $1 \le k \le m-1$ and consider a sequence*

$$\lambda_n \in (\sigma_1^{\Omega^k_{0,1}}, \sigma_1^{\Omega^{k+1}_{0,1}}), \qquad n \ge 1,$$

such that

$$\lim_{n\to\infty} \lambda_n = \sigma_1^{\Omega^{k+1}_{0,1}}.$$

For each $n \ge 1$, let u_n be a metasolution of order k of (1.1) *corresponding with $\lambda = \lambda_n$ supported in the Ω_k defined by* (3.18), *whose existence is guaranteed by Theorem 3.5. Then,*

$$\lim_{n\to\infty} u_n(x) = \infty \qquad \text{for all} \quad x \in \bigcup_{j=1}^{n_{k+1}} \overline{\Omega}^{k+1}_{0,j} \setminus \partial\Omega. \tag{3.23}$$

Moreover, if O_+ is any open subset of Ω_+ satisfying $\overline{O}_+ \subset \Omega_+$, then $u_n|_{\overline{O}_+}$, $n \ge 1$, are uniformly bounded in L^∞ and hence, there exists a subsequence of (λ_n, u_n), $n \ge 1$, relabeled again by n, and a weak solution of

$$-\Delta u = \sigma_1^{\Omega^{k+1}_{0,1}} u - a(x) u^{p+1} \qquad \text{in} \quad \Omega_+,$$

$w \in W^{2,p}_{\ell oc}(\Omega_+) \cap C^{1,\nu}(\Omega_+)$, $p > 1$, $0 < \nu < 1$, *such that*

$$\lim_{n\to\infty} \|u_n - w\|_{C^{1,\nu}(\Omega_+)} = 0$$

for each $\nu \in (0,1)$. *Therefore,* w *defines a metasolution of order* m *of* (1.1) *supported in* Ω_+ *if* $k = m - 1$.

Proof

By definition, for each $n \geq 1$ the function $u_n|_{\Omega_k}$ is a large classical solution of

$$\begin{cases} -\Delta u = \lambda_n u - a(x)u^{p+1} \qquad \text{in} \quad \Omega_k\,, \\ \\ \quad u|_{\partial\Omega\cap\partial\Omega_k} = 0\,, \quad u|_{\partial\Omega_k\setminus\partial\Omega} = \infty\,, \end{cases}$$

and in particular, $u_n|_{\Omega_k}$ is a positive strict supersolution of

$$\begin{cases} -\Delta u = \lambda_n u - a(x)u^{p+1} \qquad \text{in} \quad \Omega_k\,, \\ \\ u|_{\partial\Omega_k} = 0\,. \end{cases} \tag{3.24}$$

Thus,

$$u_n|_{\Omega_k} \geq \theta^{\Omega_k}_{[\lambda_n,a]}\,, \tag{3.25}$$

where $\theta^{\Omega_k}_{[\lambda_n,a]} \gg 0$ is the unique positive solution of (3.24), whose existence and uniqueness is guaranteed by Theorem 2.4(i). Thanks to Theorem 2.4(ii), we have

$$\lim_{n\to\infty} \theta^{\Omega_k}_{[\lambda_n,a]}(x) = \infty \qquad \text{for all} \quad x \in \bigcup_{j=1}^{n_{k+1}} \overline{\Omega}^{k+1}_{0,j} \setminus \partial\Omega_k\,. \tag{3.26}$$

Moreover, $u_n = \infty$ in $(\Omega \setminus \Omega_k) \cup (\partial\Omega_k \setminus \partial\Omega)$, since we are assuming that it is a metasolution supported in Ω_k. Thus, (3.25) and (3.26) show (3.23).

Now, let O_+ be an open subset of Ω_+ satisfying $\overline{O}_+ \subset \Omega_+$. If $\delta > 0$ is sufficiently small and O^δ_+ stands for the δ-neighborhood of O_+ we have that

$\overline{O}_+ \subset O_+^\delta$ and that $\overline{O}_+^\delta \subset \Omega_+$. Then, for each $n \geq 1$ the function $u_n|_{O_+^\delta}$ provides us with a subsolution of the problem

$$\begin{cases} -\Delta \psi = \sigma_1^{\Omega_{0,1}^{k+1}} \psi - (\min_{\overline{O}_+^\delta} a)\, \psi^{p+1} & \text{in} \quad O_+^\delta\,, \\ \psi|_{\partial O_+^\delta} = \infty\,, \end{cases} \tag{3.27}$$

and therefore,

$$u_n|_{O_+^\delta} \leq \Psi \qquad n \geq 1\,, \tag{3.28}$$

where Ψ is the unique positive solution of (3.27), whose existence and uniqueness is guaranteed by [MV97, Corollary 2.3]. By (3.28), there exists a constant $C > 0$ such that

$$\|u_n\|_{L^\infty(O_+)} \leq C \qquad n \geq 1\,.$$

The compactness argument of the proof of Theorem 3.1(i) completes the proof. $\square$

I.4. Two cases where condition (H3) is not needed

In this section we shall show that if $N = 1$ or $N \geq 2$ and (1.1) possesses radial symmetry, then condition (H3) is not necessary for the validity of the results in Sections 2, 3. It suffices to show the validity of relation (2.13) in the statement of Theorem 2.4, as this was the only property for which (H3) was needed.

Theorem 4.1. *Assume $N = 1$, $\Omega = (\alpha, \beta)$, $\alpha < \beta$, and $(H1)$, $(H2)$, $(H4)$ hold. Pick $j \in \{1, ..., n_1\}$ and let α_j, $\beta_j \in [\alpha, \beta]$, $\alpha_j < \beta_j$, such that $\Omega^1_{0,j} = (\alpha_j, \beta_j)$. Then, either $\alpha_j > \alpha$, or $\beta_j < \beta$. Moreover, if $\alpha_j > \alpha$, then*

$$\lim_{\lambda \uparrow \sigma_1^{\Omega^1_{0,1}}} \theta^{\Omega}_{[\lambda,a]}(\alpha_j) = \infty\,,$$

and if $\beta_j < \beta$, then

$$\lim_{\lambda \uparrow \sigma_1^{\Omega^1_{0,1}}} \theta^{\Omega}_{[\lambda,a]}(\beta_j) = \infty\,,$$

where $\theta^{\Omega}_{[\lambda,a]}$ is the unique strong positive solution of (1.1).

Proof

The positive solutions of (1.1) provide us with positive supersolutions of

$$-\Delta u = \lambda u - \|a\|_{L^\infty(\Omega)}\, \chi_{\Omega_+}\, u^{p+1} \qquad \text{in} \quad \Omega\,, \qquad u|_{\partial\Omega} = 0\,, \tag{4.1}$$

and hence, the strong maximum principle implies

$$\theta^{\Omega}_{[\lambda,a]} \geq \theta^{\Omega}_{[\lambda, \|a\|_{L^\infty(\Omega)}\, \chi_{\Omega_+}]} := u_\lambda\,. \tag{4.2}$$

By the assumptions, $\Omega^1_{0,j}$ is a proper subdomain of Ω and hence, either $\alpha < \alpha_j$, or $\beta > \beta_j$. Assume $\beta > \beta_j$. Then, there exists $c_j > \beta_j$ such that $(\beta_j, c_j) \subset \Omega_+$. Thanks to (4.2) it suffices to show that

$$\lim_{\lambda \uparrow \sigma_1^{\Omega^1_{0,1}}} u_\lambda(\beta_j) = \infty\,.$$

In (α_j, β_j) we have $a = 0$ and hence,

$$-u''_\lambda = \lambda u_\lambda\,.$$

Multiplying this equation by u'_λ and integrating it follows that

$$(u'_\lambda(x))^2 + \lambda u^2_\lambda(x) = (u'_\lambda(\beta_j))^2 + \lambda u^2_\lambda(\beta_j) \qquad \text{for all} \quad x \in (\alpha_j, \beta_j)\,. \tag{4.3}$$

On the other hand, by the proof of Theorem 2.4 we already know that u_λ grows to infinity uniformly on any compact subset of (α_j, β_j). Thus, we find from (4.3) that

$$\lim_{\lambda \uparrow \sigma_1^{\Omega^1_{0,1}}} [(u'_\lambda(\beta_j))^2 + \lambda u_\lambda^2(\beta_j)] = \infty\,. \tag{4.4}$$

Now, in the interval (β_j, c_j) the following holds

$$-u''_\lambda = \lambda u_\lambda - A u_\lambda^{p+1}\,, \qquad A := \|a\|_{L^\infty(\Omega)}\,,$$

and hence, multiplying this relation by u'_λ and integrating gives

$$\frac{(u'_\lambda(x))^2}{2} + \frac{\lambda}{2} u_\lambda^2(x) - \frac{A}{p+2} u_\lambda^{p+2}(x) = \frac{(u'_\lambda(\beta_j))^2}{2} + \frac{\lambda}{2} u_\lambda^2(\beta_j) - \frac{A}{p+2} u_\lambda^{p+2}(\beta_j) \tag{4.5}$$

for each $x \in (\beta_j, c_j)$. By the proof of Theorem 2.4 we already now that the solutions u_λ approach in $C^{1+\nu}$ to a strong solution on any compact subinterval of (β_j, c_j) as $\lambda \uparrow \sigma_1^{\Omega^1_{0,1}}$. In particular, the left hand side of (4.5) has a finite limit as $\lambda \uparrow \sigma_1^{\Omega^1_{0,1}}$ and hence, also does it the right hand side. Therefore, thanks to (4.4),

$$\lim_{\lambda \uparrow \sigma_1^{\Omega^1_{0,1}}} u_\lambda(\beta_j) = \infty\,.$$

If $\alpha < \alpha_j$, the previous argument can be easily adapted to show that

$$\lim_{\lambda \uparrow \sigma_1^{\Omega^1_{0,1}}} u_\lambda(\alpha_j) = \infty\,.$$

This completes the proof. $\square$

In the remaining of this section we assume that $N \geq 2$, that $\Omega = B_R$, the ball of radius $R > 0$ centered at the origin, and that $a(x)$ is radially symmetric. Then,

$$a(x) = \rho(r)\,, \qquad r = |x|\,,$$

for some bounded function $\rho : [0, R] \to \mathbb{R}_+$. Throughout the rest of this chapter given $0 < \alpha < \beta \leq R$, $A_{(\alpha,\beta)}$ stands for the annulus

$$A_{(\alpha,\beta)} := \{ x \in \mathbb{R}^N \, : \, \alpha < |x| < \beta \} \, .$$

Note that in the proof of Theorem 2.4 all the sub and supersolutions used to show the existence of the classical positive solution can be chosen to be radially symmetric. Therefore, the positive solution of (1.1) must be radially symmetric, since it is unique and the method of sub and supersolutions can be used on the corresponding spaces of radially symmetric functions.

Theorem 4.2. *Assume we are working under the previous assumptions and* $(H1)$, $(H2)$, $(H4)$ *hold. Then, for any* $j \in \{1, ..., n_1\}$ *we have*

$$\lim_{\lambda \uparrow \sigma_1^{\Omega^1_{0,1}}} \theta^{\Omega}_{[\lambda,a]}(x) = \infty \qquad \textit{for all} \quad x \in \partial\Omega^1_{0,j} \cap \Omega \, . \tag{4.6}$$

Proof

The comparison argument in the beginning of the proof of Theorem 4.1 shows that we are done if we prove the result for the case when

$$a(x) = A \, \chi_{\Omega_+} \, , \qquad A := \|a\|_{L^\infty(\Omega)} \, . \tag{4.7}$$

So, we can assume that (4.7) is satisfied. In the sequel, we use the notation

$$\varphi_\lambda(r) := \theta^{\Omega}_{[\lambda,a]}(x) \, , \qquad \sigma_1^{\Omega} < \lambda < \sigma_1^{\Omega^1_{0,1}} \, , \qquad r = |x| \, . \tag{4.8}$$

We have to distinguish between several different situations according to the structure and location of $\Omega^1_{0,j}$. Assume that $\Omega^1_{0,j} = B_\alpha$ for some $\alpha \in (0, R)$. Then, there exists $c \in (\alpha, R]$ such that $A_{(\alpha,c)} \subset \Omega_+$. For each $r \in (0, \alpha)$ we have

$$-\varphi''_\lambda(r) - \frac{N-1}{r} \, \varphi'_\lambda(r) = \lambda \, \varphi_\lambda(r)$$

and hence,

$$-(r^{N-1}\varphi_\lambda'(r))' = \lambda\, r^{N-1}\varphi_\lambda(r)\,. \tag{4.9}$$

Now, integrating (4.9) in $(0,\alpha)$ gives

$$-\alpha^{N-1}\varphi_\lambda'(\alpha) = \lambda \int_0^\alpha r^{N-1}\varphi_\lambda(r)\,dr\,. \tag{4.10}$$

Moreover, since

$$\lim_{\lambda\uparrow\sigma_1^{\Omega_{0,1}^1}} \varphi_\lambda = \infty$$

uniformly in any compact subinterval of $(0,\alpha)$, we find from (4.10) that

$$\lim_{\lambda\uparrow\sigma_1^{\Omega_{0,1}^1}} \varphi_\lambda'(\alpha) = -\infty\,. \tag{4.11}$$

On the other hand, for each $r\in(\alpha,c)$ we have

$$-(r^{N-1}\varphi_\lambda'(r))' = \lambda\, r^{N-1}\varphi_\lambda(r) - A\, r^{N-1}\varphi_\lambda^{p+1}(r)\,, \tag{4.12}$$

and hence,

$$-r^{N-1}\varphi_\lambda'(r) + \alpha^{N-1}\varphi_\lambda'(\alpha) = \lambda \int_\alpha^r s^{N-1}\varphi_\lambda(s)\,ds - A\int_\alpha^r s^{N-1}\varphi_\lambda^{p+1}(s)\,ds\,. \tag{4.13}$$

By the proof of Theorem 2.4, for any compact subinterval J of (α,c) and any $\nu\in(0,1)$ the function φ_λ converges in $C^{1+\nu}(J)$ as $\lambda\uparrow\sigma_1^{\Omega_{0,1}^1}$ to a strong solution. In particular, for each $r\in(\alpha,c)$

$$\lim_{\lambda\uparrow\sigma_1^{\Omega_{0,1}^1}} r^{N-1}\varphi_\lambda'(r)\in\mathbb{R}\,.$$

Thus, (4.11) and (4.13) imply

$$\lim_{\lambda\uparrow\sigma_1^{\Omega_{0,1}^1}} \left[\lambda\int_\alpha^r s^{N-1}\varphi_\lambda(s)\,ds - A\int_\alpha^r s^{N-1}\varphi_\lambda^{p+1}(s)\,ds\right] = -\infty\,.$$

Thanks to the mean integral value theorem, for each λ there exists $s_\lambda \in (\alpha, r)$ such that

$$\int_\alpha^r s^{N-1}\varphi_\lambda(s)[\lambda - A\varphi_\lambda^p(s)]\,ds = (r-\alpha)s_\lambda^{N-1}\varphi_\lambda(s_\lambda)[\lambda - A\varphi_\lambda^p(s_\lambda)]\,.$$

Hence,

$$\lim_{\lambda \uparrow \sigma_1^{\Omega^1_{0,1}}} (r-\alpha)s_\lambda^{N-1}\varphi_\lambda(s_\lambda)[\lambda - A\varphi_\lambda^p(s_\lambda)] = -\infty\,.$$

Thus,

$$\lim_{\lambda \uparrow \sigma_1^{\Omega^1_{0,1}}} \varphi_\lambda(s_\lambda) = \infty\,,$$

and therefore,

$$\lim_{\lambda \uparrow \sigma_1^{\Omega^1_{0,1}}} s_\lambda = \alpha\,, \qquad \lim_{\lambda \uparrow \sigma_1^{\Omega^1_{0,1}}} \varphi_\lambda(\alpha) = \infty\,.$$

This completes the proof of (4.6).

Now, assume that

$$\Omega^1_{0,j} = A_{(\alpha,\beta)}\,, \quad \text{for some } 0 < \alpha < \beta \le R\,.$$

Then, there exists $c \in (0, \alpha)$ such that $A_{(c,\alpha)} \subset \Omega_+$. For each $r \in (\alpha, \beta)$ we have

$$\varphi'_\lambda(r)\varphi''_\lambda(r) + \lambda\varphi'_\lambda(r)\varphi_\lambda(r) = -\frac{N-1}{r}(\varphi'_\lambda(r))^2 \le 0$$

and hence,

$$\frac{d}{dr}[(\varphi'_\lambda(r))^2 + \lambda\varphi_\lambda^2(r)] \le 0\,.$$

Thus,

$$(\varphi'_\lambda(\alpha))^2 + \lambda\varphi_\lambda^2(\alpha) \ge (\varphi'_\lambda(r))^2 + \lambda\varphi_\lambda^2(r) \ge \varphi_\lambda^2(r)\,. \tag{4.14}$$

Moreover, $\lim_{\lambda \uparrow \sigma_1^{\Omega^1_{0,1}}} \varphi_\lambda = \infty$ uniformly on any compact subinterval of (α, β) and therefore, (4.14) implies

$$\lim_{\lambda \uparrow \sigma_1^{\Omega^1_{0,1}}} [(\varphi'_\lambda(\alpha))^2 + \lambda \varphi^2_\lambda(\alpha)] = \infty \,. \tag{4.15}$$

On the other hand, for each $r \in (c, \alpha)$ we have

$$\varphi'_\lambda(r)\varphi''_\lambda(r) + \lambda\varphi'_\lambda(r)\varphi_\lambda(r) - A\varphi'_\lambda(r)\varphi^{p+1}_\lambda(r) = -\frac{N-1}{r}(\varphi'_\lambda(r))^2 \leq 0$$

and hence,

$$\frac{d}{dr}[\frac{(\varphi'_\lambda(r))^2}{2} + \frac{\lambda}{2}\varphi^2_\lambda(r) - \frac{A}{p+2}\varphi^{p+2}_\lambda(r)] \leq 0 \,. \tag{4.16}$$

Thus,

$$\frac{(\varphi'_\lambda(r))^2}{2} + \frac{\lambda}{2}\varphi^2_\lambda(r) - \frac{A}{p+2}\varphi^{p+2}_\lambda(r) \geq \frac{(\varphi'_\lambda(\alpha))^2}{2} + \frac{\lambda}{2}\varphi^2_\lambda(\alpha) - \frac{A}{p+2}\varphi^{p+2}_\lambda(\alpha) \,. \tag{4.17}$$

The left hand side of (4.17) is bounded above, since φ_λ approaches in $C^{1+\nu}$ to a positive solution of the equation as $\lambda \uparrow \sigma_1^{\Omega^1_{0,1}}$. Hence, the right hand side of (4.17) is bounded above and therefore, we find from (4.15) that

$$\lim_{\lambda \uparrow \sigma_1^{\Omega^1_{0,1}}} \varphi_\lambda(\alpha) = \infty \,. \tag{4.18}$$

This completes the proof if $\beta = R$. It remains to show that

$$\lim_{\lambda \uparrow \sigma_1^{\Omega^1_{0,1}}} \varphi_\lambda(\beta) = \infty \tag{4.19}$$

if $\beta < R$. Assume $\beta < R$. Then, there exists $d > \beta$ such that $A_{(\beta,d)} \subset \Omega_+$. To show (4.19) we argue by contradiction assuming that

$$\lim_{\lambda \uparrow \sigma_1^{\Omega^1_{0,1}}} \varphi_\lambda(\beta) := \varphi_1(\beta) \in \mathbb{R}_+ \,. \tag{4.20}$$

Note that the limit exists since $\lambda \to \varphi_\lambda(\beta)$ is increasing. Now, given $\eta > 0$ sufficiently small consider the family of values $\{\varphi'_\lambda(\beta)\}$, $\lambda \in [\sigma_1^{\Omega^1_{0,1}} - \eta, \sigma_1^{\Omega^1_{0,1}})$. If this family is bounded, then there exists a sequence λ_n, $n \geq 1$, such that

$$\lim_{n\to\infty} \lambda_n = \sigma_1^{\Omega^1_{0,1}}$$

and

$$\lim_{n\to\infty} \varphi'_{\lambda_n}(\beta) := \varphi'_1(\beta) \in \mathbb{R}\,. \tag{4.21}$$

For each $n \geq 1$ the couple $(\varphi_{\lambda_n}, \varphi'_{\lambda_n})$ provides us with a solution in (α, β) of a linear first order system with analytic coefficients in $[\alpha, \beta]$ and therefore, thanks to (4.20) and (4.21), we find from the theorem of continuous dependence with respect to the initial values and parameters that $\varphi_{\lambda_n}(\alpha)$, $n \geq 1$, is bounded. This contradicts (4.18) and shows that the family $\{\varphi'_\lambda(\beta)\}$ is unbounded. By Theorem 2.4, there exists a sequence λ_n, $n \geq 1$, such that $\lambda_n \uparrow \sigma_1^{\Omega^1_{0,1}}$ as $n \to \infty$ and either

$$\lim_{n\to\infty} \varphi'_{\lambda_n}(\beta) = \infty\,, \tag{4.22}$$

or

$$\lim_{n\to\infty} \varphi'_{\lambda_n}(\beta) = -\infty\,. \tag{4.23}$$

Assume that condition (4.22) is satisfied. Then, for each $n \geq 1$ and $r \in (\alpha, \beta)$ we have that

$$\varphi'_{\lambda_n}(r) > 0\,. \tag{4.24}$$

Indeed, If (4.24) fails for some $n \geq 1$, then there exists $r_0 \in (\alpha, \beta)$ such that $\varphi'_{\lambda_n}(r_0) = 0$ and $\varphi'_{\lambda_n}(r) > 0$ for all $r \in (r_0, \beta]$. Moreover,

$$-\varphi''_{\lambda_n}(r_0) = \lambda_n \varphi_{\lambda_n}(r_0) > 0$$

and hence at $r = r_0$ the function φ_{λ_n} exhibites a local maximum, which contradicts the fact that $\varphi'_{\lambda_n}(r) > 0$ for all $r \in (r_0, \beta]$. Thus, condition (4.24) holds and hence, for each $n \geq 1$ we have that

$$\varphi_{\lambda_n}(\alpha) \leq \varphi_{\lambda_n}(\beta)\,.$$

So, thanks to (4.18) we find that

$$\lim_{n\to\infty} \varphi_{\lambda_n}(\beta) = \infty\,,$$

which contradicts (4.20). Therefore, condition (4.23) holds.
Now, for each $r \in (\beta, d)$ and $n \geq 1$ we have

$$-r^{N-1}\varphi'_{\lambda_n}(r) + \beta^{N-1}\varphi'_{\lambda_n}(\beta) = \int_\beta^r s^{N-1}\varphi_{\lambda_n}(s)[\lambda_n - A\varphi^p_{\lambda_n}(s)]\,ds \tag{4.25}$$

and therefore, using (4.23) it follows from (4.25) that

$$\lim_{n\to\infty} \varphi_{\lambda_n}(\beta) = \infty\,.$$

This contradicts (4.20) as well. Therefore,

$$\lim_{\lambda \uparrow \sigma_1^{\Omega^1_{0,1}}} \varphi_\lambda(\beta) = \infty\,.$$

This completes the proof. □

I.5. Numerical computation of classical solutions

In this section we compute the curve of classical positive solutions of a two-dimensional radially symmetric prototype model of (1.1). Namely, we take $N = 2$, $\Omega = B_{0.5}$ the ball of radius 0.5 centered at the origin, $p = 4$ and

$$a(x) = \rho(r)\,, \qquad r = |x|\,,$$

where $\rho : [0, 0.5] \to [0, \infty)$ is given by

$$\rho(r) := \begin{cases} -\sin(5\pi(r + 0.5))\,, & r \in (0.1, 0.3)\,, \\ \\ 0\,, & r \in [0, 0.1] \cup [0.3, 0.5]\,, \end{cases} \tag{5.1}$$

Figure 5.1 shows a plot of $a(x)$.

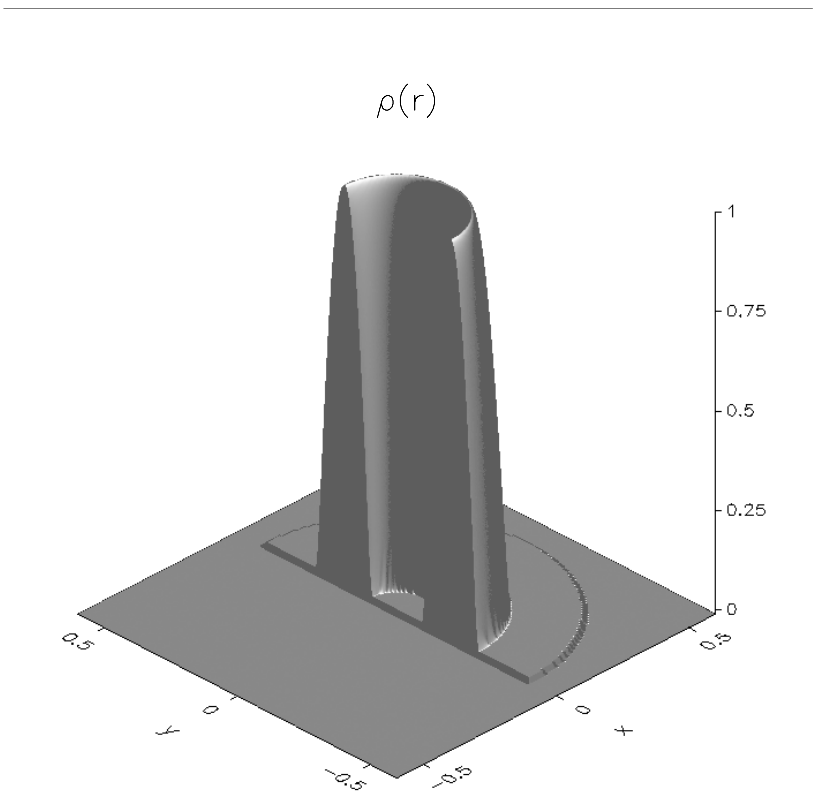

Figure 5.1: Plot of $a(x)$.

