University of La Laguna
Faculty of Sciences

BACHELOR THESIS

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Estimating General Relativistic Corrections to Angular Redshift Fluctuations

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Study programme: Physics

San Cristóbal de La Laguna, July 2021

Resumen

Una de las principales tareas de la Cosmología moderna en los últimos años ha sido la caracterización y el estudio de la estructura a gran escala del Universo. Para ello, se han estudiado principalmente las fluctuaciones de densidad en el cosmos, y obtenido así una imagen certera de como se agrupan las galaxias. Sin embargo, nuevos estudios se siguen llevando a cabo en el campo. En este trabajo se tratará de obtener la expresión relativista de un nuevo observable, las fluctuaciones angulares del desplazamiento al rojo (ARF por sus siglas en inglés), con el fin de dilucidar si aporta nueva información al estudio de estructura a gran escala en el Universo.

En primer lugar, después de un breve repaso histórico, se expondrán los principales métodos de análisis espectral usados en cosmología observacional tales como funciones de correlación y espectros de potencia, con el fin de introducir al lector no experto en la materia. Tras una breve exposición de los objetivos y motivaciones del trabajo, se introducirá el formalismo que se seguirá a lo largo de todo el documento, para de esta manera facilitar el seguimiento de las derivaciones por parte del lector. Este formalismo es similar al adoptado por J. Yoo, A. L. Fitzpatrick y M. Zaldarriaga en su artículo *New perspective on galaxy clustering as a cosmological probe: general relativistic effects* (1), trabajo sobre el que nos basaremos principalmente para la obtención de las correcciones a las ARF.

Una vez sentadas las bases sobre las que basarnos, se presentará el observable en su versión no relativista, obtenida por Carlos Hernández-Monteagudo, Jonás Chaves-Montero y Raúl E. Angulo en *Angular Redshift Fluctuations: a New Cosmological Observable* (2), y se definirá el *redshift* observado z_g y la densidad de galaxias n_g como invariantes relativistas. De esta manera se podrán hallar las correcciones a ambos observables usando teoría de perturbaciones a primer orden para una métrica FLWR en forma general (sin particularizar a ningún gauge en específico). Este método, que se denomina método *gauge-ready*, representa una ventaja en muchos casos ya que nos permite hallar expresiones generales que pueden particularizarse al gauge más conveniente en función del problema a resolver.

La obtención de estas correcciones supondrá el grueso de nuestro trabajo. Las correcciones al *redshift* observado (Δz) aparecerán simplemente con el tratamiento relativista del mismo. Sin embargo, para la densidad de galaxias (cuyas correc-

ciones denominaremos δ_g) tendremos en cuenta diversos factores: correcciones debidas al efecto de lente gravitacional (desplazamientos radiales, angulares y efectos de convergencia) que afectarán a la distancia de luminosidad, correcciones al volumen observado, correcciones debidas a efectos de selección (sólo se tienen en cuenta galaxias con luminosidad mayor de una luminosidad límite L_{thr}) y correcciones debida al sesgo (en inglés *bias*) de las galaxias al trazar posiciones dadas por la densidad de materia oscura. Todas estas correcciones afectan a la densidad de galaxias, y por ende, conformarán lo que denominaremos *galaxy fluctuation field* δ_g. Teniendo en cuenta estos factores, se darán unas expresiones finales para Δz y δ_g.

Como siguiente paso, estas expresiones serán incorporadas a la definición de las ARF y así hallar la expresión a primer orden en perturbaciones de las mismas. De esta manera se obtendrá la ecuación fundamental de nuestro trabajo en forma general. El último paso consistirá en tomar esta fórmula y particularizarla a unos gauges dados con el fin de mostrar la utilidad del método *gauge-ready*. Dado que los dos gauges más importantes en cosmología son el *synchronous gauge* y el *conformal-Newtonian gauge*, se tomarán estos como ejemplo.

Como conclusiones, se valorarán los resultados obtenidos y se introducirá brevemente el trabajo realizado en las prácticas realizadas en el Instituto de Astrofísica de Canarias entre marzo y mayo de este año, en las que se realizaron simulaciones numéricas sobre las expresiones relativistas de las ARF en algunos casos simples. Dado que las expresiones obtenidas analíticamente no nos permiten apreciar si se aporta información de utilidad al estudio de la estructura a gran escala del universo, sucesivos trabajos tales como el realizado en las prácticas están siendo actualmente llevados a cabo, y esperamos puedan aportar nueva información sobre las ARF en un futuro próximo.