Note that we have represented the half of its graphic. So, the diameter slice provides us with the profile of $\rho(x)$. The corresponding model fits into the abstract setting of Section 4 with

$$\Omega_+ = A_{(0.1,0.3)}\,, \qquad \Omega_0 = B_{0.1} \cup A_{(0.3,0.5)}\,.$$

In Table 5.2 we are giving the theoretical and numerical values of the principal eigenvalues of $-\Delta$ in some of the relevant subdomains of Ω. Namely, Ω itself, and each of the components of Ω_0. The theoretical values are calculated from the estimate 2.4048 for the first zero of the Bessel function J_0. The numerical values are the unique values of λ for which bifurcation to positive solutions from $u = 0$ occurs. These values have been computed by means of the pseudo-spectral method described bellow using 125 modes. Note that the principal eigenfunctions associated with each of these subdomains are radially symmetric and hence we are actually dealing with one-dimensional linear eigenvalue problems. Also, note that the ***numerical principal eigenvalue*** might provide us with a sharper estimate to the theoretical bifurcation value than the one obtained from a bad approximation to the first zero of J_0.

Subdomain	Theoretical σ_1	Computed σ_1
$B_{0.5}$	23.132252	23.131769
$B_{0.1}$	578.306304	578.294226
$A_{(0.3,0.5)}$		245.138590

Table 5.2: Principal eigenvalues of some relevant subdomains.

Thanks to the values given in Table 5.2, we have

$$\Omega_{0,1}^1 = A_{(0.3,0.5)}\,, \qquad \Omega_{0,1}^2 = B_{0.1}\,,$$

since $\sigma_1^{A_{(0.3,0.5)}} < \sigma_1^{B_{0.1}}$.

By the comments before the statement of Theorem 4.2,

$$\theta^{\Omega}_{[\lambda,a]}(x) = u_\lambda(r)\,, \qquad r = |x|\,, \tag{5.2}$$

where $u_\lambda(r)$ is the unique positive solution of the one-dimensional problem

$$\begin{cases} -u''(r) - \frac{1}{r}\,u'(r) = \lambda u(r) - \rho(r)u^5\,, \quad r \in (0,0.5)\,, \\ u'(0) = 0\,, \quad u(0.5) = 0\,. \end{cases} \tag{5.3}$$

These solutions are the restriction to $[0,0.5]$ of the positive solutions of

$$\begin{cases} -u''(r) - \frac{1}{r}\,u'(r) = \lambda u(r) - \hat\rho(r)u^5\,, \quad r \in (-0.5,0.5)\,, \\ u(-0.5) = 0\,, \quad u(0.5) = 0\,, \end{cases} \tag{5.4}$$

where $\hat\rho(r) = \rho(-r)$ for each $r \in [-0.5,0]$, and $\hat\rho(r) = \rho(r)$ if $r \in [0,0.5]$. Instead of (5.4) we will consider its phase translation to the interval $[0.1]$

$$\begin{cases} -u''(x) - \frac{1}{x-0.5}\,u'(x) = \lambda u(x) - \hat\rho(x-0.5)u^5\,, \quad x \in (0,1)\,, \\ u(0) = 0\,, \quad u(1) = 0\,. \end{cases} \tag{5.5}$$

To compute the bifurcation diagram of positive solutions of (5.5) we use spectral collocation methods coupled with path-following techniques. This gives high accuracy with low computational work. In the numerical computations we have used trigonometric modes and the collocation points have been taken to be equidistant, with the number of modes equal to the number of collocation points. Let M denote the number modes and

$$x_i = \frac{i}{1+M}\,, \qquad 1 \le i \le M\,,$$

the collocation points. Then, the solutions $u(x)$ of (5.5) are approximated by

$$u_M(x) = \sum_{j=1}^{M} c_j \sin(j\pi x)\,,$$

being $C = (c_1, ..., c_M)^T$ (T =transposition) a solution of

$$BC - EDC = \lambda JC - A(JC)^5 \tag{5.6}$$

where

$$\begin{cases} J = (\sin(j\pi x_i))_{1\le i,j\le M} \, , \\ B = ((j\pi)^2 \sin(j\pi x_i))_{1\le i,j\le M} \, , \\ A = \text{diag } (\hat{\rho}(x_i - 0.5))_{1\le i\le M} \, , \\ E = \text{diag } (e_j)_{1\le j\le M} \, , \\ D = (d_{ij})_{1\le i,j\le M} \, , \end{cases}$$

$$e_j = \begin{cases} \frac{1}{x_j - 0.5} \, , \text{ if } x_j \neq 0.5 \, , \\ 1 \, , \text{ if } x_j = 0.5 \, , \end{cases}$$

and

$$d_{ij} = \begin{cases} j\pi \cos(j\pi x_i) \, , \qquad 1 \le j \le M \, , \text{ if } x_i \neq 0.5 \, , \\ -(j\pi)^2 \sin(j\pi/2) \, , \qquad 1 \le j \le M \, , \text{ if } x_i = 0.5 \, . \end{cases}$$

Making these choices preserves the zero solution of (1.1), although the bifurcation values to positive solutions for the continuous and the discrete models will not be equal, since we are working with trigonometric series instead of with Fourier series of Bessel functions.

It is well known that the set of positive solutions of both the discrete and the continuous counterparts of the model have the same structure around any regular or simple bifurcation point (cf. [RBR80], [RBR181], [RBR281]). In fact, this is true for a wider class of models than (1.1), although thanks to the special structure of (1.1) it can be shown that not only the local but also the global structure of the set of positive solutions is preserved. Actually, the set of positive solutions of the discrete model approaches the set of positive solutions of its continuous counterpart as the number of modes grows to infinity. Being outside the scope of this memoir, this global convergence analysis will appear elsewhere.

By the continuity of the coefficient $a(x)$, any solution of (5.5) is of class C^2 and hence its jth Fourier coefficient, denoted by $\hat{u}_j$, decays as $O(j^{-2})$ when $j \uparrow \infty$ (cf. [CHQZ88, pp. 35]). Thus,

$$\max_{0\le x\le 1} |\sum_{j=1}^{M+1} \hat{u}_j \sin(j\pi x) - \sum_{j=1}^{M} \hat{u}_j \sin(j\pi x)| = O(M^{-2}) \qquad \text{as} \;\; M \uparrow \infty \,.$$

Thanks to this feature we have adopted the following criterion to choose the number of modes in our computations

$$|c_k| \le \frac{1}{2} 10^{-4} \,, \qquad M - 10 \le k \le M \,. \tag{5.7}$$

By the radial symmetry of the problem $c_{2\ell} = 0$ for all $\ell \in \mathbb{N}$. Therefore, (5.7) will be satisfied for any even k.

To compute the global curve of classical positive solutions of (5.5), whose existence is guaranteed from Theorem 2.4, as well as the principal eigenvalues of the linearizations around each of those solutions, we have used standard path-following techniques as in [DK81], [Ei86], [LEDM92] and the references there in, where we send for further details.

Figure 5.3 shows the numerical bifurcation diagram of positive solutions that we have obtained. In Figure 5.3 we have represented the L_2-norm of each non-negative solution versus the parameter λ, $\lambda \in (0.0, 250.0)$. Continuous lines are filled in by stable solutions and dashed lines by unstable solutions. Each of the points on these curves represents a non-negative solution of (1.1). The diagram shows two curves, one of them filled in by positive solutions which are linearly asymptotically stable, and the other is the trivial branch $(\lambda, u) = (\lambda, 0)$. The solution $u = 0$ is stable until the bifurcation value $\lambda = 23.131769$ where it becomes unstable for any further value of the parameter.

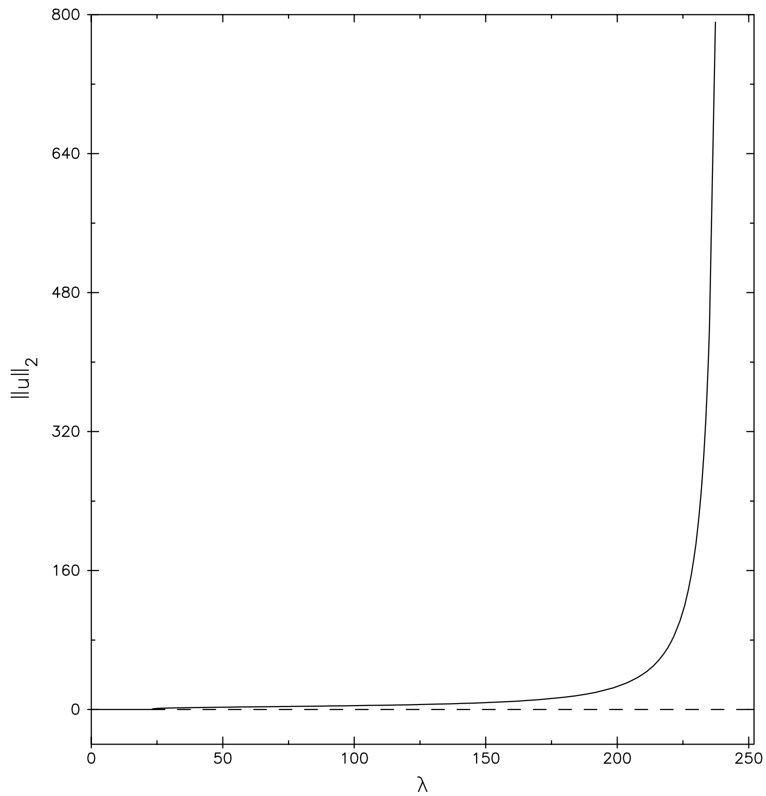

Figure 5.3: Bifurcation diagram of classical positive solutions.

Table 5.4 shows the number of modes needed to reach the convergence criterion (5.7). If $\lambda \in (203.799461, \sigma_1^{A(0.3,0.5)})$, then (5.7) can not be attained with 475 modes, and we could not increase the number of modes to reach (5.7) since our computer does not have enough memory to do it.

Interval of λ	Number of modes
$(23.131769, 71.283046)$	65
$[71.283046, 101.229218)$	95
$[101.229218, 126.233139)$	125
$[126.233139, 151.061935)$	185
$[151.061935, 179.621582)$	245
$[179.621582, 191.845247)$	335
$[191.845247, 203.799461)$	475

Table 5.4: The number of modes needed to reach (5.7).

In fact, using 475 modes our computer spend around 20 minutes to calculate each of the positive solutions in the bifurcation diagram. Therefore, to compute the curve of positive solutions for further values of λ we must forget about the criterion (5.7). If $\lambda \in [203.799461, 238.043512]$, then, instead of (5.7), the following weaker criterion was reached

$$|c_k| \leq \frac{1}{2}10^{-1}\,, \qquad M-10 \leq k \leq M\,. \tag{5.8}$$

If $\lambda \in (238.043512, 240.988748)$, then the solutions do not satisfy (5.8) either. Actually, the slope of the curve of positive solutions becomes very large as $\lambda \uparrow 240.988748$, making a very hard task following the curve for further values of λ, since very small changes in λ provoc drastical variations of the length of arc along the curve. In fact, the length of the arc of curve corresponding to the interval $\lambda \in (240.239243, 240.335274)$ is 500, while its own length is 0.096031. The L^∞-norm of the positive solution grew from 7887.99 up to 8674.60. Also, note that even using a larger number of modes, we will never reach the exact value $\sigma_1^{A_{(0.3,0.5)}} \simeq 245.138590$ where bifurcation from infinity occurs, since discretizing the model slightly enlarges each of the components of Ω_0. In fact, $A_{(0.3,0.5)}$ becomes into $A_{(0.298319,0.5)}$ and since

$$\sigma_1^{A_{(0.298319,0.5)}} \simeq 241.035767\,,$$

the numerical value where the bifurcation from infinity occurs will be 241.035767 which is smaller than the theoretical one, 245.138590.

Figure 5.5 shows the profiles of some of the positive solutions that we have computed. All of them satisfy the convergence criterion (5.7). Note how these solutions grow in $\overline{\Omega}^1_{0,1} = \overline{A}_{(0.3,0.5)}$, while they stabilize in $\Omega \setminus \overline{\Omega}^1_{0,1}$, as λ increases. As λ moves away from the bifurcation value 23.131769, the principal eigenvalue of the linearization around the positive solution grows from zero up to reach its maximum value at a critical λ, where it becomes decreasing for any further value of λ up to approach the critical value where the bifurcation from infinity takes place, where it converges to zero.

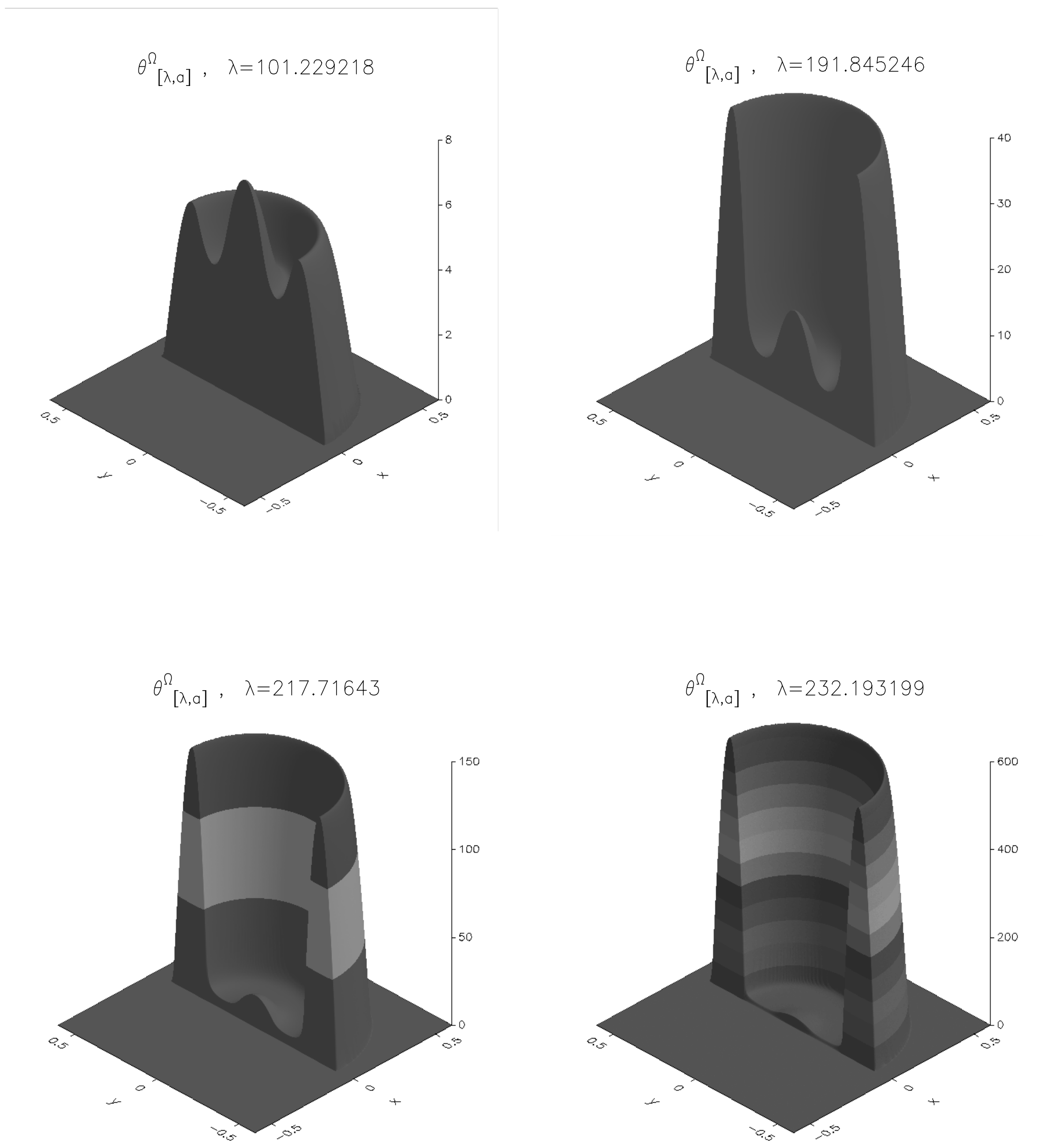

Figure 5.5. Profiles of the classical positive solutions.

In Table 5.6 we have collected some representative values of λ together with the L^∞-norms of the corresponding positive solutions and the principal eigenvalues of their linearizations (p.e.l). For further values of λ the determinant of the linearization at the discrete positive solution becomes oscillating and so we stop the continuation here in. In fact the numerical solutions ontained for further values of λ become negative around the interior relative minima and therefore, they are not admissible.

Value of λ	p.e.l.	$\lVert\theta^{\Omega}_{[\lambda,a]}\rVert_\infty$
221.654455	12.68	184.65
229.914607	7.92	418.90
233.646062	5.60	732.53
237.357723	3.13	1753.84
239.509913	1.59	4505.86
240.988748	0.45	22598.95

Table 5.6: The principal eigenvalue of the linearization.

Actually, we should have stopped the continuation at the value $\lambda = 237.357723$, since for greater values of λ the solutions become decreasing around their interior relative minima in strong contradiction with the monotonicity given by Theorem 2.4.

To complete this section, it should be pointed out that our numerical computations strongly suggest that the absolute value of infinitely many of the odd Fourier coefficients will grow to infinity as $\lambda \uparrow \sigma_1^{A(0.3,0.5)}$ making impossible the global approximation to the whole curve of positive solutions of (1.1) with finitely many modes.

I.6. Numerical computation of metasolutions

In this section we compute the curve of minimal radially symmetric metasolutions of order 1 of the model analyzed in Section 5. In the proof of Theorem 3.5 the metasolutions of order one were constructed as point-wise limits of radially symmetric classical solutions. Therefore, in our present setting Theorem 3.5 actually shows the existence of radially symmetric metasolutions up to order two. We will denote to the minimal metasolution of order one by u_λ. According to Theorem 3.5 they exist if

$$\sigma_1^{A(0.3,0.5)} \simeq 245.138590 \leq \lambda < \sigma_1^{B_{0.1}} \simeq 578.306304\,.$$

These metasolutions are given by the solutions of the problem

$$\begin{cases} -u''(x) - \frac{1}{x}u'(x) = \lambda u - a(x)u^5 & \text{in } \ B_{0.3}\,, \\ u(0.3) = \infty\,, \quad u'(0) = 0\,. \end{cases} \tag{6.1}$$

It should be pointed out that we do not know if any metasolution of the problem should be radially symmetric or not and that even the uniqueness of the positive solution of (6.1) is an open problem. The main difficulty coming from the fact that the uniqueness results of [MV97] do not apply to our models, since the coefficient $a(x)$ is not bounded away from zero. As a result of our numerical analysis in this section the uniqueness of the radially symmetric metasolution of order one will become clear.

The minimal metasolution corresponding with the value $\lambda = \sigma_1^{A(0.298319,0.5)}$ for the pure espectral discretization of the continuous model can be taken to be as the last classical solution computed in Section 5. In general, this solution will not be infinity on the boundary of the disk $B_{0.3}$, but sufficiently large, say α. Then, a possible strategy to approach the metasolutions of order one would be to compute the unique positive solution of

$$\begin{cases} -u''(x) - \frac{1}{x}u'(x) = \lambda u - a(x)u^5 & \text{in } B_{0.3}\,, \\ \\ u(0.3) = \alpha\,, \quad u'(0) = 0\,. \end{cases} \tag{6.2}$$

for $\lambda \in (\sigma_1^{B_{0.3}}, \sigma_1^{B_{0.1}})$, whose existence is guaranteed by Theorem 3.1. Solving (6.2) is substantially more involved than computing the classical solutions for the original problem since now in order to adjust the boundary condition we will have to take a large number of modes from the very beginning, because α is assumed to be very large, and actually the number of modes should be further increased to compute the solutions of (6.2) for values of λ close to $\sigma_1^{B_{0.1}}$. The main trouble from this method comes from the fact that since we are changing the boundary condition it might happen the solutions of (6.1) and (6.2) are far away. Thus, this strategy does not look the best possible.

By the one-dimensional nature of (6.1) to compute its solutions we use the explicit fourth order Runge Kutta method (ERK) described in [Is96, pp. 41]. Writing down the equation as a first order system in the unknowns (u, u') we should find out the value of $u_0 := u_\lambda(0)$ for which the solution subject to the initial conditions

$$u_\lambda(0) = u_0\,, \qquad u'_\lambda(0) = 0\,,$$

satisfies

$$\lim_{x \uparrow 0.3} u_\lambda(x) = \infty\,. \tag{6.3}$$

To compute u_0 we propose an initial interval $[u_I, u_F]$, pick $u_0^1 \in [u_I, u_F]$ and use the (ERK) method with M_1 nodes to compute the solution of the system with the initial conditions $(u_0^1, 0)$, say $u_\lambda^{M_1}$. Then, using a bisection procedure we change the u_0^1 according to the following criterion: If

$$\frac{du_\lambda^{M_1}}{d\lambda}(0.3) < 10^{10}\,,$$

then we increment u_0^1, while if there exists $x_1 \in (0, 0.3)$ for which

$$\lim_{x \uparrow x_1} u_\lambda^{M_1}(x) = \infty \,,$$

then we decrement u_0^1. The process is stopped when

$$\frac{du_\lambda^{M_1}}{d\lambda}(0.3) > 10^1 0 \,.$$

This procedure provides us with a relatively good approximation of u_0. To refine it we repeat he process but this time using a higher number of nodes, say M_2, up to reach the criterion

$$u_\lambda^{M_2}(0.3) \geq u_\lambda^{M_1}(0.3) \,. \tag{6.4}$$

If

$$\frac{|u_\lambda^{M_1}(0) - u_\lambda^{M_2}(0)|}{u_\lambda^{M_1}(0)} < 0.5 \times 10^{-5} \tag{6.5}$$

then we stop the process and propose

$$u_\lambda(0) := u_\lambda^{M_2}(0) \,.$$

If condition (6.5) fails, then we repeat the process increasing the number of nodes up to reach (6.5). As far as to the choice of the interval $[u_I, u_F]$ concerns, we proceed as follows. As u_I we propose the $u_{\lambda - \Delta\lambda}(0)$ computed at the previous λ, $\lambda - \Delta\lambda$. At $\sigma_1^{A(0.3,0.5)} + \Delta\lambda$, we take $u_I = u_{\hat{\lambda}}(0)$, where $\hat{\lambda}$ stands for the last value of λ for which we have computed the classical solution. These choices are based on the fact that the minimal solutions of (6.1) are increasing with λ. This can be easily accomplished from Theorem 2.5 by passing to the limit as $\alpha \uparrow \infty$. As to u_F concerns, it can be taken as large as wished. A relevant feature is the fact that the previous iterative scheme always converged, strongly suggesting the uniqueness of the positive solution of (6.1). Note that if (6.1) would possess two solutions, say u_1 and u_2, with $u_1(0) < u_2(0)$ then for any $u_0 \in (u_1(0), u_2(0))$ the solution of the Cauchy problem associated with the underlying system provides us with a solution of (6.1). Otherwise, we would contradict the uniqueness obtained from Theorem 3.1. Therefore, if uniqueness does not occur, then there is a continuum of values of u_0 for which the corresponding solutions are metasolutions.

An important feature is the fact that as larger is taken the initial value of u as sooner is going to blow up the solution of the Cauchy problem for the associated system. Conversely, if u_0 stands for the initial value of u_λ and $u(0) > u_0$, then as closer $u(0)$ and u_0 are as closer will be the blow up time of the solution with initial data $(u(0), 0)$ to 0.3. If $u(0) < u_0$ is sufficiently close to u_0, then $u(x)$ is globally defined in $[0, 0.3]$. If $u(0) \simeq 0$, $u(0) > 0$, then the solution will become negative and after some time will blow up to $-\infty$. All these features strongly suggest the uniqueness of the positive solution of (6.1).

Instead of five significative digits, we could have taken a higher number of them, but five are sufficient for our purposes here in. In fact, it should be remarked that if $u_\lambda^{M_1}(0.3)$ in the test (6.4) is huge and we increase drastically the number of nodes, then the number of significative digits grows.

To compute the metasolution for values of λ close to $\sigma_1^{B_{0.1}}$ the number of nodes in the scheme above should be taken sufficiently large so that the step in the (ERK) be smaller than the jump to infinity of the metasolution.

Table 6.1 shows the values of $u(0)$, M, $u(0.3)$ and $u'(0.3)$ for the corresponding values of λ in the first column of the table. In all cases, small variations in $u(0)$ provoke drastical changes in $u(0.3)$ and $u'(0.3)$. Precisely, if $u(0)$ is smaller than the real value of the metasolution, then the computed solution is finite at 0.3, while if $u(0)$ is greater than the initial value of the computed metasolution, then the solution $u(x)$ blows up to ∞ before 0.3. According to Lemma 3.6, this means that we have computed the metasolution of another problem corresponding with a larger $\Omega^2_{0,1}$ than $B_{0.1}$. In Figure 6.2 we have represented the plots of the metasolutions that we have computed for each of the first four values of λ in the first column of Table 6.1.

λ	M	$u(0)$	$u(0.3)$	$u'(0.3)$
300.0	601	32.627686	$0.67 \times 10^{4}4$	0.41×10^{132}
300.0	6001	32.627656	0.43×10^{57}	0.40×10^{172}
450.0	600	172.746567	0.10×10^{8}	0.65×10^{11}
450.0	6001	172.746426	0.67×10^{15}	0.97×10^{37}
500.0	601	449.299480	0.15×10^{46}	0.60×10^{136}
500.0	6001	449.295567	0.14×10^{83}	0.42×10^{251}
525.0	601	920.604016	0.49×10^{46}	0.25×10^{138}
525.0	6001	920.555679	0.14×10^{132}	Infinity
550.0	601	2912.833978	0.15×10^{52}	0.24×10^{155}
550.0	6001	2909.932954	0.44×10^{164}	Infinity
550.0	10001	2909.929864	0.13×10^{172}	Infinity

Table 6.1: (ERK) results.

According to Theorems 3.5, 3.7 the metasolutions of order 1 supported in $B_{0.3}$ are point-wise increasing with λ in $B_{0.3}$, and they do it faster in $B_{0.1}$, where $a(x)$ vanish, than in $A_{(0.1,0.3)}$, where $a(x)$ is positive. Note that each of these metasolutions takes the value ∞ on $\partial B_{0.3}$. Moreover, as $\lambda \uparrow \sigma_1^{B_{0.1}}$, the corresponding metasolutions entirely blow up in $\overline{B}_{0.1}$, while they stabilize in $A_{(0.1,0.3)}$.

As already predicted by Theorem 3.7, along some subsequence the point-wise limit of these metasolutions as $\lambda \uparrow \sigma_1^{B_{0.1}}$ provides us with a metasolution of order 2 supported in $A_{(0.1,0.3)}$. Note that in the model we are dealing with the highest order of the metasolutions is $m = 2$, and hence, Theorem 3.5 guarantees the existence of a metasolution of order 2 supported in $A_{(0.1,0.3)}$ for each $\lambda \geq \sigma_1^{B_{0.1}}$.

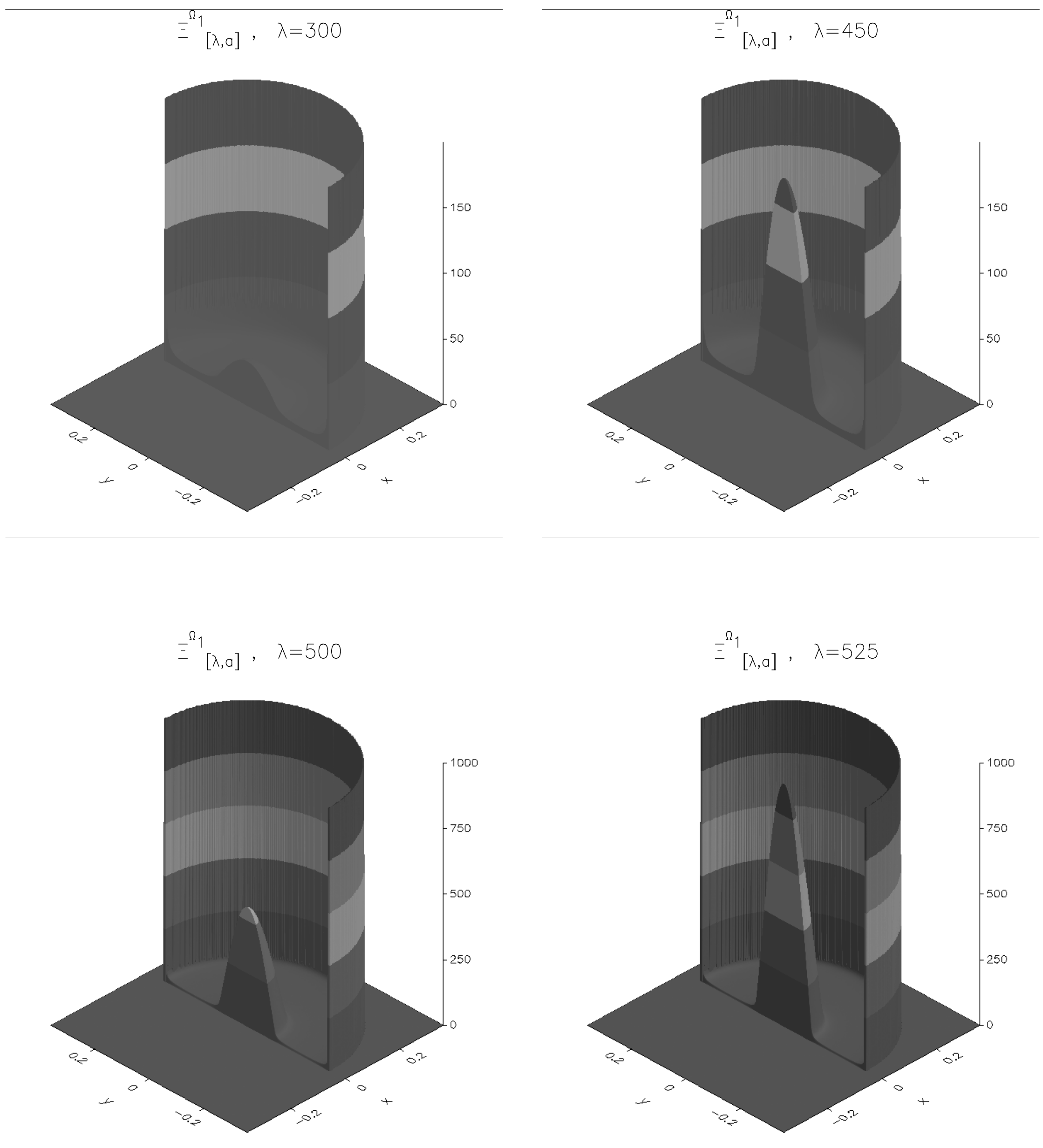

Figure 6.2: Metasolutions of order 1.

Computing the metasolutions of order one for values of $\lambda < \sigma_1^{B_{0.1}}$ sufficiently close to $\sigma_1^{B_{0.1}}$ is a very difficult task, mainly due to the highest number of digits needed to get their initial values. The following numerical experiment illustrates that fact. Taking $\lambda = 575$, $M = 50001$ and

$$u(0) = 129396.909009847150 \tag{6.6}$$

the (ERK) gives the following value

$$u(0.3) = 17.34\,,$$

while if we take

$$u(0) = 129396.909009847877\,, \tag{6.7}$$

then the (ERK) scheme gives

$$u(0.295524) = \infty\,.$$

Therefore, a change in the 16th digit provoke s a drastical change in the value of $u(0.3)$. Although these features tell us that the value of the minimal metasolution at 0, u_0, should lye between the values (6.6) and (6.7), it is rather clear that we will never be able to reach it, since 16 significative digits is the limit of the precision of our computer. Increasing the number of nodes will not overcome this trouble. Computing the metasolutions of order two is a much harder task and we think that some new idea, or scheme, is needed to compute these metasolutions. At present we do not know how to overcome all these difficulties and so, we stop our analysis here.

In Figure 6.3 we have represented the value at 0.2 of the classical solutions and the metasolutions of order one that we have computed versus the parameter λ. The continuous line stands for the curve of classical positive solutions. The dashed line represents the metasolutions of order one. Note that 0.2 is the middel point of the support of the coefficient $\rho(r)$. In general, it does not equal the point where the interior minima of these solutions is taken, but it is very close to it in most cases. Classical solutions have been represented up to the value $\lambda = 237.3577$, and the metasolutions have been represented from $\lambda = 250.0$ up to $\lambda = 570.0$.

Based on this diagram, it seems that the set of classical and non-classical solutions of our model possesses the structure of a continuous curve connecting $(\lambda, u) = (\sigma_1^\Omega, 0)$ with $(\lambda, u) = (\infty, \infty)$, although at present it is not clear what are the functional spaces where these models should be analyzed mathematically.

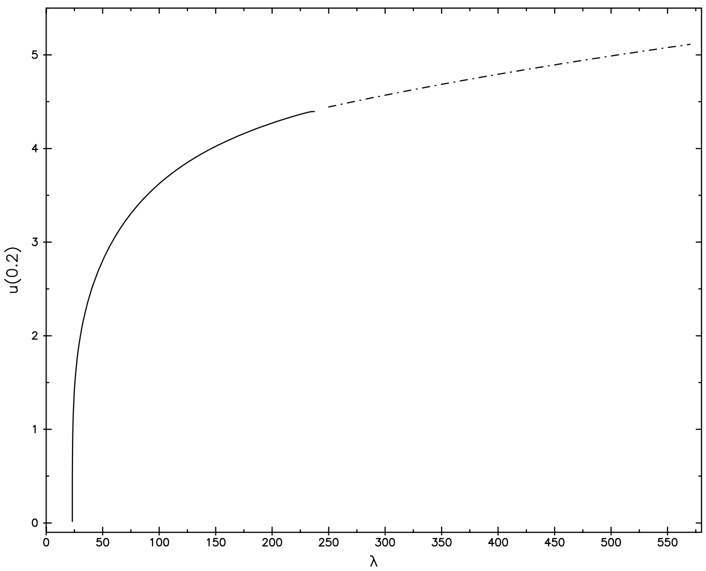

Figure 6.3: Global bifurcation diagram.

Chapter II
The Superlinear Indefinite Problem

Chapter II

The Superlinear Indefinite Problem

II.1. Introduction

In this Chapter we analyze the structure of the set of positive solutions of

$$-\Delta u = \lambda u - a(x)|u|^p u \quad \text{in } \Omega\,, \qquad u|_{\partial\Omega} = 0\,, \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 1$, of class C^2, $\lambda \in \mathbb{R}$ is regarded as a continuation parameter and $p \in (0,\infty)$. We suppose that $a \neq 0$ is a bounded measurable function on $\overline{\Omega}$ and put

$$\Omega_\pm := \{\, x \in \Omega \,:\, a^\pm(x) > 0 \,\}\,,$$

where $a^+ := \max\{a,0\}$ is the positive, and $a^- := a^+ - a$ is the negative part of a. In addition we assume that Ω_+ and Ω_- are open sets of class C^2 and that $a^\pm$ is bounded away from zero on compact subsets of $\Omega_\pm$. Note that Ω_+ and Ω_- have only finitely many components.

Problem (1.1) provides us with the steady states of the parabolic model

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = \lambda u - a(x)|u|^p u \quad (x,t) \in \Omega \times (0,\infty)\,, \\ u(\cdot,t)|_{\partial\Omega} = 0 \quad t > 0\,, \\ u(\cdot,0) = u_0 \geq 0 \quad \text{in } \ \Omega\,, \end{cases} \tag{1.2}$$

Although in mathematical biology the coefficient $a(x)$ is usually taken to be non-negative, (1.2) has a biological meaning even in the general case case when $a(x)$ changes of sign. Typically, $u(\cdot,t)$ is the density at time t of a single species inhabiting Ω, λ is the net growth rate of the species, u_0 is the initial population density, and the coefficient $a(x)$ measures the saturation effect responses to the population stress in Ω_+ (cf. I.1), while in Ω_- it measures the symbiosis effects due to the intraspecific cooperation. In the region

$$\Omega_0 := \Omega \setminus (\overline{\Omega}_+ \cup \overline{\Omega}_-)\,,$$

the individuals of the species are free from other effects than diffusion and hence the components of Ω_0 (at most finitely many) can be though as refuges.

Recent references about the existence, stability and multiplicity of positive solutions for (1.1) are [BO86], [Ou91], [Ou92], [BCN94], [BCN95], [FKLM96], [AT96], [Lo97], [AL98], [LS98], [GGLS98], and the references there in, where we send for further information. In Section 2 we collect some of the results obtained in these references, mostly those needed for the mathematical analysis carried out in Section 3, where it will be shown that (1.2) possesses a stable positive steady-state if, and only if,

$$\int_\Omega a\Phi^{p+2} > 0\,, \tag{1.3}$$

and that (1.1) does not admit a stable positive solution if $\lambda \leq \sigma_1^\Omega$. Hereafter, $\Phi > 0$ stands for the principal eigenfunction associated to σ_1^Ω, the principal eigenvalue of $-\Delta$ in Ω under homogeneous Dirichlet boundary conditions. This result is easily obtained if $a \leq 0$, but it is far from easy in the general case when $a(x)$ changes of sign. From the biological point of view, it means that if either $\lambda \leq 0$, or $\lambda > 0$ but the habitat is not sufficiently large so that $\lambda > \sigma_1^{\Omega}$, then the species u does not have enough room to avoid extinction. The main result of Section 3 is the following.

Theorem 1.1. *Assume* $\Omega_+, \Omega_- \neq \emptyset$, (1.3), *and that the positive solutions of* (1.1) *possess uniform* L_∞ *a priori bounds for* λ *in compact subintervals of* $\mathbb{R}$. *Then, there exists*

$$\sigma_1^{\Omega} < \lambda^* < \sigma_1^{\Omega \setminus \overline{\Omega}_+}$$

such that the set of λ*'s for which* (1.1) *possesses a positive solution is* $(-\infty, \lambda^*]$. *Moreover, for each* $\lambda \in (\sigma_1^{\Omega}, \lambda^*]$, (1.1) *possesses a unique linearly stable positive solution. Furthermore, if* $\lambda \in (\sigma_1^{\Omega}, \lambda^*)$, *then this solution is linearly asymptotically stable and* (1.1) *possesses at least two positive solutions.*

The uniqueness of the stable positive solutions is quite striking, since (1.1) possesses at least two positive solutions for each $\lambda \in (\sigma_1^{\Omega}, \lambda^*)$. In fact, it will become clear later how playing around with the shape of Ω and the nodal behavior of $a(x)$ the problem (1.1) might have as many positive solutions as we wish. The relevance of Theorem 1.1 comes from the fact that the uniqueness of the stable positive solution is an universal property independent of the shape of Ω and the nodal behavior of $a(x)$.

It follows from Theorem 1.1 that if $\lambda \leq \sigma_1^{\Omega}$, then $u = 0$ is the unique stable non-negative solution of (1.1). Therefore, the species might be driven to extinction, although since Theorem 1.1 provides us with a positive solution, necessarily unstable, the species can avoid extinction if the initial population lies on the stable manifold of some of these positive states. When this last phenomenology occurs, a very natural question arises. How does distribute the species in Ω as $\lambda \downarrow -\infty$?. From the biological point of view, it is quite clear that in order to avoid extinction the species should concentrate in the regions where intraspecific cooperation takes place, and within these regions such concentration should be more emphasized around the values where the symbiosis rate is higher. This strongly suggests that the positive solutions of (1.1) should exhibit a spike layer behavior as $\lambda \downarrow -\infty$ around the negative local minima of $a(x)$, exhibiting a single or multiple peak profile accordingly with the nodal behavior of $a(x)$, and the number of components of Ω_-. In the very special case when $\Omega_- = \Omega$ variational methods have proven to be useful to show the existence of some spike layer solutions of (1.1), but unfortunately no result is available for superlinear indefinite problems. The main difficulty coming from the fact that the standard variational methods can not be applied to treat the case when $a(x)$ changes of sign, as already pointed out in [BCN95] (cf. [NW95], [DF97], [DF98], [Li97], and the references there in).

To gain insight into this and some related problems, in Sections 4-8 we use spectral methods with collocation coupled with path following techniques to compute the set of positive solutions of (1.1) for some special one dimensional prototype models. The numerics is of interest on its own right, since in the case when $\Omega_- = \emptyset$ and $\Omega_0 \neq \emptyset$ the population might exhibit entire blow up in some of the components of the refuge Ω_0. In fact, the set of positive solutions of (1.1) bifurcates from infinity at the value $\lambda = \sigma_1^{\Omega_0}$ (cf. Theorem 2.4 in Section 2). Although Theorem 2.6 of Section 2 shows that this can not occur if $a(x)$ changes of sign, since a priori bounds for the positive solutions of (1.1) in one space dimension are available, one can easily imagine that (1.1) might exhibit arbitrarily large positive solutions by chosing Ω_-, or $\|a^-\|_\infty$, sufficiently small.

Our numerical computations strongly suggest that if $N = 1$, Ω_- has n components, and $a(x)$ has a unique local minimum on each of these components, then (1.1) possesses

$$\sum_{j=1}^{n} \binom{n}{j} = 2^n - 1 \tag{1.4}$$

positive solutions for each $\lambda < 0$ sufficiently small. Among them, n solutions will have one single peak on each of the minima of a^-, $n(n-1)/2$ solutions will have two peaks, and in general $\binom{n}{j}$ solutions will exhibit j peaks.

As far as to the structure of the set of positive solutions of (1.1) concerns, the numerics shows that it is strongly based upon the symmetry properties of $a(x)$. More precisely, if we take $\Omega = (0,1)$, $a > 0$ in $(1/3, 2/3)$, $a < 0$ in $(0,1/3) \cup (2/3,1)$, and $a(x)$ is symmetric, then the set of positive solutions consists of a unique global component emanating from $u = 0$ at $\lambda = \pi^2$ and exhibiting a subcritical pitchfork bifurcation at a certain value $\lambda = \lambda_c$ in such a way that (1.1) possesses three positive solutions for each $\lambda < \lambda_c$, one of them with two peaks on each of the negative minima of $a(x)$ and the remaining two with a single peak on each of the negative minima of a. It turns out that breaking the symmetry of the coefficient $a(x)$ results into an imperfect bifurcation at $\lambda = \lambda_c$ and therefore the structure of the solution set of (1.1) suffers a drastical change. Now, the solution set possesses two global components. One of them is the component bifurcating from $u = 0$ at $\lambda = \pi^2$, denoted by $\mathcal{C}^+$, and the other is a *global folding* bounded away from $\mathcal{C}^+$, referred to as $\mathcal{F}^+$. The separation between these two components is as much emphasized as much separated from its original profile is $a(x)$. Nevertheless, however the number of components varied as a result of the symmetry breaking, the number of positive solutions as well as their profiles did not change for $\lambda < \lambda_c$ sufficiently negative.

A really striking feature is the divergence to infinity of $\mathcal{F}^+$ as we make a^- converge to zero in some of the components of Ω_-, e.g. $(0,1/3)$. In this case, keeping fixed $a(x)$ in $(1/3,1]$, the component $\mathcal{C}^+$ converges to the corresponding component of the problem with $\Omega_0 = (0,1/3)$, while the solutions of $\mathcal{F}^+$ grow up to infinity in $(0, 1/3]$. Beside supporting the validity of formula (1.4), these computational experiments confirm the robustness of our numerical schemes and suggest that in order to realize the dynamics of (1.2) the functional spaces where these superlinear indefinite problems are studied should be up-dated to include solutions whose value is infinity in some of the components of Ω_0.

The fact that the number of positive solutions in superlinear subcritical problems can suffer drastical changes as the shape of Ω changes is well documented in the literature, [HV84], [Da88], [Da90],[Ce95]. In fact, breaking down the convexity of Ω can result into an arbitrarily large number of solutions. Our work shows that the same effect arises by varying the coefficients of the model, instead of the shape of the domain, even in the simplest one dimensional models.

One can easily imagine that varying coefficients in higher dimensional problems the complexity of the bifurcation diagrams will increase as much as we wish, since we can play around not only with the shape of Ω but also with the nodal behavior of $a(x)$. Now, it becomes clear why Theorem 1.1 is so relevant. As already commented above, the uniqueness of the stable state does not depend on the geometry of the domain, it is a universal property.

We have already sketched most of the contain of Sections 2, 3. Beside the uniqueness of the stable positive solution, when it exists, in Section 3 we shall obtain some general multiplicity results by using the fixed point index in cones. In Section 4 we introduce the numerical schemes used in the remaining sections. In Section 5 we analyze a pure sublinear model, in Section 6 we analyze an asymmetric superlinear indefinite problem, in Section 7 we analyze a symmetric superlinear indefinite problem, and in Section 8 we analyze the symmetry breaking we were talking before by adding a parameter to the coefficient $a(x)$ of the model treated in Section 7 and studying how vary the corresponding bifurcation diagrams as the parameter varies up to provide us with the asymmetric model of Section 6.

II.2. On the existence of positive solutions

Since λ is regarded as a continuation parameter, the positive solutions of (1.1) will be though as couples (λ, u). Any weak solution of (1.1) lies in $W_p^2(\Omega)$ for all $p > N$, and hence in $C^{2-N/p}(\overline{\Omega})$. In particular, any weak solution is a.e. in Ω twice classically differentiable (e.g. [St70, Theorem VIII.1). In other words, any weak solution is ***strong*** (e.g. [GT83, Chapter 9]). By the strong maximum principle any strong non-negative solution of (1.1) $u \neq 0$ satisfies $u(x) > 0$ for all $x \in \Omega$ and $\frac{\partial u}{\partial n}(x) < 0$ for all $x \in \partial\Omega$, that is, it lies in the interior of the cone P of positive functions of $U := C_0^1(\overline{\Omega})$. It is well known that $\lambda = \sigma_1^\Omega$ is the unique bifurcation value of (1.1) to positive solutions from the trivial state $u = 0$. Moreover, as a consequence from the main theorem of [CR71] a curve of positive solutions emanates from $(\lambda, u) = (\lambda, 0)$ at $\lambda = \sigma_1^\Omega$. More precisely, if Φ stands for the principal eigenfunction associated with σ_1^Ω, normalized so that

$$\int_\Omega \Phi^2 = 1\,,$$

then the following result is satisfied.

Proposition 2.1. *There exist $s_0 > 0$ and two unique mappings of class C^1*

$$\mu : \ (-s_0, s_0) \to \mathbb{R}\,, \qquad v : \ (-s_0, s_0) \to U\,,$$

such that $\mu(0) = 0$, $v(0) = 0$, and for each $s \in (-s_0, s_0)$ $\int_\Omega v(s)\Phi = 0$ and the couple

$$(\lambda(s), u(s)) := (\sigma_1^\Omega + \mu(s), s(\Phi + v(s))) \tag{2.1}$$

is a solution of (1.1). *Moreover, if s_0 is sufficiently small, then these are the unique non-trivial solutions of* (1.1) *in a neighborhood of* $(\lambda, u) = (\sigma_1^\Omega, 0) \in \mathbb{R} \times U$. *Furthermore,*

$$\lim_{s\to 0} \frac{\mu(s)}{|s|^p} = \int_\Omega a\Phi^{p+2}\,. \tag{2.2}$$

Proof

It suffices to check (2.2). The remaining assertions are easily obtained from the main theorem of [CR71]. Substituting (2.1) into (1.1), dividing by s, using the definition of Φ and rearranging terms gives

$$(-\Delta-\sigma_1^{\Omega})v(s)=\mu(s)(\Phi+v(s))-|s|^p a(\Phi+v(s))^{p+1}\,,$$

for each $s\in(-s_0,s_0)$. Now, multiplying this equation by Φ, integrating over Ω and applying the formula of integration by parts yields

$$\mu(s)\int_\Omega \Phi(\Phi+v(s))=|s|^p\int_\Omega a\Phi(\Phi+v(s))^{p+1}\qquad s\in(-s_0,s_0)\,.$$

Dividing this relation by $|s|^p$ and passing to the limit as $s\to 0$ gives (2.2). This completes the proof. $\square$

Since $v(0)=0$ and $\Phi\in\operatorname{int}P$, the couple $(\lambda(s),u(s))=(\sigma_1^{\Omega}+\mu(s),s(\Phi+v(s)))$ provides us with a positive solution of (1.1) for each $s\in(0,s_0)$, if $s_0>0$ is sufficiently small. Therefore, setting

$$\mathcal{D}:=\int_\Omega a\Phi^{p+2} \tag{2.3}$$

the bifurcation to positive solutions is supercritical if $\mathcal{D}>0$ (i.e. $\lambda(s)>\sigma_1^{\Omega}$), while it is subcritical if $\mathcal{D}<0$ (i.e. $\lambda(s)<\sigma_1^{\Omega}$).
By the global bifurcation theorems of [Ra71] and [Da74] the component (maximal closed and connected set) of positive solutions of (1.1) emanating from $u=0$ at $\lambda=\sigma_1^{\Omega}$ is unbounded in $\mathbb{R}\times U$. Let $\mathcal{C}^+$ denote it. The behavior of $\mathcal{C}^+$ is based upon the nodal behavior of $a(x)$ as well as on the nature of the local bifurcation to positive solutions from $u=0$, as the following result illustrates.