Contents

1 Introduction

Abstract:
En este primer capítulo introducimos los conceptos fundamentales sobre los que se basa el trabajo realizado. Tras un breve repaso sobre la historia de la cosmología, nos centraremos en introducir el análisis espectral y definir el nuevo observable sobre el que basaremos nuestro trabajo: las ARF. Finalmente, se presentará el objetivo del trabajo, hallar las correcciones relativistas a este observable.

From the early decades of the last century, the discovery of extra-galactic structures (so-called *nebulae*, later galaxies) has given rise to a new understanding of the cosmos and its nature. Measurements of the NGC6822, M31 and M33 cepheids by Hubble in 1924 (3), and the later first correlation between distance and redshift (along with the suggestion of an expanding universe, framed in the formulation of Einstein's general theory of relativity) by Lemaître in 1927 (4) set the fundamental basis for modern cosmology. From that point, galaxies have been used as distance estimators[1] in the universe by using clustering statistics.

There are several statistical methods which can be used to study galaxy samples and obtain information about structure formation and background cosmology. Taking into account selection effects, which limit the galaxy sample as only galaxies with luminosity greater than a threshold apparent magnitude m_{thr} are considered, qualitative information about the universe structure can be inferred by means of statistical measures having the capacity of distinguish between different point patterns (6).

To this end we can use *2-point correlation functions* $\xi(r)$, defined as a measure of the excess probability dP, above what is expected for an unclustered random Poisson distribution (with $\xi(r) = 0$), of finding a galaxy in a volume element dV at a distance r from another arbitrary chosen galaxy

$$dP = n[1 + \xi(r)]dV. \tag{1.1}$$

[1] Note here that the definition of distance in cosmology is ambiguous. For the luminosity selection effects one has to use the luminosity distance $\mathcal{D}_L$, for the angular selection effects the angular diameter distance $\mathcal{D}_A$, and in order to describe spatial clustering the comoving distance r is commonly used. See (5) for more information.

However, what it is usually computed is its Fourier transform, the *linear power spectrum* ($\mathcal{P}(\mathbf{k})$ in $\mathbf{k}$-space, which despite of being formally equivalent to $\xi(\mathbf{r})$, it is more physically intuitive as it decomposed the probability into characteristic lengths ($k = 2\pi/L$), and hence it differentiates processes on different scales. Alternatively, if we consider the angular distribution instead of the spatial distribution and hence we define the excess probability as $\omega(\hat{\mathbf{n}})$ with $\hat{\mathbf{n}}$ the direction on the sky, we can also work with the *angular power spectrum* C_ℓ.

Using these tools, it is useful to study the **density contrast**

$$\delta(\mathbf{r}) \equiv \frac{\rho(\mathbf{r}) - \bar{\rho}}{\bar{\rho}}, \tag{1.2}$$

with $\bar{\rho}$ the universe mean density and $\rho(\mathbf{r})$ the density at (comoving) position $\mathbf{r}$, as fluctuations over a smooth mean density $\bar{\rho}$. From this definition we can construct a galaxy density contrast δ_g for the galaxy clustering and a matter density contrast δ_m describing the distribution of matter in the Universe. These two quantities will be related by a *bias* function (7)

$$\delta_g = F\left[\delta_m\right] \simeq \sum_{i=0}^{N} \frac{b_i}{i!}\delta_m^i, \tag{1.3}$$

with b_i the *bias parameters*, which if we consider only local effects gives a linear relation between δ_g and δ_m[2].

Hence considering this density contrast we define the (linear) matter power spectrum $\mathcal{P}(\mathbf{k})$ as

$$\langle \delta(\mathbf{k})\delta^{\star}(\mathbf{k}')\rangle = (2\pi)^3 \delta_D\left(\mathbf{k} - \mathbf{k}'\right) \mathcal{P}(\mathbf{k}) \tag{1.4}$$

for $\delta(\mathbf{k})$ the amplitude of the density contrast in Fourier space and $\delta_D(\mathbf{k})$ the Fourier-space Dirac delta function ; and the angular power spectrum

$$C_\ell \equiv \left\langle a_{l,m} a^*_{l,m} \right\rangle \tag{1.5}$$

with a_{lm} the coefficients of the expansion of $\delta^{2D}(\hat{\mathbf{n}})$[3] in spherical harmonics over the celestial sphere (8)

$$\delta^{2D}(\hat{\mathbf{n}}) = \sum_{l,m} a_{l,m} Y_{l,m}(\theta, \phi). \tag{1.6}$$

[2]More information on the local bias parameter will be given later when considering the relativistic expression of δ_g.