Proposition 2.2. *If* (1.1) *possesses a positive solution* (λ,u) *with* $\lambda\geq\sigma_1^{\Omega}$, *then*

$$\mathcal{D}:=\int_\Omega a\Phi^{p+2}>0\,. \tag{2.4}$$

In particular, if $\mathcal{D} \leq 0$, then (1.1) *does not admit a positive solution for $\lambda \geq \sigma_1^\Omega$ and hence, $\mathcal{C}^+ \subset (-\infty, \sigma_1^\Omega) \times U$.*

The proof is based on the following lemma, whose proof can be found in [BCN95] and [Lo97].

Lemma 2.3. *Let $p > N$ and u, $v \in W_p^2(\Omega)$ two arbitrary functions such that $u = v = 0$ on $\partial\Omega$ and $\frac{v}{u} \in C^1(\Omega) \cap C(\overline{\Omega})$. Then, for any function $f : [0, \infty) \to \mathbb{R}$ of class C^1 the following Picone's identity holds*

$$\int_\Omega f(\frac{v}{u})(-v\Delta u + u\Delta v) = -\int_\Omega f'(\frac{v}{u})u^2|\nabla\frac{v}{u}|^2 . \qquad (2.5)$$

Proof of Proposition 2.2

Let (λ, u) be a positive solution of (1.1). Then, since Φ, $u \in \operatorname{int} P$, we have $\frac{\Phi}{u} \in C^1(\overline{\Omega})$ and

$$\left(\frac{\Phi}{u}\right)^{p+1}(-\Phi\Delta u + u\Delta\Phi) = \left(\frac{\Phi}{u}\right)^{p+2} u^2(\lambda - \sigma_1^\Omega) - a(x)\Phi^{p+2} . \qquad (2.6)$$

On the other hand, (2.5) gives

$$\int_\Omega (\frac{\Phi}{u})^{p+1}(-\Phi\Delta u + u\Delta\Phi) < 0 ,$$

since u can not be a multiple of Φ, unless $a = 0$, and $f(t) = t^{p+1}$ is increasing. Therefore, we find from (2.6) that

$$(\lambda - \sigma_1^\Omega)\int_\Omega \left(\frac{\Phi}{u}\right)^{p+2} u^2 < \int_\Omega a\Phi^{p+2} .$$

This completes the proof. $\square$

The next result provides us with the structure of $\mathcal{C}^+$ when $a(x)$ is non-negative (see Theorem I.2.4 in Chapter I). Note that in this case $\mathcal{D} > 0$ and hence, $\mathcal{C}^+$ emanates supercritically from $u = 0$.

Theorem 2.4. *Assume $a \geq 0$, $a \neq 0$. Then,* (1.1) *possesses a positive solution if, and only if, $\sigma_1^{\Omega} < \lambda < \sigma_1^{\Omega_0}$, where $\Omega_0 := \Omega \setminus \overline{\Omega}_+$ and $\sigma_1^{\Omega_0} = \min_{1 \leq j \leq n_0} \sigma_1^{\Omega_0^j}$, being Ω_0^j, $1 \leq j \leq n_0$, the components of Ω_0. Moreover, the positive solution is unique if it exists, and if we denote it by θ_λ, then the map $(\sigma_1^{\Omega}, \sigma_1^{\Omega_0}) \to C(\overline{\Omega})$, $\lambda \to \theta_\lambda$, is C^1, increasing, $\lim_{\lambda \downarrow \sigma_1^{\Omega}} \|\theta_\lambda\|_U = 0$ and $\lim_{\lambda \uparrow \sigma_1^{\Omega_0}} \theta_\lambda = \infty$ uniformly on any compact subset of each of the components Ω_0^j of Ω_0 for which $\sigma_1^{\Omega_0} = \sigma_1^{\Omega_0^j}$.*
Furthermore, if $u_0 \geq 0$, $u \neq 0$, and we denote by $u(x,t;u_0)$ the unique solution of the evolutionary model (1.2), *then $\lim_{t \uparrow \infty} \|u(\cdot,t;u_0)\|_U = 0$ if $\lambda \leq \sigma_1^{\Omega}$, and $\lim_{t \uparrow \infty} \|u(\cdot,t;u_0) - \theta_\lambda\|_U = 0$ if $\sigma_1^{\Omega} < \lambda < \sigma_1^{\Omega_0^j}$, while $\lim_{t \uparrow \infty} \|u(\cdot,t;u_0)\|_{C(\overline{\Omega})} = \infty$ if $\lambda \geq \sigma_1^{\Omega_0}$.*

As an immediate consequence it follows from this result that the set of positive solutions of (1.1) consists of $\mathcal{C}^+$, which is a C^1-curve emanating supercritically from $u = 0$ at $\lambda = \sigma_1^{\Omega}$ and growing up to infinity at $\lambda = \sigma_1^{\Omega_0}$. In particular, L_∞ a priori bounds for the positive solutions are not available, since these solutions bifurcate from infinity at $\lambda = \sigma_1^{\Omega_0}$. Figure 5.1 in Section 5 shows a typical bifurcation diagram under the assumptions of Theorem 2.4.

As far as concerns to the limiting profile of the positive solutions of (1.1) in Ω_+ as $\lambda \uparrow \sigma_1^{\Omega_0}[-\Delta]$, the following result is known (cf. [GGLS98, Theorem 6.4] and [LS98, Theorem 4.3]).

Theorem 2.5. *Under the same assumptions of Theorem 2.4, suppose in addition that Ω_0 is connected and that $a \in C^1(\overline{\Omega})$. Then, $\lim_{\lambda \uparrow \sigma_1^{\Omega_0}} \theta_\lambda(x) = \theta_\infty(x)$ for each $x \in \Omega_+$, where θ_∞ is the minimal positive solution of*

$$-\Delta u = \lambda u - a(x)|u|^p u \quad \text{in } \Omega_+ , \qquad u|_{\partial\Omega_+} = \infty , \tag{2.7}$$

Problems of the same type as (2.7) has been dealt with in [MV97] and the references there in, where the uniqueness of the positive solution of (2.7) was

shown to occur when a is a positive constant. The problem of the uniqueness of the positive solution for (2.7) in our general setting seems to be open.
In the most general case when $a(x)$ changes of sign, the structure of $\mathcal{C}^+$ might change drastically. In this case the existence of L_∞ a priori bounds for the positive solutions of (1.1) depends on how large is the exponent p, and it looks like it also depends on how fast $a^-(x)$ decays to zero on $\partial\Omega_-$. More precisely, the following result was found in [AL98] (cf. Theorem 4.3 and Theorem 5.2 there in).

Theorem 2.6. *Assume that $a(x)$ changes of sign and that some of the following three conditions is satisfied:*

(C1) $N = 1\,,\,2$.

(C2) $N \geq 3$ *and there exist* $\alpha^- : \overline{\Omega}_- \to \mathbb{R}$, *continuous and bounded away from zero in a neighborhood of* $\partial\Omega_-$, *and a constant* $\gamma \geq 0$ *such that*

$$a^-(x) = \alpha^-(x)[\mathit{dist}\ (x, \partial\Omega_-)]^\gamma \qquad x \in \Omega_-\,, \tag{2.8}$$

and $p + 1 < \min\{\,(N+1+\gamma)/(N-1)\,,\,(N+2)/(N-2)\,\}$.

(C3) $N \geq 3$, $\overline{\Omega}_- \subset \Omega$, $\overline{\Omega}_+ \cap \overline{\Omega}_- = \emptyset$, *and* $p + 1 < N/(N-2)$.

Then, any set of positive solutions of (1.1) $\mathcal{S}$ *with* $\Lambda_{\mathcal{S}} := \{\,\lambda \in \mathbb{R} \,:\, (\lambda, u) \in \mathcal{S}\,\}$ *bounded is bounded in* $\mathbb{R} \times U$.

Combining Proposition 2.2 here in with Theorem 3.3, Theorem 7.1 and Theorem 7.4 of [AL98], the following result is easily obtained.

Theorem 2.7. *Assume that $a(x)$ changes of sign and that if $\mathcal{S}$ is any set of positive solutions of* (1.1) *with* $\Lambda_{\mathcal{S}} := \{\,\lambda \in \mathbb{R} \,:\, (\lambda, u) \in \mathcal{S}\,\}$ *bounded, then $\mathcal{S}$ is bounded in $\mathbb{R} \times U$. Let Λ denote the set of λ's for which* (1.1) *possesses a positive solution. Then,* $\Lambda = (-\infty, \sigma_1^\Omega)$ *if* $\mathcal{D} \leq 0$, *and there exists* $\sigma_1^\Omega < \lambda^* < \sigma_1^{\Omega\setminus\overline{\Omega}_+}$ *such that* $\Lambda = (-\infty, \lambda^*]$ *if* $\mathcal{D} > 0$. *Moreover, in this case* (1.1) *possesses two positive solutions (at least) for each* $\lambda \in (\sigma_1^\Omega, \lambda^*)$.

As an immediate consequence from this result, the component $\mathcal{C}^+$ possesses a positive solution for each $\lambda \in (-\infty, \sigma_1^\Omega)$. As suggested by the numerical computations carried out in the next sections, $\mathcal{C}^+$ might be constituted by one or several branches depending on the nodal behavior of $a(x)$, in strong contrast with the situation described by Theorem 2.4. In fact, the set of positive solutions of (1.1) might possess two, or more, components as the numerical computations of Section 8 show.

II.3 Abstract multiplicity results
The existence and the uniqueness of the stable positive solution

Given a positive solution (λ_0, u_0) of (1.1) its stability as an steady-state of (1.2) is given by the spectrum of the linearization

$$\mathcal{L}_{(\lambda_0,u_0)} := -\Delta + (p+1)a(x)u_0^p - \lambda_0 \tag{3.1}$$

under homogeneous Dirichlet boundary conditions. In fact, if (λ_0, u_0) is hyperbolic, then the dimension of its unstable manifold is the sum of the algebraic multiplicities of all the negative eigenvalues of $\mathcal{L}_{(\lambda_0,u_0)}$ (finitely many). If $\sigma_1^\Omega[\mathcal{L}_{(\lambda_0,u_0)}] > 0$, then (λ_0, u_0) is exponentially asymptotically stable, while it is unstable if $\sigma_1^\Omega[\mathcal{L}_{(\lambda_0,u_0)}] < 0$. It will be said that (λ_0, u_0) is ***neutrally stable*** if $\sigma_1^\Omega[\mathcal{L}_{(\lambda_0,u_0)}] = 0$.
The following results provide us with the structure of the solution set of (1.1) around any asymptotically, or neutrally, stable positive solution.

Lemma 3.1. *Let (λ_0, u_0) be a positive solution of* (1.1) *satisfying*

$$\sigma_1^\Omega[\mathcal{L}_{(\lambda_0,u_0)}] > 0\,. \tag{3.2}$$

Then, there exist $\varepsilon > 0$ and an analytic mapping $u : (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \to U$ such that $u(\lambda_0) = u_0$ and $(\lambda, u(\lambda))$ is a positive solution of (1.1) *for each*

$\lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$. Moreover, the map $(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \to C(\overline{\Omega})$, $\lambda \to u(\lambda)$, is increasing and there exists a neighborhood $\mathcal{N}$ of (λ_0, u_0) in $\mathbb{R} \times U$ such that if $(\lambda, u) \in \mathcal{N}$ is a solution of (1.1)*, then $(\lambda, u) = (\lambda, u(\lambda))$ for some $\lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$.*

Proof

The solutions of (1.1) are the zeros of the operator $\mathcal{H} : \mathbb{R} \times U \to U$ defined by

$$\mathcal{H}(\lambda, u) = u - (-\Delta)^{-1}[\lambda u - a|u|^p u] . \tag{3.3}$$

Since $u_0 \in \operatorname{int} P$, for any $u \in U = C_0^1(\overline{\Omega})$ with $\|u\|_U \simeq 0$ we have

$$|u_0 + u|^p (u_0 + u) = u_0^{p+1} (1 + \frac{u}{u_0})^{p+1} .$$

Moreover, $u/u_0 \simeq 0$ in $C(\overline{\Omega})$. Hence, $\mathcal{H}(\lambda, u)$ is real analytic in both arguments at (λ_0, u_0).

We are assuming that $\mathcal{H}(\lambda_0, u_0) = 0$, and (3.2) implies that $D_u \mathcal{H}(\lambda_0, u_0)$ is an isomorphism. Thus, the local existence and uniqueness of the solution curve passing through by (λ_0, u_0) follows from the implicit function theorem. Moreover, by implicit differentiation

$$\mathcal{L}_{(\lambda, u(\lambda))} u'(\lambda) = u(\lambda) > 0$$

and hence, we find from the strong maximum principle that $u'(\lambda) > 0$. Note that (3.2) implies that $\sigma_1^\Omega [\mathcal{L}_{(\lambda, u(\lambda))}] > 0$ for $\lambda \simeq \lambda_0$, since the principal eigenvalue varies continuously with the potential. This completes the proof. $\square$

Proposition 3.2. *Let (λ_0, u_0) be a positive solution of* (1.1) *such that*

$$\sigma_1^\Omega [\mathcal{L}_{(\lambda_0, u_0)}] = 0 . \tag{3.4}$$

Let $\psi_0 > 0$ denote the principal eigenfunction associated with $\sigma_1^\Omega [\mathcal{L}_{(\lambda_0, u_0)}]$. Then, there exist $\varepsilon > 0$ and a real analytic mapping $(\lambda, u) : (-\varepsilon, \varepsilon) \to \mathbb{R} \times U$

such that $(\lambda(0), u(0)) = (\lambda_0, u_0)$ *and* $(\lambda(s), u(s))$ *is a positive solution of* (1.1) *for each* $s \in (-\varepsilon, \varepsilon)$. *Moreover,*

$$u(s) = u_0 + s\psi_0 + v(s)\,, \qquad \lambda(s) = \lambda_0 + s^2\lambda_2 + O(s^3)\,, \tag{3.5}$$

where $v(s) = O(s^2)$ *as* $s \to 0$, $\int_\Omega v(s)\psi_0 = 0$ *for each* $s \in (-\varepsilon, \varepsilon)$, *and*

$$\lambda_2 = \frac{p(p+1)}{2} \frac{\int_\Omega a u_0^{p-1} \psi_0^3}{\int_\Omega u_0 \psi_0} < 0\,. \tag{3.6}$$

Furthermore, there exists a neighborhood $\mathcal{N}$ *of* (λ_0, u_0) *in* $\mathbb{R} \times U$ *such that if* $(\lambda, u) \in \mathcal{N}$ *is a solution of* (1.1), *then* $(\lambda, u) = (\lambda(s), u(s))$ *for some* $s \in (-\varepsilon, \varepsilon)$. *Also,*

$$\mathit{sign}\ \lambda'(s) = \mathit{sign}\ \sigma_1^\Omega[\mathcal{L}_{(\lambda(s), u(s))}]\,. \tag{3.7}$$

Summarizing, the set of solutions of (1.1) *around* (λ_0, u_0) *possesses the structure of a quadratic subcritical turning point. Moreover, the solutions on the upper half branch are linearly unstable, while the solutions on the lower one are exponentially asymptotically stable.*

The existence and the uniqueness of the curve $(\lambda(s), u(s))$ as well as the relation (3.7) were shown in [Am76, Proposition 20.8]. They follow by applying the implicit function theorem to a certain operator related to (3.3) through by a Lyapunov-Schmidt decomposition of U. It remains to prove the validity of the expansion of $\lambda(s)$ in (3.5), as well as the relation (3.6) and the last assertion in the statement.

Proof

For each $s \in (-\varepsilon, \varepsilon)$ we have

$$-\Delta u(s) = \lambda(s) u(s) - a u_0^{p+1} [1 + s\frac{\psi_0}{u_0} + \frac{v(s)}{u_0}]^{p+1}\,. \tag{3.8}$$

Differentiating (3.8) with respect to s at $s=0$ and rearranging terms gives

$$0=\mathcal{L}_{(\lambda_0,u_0)}\psi_0=\lambda'(0)u_0 \tag{3.9}$$

and hence, $\lambda'(0)=0$. Now, differentiating twice with respect to s the relation (3.8), particularizing the resulting equation at $s=0$ and rearranging terms gives

$$\mathcal{L}_{(\lambda_0,u_0)}v''(0)=\lambda''(0)u_0-p(p+1)au_0^{p-1}\psi_0^2\,. \tag{3.10}$$

Note that for any $p>0$ the function $u_0^{p-1}\psi_0^2=u_0^p\psi_0\frac{\psi_0}{u_0}$ is well defined, since ψ_0, $u_0\in\operatorname{int}P$. Multiplying (3.10) by ψ_0, integrating over Ω and applying the formula of integration by parts we find that

$$\lambda''(0)=p(p+1)\frac{\int_\Omega au_0^{p-1}\psi_0^3}{\int_\Omega u_0\psi_0}\,.$$

Thus, to complete the proof of (3.6) it remains to show that

$$\int_\Omega au_0^{p-1}\psi_0^3<0\,. \tag{3.11}$$

Thanks to Lemma 2.3 we obtain

$$\int_\Omega\left(\frac{\psi_0}{u_0}\right)^2(-\psi_0\Delta u_0+u_0\Delta\psi_0)=-2\int_\Omega\psi_0u_0|\nabla\frac{\psi_0}{u_0}|^2<0\,, \tag{3.12}$$

since ψ_0 can not be a multiple of u_0. Moreover,

$$-\psi_0\Delta u_0+u_0\Delta\psi_0=pau_0^{p+1}\psi_0\,,$$

and hence, (3.11) follows from (3.12).

Finally, using (3.7) it is easily seen that $\sigma_1^\Omega[\mathcal{L}_{(\lambda(s),u(s))}]>0$ if $s<0$, while $\sigma_1^\Omega[\mathcal{L}_{(\lambda(s),u(s))}]<0$ if $s>0$. This completes the proof. $\square$

Corollary 3.3.

Let (λ_0, u_0) *be a positive solution of* (1.1) *with* $\sigma_1^\Omega[\mathcal{L}_{(\lambda_0,u_0)}] = 0$. *Then, there exists* $\varepsilon > 0$ *such that for each* $\lambda \in (\lambda_0 - \varepsilon, \lambda_0)$ *the problem* (1.1) *possesses two positive solutions, one of them, say* (λ, u), *satisfying* $\sigma_1^\Omega[\mathcal{L}_{(\lambda,u)}] > 0$. *Moreover,* (1.1) *does not admit a positive solution if* $\lambda > \lambda_0$, $|\lambda - \lambda_0| + \|u - u_0\| < \varepsilon$.

The following result shows that $u = 0$ is the unique non-negative stable solution of (1.1) for $\lambda \leq \sigma_1^\Omega$.

Theorem 3.4. *Let* (λ_0, u_0) *be a positive solution of* (1.1) *with* $\lambda_0 \leq \sigma_1^\Omega$. *Then,* $\sigma_1^\Omega[\mathcal{L}_{(\lambda_0,u_0)}] < 0$.

Proof

If $a \leq 0$, $a \neq 0$, then the proof is very easy. Indeed, by the monotonicity of the principal eigenvalue with respect to the potential and the Krein-Rutman theorem we have

$$\sigma_1^\Omega[-\Delta + (p+1)au_0^p - \lambda_0] < \sigma_1^\Omega[-\Delta + au_0^p - \lambda_0] = 0\,,$$

since $(-\Delta + au_0^p - \lambda_0)u_0 = 0$. The proof in the general case when $a(x)$ changes of sign is far from elementary and it will follow by contradiction. Assume that (1.1) possesses a positive solution (λ_0, u_0) such that $\lambda_0 \leq \sigma_1^\Omega$ and $\sigma_1^\Omega[\mathcal{L}_{(\lambda_0,u_0)}] \geq 0$. Then, thanks to Corollary 3.3, (1.1) possesses a positive solution (λ_1, u_1) satisfying $\lambda_1 \leq \sigma_1^\Omega$ and $\sigma_1^\Omega[\mathcal{L}_{(\lambda_1,u_1)}] > 0$. By Lemma 3.1, through by (λ_1, u_1) passes a regular curve $(\lambda, u(\lambda))$ of positive solutions of (1.1) such that $\sigma_1^\Omega[\mathcal{L}_{(\lambda,u(\lambda))}] > 0$, $\lambda \simeq \lambda_1$. By global continuation to the left of λ_1, some of the following complementary options occurs: Either (i) $u(\lambda) > 0$ and $\sigma_1^\Omega[\mathcal{L}_{(\lambda,u(\lambda))}] > 0$ for each $\lambda < \lambda_1$, or (ii) there exists $\lambda_b < \lambda_1$ such that $u(\lambda) > 0$ and $\sigma_1^\Omega[\mathcal{L}_{(\lambda,u(\lambda))}] > 0$ for each $\lambda \in (\lambda_b, \lambda_1)$, while $u(\lambda_b) = 0$, or (iii) there exists $\lambda_t < \lambda_1$ such that $u(\lambda) > 0$ and $\sigma_1^\Omega[\mathcal{L}_{(\lambda,u(\lambda))}] > 0$ for each $\lambda \in (\lambda_t, \lambda_1)$, while $u(\lambda_t) > 0$ and $\sigma_1^\Omega[\mathcal{L}_{(\lambda_t,u(\lambda_t))}] = 0$.

The option (ii) is excluded, since $\lambda_b < \lambda_1 \leq \sigma_1^\Omega$ and $\lambda = \sigma_1^\Omega$ is the unique bifurcation value to positive solutions from $u = 0$. By Corollary 3.3, the option (iii) can not occur either. Therefore, the option (i) occurs. Note that Lemma 3.1 implies $u(\lambda) < u(\lambda_1) = u_1$ for each $\lambda < \lambda_1$. Now, given $\lambda < \min\{0, \lambda_1\}$, let $x_\lambda \in \Omega$ denote a point where

$$u(\lambda)(x_\lambda) = \|u(\lambda)\|_{L_\infty(\Omega)}\,.$$

Then, since $-\Delta u(\lambda)(x_\lambda) \geq 0$, we find from (1.1) that

$$\lambda - a(x_\lambda)[u(\lambda)(x_\lambda)]^p \geq 0\,.$$

Therefore, $a(x_\lambda) < 0$ and $\lim_{\lambda \downarrow -\infty} u(\lambda)(x_\lambda) = \infty$. This contradicts $u(\lambda) < u_1$ and completes the proof. $\square$

Corollary 3.5. *If* $\mathcal{D} = \int_\Omega a\Phi^{p+2} \leq 0$, *then any positive solution of* (1.1) *is linearly unstable.*

Proof

Assume $\mathcal{D} \leq 0$. By Proposition 2.2, (1.1) does not admit a positive solution if $\lambda \geq \sigma_1^\Omega$. Theorem 3.4 completes the proof. $\square$

Theorem 3.6. *The problem* (1.1) *possesses a stable positive solution if, and only if,* $\mathcal{D} > 0$.

Proof

By Corollary 3.5, $\mathcal{D} > 0$ if (1.1) possesses a stable positive solution. Assume $\mathcal{D} > 0$. Then, the component $\mathcal{C}^+$ of the set of positive solutions emanates from $u = 0$ towards the right of σ_1^Ω. Moreover, thanks to Proposition 2.1, in a neighborhood of $(\lambda, u) = (\sigma_1^\Omega, 0)$ it entirely consists of the regular curve $(\lambda(s), u(s))$, $s > 0$, defined in (2.1). It follows from (2.2) that $\lambda'(s) > 0$ if $s \simeq 0$. Hence, differentiating (1.1) with respect to s gives

$$\mathcal{L}_{(\lambda(s),u(s))} u'(s) = \lambda'(s) u(s) > 0\,.$$

Multiplying this relation by the principal eigenfunction of $\mathcal{L}_{(\lambda(s),u(s))}$, which will be denoted by Φ_s, integrating over Ω and applying the formula of integration by parts we find that

$$\sigma_1^\Omega[\mathcal{L}_{(\lambda(s),u(s))}]\int_\Omega \Phi_s u'(s) = \lambda'(s)\int_\Omega \Phi_s u(s) > 0\,.$$

Therefore,

$$\sigma_1^\Omega[\mathcal{L}_{(\lambda(s),u(s))}] > 0\,,$$

since $u'(s) = \Phi + O(s) > 0$. This shows that in a neighborhood of the bifurcation point the component $\mathcal{C}^+$ entirely consists of asymptotically stable solutions and completes the proof. $\square$

The following result shows that the value λ^* arosen in the statement of Theorem 2.7 equals the λ-coordinate of the first turning point, necessarily subcritical, along the component $\mathcal{C}^+$. It also shows the uniqueness of the stable solution for each $\lambda \in (\sigma_1^\Omega[-\Delta],\lambda^*]$.

Theorem 3.7. *Assume that $a(x)$ changes of sign, that $\mathcal{D} > 0$ and that if $\mathcal{S}$ is any set of positive solutions of* (1.1) *with $\Lambda_S := \{\,\lambda\in\mathbb{R} \,:\, (\lambda,u)\in\mathcal{S}\,\}$ bounded, then $\mathcal{S}$ is bounded in $\mathbb{R}\times U$. Thanks to Theorem 2.7, there exists $\lambda^* > \sigma_1^\Omega$ such that the set of λ's for which* (1.1) *possesses a positive solution is $(-\infty,\lambda^*]$.*
Then, there exists a real analytic map

$$u \,:\, (\sigma_1^\Omega[-\Delta],\lambda^*] \to C(\overline{\Omega})\,, \qquad \lambda \to u(\lambda)\,,$$

such that for each $\lambda \in (\sigma_1^\Omega[-\Delta],\lambda^*]$ the couple $(\lambda,u(\lambda))$ is a positive solution of (1.1) *satisfying $\sigma_1^\Omega[\mathcal{L}_{(\lambda,u(\lambda))}] > 0$ if $\lambda < \lambda^*$, $\sigma_1^\Omega[\mathcal{L}_{(\lambda^*,u(\lambda^*))}] = 0$, and $\lim_{\lambda\downarrow\sigma_1^\Omega} u(\lambda) = 0$. Moreover, the mapping $\lambda \to u(\lambda)$ is increasing and these are the unique positive solutions of* (1.1) *which are not linearly unstable.*

Proof

By the proof of Theorem 3.6, (1.1) possesses a positive solution (λ_0, u_0) with $\lambda_0 \in (\sigma_1^\Omega, \lambda^*)$ and $\sigma_1^\Omega[\mathcal{L}_{(\lambda_0,u_0)}] > 0$. Thanks to Lemma 3.1, through by (λ_0, u_0) passes a real analytic curve $(\lambda, u(\lambda))$ of positive solutions of (1.1) satisfying $\sigma_1^\Omega[\mathcal{L}_{(\lambda,u(\lambda))}] > 0$, $\lambda \simeq \lambda_0$. By Lemma 3.1, Corollary 3.3 and Theorem 3.4 it is easily seen how this local curve can be prolongated to the left of λ_0 up to reach the value $\lambda = \sigma_1^\Omega$ where it degenerates to $u = 0$. Note that $\sigma_1^\Omega[\mathcal{L}_{(\lambda,u(\lambda))}] > 0$ for each $\lambda \in (\sigma_1^\Omega, \lambda_0]$. Similarly, it can be prolongated to the right of λ_0 up to reach a value $\lambda_t \leq \lambda^*$ where $\sigma_1^\Omega[\mathcal{L}_{(\lambda_t,u(\lambda_t))}] = 0$, while $\sigma_1^\Omega[\mathcal{L}_{(\lambda,u(\lambda))}] > 0$ for each $\lambda \in [\lambda_0, \lambda_t)$. By Proposition 3.2, $(\lambda_t, u(\lambda_t))$ is a quadratic subcritical turning point.

Thanks to the local uniqueness guaranteed by Propositions 2.1, 3.2 and using the nondegeneration of the positive solutions on the compact arcs $(\lambda, u(\lambda))$, $\sigma_1^\Omega + \varepsilon \leq \lambda \leq \lambda_t - \varepsilon$, $\varepsilon > 0$, $\varepsilon \simeq 0$, it is easily seen that there exists $\delta > 0$ such that the open δ-neighborhood in $\mathbb{R} \times U$ of the set $(\lambda, u(\lambda))$, $\sigma_1^\Omega \leq \lambda \leq \lambda_t$, denoted by $\mathcal{N}_\delta$, does not contain any positive solution of (1.1) on its boundary. Note that the arcs $(\lambda, u(\lambda))$, $\sigma_1^\Omega + \varepsilon \leq \lambda \leq \lambda_t - \varepsilon$, are compact since the positive solutions of (1.1) are fixed points of a compact operator. We have just isolated the curve of stable positive solutions that we have constructed above.

Assume $\lambda_t < \lambda^*$ and pick $\lambda_1 \in (\lambda_t, \lambda^*)$. Then, by [Am76, Proposition 20.3] the minimal positive solution (λ_1, u_1) of (1.1) is well defined. Moreover, thanks to [Am76, Proposition 20.4], $\sigma_1^\Omega[\mathcal{L}_{(\lambda_1,u_1)}] \geq 0$. By construction, $(\lambda_1, u_1) \in \mathbb{R} \times U \setminus \bar{\mathcal{N}}_\delta$. Now, combining Lemma 3.1 and Proposition 3.2 together with a global continuation argument to the left of λ_1, it is easily seen that (1.1) must have a positive solution linearly asymptotically stable for $\lambda = \sigma_1^\Omega$, since the curve through by (λ_1, u_1) lies within $\mathbb{R} \times U \setminus \bar{\mathcal{N}}_\delta$ and hence, it can not degenerate to $u = 0$. The existence of a stable positive solution for $\lambda = \sigma_1^\Omega$ contradicts to Theorem 3.4. This contradiction shows that $\lambda_t = \lambda^*$.

The fact that (1.1) does not admit a stable positive solution in $\mathbb{R} \times U \setminus \bar{\mathcal{N}}_\delta$ follows by contradiction with the same continuation argument that we have just used to show that $\lambda_t = \lambda^*$.

It remains to show the monotonicity of $\lambda \to u(\lambda)$. Differentiating (1.1) with respect to λ gives

$$\mathcal{L}_{(\lambda,u(\lambda))} u'(\lambda) = u(\lambda) > 0$$

and since $\mathcal{L}_{(\lambda,u(\lambda))}$ satisfies the strong maximum principle we find from this relation that $u'(\lambda) > 0$. This completes the proof. $\square$

In the rest of this section we give some multiplicity results by means of the fixed point index in positive cones. We will assume that $\Omega_- \neq \emptyset$, to avoid the situation described by Theorem 2.4, and that any set of positive solutions $\mathcal{S}$ of (1.1) with $\Lambda_S := \{\, \lambda \in \mathbb{R} \,:\, (\lambda, u) \in \mathcal{S} \,\}$ bounded is bounded in $\mathbb{R} \times U$.

Let $\beta > \sigma_1^{\Omega \setminus \overline{\Omega}_+}$ and $\alpha < \sigma_1^{\Omega}$ be and consider the interval $\Lambda_b := [\alpha, \beta]$. Thanks to Theorem 2.7, (1.1) does not admit a positive solution if $\lambda \geq \beta$. Moreover, we are assuming that the set of positive solutions of (1.1) in Λ_b is bounded in $\mathbb{R} \times U$. Thus, there exists $M > 0$ such that

$$au^p < \lambda + M$$

for any positive solution (λ, u) of (1.1) with $\lambda \in \Lambda_b$.

Let e denote the unique positive solution of

$$(-\Delta + M)e = 1 \quad \text{in } \ \Omega\,, \qquad e|_{\partial\Omega} = 0\,.$$

By the strong maximum principle, $e \in \operatorname{int} P$. Let $C_e(\overline{\Omega})$ denote the ordered Banach space consisting of all functions $u \in C(\overline{\Omega})$ for which there exists a constant $\gamma > 0$ such that $-\gamma e \leq u \leq \gamma e$ endowed with the norm

$$\|u\|_e := \inf \{\gamma > 0 \,:\, -\gamma e \leq u \leq \gamma e \,\}$$

and ordered by the cone of positive functions, P_e. Then, the operators $\mathcal{K}_\lambda : C_e(\overline{\Omega}) \to C_e(\overline{\Omega})$ defined by

$$\mathcal{K}_\lambda u := (-\Delta + M)^{-1}[(\lambda + M)u - a(x)|u|^p u]\,, \qquad \lambda \in \Lambda_b\,, \tag{3.13}$$

are compact and strongly order preserving in the sense of [Am76]. Moreover, the solutions of (1.1) are the fixed points of $\mathcal{K}_\lambda$. Let B_1 denote the unit ball in $C_e(\overline{\Omega})$ and for any $\rho > 0$ set $P_\rho := \rho B_1 \cap P_e$. Since we have uniform a priori bounds for the positive solutions of (1.1) in Λ_b, the fixed point index of $\mathcal{K}_\lambda$ in P_ρ, denoted by ind $(\mathcal{K}_\lambda, P_\rho)$, makes sense if ρ is sufficiently large. Moreover, the following result is satisfied.

Proposition 3.8. *If $\lambda \neq \sigma_1^\Omega$, then $u = 0$ is an isolated fixed point of $\mathcal{K}_\lambda$ in P_e such that*

$$\textit{ind}\ (\mathcal{K}_\lambda, 0) = 1 \quad \textit{if}\ \ \lambda < \sigma_1^\Omega\,, \tag{3.14}$$

and

$$\textit{ind}\ (\mathcal{K}_\lambda, 0) = 0 \quad \textit{if}\ \ \lambda > \sigma_1^\Omega\,. \tag{3.15}$$

Moreover, for all $\lambda \in \Lambda_b$

$$\textit{ind}\ (\mathcal{K}_\lambda, P_\rho) = 0\,. \tag{3.16}$$

Proof

If $\lambda \neq \sigma_1^\Omega$, then the operator $I - D_u\mathcal{K}_\lambda(0)$ is invertible on P_e, i.e. 1 is not an eigenvalue to a positive eigenfunction of $D_u\mathcal{K}_\lambda(0)$. Indeed, since

$$D_u\mathcal{K}_\lambda(0)\Phi = (-\Delta + M)^{-1}[(\lambda + M)\Phi] = \frac{\lambda + M}{\sigma_1^\Omega[-\Delta] + M}\,\Phi\,,$$

we find that spr $D_u\mathcal{K}_\lambda(0) > 1$ if $\lambda > \sigma_1^\Omega$, while spr $D_u\mathcal{K}_\lambda(0) < 1$ if $\lambda < \sigma_1^\Omega$. Thus, Lemma 13.1 of [Am76] completes the proof of (3.14) and (3.15). The relation (3.16) follows from the homotopy invariance of the index using in addition the fact that $u = 0$ is the unique non-negative solution of (1.1) for $\lambda = \beta$. This completes the proof. $\square$

Theorem 3.9. *(i) If $\lambda < \sigma_1^\Omega$, then* (1.1) *possesses a positive solution. If in addition we assume* (1.1) *to have a finite number of non-degenerate positive solutions, say u_1,...,u_N, then $N = 2k+1$ for some $k \geq 0$, and exactly $k+1$ among them have index -1, while the remaining k have index* 1.
(ii) If $\mathcal{D} > 0$, then (1.1) *possesses two positive solutions (at least) for each $\lambda \in (\sigma_1^\Omega, \lambda^*)$. Moreover, if we assume* (1.1) *to have a finite number of non-degenerate positive solutions, say u_1,...,u_N, then $N = 2k$ for some $k \geq 1$, and exactly k among them have index -1, while the remaining k have index* 1.

Proof

(i) Without lost of generality we can assume that α has been chosen so that $\alpha < \lambda$. It follows from (3.14) and (3.16), using the additivity of the fixed point index, that

$$\text{ind}\,(\mathcal{K}_\lambda, P_\rho \setminus \bar{P}_\delta) = -1\,, \tag{3.17}$$

provided $\delta > 0$ is sufficiently small. Therefore, (1.1) possesses another solution in $P_\rho \setminus \bar{P}_\delta$, necessarily positive.
Now, assume that (1.1) possesses $N \geq 1$ nondegenerate positive solutions, u_j, $1 \leq j \leq N$. Then, each of them is isolated and Leray-Schauder formula implies

$$\text{ind}\,(\mathcal{K}_\lambda, u_j) = (-1)^{n_j}\,, \qquad 1 \leq j \leq N\,,$$

where n_j is the sum of the algebraic multiplicities of all the eigenvalues greater than one of $D_u\mathcal{K}_\lambda(u_j)$. In particular, each of these indices equals 1 or -1. Moreover, it follows from (3.17) that

$$\sum_{j=1}^{N} (-1)^{n_j} = -1\,.$$

This completes the proof of Part (i).

(ii) Assume $\mathcal{D} > 0$ and pick $\lambda \in (\sigma_1^\Omega, \lambda^*)$. Then, thanks to Theorem 3.7 the problem (1.1) possesses a unique positive solution (λ, u) such that $\sigma_1^\Omega[\mathcal{L}_{(\lambda,u)}] > 0$. By Leray Schauder formula,

$$\mathrm{ind}\,(\mathcal{K}_\lambda, u) = 1\,,$$

and hence, it follows from the additivity of the fixed point index that

$$\mathrm{ind}\,(\mathcal{K}_\lambda, P_\rho \setminus (u + \delta \bar{B}_1)) = -1\,, \tag{3.18}$$

provided $\delta > 0$ is sufficiently small. Since $u = 0$ is an isolated solution of (1.1) with index zero, once again the additivity of the fixed point together with (3.18) imply the existence of a further positive solution. The last assertion of Part (ii) follows adapting the corresponding argument of the proof of Part (i). $\square$

Remark 3.10. *In the proof of Theorem 3.9, each n_j equals the dimension of the unstable manifold of the corresponding u_j. Therefore, the index is one if, and only if, the dimension of the unstable manifold is odd.*

Note that Theorem 3.9 provides us with the multiplicity result of Theorem 2.7 by using an striking argument substantially simpler than the one given in [AL98, Theorem 7.4]. This is so because as a result of Theorem 3.7 the minimal solution always has index one, except at the value λ^* where it has index zero. This is not necessarily true under the general assumptions of [Am76].

II.4. Pseudo-spectral methods coupled with path following

In the remaining of this Chapter we solve a representative class of one dimensional prototype models of (1.1) by using spectral collocation methods coupled with path-following techniques. This gives high accuracy with low computational work. In all our numerical computations we have used trigonometric modes and the collocation points have been taken to be equidistant, with the number of modes equal to the number of collocation points. Let N denote the number modes and

$$x_i = \frac{i}{1+N}\,, \qquad 1 \le i \le N\,,$$

the collocation points. Then, the solutions $u(x)$ of (1.1) are approximated by

$$u_N(x) = \sum_{j=1}^{N} c_j \sin(j\pi x)\,, \tag{4.1}$$

being $C = (c_1, ..., c_N)^T$ a solution of

$$BC = \lambda JC - A(x)(JC)^{r+1} \tag{4.2}$$

where

$$J = (\sin(j\pi x_i))_{1\le i,j\le N}\,, \qquad B = ((j\pi)^2 \sin(j\pi x_i))_{1\le i,j\le N}\,,$$

$$A(x) = \text{diag }(a(x_i))_{1\le i\le N}\,.$$

For this choice the zero solution of (1.1) as well as the first N bifurcation values from it are preserved. In fact, for regular, turning and simple bifurcation points from the trivial solution the local topological structure of the solution set is known to be coincident for both the discrete and the continuous counterparts of the model (cf. [RBR80], [RBR181], [RBR281]). Therefore, also is preserved the global structure of any compact component of the solution set of (1.1) filled in by regular, turning and bifurcation points, since in one space dimension any bifurcation point is simple. Furthermore, as the number of modes increases the approximated compact component converge to the corresponding continuous one (cf. [LEDM92] and the references there in).

If the coefficient $a(x)$ is continuous, then any solution of (1.1) is of class C^2 and hence its jth Fourier coefficient, say $\hat{u}_j$, decays as $O(j^{-2})$ if $j \uparrow \infty$ (cf. [CHQZ88, pp. 35]) and hence,

$$\max_{0\leq x\leq 1} |\sum_{j=1}^{N+1} \hat{u}_j \sin(j\pi x) - \sum_{j=1}^{N} \hat{u}_j \sin(j\pi x)| = O(N^{-2}) \qquad \text{as} \;\; N \uparrow \infty \, .$$

Due to these features we have used the following criteria to choose the number of modes in our computations

$$|c_N| \leq \frac{1}{2} 10^{-4} \, .$$

To compute the global solution curves of (4.1) as well as the dimension of the unstable manifolds of their solutions we have used standard path-following techniques as in [DK81], [Ei86], [LEDM92] and the references there in, where we send for further details.

II.5. A pure sublinear problem

In this section we consider (1.1) with

$$\Omega = (0,1)\,, \quad p = 4\,, \qquad a(x) = \max\{0, -\sin(3\pi x)\}\,. \tag{5.1}$$

Note that $a(x) = 0$ if, and only if, $x \in [0,1/3] \cup [2/3,1]$ and that $a(x) > 0$ if $x \in (1/3,2/3)$. To short notations we shall denote

$$I_j = ((j-1)/3\,,\, j/3)\,, \qquad 1 \leq j \leq 3\,. \tag{5.2}$$

For this choice (1.1) becomes a pure sublinear problem with a vanishing coefficient $a(x)$ and hence it fits into the framework of Theorem 2.4. Now, we have $\Omega_+ = I_2$ and $\Omega_0 = I_1 \cup I_3$. Thus, $\sigma_1^{\Omega} = \pi^2$ and $\sigma_1^{\Omega_0} = (3\pi)^2 \simeq 88.8264$. As in the remaining sections, to solve this example we have coupled a pure spectral method with collocation and a path continuation solver as already commented in Section 4. Figure 5.1 shows the bifurcation diagram of positive solutions that we have obtained. In the bifurcation diagram we have represented the L_2-norm of the non-negative solutions versus the parameter λ and we have divided it into two pieces, those on the first column, each of them corresponding to a different range of values of λ, from the top to the bottom $(0,60)$ and $(0,90)$, respectively. In the second picture we did not represented the solution $u = 0$. Continuous lines are filled in by stable solutions and dashed lines by unstable solutions, each point on these curves representing a non-negative solution of (1.1). The first diagram represents to the trivial state $u = 0$ together with the curve of positive solutions emanating from it at π^2, where it loses stability becoming unstable. Since this bifurcation is supercritical, by the exchange stability principle the positive solutions are stable for $\lambda \simeq \pi^2$. Theorem 2.4 shows that in fact any positive solution is linearly asymptotically stable, and this agrees with our numerical computations. Each of the pictures on the second column of

Figure 5.1 shows the profiles of some of the solutions along the corresponding piece of the diagram on its left.

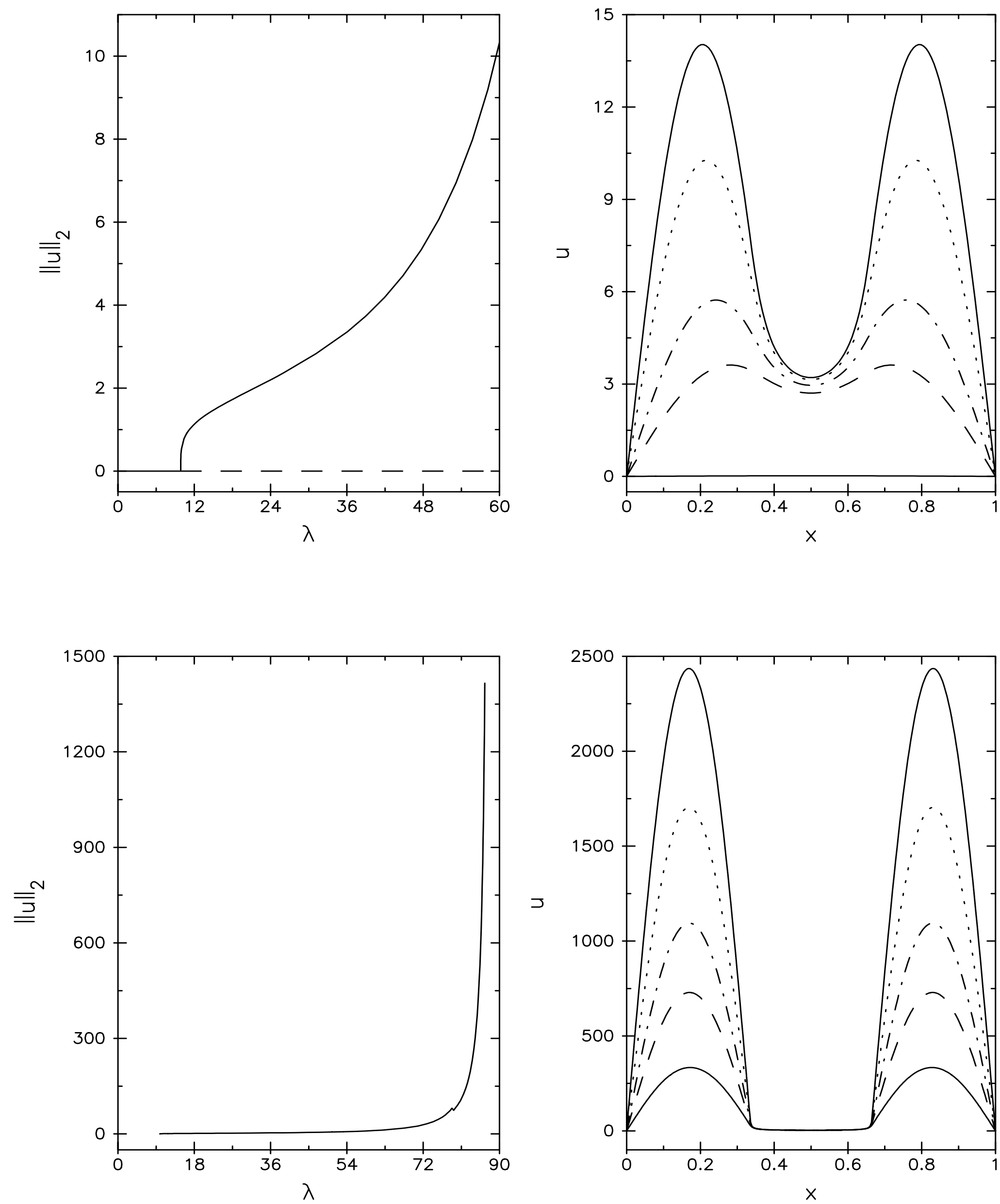

Figure 5.1: Bifurcation diagram and profiles of its solutions.

The picture on the first row shows the positive solutions for each of the following values: $\lambda = 9.8697$, almost zero in the diagram, since this value is very close to the bifurcation value, and $\lambda = 31.1351$, 41.9317, 53.1735, 58.1769. The second picture on the second column shows the profiles of the positive solutions for $\lambda = 83.1139$, 85.0022, 85.6685, 86.2134, 86.5453. Note that the solutions grow as λ increases in complete agreement with Theorem 2.4 and Theorem 2.5.

To carry out the numerical computations we have used 62 modes to complete the calculation of the solutions up to $\lambda = 31.1351$, 143 modes up to $\lambda = 80.0872$, and for larger values of lambda 242 modes in order to increase the accuracy near the bifurcation point from infinity. A lower number of modes makes impossible the calculation of the positive solutions for values of λ close to $\sigma_1^{\Omega_0} = (3\pi)^2$. In few moments it will become clear why. The numerical Jacobian of the linearization of the discrete approximation of (1.1) at the computed solution increases along the curve up to the value $\lambda = 85.6685$, where it becomes decreasing up to $\lambda = 86.5453$. From this value the path following solver gives slow convergence and a higher number of modes should be taken if we would want to compute the solutions for values of λ closer to $(3\pi)^2$, but this is outside the scope of this work. Therefore, we stop the computations here and propose the following numerical value for the point where bifurcation from infinity occurs

$$\text{numerical } \sigma_1^{\Omega_0} \simeq 86.5453\,.$$

At first glance this value is far away from the value given by Theorem 2.4, $\sigma_1^{\Omega_0} \simeq 88.8264$. Fortunately, this difference can be explained from the fact that in the numerical calculations we are reducing the support of $a(x)$ to the interval $(1/3+1/243, 2/3-1/243)$, because $1/3+1/243$ and $2/3-1/243$ are the first collocation points after 1/3 and before 2/3, respectively, and therefore the numerical $\sigma_1^{\Omega_0}$ should approximate to the value

$$(\pi/(1/3+1/243))^2 \simeq 86.6732\,,$$

rather than to 88.8264. Of course, 86.5453 is a quite good approximation of 86.6732. This explains as well why a high number of modes might be

necessary to get a reasonable approximation when spatially varying coefficients arise in the formulation of the model, in strong contrast with the case of constant coefficients where in general a low number of modes are sufficient to get good numerical approximations even when dealing with systems (cf. [LEDM92] and the references there in).
We point out that the numerical positive solution grows to infinity in Ω_0 while it stabilizes in Ω_+, the support of $a(x)$, as λ approaches 86.5453. Thus, it exhibits the same limiting behavior described in Theorems 2.4, 2.5.

II.6. A superlinear indefinite problem

In this section we consider (1.1) with

$$\Omega = (0,1)\,, \quad p = 4\,, \quad a(x) = \begin{cases} 0\,, & x \in \bar{I}_1\,, \\ -\sin(3\pi x)\,, & x \in \bar{I}_2\,, \\ -\frac{1}{2}\sin(3\pi x)\,, & x \in \bar{I}_3\,, \end{cases} \tag{6.1}$$

where I_j, $1 \le j \le 3$, are the intervals defined in (5.2). In this example $\Omega_0 = I_1$, $\Omega_+ = I_2$ and $\Omega_- = I_3$. So, we are working under the assumptions of Theorem 2.6 and hence, uniform a priori bounds for λ in compact intervals of $\mathbb{R}$ are available. Thus, the conclusions of Theorem 2.7 hold. Moreover, $\sigma_1^{\Omega} = \pi^2$, $\sigma_1^{\Omega\setminus\overline{\Omega}_+} = (3\pi)^2$ and $\int_0^1 a(x)\sin^6(\pi x)\,dx \simeq 0.1781 > 0$. Therefore, the conclusions of Theorem 3.7 are satisfied. Figure 6.1 shows the bifurcation diagram that we have computed (left column) as well as the profiles of some of the positive solutions along it (right column). In the bifurcation diagram continuous lines are filled in by stable solutions and dashed lines by unstable ones.

The first picture on the first column shows the bifurcation diagram for $\lambda \in (0, 20)$ and the second one for $\lambda \in (-1000, 100)$. The numerical results agree with the predictions make by Theorem 3.7.

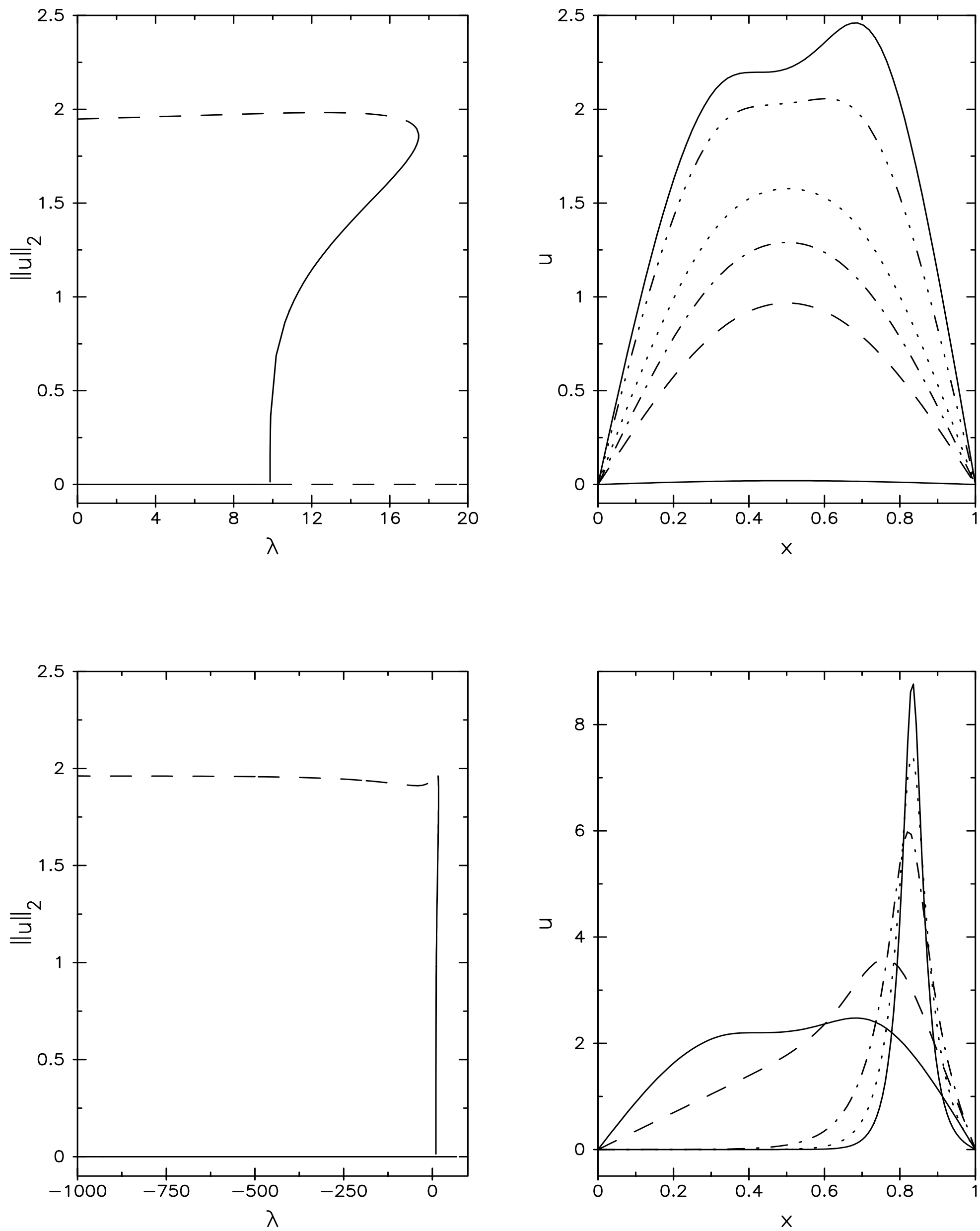

Figure 6.1. Bifurcation diagram and profile of its solutions.

The continuum $\mathcal{C}^+$ of positive solutions introduced in Section 2 emanates supercritically from $u = 0$ at $\lambda = \pi^2$ and exhibits the subcritical turning point at the value $\lambda^* = 17.4790$ where it turns backwards. By Theorem 3.7, the problem does not admit a positive solution if $\lambda > \lambda^*$. The whole curve consists of two arcs, one of them joining the bifurcation point with the turning point, refereed to as the lower branch, and its complement, referred to as the upper branch. Along the lower branch the solutions are linearly asymptotically stable, becoming unstable at the turning point. The solutions on the upper branch have one dimensional unstable manifold in the whole interval where we have computed them $(-1000, 17.4790)$. Therefore, the numerical results fully agree with fixed point index calculations of Section 3.

In the first picture of the second column we have represented the positive solutions along the lower branch corresponding with the values $\lambda = 9.8697$, 10.1817, 10.8567, 12.0639, 15.7601 and 17.4789. Note that the solution corresponding with $\lambda = 9.8697$ is very small, since this value of λ is close to π^2, the bifurcation value. These solutions increase with λ, as predicted by Theorem 3.7. This monotonicity illustrates how the model exhibits a typical sublinear behavior along the lower branch. This behavior is rather natural, since for small positive solutions $a^+(x)$ is the dominant part of $a(x)$. Indeed, it is sufficiently large so that $\mathcal{C}^+$ bifurcates supercritically from $u = 0$. At the turning point the principal eigenvalue of the positive solution vanishes becoming negative along the upper branch, while the remaining eigenvalues of the linearization are positive. In the second picture of the second column we have represented the positive solutions along the upper branch corresponding with the values $\lambda = 17.4774$, 0.0, -200.0, -500.0, -1000.0. Along the upper branch the point-wise monotonicity of the solutions along the lower one is lost. Now, the solutions exhibit a single peak around the minimum of $a(x)$, this peak being as much emphasized as smaller is λ. Roughly speaking, this spike layer behavior is due to the fact that for large solutions (in L_∞-norm) $a^-(x)$ is the dominant part of $a(x)$, instead of $a^+(x)$. This shows how the model exhibits a genuine superlinear behavior along the upper branch.

Our numerical results predict the existence of a unique positive solution for each $\lambda < \pi^2$. Although this uniqueness agrees with Theorem 3.9, the existence of some further solution can not be excluded. In fact, it might happen the solution set to have more than one component, as the example analyzed in Section 8 shows. Nevertheless, we believe that $\mathcal{C}^+$ is the unique component of the set of positive solutions of the present example, and that none secondary bifurcation can occur along it.

II.7. A symmetric superlinear indefinite problem

In this section we make the choice

$$\Omega = (0,1)\,, \quad p = 4\,, \quad a(x) = \begin{cases} -\frac{1}{2}\sin(3\pi x)\,, & x \in \bar{I}_1 \cup \bar{I}_3\,, \\ \\ -\sin(3\pi x)\,, & x \in \bar{I}_2\,, \end{cases} \tag{7.1}$$

where I_j, $1 \leq j \leq 3$, are the intervals defined in (5.2). In this example $\Omega_0 = \emptyset$, $\Omega_+ = I_2$ and $\Omega_- = I_1 \cup I_3$. So, as for the choice (6.1), we are working under the assumptions of Theorem 2.6 and hence, uniform a priori bounds for λ in compact intervals of $\mathbb{R}$ are available. Thus, the conclusions of Theorem 2.7 hold. Moreover, $\sigma_1^\Omega = \pi^2$, $\sigma_1^{\Omega\setminus\bar{\Omega}_+} = (3\pi)^2$ and $\int_0^1 a(x)\sin^6(\pi x)\,dx \simeq 0.1727 > 0$. Therefore, the conclusions of Theorem 3.7 are satisfied. Figure 7.1 shows the bifurcation diagram that we have computed as well as the profiles of some of the positive solutions along it. In the bifurcation diagram continuous lines are filled in by stable solutions and dashed lines by unstable ones. The first picture on the first row shows the bifurcation diagram for $\lambda \in (0,20)$ and the second one for $\lambda \in (-1000,100)$.

Note that specific features shown in the first figure are obviously lost in the second one, since we are using another magnification scale.

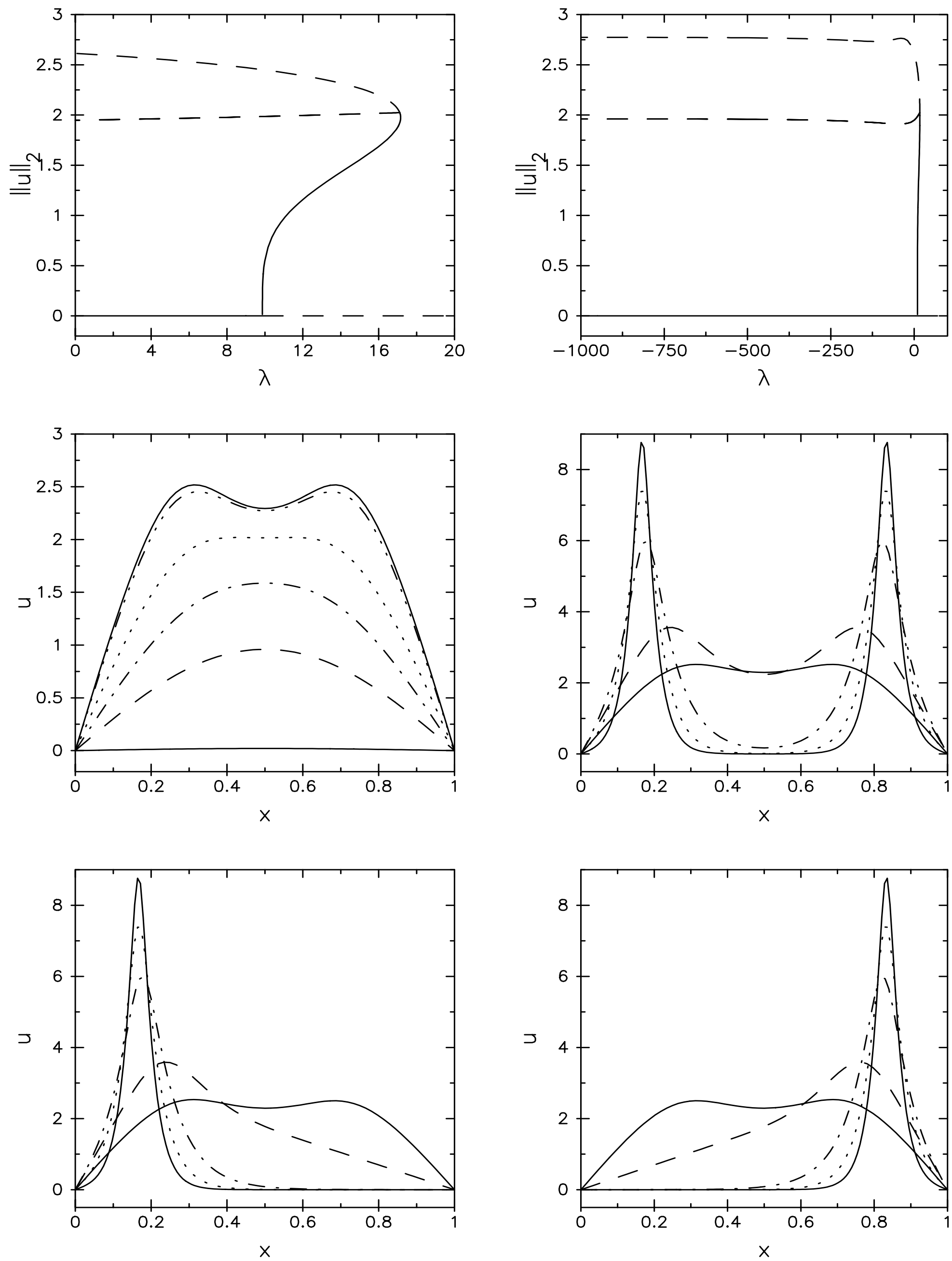

Figure 7.1. Bifurcation diagram and profile of its solutions.

As in Section 6, the numerical results agree with the predictions make by Theorem 3.7. The continuum $\mathcal{C}^+$ emanates supercritically from $u = 0$ at

$\lambda = \pi^2$ and exhibits the subcritical turning point at the value $\lambda^* = 17.1615$ where it turns backwards. By Theorem 3.7, the problem does not admit a positive solution if $\lambda > \lambda^*$. As in the example considered in Section 6, the solutions along the continuum $\mathcal{C}^+$ are linearly asymptotically stable and increase with λ up to reach the turning point where they become unstable, but now a new feature arises. Namely, the unstable manifolds on the upper branch are one-dimensional from $\lambda^* = 17.1615$ up to $\lambda_b := 17.1142$ where they become two-dimensional. So, a secondary bifurcation occurs along the upper branch of $\mathcal{C}^+$. As a result of the symmetry of $a(x)$ the secondary bifurcation is one-sided, and it turns out to be subcritical. Each of the solutions on the secondary branches possesses a one-dimensional unstable manifold. We have computed the solutions on each of the three branches, the primary and the secondary ones, up to the value $\lambda = -1000$, although the bifurcation diagram in the first row of Figure 7.1 only shows two branches, the primary and one of the secondaries. This trouble coming from the fact that the solutions on any of the secondary branches can be obtained by reflection around 0.5 from the corresponding solutions along the other secondary branch, since $a(x)$ is symmetric, and hence the $\| \cdot \|_2$-norm is not able to distinguish between them. Our computations show that the secondary branches are bounded away from the primary one, since for the range of λ's for which we have computed them the principal eigenvalue of the discrete linearizations stayed negative and bounded away from zero while their second eigenvalues always stayed positive and bounded away from zero and the Jacobians of such linearizations grew when λ decreased from 17.1142 up to -1000. We should point out that all these features are completely consistent with the multiplicity results of Section 3. In the second row of Figure 7.1 the first figure shows the plots of the solutions on the lower branch of $\mathcal{C}^+$ (those emanating supercritically from $(\lambda, u) = (\lambda, 0)$ at $\lambda = \pi^2$) corresponding with the values $\lambda = 9.8697$, 10.1620, 12.0291, 15.2227, 17.1614, 17.1137, and the second figure shows the plots of the solutions on the upper half of the primary branch corresponding with $\lambda = 17.1137$, 0.0, -200.0, -500.0, -1000.0. It is rather clear how the last ones exhibit a two-peak layer behavior as $\lambda \downarrow -\infty$ with the peaks located around each of the minima of $a(x)$. Far away from these minima, the solutions converge to zero as $\lambda \downarrow -\infty$. The third row shows the plots of the solutions on each of the two secondary branches corresponding with the values $\lambda = 17.1101$, 0.0, -200.0, -500.0, -1000.0. Each of the plots in the left side figure can be obtained by reflection from the corresponding one in the right side figure, as pointed out above. These solutions exhibit a single peak behavior as $\lambda \downarrow -\infty$. Accordingly with the secondary branch where the solution lies, the peak choses the minimum of $a(x)$ where it is localized.

In strong contrast with the case when $a(x)$ is a negative constant and $N = 1$, where elementary phase portrait techniques apply to show that (1.1) possesses a unique positive solution for each $\lambda < \pi^2$, our numerical computations show that there exists $\lambda_b \in \mathbb{R}$ such that for each $\lambda \in (-\infty, \lambda_b)$ the problem (1.1), (7.1) possesses three positive solutions (at least). Two of them with a single peak on each of the minima of $a(x)$ and the third one with two peaks, each of them on each of these minima. Further numerical computations strongly suggest that one can have as many positive solutions as wanted by chosing a sufficiently wavy $a(x)$. Therefore, varying coefficients in semilinear reaction diffusion equations might provide us with very complex bifurcation diagrams even in one spatial dimension. Our analysis suggests that varying coefficients is sort of equivalent to varying domains in higher dimensional reaction diffusion equations (cf. [HV83], [Da88], [Da90], [Ce95], and the references there in). In higher dimensional superlinear problems, it is well known that breaking down the convexity of the domain can result into multiple positive solutions even for autonomous kinetics. More precisely, joining up two balls of the same radius by a thin narrow strep originates into the superlinear N-dimensional model with $a(x) < 0$ constant the same effect as the coefficient $a(x)$ given by (7.1) does in one space dimension. At the end of the day, the width of the connecting strep can be regarded as a one-dimensional parameter.

One can easily imagine that varying coefficients in higher dimensional problems the complexity of the bifurcation diagrams will increase as much as we wish, since one can play around not only with the shape of the support domain but also with the nodal behavior of $a(x)$. Now, it becomes clear why Theorem 3.9 is so relevant, as the uniqueness of the stable state does not depend on the geometry of the domain nor on the nodal behavior of $a(x)$ but it is a universal property. Note that for each $\lambda \in (\pi^2, \lambda_b)$ our example possesses four positive solutions. since $\pi^2 < \lambda_b = 17.1142$.

II.8. Symmetry breaking towards imperfect bifurcation

In this section we make the choice

$$\Omega = (0,1)\,, \quad p = 4\,, \quad a(x) = \begin{cases} -\varepsilon \sin(3\pi x)\,, & x \in \bar{I}_1\,, \\ -\sin(3\pi x)\,, & x \in \bar{I}_2\,, \\ -\frac{1}{2}\sin(3\pi x)\,, & x \in \bar{I}_3\,, \end{cases} \tag{8.1}$$

where I_j, $1 \leq j \leq 3$, are the intervals defined in (5.2), and $\varepsilon \in [0, \frac{1}{2}]$ is as a real parameter. This family of problems provides us with an homotopy between the problems dealt with in Section 6 and Section 7. If $\varepsilon = 0.5$, then (8.1) becomes (7.1), while it gives (6.1) if $\varepsilon = 0$. Our main goal in this section is analyzing how change the global bifurcation diagrams as the parameter ε varies from 0.5 to 0. If $\varepsilon \in (0, 0.5)$, then $\Omega_0 = \emptyset$, $\Omega_+ = I_2$ and

$\Omega_- = I_1 \cup I_3$. So, as in the previous sections, we have the conclusions of Theorem 2.6 and Theorem 3.7.

Our numerical computations show that an imperfect bifurcation arises as a consequence from the symmetry breaking of the coefficient $a(x)$. Indeed, if $\varepsilon = 0.499995$, then the global bifurcation diagram that we computed looks like the one already analyzed in Section 7 for $\varepsilon = 0.5$, being the same the profile of the solutions on each of its branches, while for $\varepsilon = 0.49995$ the diagram exhibits an imperfect bifurcation at the old pitchfork bifurcation. To detect it the continuation step must be taken sufficiently small and the number of modes sufficiently large. Otherwise, the numerical scheme will provide us with the same diagram obtained for $\varepsilon = 0.5$. The imperfect bifurcation arises at the first value of the parameter $0.49995 \leq \varepsilon \leq 0.5$ where the one-sided bifurcation on the primary branch is lost. For $\varepsilon = 0.49995$ the computations show that the component $\mathcal{C}^+$ is a regular curve possessing a subcritical turning point at $\lambda = 17.1617$ as the unique relevant feature. Figure 8.1 shows a plot of $\mathcal{C}^+$ for $\varepsilon = 0.49995$,

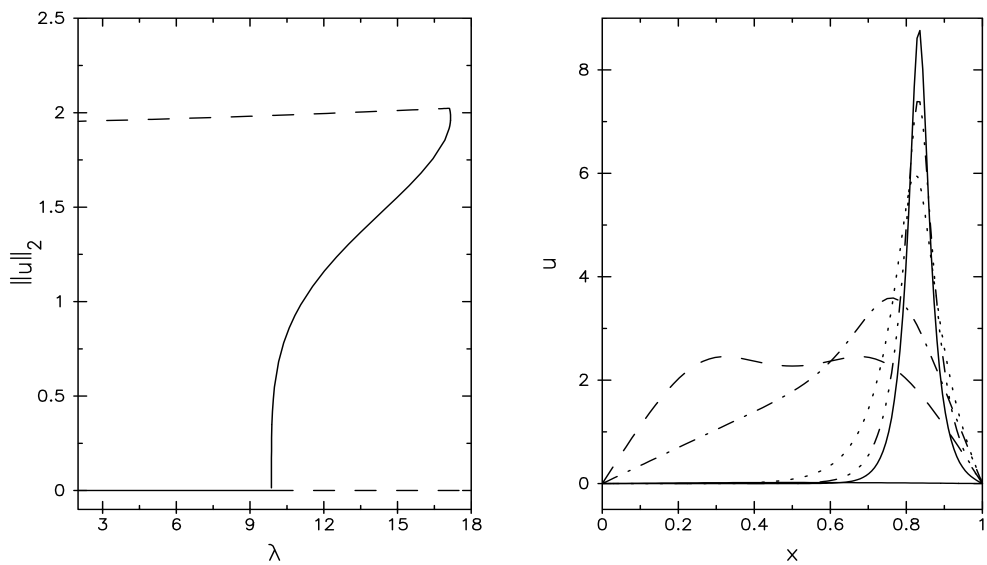

Figure 8.1. $\mathcal{C}^+$ for $\varepsilon = 0.49995$, and the profile of some solutions along it.

as well as the profiles of the solutions corresponding with $\lambda = 17.1617$, 0.0, -200.0, -500.0, -1000.0. An important feature is the spike layer behavior of the solutions along $\mathcal{C}^+$ as $\lambda \downarrow -\infty$. These solutions possess a unique peak around 5/6, which is the point where the absolute minimum of $a(x)$ is reached. For $\varepsilon = 0.5$ and $\varepsilon = 0.499995$, the pitchfork bifurcation occurs at $\lambda = 17.1142$ and $\lambda = 17.1136$, respectively. So, it did not move away to $-\infty$ and since all the solutions along the secondary branches are non-degenerate the implicit function theorem strongly suggests that even for $\varepsilon = 0.49995$ the model should have at least three solutions for a certain interval of λ's to the left of $\lambda = 17.1136$. To compute the two solutions outside $\mathcal{C}^+$ we proceeded by taking $\varepsilon = 0.5$, picking up $\lambda = 17.0087$, which is sufficiently far away from the bifurcation value, and then using ε as the main continuation parameter, instead of λ, in order to compute the perturbation of the unique solution with two peaks. Note that the solution having the single peak around 5/6 perturbs into a solution lying in $\mathcal{C}^+$ for $\varepsilon = 0.49995$. Once calculated the solution with the two peaks corresponding with $\lambda = 17.0087$ and $\varepsilon = 0.49995$, ε is kept fixed and λ is again used as the main continuation parameter to compute the whole component of solutions passing though by it.

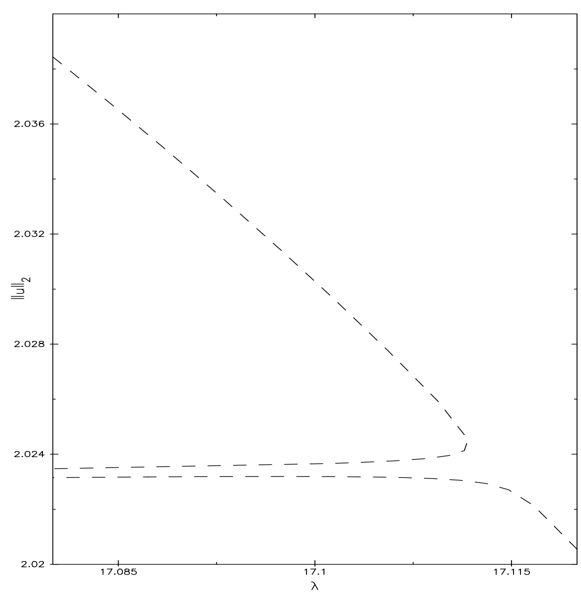

Figure 8.2. Bifurcation diagram for $\varepsilon = 0.49995$.

Figure 8.2 shows a magnified piece of the up-dated bifurcation diagram for $\varepsilon = 0.49995$. It illustrates the imperfect bifurcation originated by the lost symmetry of $a(x)$. To magnify the difference between the branches, we have represented the L_2-norm of the solutions versus the parameter, since the L_∞ norms of the solutions along them are very similar. The bifurcation diagram exhibits two components. Namely, $\mathcal{C}^+$, and a ***global subcritical folding***, referred to as $\mathcal{F}^+$, with the turning point located at $\lambda = 17.1117$. The solutions on the upper half-branch of $\mathcal{F}^+$ have two dimensional unstable manifolds and exhibit two peaks, each of them around each of local minima of $a(x)$, while the solutions on its lower half-branch have one-dimensional unstable manifolds and possess one peak around 1/6, which is a local minimum of $a(x)$.

In Figure 8.3 we have represented the profiles of the solutions along the upper half-branch of $\mathcal{F}^+$ corresponding with $\lambda = 17.0974$, 0.0, -200.0, -500.0, -1000.0, and the solutions on the lower half-branch corresponding with the same values of λ.

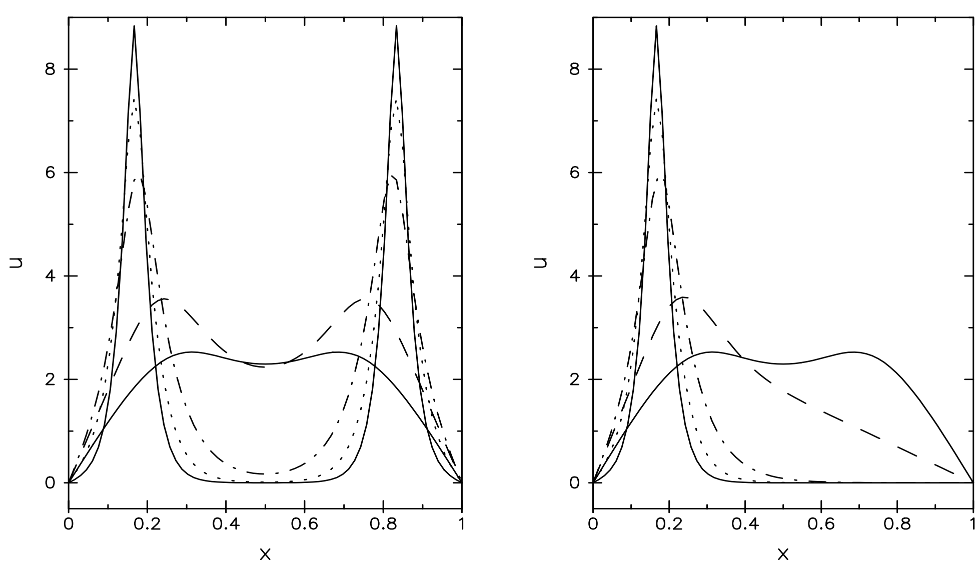

Figure 8.3. The profiles of the solutions along $\mathcal{F}^+$.

The Jacobians of the linearizations along $\mathcal{F}^+$ always increased as long as λ decreased, strongly suggesting that the bifurcation diagram shown in Figure 8.2 will be the right one even far away from the turning point. As in Section 7, we expect the model to have exactly three positive solutions for λ bellow the λ-coordinate of the turning point of $\mathcal{F}^+$. This agrees with the multiplicity results of Section 3.

Imperfect bifurcation phenomena are very well documented in the literature, [GS85], [Ke87],[Lo88]. Our example illustrates why singularity theory has proven so useful in the analysis of nonlinear problems, because from merely local information one can make predictions about the global behavior of the several components of the solution set.

For $\varepsilon < 0.49995$ the bifurcation diagram shown in Figure 8.2 is persistent, although as smaller is taken ε as larger is the separation between $\mathcal{C}^+$ and $\mathcal{F}^+$. Moreover, as $\varepsilon \downarrow 0$ the primary components $\mathcal{C}^+$ approach to the primary component already computed in Section 6 for the case $\varepsilon = 0$, in the sense that the positive solutions along $\mathcal{C}^+$ are point-wise convergent as $\varepsilon \downarrow 0$ to the corresponding positive solutions of the model with $\varepsilon = 0$.

Figure 8.4 illustrates this fact. The first plot of the first row shows the profiles of the solutions of $\mathcal{C}^+$ obtained for $\lambda = 11.9667$ and each of the values $\varepsilon = 0.5,\ 0.5 \times 10^{-7}$. The limiting profile of these solutions as $\varepsilon \downarrow 0$ is almost the same as the profile obtained in Section 6 for $\lambda = 12.0060$, the one on the right side picture. The first plot of the second row shows the profiles of the solutions of $\mathcal{C}^+$ obtained for $\lambda = 17.1101$ and each of the values $\varepsilon = 0.5,\ 0.4581,\ 0.4016,\ 0.2872,\ 0.1137,\ 0.0250$. As for the previous choice of λ, the limiting profile of these solutions as $\varepsilon \downarrow 0$ is almost the same as the profile obtained in Section 6 for $\lambda = 17.0854$, represented on the right side figure. Note that the convergence in the first case is faster than the convergence in the second one. As far as to the solutions in $\mathcal{F}^+$ concerns, the numerics show how they grow to infinity all over $(0, 1/3)$ as $\varepsilon \downarrow 0$, while they stabilize to a bounded profile in $(1/3, 1)$. In particular, the whole component grows to infinity as the parameter $\varepsilon \downarrow 0$ in any L_p-norm, $1 \leq p \leq \infty$.

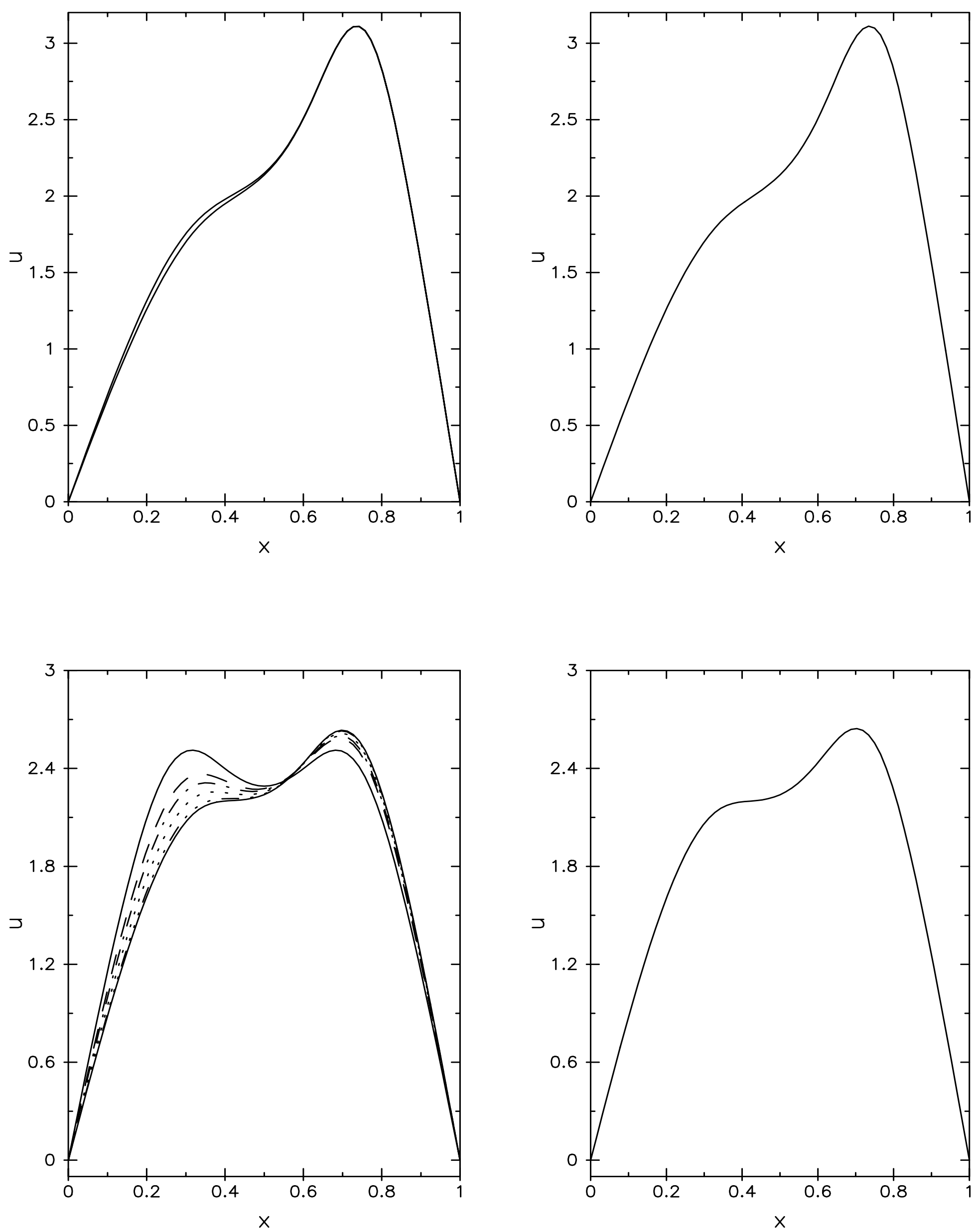

Figure 8.4. The stabilization of the solutions along the primary branches.

In the first row of Figure 8.5 we have represented the plots of the solutions in the upper half-branch of $\mathcal{F}^+$ corresponding with $\lambda = 11.8941$ and each of the following values of ε: $\varepsilon = 0.5,\ 0.1247,\ 0.1,\ 0.001,\ 0.12 \times 10^{-3}$ (left side figure), and $\varepsilon = 0.11 \times 10^{-4},\ 0.1 \times 10^{-5},\ 0.51 \times 10^{-6},\ 0.20 \times 10^{-6},$

0.11×10^{-6} (right side figure).

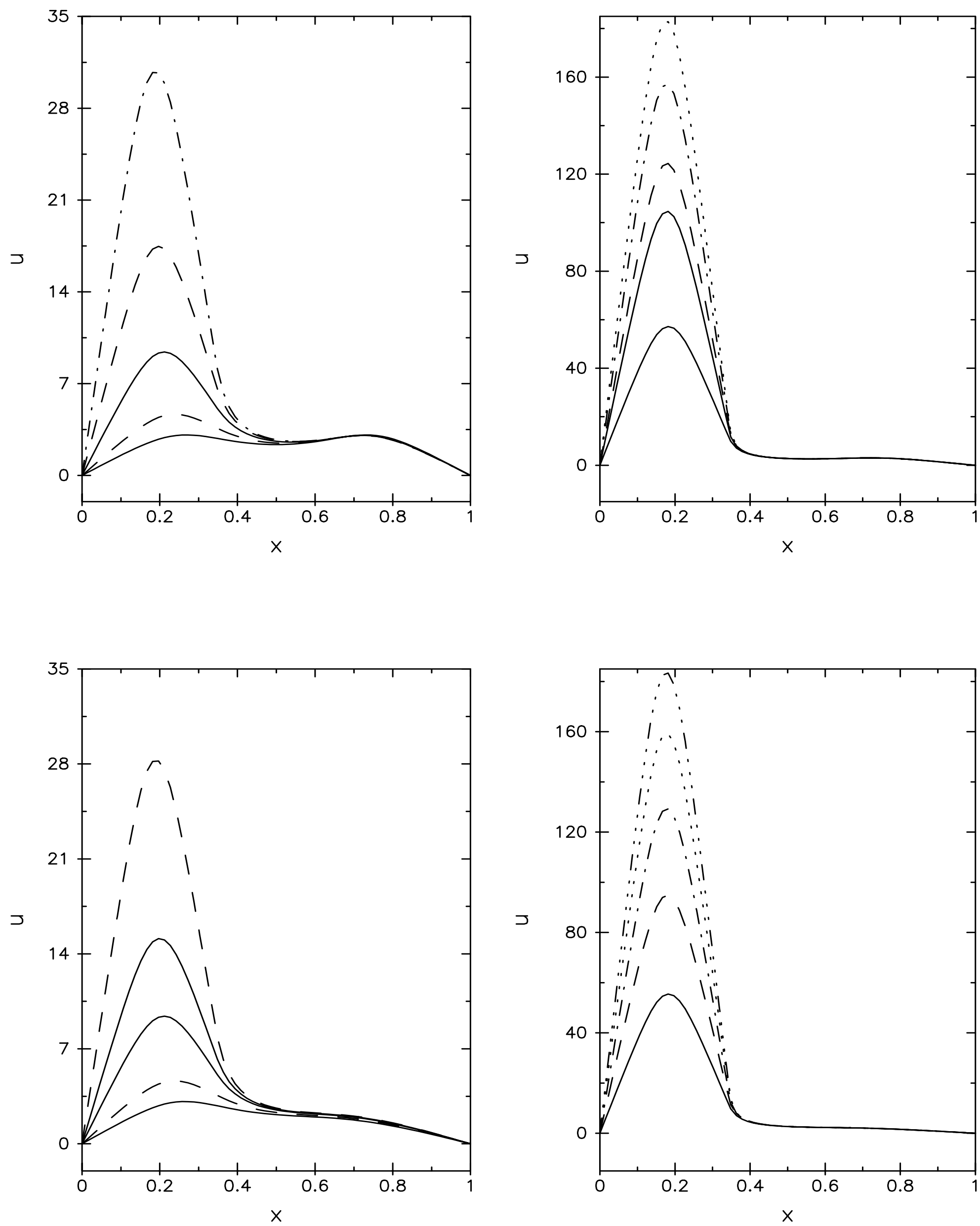

Figure 8.5. Limiting profiles of the solutions on the folding component.

In the second row of Figure 8.5 we have represented the plots of the solutions in the lower half-branch of $\mathcal{F}^+$ corresponding with $\lambda = 11.9364$ and each of the following values of ε: $\varepsilon = 0.5$, 0.1275, 0.01, 0.0017, 0.16×10^{-3} (left side figure), and 0.12×10^{-4}, 0.15×10^{-5}, 0.43×10^{-6}, 0.19×10^{-6}, 0.11×10^{-6}

(right side figure).

The solutions on the lower half-branch of $\mathcal{F}^+$ stabilize to zero in $(1/3, 1)$, while the solutions on the upper half-branch stabilize to a positive solution of the boundary value problem

$$-\theta'' = \lambda\theta - a(x)\theta^5\,, \qquad x \in (1/3, 1)\,, \qquad \theta(1/3) = \infty\,,\ \ \theta(1) = 0\,. \quad (8.2)$$

Beside showing that the a priori bounds of the positive solutions are lost as $\varepsilon \downarrow 0$, these computations illustrate how the choice of the functional space is pivotal for the multiplicity results. If we restrict ourselves to consider classical solutions, then for each $\lambda < \pi^2$ and $\varepsilon \in (0, 0.5]$ the problem (1.1) with the choice (8.1) possesses three classical solutions, two of them with one peak around each of the two local minima of $a(x)$ and the remaining one with two peaks around each of these minima, while the problem with $\varepsilon = 0$ only possesses one solution with one peak around the absolute minimum of $a(x)$. In particular, the folding component bifurcates subcritically from infinity at $\varepsilon = 0$. Things become much more suggestive if we consider functional spaces including functions which are allowed to be infinity on sets of positive measure, as for instance I_1, the intervals where $a(x)$ vanishes for $\varepsilon = 0$, i.e. if we enlarge $\mathbb{R}$ with the infinity point. If we proceed in this way, then (1.1) possesses three solutions for $\varepsilon = 0$ as well. Namely, one on the primary branch, which exhibits a single peak around the value where the absolute minimum of $a(x)$ is taken $(5/6)$, and two non-classical solutions, say u_1 and u_2, which are defined by

$$u_1(x) = \begin{cases} \infty\,, & x \in (0, \frac{1}{3}]\,, \\ 0\,, & x \in (\frac{1}{3}, 1]\,, \end{cases} \qquad u_2(x) = \begin{cases} \infty\,, & x \in (0, \frac{1}{3}]\,, \\ \theta(x)\,, & x \in (\frac{1}{3}, 1]\,, \end{cases}$$

where $\theta(x)$ is the minimal positive solution of (8.2). If, instead of representing the bifurcation diagrams with the L_p-norm of the solutions versus λ, we represent the minus derivative at 1 of the solutions versus λ, for instance, then the bifurcation diagrams of (1.1) with the choice (8.1) will approach as $\varepsilon \downarrow 0$ to the bifurcation diagram of (1.1) with the choice (8.1) and $\varepsilon = 0$. Obviously, this diagram will contain one more component than the one computed in Section 6, where only classical solutions were considered. In particular, structural stability results are strongly based upon the choice of the functional spaces. The implicit function theorem never fails if one choses the right functional space (cf. [BV97]).

Chapter III
The Critical Superlinear Problem

Chapter III

The Critical Superlinear Problem

III.1. Introduction

In this chapter we address the problem of the existence of a priori bounds for the radially symmetric positive solutions of

$$-\Delta u = \lambda u - a(x)|u|^p u \qquad \text{in} \quad \Omega\,, \qquad u|_{\partial\Omega} = 0\,, \tag{1.1}$$

where $\Omega = B_R$ is the ball of center 0 and radius R of $\mathbb{R}^N$, $N \geq 3$, $\lambda \in \mathbb{R}$, $p > 0$ and $a(x)$ is a bounded measurable and radially symmetric non-positive function in $\overline{\Omega}$ such that

$$\Omega_- := \{x \in \Omega \,:\; a(x) < 0\}$$

is open and it possesses a finite number of components, which are annular regions plus eventually some ball centered at the origin. The existence of a priori bounds for the positive solutions is strongly based on the growth of the nonlinearity at infinity, i.e. the size of p. In fact, if $p + 1 < \frac{N+2}{N-2}$ then the results of Chapter II and the references there in show the existence of a priori bounds uniform on compact subintervals of λ, while things change drastically if $p+1 \geq \frac{N+2}{N-2}$, even in the case when a is a negative constant. Assume that this is the case. Then, the following identity by Pohozaev ([Po65]) is satisfied by any classical positive solution of (1.1)

$$N \int_\Omega \int_0^{u(x)} (\lambda s - a s^{p+1})\, ds\, dx + \frac{2-N}{2} \int_\Omega u(x)(\lambda u(x) - a u^{p+1}(x))\, dx$$
$$= \frac{1}{2} \int_{\partial\Omega} \left(\frac{\partial u}{\partial n}\right)^2 < x, n > \, d\sigma\, .$$

Thus,

$$\lambda \int_\Omega u^2(x)\, dx - \left(\frac{2-N}{2} + \frac{N}{p+2}\right) a \int_\Omega u^{p+2}(x)\, dx > 0\, ,$$

and therefore, when $\lambda \leq 0$ the problem (1.1) does not admit a classical positive solution if

$$\frac{2-N}{2} + \frac{N}{p+2} \leq 0\, ,$$

which is equivalent to

$$p+1 \geq \frac{N+2}{N-2}\, .$$

Therefore, the global continuum of positive solutions emanating from $(\lambda, u) = (\lambda, 0)$ at the principal eigenvalue $\lambda = \sigma_1^\Omega[-\Delta]$ blows up at some $\lambda = \lambda^* \in [0, \sigma_1^\Omega[-\Delta]]$, in the sense that there exists a sequence of positive solutions (λ_n, u_n) with $\lambda_n \to \lambda^*$ such that $||u_n||_\infty \to \infty$ as $n \to \infty$. Indeed, if u is a positive solution of (1.1), then

$$\lambda = \sigma_1^\Omega[-\Delta + a u^p] < \sigma_1^\Omega[-\Delta]$$

and hence, if a is assumed to be a negative constant, then a necessary condition so that (1.1) admits a positive solution is $\lambda \in (0, \sigma_1^\Omega[-\Delta])$.

To gain insight into the problem of the search for a priori bounds, in this chapter we compute the curve of classical positive solutions of a three-dimensional radially symmetric prototype model of (1.1). Througout this chapter we assume that $N = 3$, $\Omega = B_{0.5}$ is the ball of radius 0.5 centered at the origin, $r = |x|$ and $u(r)$ is a radially symmetric positive solution of (1.1). With these assumptions problem (1.1) becomes into,

$$\begin{cases} -u''(r) - \frac{2}{r}\, u'(r) = \lambda u(r) - a(r) u^{p+1}\,, & r \in (0, 0.5)\,, \\ u'(0) = 0\,, \quad u(0.5) = 0\,. \end{cases} \tag{1.2}$$

Note that the solutions of (1.2) are the restrictions to $[0, 0.5]$ of the positive solutions of

$$\begin{cases} -u''(r) - \frac{2}{r}\, u'(r) = \lambda u(r) - \hat{a}(r) u^{p+1}\,, & r \in (-0.5, 0.5)\,, \\ u(-0.5) = 0\,, \quad u(0.5) = 0\,, \end{cases} \tag{1.3}$$

where $\hat{a}(r) = a(-r)$ for each $r \in [-0.5, 0]$, and $\hat{a}(r) = a(r)$ if $r \in [0, 0.5]$. As in Section $I.5$, instead of (1.3) we will consider its phase translation to the interval [0.1]

$$\begin{cases} -u''(x) - \frac{2}{x-0.5}\, u'(x) = \lambda u(x) - \hat{a}(x - 0.5) u^{p+1}\,, & x \in (0, 1)\,, \\ u(0) = 0\,, \quad u(1) = 0\,. \end{cases} \tag{1.4}$$

To compute the bifurcation diagram of positive solutions of (1.4) we use the same spectral collocation methods coupled with path-following techniques introduced in the previous chapters. So, we use trigonometric modes and equidistant collocation points with the number of modes equal to the number of collocation points. Let M denote the number of modes and $x_i = \frac{i}{1+M}$, $1 \le i \le M$, the collocation points. Then, the solutions $u(x)$ of (1.4) are approximated by

$$u_M(x) = \sum_{j=1}^{M} c_j \sin(j\pi x)\,,$$

being $C = (c_1, ..., c_M)^T$ (T =transposition) a solution of

$$BC - EDC = \lambda JC - A(JC)^{p+1} \tag{1.5}$$

where

$$\begin{cases} J = (\sin(j\pi x_i))_{1\le i,j\le M} \,, \\ B = ((j\pi)^2 \sin(j\pi x_i))_{1\le i,j\le M} \,, \\ A = \text{diag } (\hat{a}(x_i - 0.5))_{1\le i\le M} \,, \\ E = \text{diag } (e_j)_{1\le j\le M} \,, \\ D = (d_{ij})_{1\le i,j\le M} \,, \end{cases}$$

$$e_j = \begin{cases} \frac{2}{x_j - 0.5} \,, \text{ if } x_j \neq 0.5 \,, \\ 2 \,, \text{ if } x_j = 0.5 \,, \end{cases}$$

and

$$d_{ij} = \begin{cases} j\pi \cos(j\pi x_i) \,, \qquad 1 \le j \le M \,, \text{ if } x_i \neq 0.5 \,, \\ -(j\pi)^2 \sin(j\pi/2) \,, \qquad 1 \le j \le M \,, \text{ if } x_i = 0.5 \,. \end{cases}$$

Making these choices, the zero solution of (1.2) is preserved, although the bifurcation values to positive solutions for the continuous and the corresponding discrete models will not be equal, since we are working with trigonometric series instead of with Fourier series of Bessel functions (see Section $I.5$). The criterion to choose the number of modes in our computations is the same that in Section $I.5$. Namely,

$$|c_k| \le \frac{1}{2} 10^{-4} \,, \qquad M - 10 \le k \le M \,. \tag{1.6}$$

Since the problem (1.3) is an extension of the problem (1.2), we can obtain positive solutions of (1.3) that are not radially symmetric and which are not solutions of the problem (1.1). In this work we are only interested on the radially symmetric positive solutions of (1.1) and, hence, $c_{2\ell} = 0$ for all $\ell \in \mathbb{N}$ and (1.6) will be satisfied for any even k.

All our numerical computations have been carried out by taking $a(r) = -1.0$ in Ω_- for all our choices of Ω_-, and $p + 1 = 5$, which is the critical exponent for $N = 3$. Also we have solved the problem (1.3), for each choice of Ω_-, with $p + 1 = 4$, bellow the critical exponent, to make the comparison between both cases. In Section 2 we solve and discuss the problem taking $a(r) = -1.0$ in a ball centered at the origin with radius ρ, $0 < \rho \leq 0.5$; namely, $\Omega_- = \Omega = B_{0.5}$ and $\Omega_- = B_{1/6}$. In Section 3, we choose $a(r) = -1.0$ in an annulus centered at the origin with interior radius ρ, $0 < \rho < 0.5$ and exterior radius 0.5, $\Omega_- = A_{(\rho,0.5)}$. A really striking feauture is the fact that if $a(r)$ vanishes on some small ball B_ε while it is kept as a negative constant on its complement then our numerical calculations show that the model possesses a positive solution for a range of values $\lambda < 0$ bounded away from zero when $p+1 = 5$, the critical exponent, so suggesting that in this case the radially symmetric positive solutions of the model should have a priori bounds. While if $a(r)$ remains negative on any ball B_ρ, $0 < \rho \leq 0.5$, then the positive solutions grow to infinity at some $0 < \lambda^* < \sigma_1^\Omega[-\Delta]$, as it was predicted by Pohozaev's identity.

III.2. Case $\Omega_- = B_\rho$, $0 < \rho \leq 0.5$

In this section we solve the problem,

$$\begin{cases} -u''(r) - \frac{2}{r}\, u'(r) = \lambda u(r) - \chi_{B_\rho} u^{p+1} \,, & r \in (-0.5, 0.5)\,, \\ u(-0.5) = 0\,, \quad u(0.5) = 0\,. \end{cases} \tag{2.1}$$

where χ_C, $C \subset (-0.5, 0.5)$, is defined by

$$\chi_C = \begin{cases} -1.0\,, & if \quad r \in C\,, \\ 0.0\,, & if \quad r \in (-0.5, 0.5) \setminus \overline{C}\,. \end{cases} \tag{2.2}$$

and $B_\rho = (-\rho, \rho)$. The values of $p+1$ will be 4, bellow of critical exponent, and $p+1 = 5$, the critical exponent.

The theoretical and numerical values of the principal eigenvalue of $-\Delta$ in $B_{0.5} \subset \mathbb{R}^3$ under Dirichlet boundary conditions are given in the following table.

Theoretical σ_1	Computed σ_1
39.478418	39.477579

The theoretical value is calculated from the estimate 3.14159265 for the first zero of the Bessel function $J_{1/2}$. The numerical value is the unique value of λ for which bifurcation to positive solutions from $u = 0$ occurs. This value has been computed by means of the pseudo-spectral method described in Section $III.1$ using 201 modes. Note that the principal eigenfunction associated with $B_{0.5}$ is radially symmetric and therefore, we are actually dealing with one-dimensional linear eigenvalue problems.

At this point one should make some comments. If an even number of collocation points are used, then we can not save the zero singularity and in fact the numerical program give us a first bifurcation point from trivial branch at a value of λ which is not the first zero of $J_{1/2}$. At this value the zero solution becomes unstable, with one-dimensional unstable manifold, until the second bifurcation point, which correspond just with first zero of $J_{1/2}$, where the unstable manifold becomes two-dimensional. Table 2.1 shows some of the computed values of the parameter λ where bifurcation from $u = 0$ occurs when we use an even number of collocation points.

λ	Number of modes
10.385976	124
10.186491	200
10.079563	300
10.026552	400
9.994900	500
9.958875	700

Table 2.1 First bifurcation point with an even number of points.

Note that the values of λ in Table 2.1 seem to stabilize to some value, but the convergence is very slow, in strong contrast with the values found from the first zero of $J_{1/2}$ which are always very close to the theoretical one. These features can be explained from the fact that the linearization at $u = 0$, which is given by

$$\begin{cases} -u''(r) - \frac{2}{r}\, u'(r) = \lambda u(r)\,, \quad r \in B_{0.5} \subset \mathbb{R}^3\,, \\ \\ u = 0\,, \quad in\ \partial B_{0.5}\,. \end{cases} \tag{2.3}$$

posseses two independent solutions defined by

$$x^{-1/2} J_{1/2} = A \frac{sinx}{x}\ , \quad x^{-1/2} J_{-1/2} = A \frac{cosx}{x}\ , \quad A = \sqrt{2/\pi}\ ,\ x = r\sqrt{\lambda}\ .$$

The function $x^{-1/2} J_{-1/2}$ has a singularity at 0 and its first zero is located at $x = \frac{\pi}{2}$. The principal eigenvalue of problem (2.3) associated with the first zero of $J_{-1/2}$ will be $\lambda = [\frac{(\pi/2)}{(1/2)}]^2 \simeq 9.869604$. Hence, the values at Table 2.1 are a not very good approximation to this principal eigenvalue. Clairly we would need a much larger number of modes to obtain a good aproximation. While the function $J_{1/2}$ is a regular solution with its first zero located at $x = \pi$, and hence, the principal eigenvalue of (2.3) is $\lambda = [\frac{\pi}{(1/2)}]^2 \simeq 39.478418$.

To obtain this principal eigenvalue we have to save the singularity 0, and this is why we will work with an odd number of collocation points.
First, we have solved equation (2.1) making the choice $\rho = 0.5$. In this case (2.1) becomes into,

$$\begin{cases} -u''(r) - \frac{2}{r}\,u'(r) = \lambda u(r) + u^{p+1}\,, \quad r \in (-0.5, 0.5)\,, \\ \\ u(-0.5) = 0\,, \quad u(0.5) = 0\,, \end{cases} \tag{2.4}$$

Figure 2.2 shows the bifurcation diagram that we have found and some of the profiles of the radially symmetric positive solutions for several values of λ, in the special case when $p+1 = 4$. The first plot of Figure 2.2 represents the L_∞-norm of each of the computed non-negative solutions versus the parameter λ, $\lambda \in (-100.0, 50.)$. Continuous lines are filled in by stable solutions and dashed lines by unstable solutions. Each of the points on these curves represents a radially symmetric non-negative solution of (2.4).

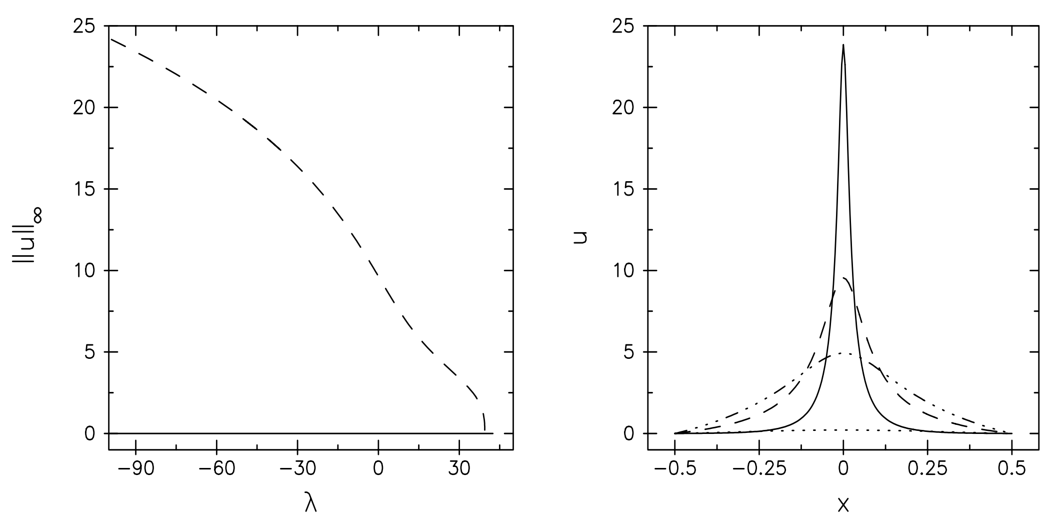

Figure 2.2: Bifurcation diagram and solution profiles in case p+1=4.

The diagram shows two curves, one of them filled in by positive solutions which are unstable, and the other is the trivial branch $(\lambda, u) = (\lambda, 0)$. The solution $u = 0$ is stable until the bifurcation value $\lambda = 39.477579$ where it becomes unstable for any further value of the parameter. The branch of positive solutions emanates subcritically from the trivial branch at the bifurcation value being the unstable manifold of the positive solutions on this branch one-dimensional until the value $\lambda = 39.389274$ where it becomes two-dimensional; at this value of λ there exists a subcritical secondary bifurcation to a new branch of positive solutions which is not included in the plot; positive solutions along this branch are not radially symmetric and we shall not focus our attention on them here in. Observe that there exist positive solutions for values of $\lambda \leq 0$ and that $p+1 = 4$ is bellow of critical exponent. So that the positive solutions satisfy the convergence criterion (1.6) we have needed 125 modes for $\lambda \in (-36.249395, 39.472111]$ and 201 modes for $\lambda \in [-95.230062, -36.249395]$. The second plot in Figure 2.2 shows the profiles of the positive solutions for the values of the parameter $\lambda = 39.472111$, 20.225725, 0.329421 and -95.230062; the profiles of the solutions exhibit a single peak around 0, this peak being as much emphasized as smaller is λ, and converge to 0 in the rest of the domain. The results completely agree with the analytical mathematical results of Chapter II.

Now we take $p + 1 = 5$. Then, the situation is completely different. Figure 2.3 shows the bifurcation diagram that we have obtained in this case and some of the profiles of the corresponding radially symmetric solutions. As above, in the first plot of Figure 2.3 we have represented the L_∞-norm of each of the non-negative solutions versus the parameter λ, $\lambda \in (10.0, 50.0)$. The diagram shows two curves, one of them filled in by the radially symmetric positive solutions which are unstable, and the other is the trivial branch $(\lambda, u) = (\lambda, 0)$. The solution $u = 0$ is stable until the bifurcation value $\lambda = 39.477579$ where it becomes unstable. The branch of positive solutions emanates subcritically from the trivial branch at $\lambda = 39.77579$ with one-dimensional unstable manifold until the value $\lambda = 39.410638$, where they become two-dimensional; at $\lambda = 39.410638$ there exists a secondary bifurcation to positive solutions not radially symmetric which were not plotted here. The results look like before, but now a new feature arises. The maximum of each of the positive solutions, located at the center of the domain, grows as λ decreases approaching a positive value of λ, $\lambda^* \simeq 10.407598$.

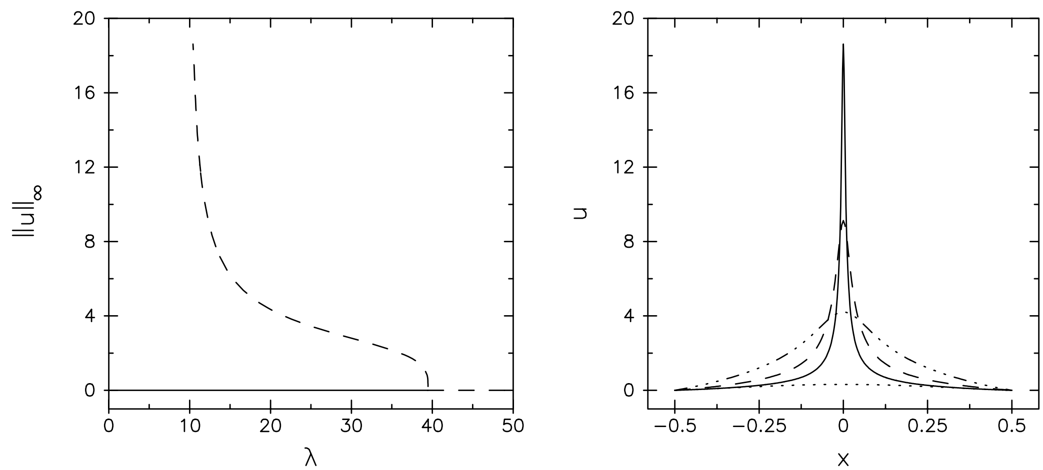

Figure 2.3: Bifurcation diagram and solution profiles in case p+1=5.

To calculate the positive solutions as λ approaches λ^* we needed increasing the number of modes drastically to reach the convergence criterion (1.6). Table (2.4) shows the number of modes needed to reach it.

Interval of λ's	Number of modes
$(13.228650, 39.474591]$	125
$(11.346510, 13.228650]$	201
$[10.407598, 11.346510]$	501

Table 2.4: The number of modes needed to reach (1.6).

We are close to reach the limit of alowable memory of our computer and so we must stop our calculations here in. In the second picture of Figure 2.3 we have plotted the solution profiles corresponding with the values of the parameter $\lambda = 39.472611$, 20.615684, 12.297267 and 10.407598. As in the case $p+1 = 4$, the solutions exhibit a one-pike layer behavior with the peak located around 0 and converge to 0 in the rest, but this time this behavior happens for $0 < \lambda^* \leq \lambda$. So, as it was predicted by the theoretical results, the a priori bounds are lost.

The situation is quite similar if we take $0 < \rho < 0.5$. Indeed, if we consider (2.1) with $\rho = 1/6$, then, we are concerned with the problem

$$\begin{cases} -u''(r) - \frac{2}{r}\, u'(r) = \lambda u(r) - \chi_{B_{1/6}} u^{p+1}\,, \quad r \in (-0.5, 0.5)\,, \\ \\ u(-0.5) = 0\,, \quad u(0.5) = 0\,, \end{cases} \tag{2.5}$$

If $p+1 = 4$, we have a priori bounds and the results that we have obtained are rather similar to the ones found in the previous case, so we solve (2.5) taking $p+1 = 5$. Figure 2.5 shows the bifurcation diagram that we have obtained and some of the profiles of the corresponding radially symmetric solutions.

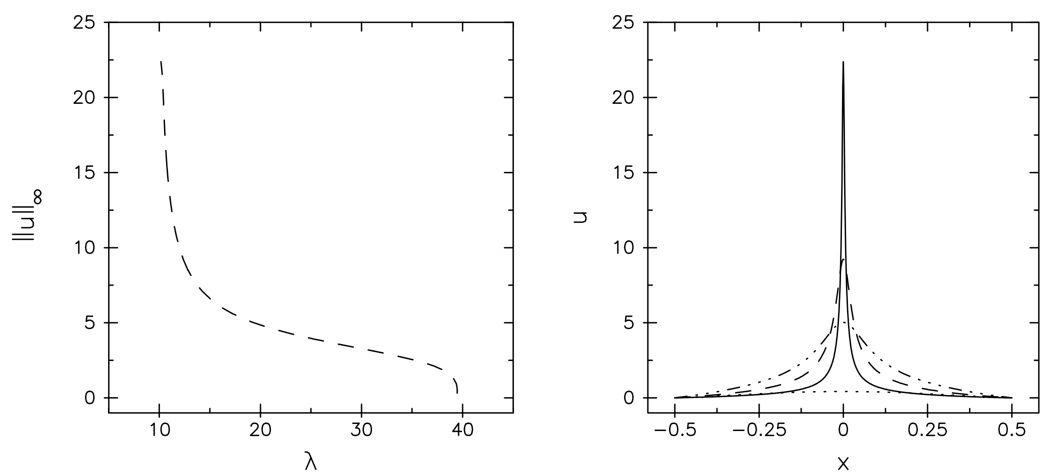

Figure 2.5: Bifurcation diagram and solution profiles for (2.5), p+1=5.

In the first plot of Figure 2.5 we have represented the L_∞-norm of each of the non-negative solutions versus the parameter λ, $\lambda \in (10.0, 50.0)$. The diagram that we have obtained looks like the one found for problem (2.4). We have two curves, one of them filled in by radially symmetric positive solutions which are unstable, and the other is the trivial branch $(\lambda, u) = (\lambda, 0)$. The solution $u = 0$ is stable until the bifurcation value $\lambda = 39.477579$ where it becomes unstable. The branch of positive solutions emanates subcritically from the trivial branch at $\lambda = 39.477579$ with one-dimensional unstable manifold until the value $\lambda = 39.423582$ where they become two-dimensional; at $\lambda = 39.410638$ there exists a secondary bifurcation to non radially symmetric positive solutions not radially symmetric which were not plotted. The maximum of each of the positive solutions, also located at the center of the domain, grows as λ decreases to a certain positive value of λ that we have denoted by λ^*; this time we have used a larger number of modes in order to give a better aproximation for λ^*. Table (2.6) shows the number of modes needed to calculate the radially symmetric solutions of problem (2.5) inppossing the convergence criterion (1.6).

Interval of λ	Number of modes
$(12.827966, 39.473845]$	201
$(10.751210, 12.827966]$	401
$[10.364946, 10.751210]$	701

Table 2.6: The number of modes needed to reach (1.6).

We have calculated some additional solutions using 701 modes until the value $\lambda = 10.155838$ under the weaker convergence criterion

$$|c_k| \leq \frac{1}{2} 10^{-3}\,, \qquad M - 10 \leq k \leq M\,.$$

So, the numerical value proposed for λ^* is 10.155838. In the second picture of Figure 2.5 we have plotted the profiles of the radially symmetric solutions corresponding with the values $\lambda = 39.471180$, 19.221888, 12.361597 and 10.155838.

Observe that the last value of λ for which we have calculated a solution satisfying the criterion (1.6) is $\lambda = 10.407598$ for problem (2.4) when we use

501 modes, and $\lambda = 10.364946$ for problem (2.5) using 701 modes. These values are close to the values $\lambda = 9.994900$ and $\lambda = 9.958875$ of the Table 2.1 corresponding to the approximation of the principal eigenvalue $\sigma_1^{B_{0.5}}[-\Delta]$ using 500 and 700 collocations points, respectively. These numerical results strongly suggest that the value of the parameter where the solutions blow-up to ∞ is the value $\lambda = 9.869604$ corresponding to the first zero of the Bessel function $J_{-1/2}$, independently on the size of the ball B_ρ where a(r) is negative, although any theoretical result does not exist about this. We have tried to continue the branch bifurcating from trivial branch at $\lambda = 9.869604$, but we could not. This analysis is in progress.

III.3. Case $\Omega_- = A_{(\rho,0.5)}$, $0 < \rho < 0.5$

In this section we solve the problem,

$$\begin{cases} -u''(r) - \frac{2}{r}\, u'(r) = \lambda u(r) - \chi_{A_{(\rho,0.5)}}\, u^{p+1} \,, \quad r \in (-0.5, 0.5)\,, \\ u(-0.5) = 0\,, \quad u(0.5) = 0\,. \end{cases} \tag{3.1}$$

where, $A_{(\rho,0.5)} = (-0.5, -\rho) \cup (\rho, 0.5)$ is an annulus centered at the origin with interior radius equal to ρ and exterior radius 0.5; $\chi_{A_{(\rho,0.5)}}$ is given by (2.2).

The theoretical and numerical values of the principal eigenvalue of $-\Delta$ in $B_{0.5}$ of $\mathbb{R}^3$ under Dirichlet boundary conditions are the same as in Section 2:

Theoretical σ_1	Computed σ_1
39.478418	39.477579

The same comments of Section 2 about the even number of collocation points are valid as well in our current situation. So, we will work with an odd number of collocation points.

We have solved equation (3.1) taking $\rho = 1/6$. In this case problem (3.1) becomes into,

$$\begin{cases} -u''(r) - \frac{2}{r}\, u'(r) = \lambda u(r) - \chi_{A_{(1/6,0.5)}} u^{p+1}\,, \quad r \in (-0.5, 0.5)\,, \\ u(-0.5) = 0\,, \quad u(0.5) = 0\,, \end{cases} \tag{3.2}$$

Firstly we have solved the equation (3.2) taking $p+1 = 4$. Figure 3.1 shows the bifurcation diagram and some of the profiles of the radially symmetric solutions that we have computed. As in Section 2 the diagram shows two curves, one of them filled in by radially symmetric positive solutions which are unstable, and the other is the trivial branch $(\lambda, u) = (\lambda, 0)$. The solution $u = 0$ is stable until the bifurcation value $\lambda = 39.477579$ where it becomes unstable.

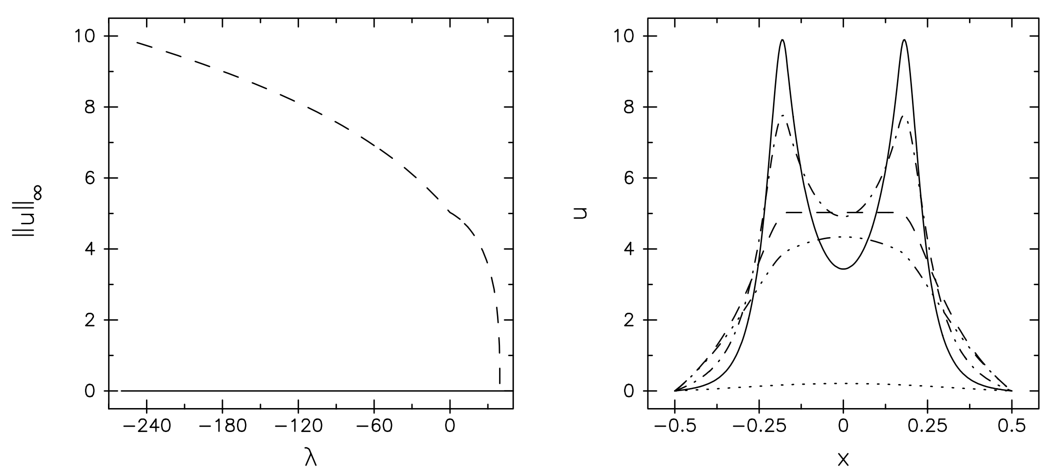

Figure 3.1: Bifurcation diagram and solution profiles for (3.2), p+1=4.

The branch of positive solutions emanates subcritically from the trivial branch at $\lambda = 39.77579$ with one-dimensional unstable manifold until the value $\lambda = 39.390170$ where it becomes two-dimensional; at $\lambda = 39.390170$ there exists a secondary bifurcation to non radially symmetric positive solutions which were not plotted. The number of modes that we have needed to reach the convergence criterion (1.6) is given in Table 3.2.

Interval of λ	Number of modes
$(8.009557, 39.473384]$	125
$(-60.020510, 8.009557]$	201
$[-255.012323, -60.020510]$	301

Table 3.2: The number of modes needed to reach (1.6) with p+1=4.

Observe that, as predicted, there exist positive solutions for negative values of λ bounded away from zero, since $p+1=4$ is bellow of critical exponent. The second plot of the Figure 3.1 shows the profiles of the positive solutions for the values of the parameter $\lambda = 39.473384$, 19.993987, 0.93×10^{-5}, -100.017817 and -255.012323. The profiles of the solutions grow until λ approaches zero. At $\lambda = 0.93 \times 10^{-5}$ the profile of solution seems constant in the interval (-1/6,1/6). As λ decreases from 0, the profiles of the solutions exhibit a two-peaks layer behavior with the peaks located around $-1/6$ and $1/6$. These results completely agree with the theoretical results of Chapter II.

Now, we solve (3.2) taking $p+1=5$. Quite surprisingly the results do not substantially differ from the ones found for the previous case. Contrarily to our hopes, the a priori bounds were not lost. Figure 3.3 shows the bifurcation diagram that we have obtained and some profiles of the radially symmetric solutions. As above, the first plot of Figure 3.3 representes the bifurcation diagram with $\lambda \in (-270.0, 50.0)$. We have used in Figure 3.1 and Figure 3.3 the same scales. We have found the secondary subcritical bifurcation to non radially symmetric positive solutions at $\lambda = 39.411358$.

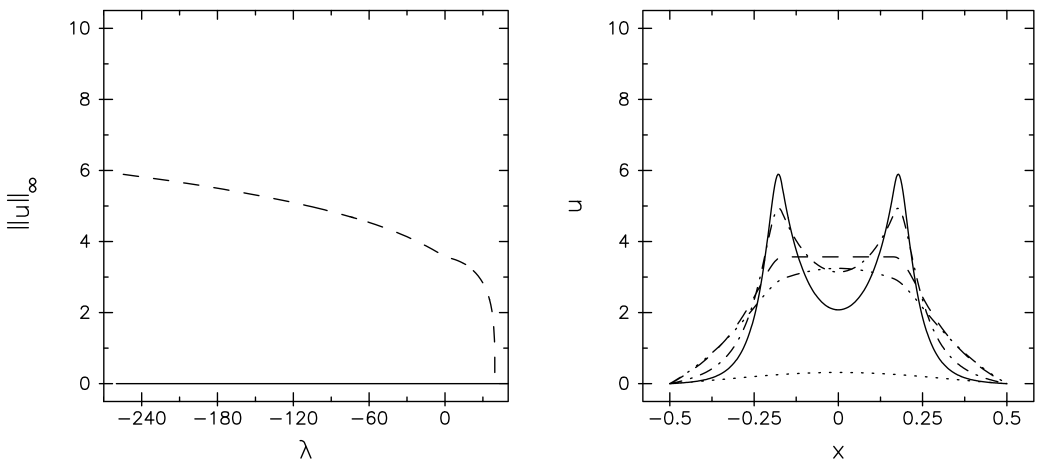

Figure 3.3: Bifurcation diagram and solution profiles for (3.2), p+1=5.

The number of modes that we needed to reach the convergence criterion (1.6) is given in Table 3.4.

Interval of λ	Number of modes
$(16.318628, 39.473844]$	125
$(-70.026805, 16.318628]$	201
$[-255.024261, -70.026805]$	301

Table 3.4: The number of modes needed to reach (1.6) with p+1=5.

We could have follow working with 301 modes, but we stop here the calculations in order to have the same range of values of λ than the case $p+1=4$. Observe that even in the critical case $p+1=5$ the model posseses a positive solution for a range of negative values of λ bounded away from zero, strongly suggesting the existence of a priori bounds for the radially symmetric positive solutions. So it looks like that in the case when $a(r)$ vanishes the existence of a priori bounds will be strongly based on the nodal structure of the weight function $a(r)$ rather on the growth at infinity of the nonlinearity (size of p). The second plot in Figure 3.3 shows the profiles of some of the positive solutions for the values of $\lambda = 39.473844$, 20.316726, 0.17×10^{-4}, -100.026176 and -255.024261. As above, the profiles of the solutions grow until λ approaches the value 0; at $\lambda = 0.17 \times 10^{-4}$ the profile of the solution seems to be constant in the interval (-1/6,1/6) and, as λ decreases becoming negative, the profiles of the solutions exhibit a two-peaks layer behavior with the peaks located around $-1/6$ and $1/6$. Not only the a priori bounds are not lost, but the profiles of the solutions corresponding with $p + 1 = 5$ grow even more slowly than in case $p + 1 = 4$.

We have solved the equation (3.1) taking different values of ρ and the behavior of the profiles of the solutions were the same. It looks like the same behavior occurs when $a(r)$ vanishes on some ball centered at the origin, even in the case when $p+1 > 5$ is bounded away from the critical exponent 5, but this analysis is at present in progress and it will be completed and included elewhere.

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