[3]2D density contrast field at fixed comoving distance r, also called Angular Density Fluctuations (ADF) and 2D clustering.

The study of these power spectra of density fluctuations can be used to constrain cosmology, as they provide information about both the amount and nature of different forms of energy in the Universe, and hence about the formation of large-scale structure (9). Moreover, it allows for a test of observables with theoretical predictions for any cosmological setup. For example, the study the galaxy power spectrum can be used to infer the form of the primordial matter power spectrum, and in that way help to understand the initial conditions of the early universe (10).

The angular power spectrum C_ℓ is of wide use in modern cosmology. It is a natural tool for data analysis of the CMB (when accounting for temperature fluctuations). Moreover, as introduced in Eq. (1.5), it can also provide information of Angular Density Fluctuations (ADF) in the celestial sphere on scales of order $\approx \pi/\ell$. That is the reason of the importance of ADF as a cosmological probe. As an alternative, C. Hernández Monteagudo, J. Chavez Montero & Raúl E. Angulo (2019) (2) propose a new observable: Angular Redshift Fluctuations (ARF) in the galaxy redshift field as a new cosmological probe to extract cosmological information in the Universe.

This new observable arises from considering the cosmological redshift as a field, and expressing the angular anisotropies of the redshift field as

$$\delta z(\hat{\mathbf{n}}) = \frac{\sum_j (z_j - \bar{z}) W_j}{\langle \sum_j W_j \rangle}, \tag{1.7}$$

for a given galaxy sample under a Gaussian window function $W\left(z_{obs} - z_g\right) = \exp\left[-\left(z_{obs} - z_g\right)^2 / 2\sigma_z^2\right]$, centered at a central redshift z_{obs} set by the observer. Here the summation is over all galaxies. In the same way as it was done for ADF in Eq. (1.5), we can obtain the C_l for the ARF and hence study the anisotropies in the redshift field of the observed galaxies at different scales, so that cosmological information present in the galaxy field can be inferred.

1.1 Aim of this work

As the results found by Hernández-Monteagudo *et al.* for the ARF do not account for general relativistic effects, in this work we will try a relativistic approach with the aim of revealing the new information that can be extracted from this new observable. In order to do this, we shall start from the expression for the ARF obtained by Hernández-Monteagudo *et al.* and express it in term of covariant quantities. Working to linear order (we suppose small corrections) under a general Friedmann-Lemaître-Robertson-Walker (FLWR) metric, we will derive all the relativistic corrections affecting the angular power spectrum for the ARF in a *gauge-ready* form (*i.e.* without imposing any gauge condition). Finally, we will characterize the solutions for both the conformal-Newtonian gauge and the synchronous gauge[4].

With the results found, we want to set the analytical basis for future works on this matter that will potentially reveal new information in large scale surveys, or at least prove the ARF to be a complementary and useful observable in standard cosmological analyses. This work will set the expressions for general relativistic corrections, whose numerical computation are being estimated in a parallel work. Indeed, some of the results of this project are linked to the efforts developed as the main project of my internship at the Instituto de Astrofísica de Canarias (IAC), and will be shortly discussed in Chapter 4.

[4]More information about gauges in Cosmology will be given in Subsection 2.2.2.5. However, for a first contact on the matter the lectures on cosmological dynamics given by E. Bertschinger at Les Houches in August 1993 are highly recommended, specifically the section **4.Relativistic cosmological perturbation theory** (11).

2 Methodology

Abstract:
En este capítulo introducimos la notación que seguiremos a lo largo del documento. Tras esto, se hallarán las ecuaciones correspondientes a las correcciones relativistas al redshift observado y al número de galaxias, de manera que podamos obtener las ARF en su expresión relativista. Además, se incluye un breve inciso introductorio a los gauges en cosmología, con el fin de justificar el método *gauge-ready* adoptado en este trabajo.

2.1 Standard formalism and FLWR metric

As mentioned in Chapter 1, this work has been developed in a general case, without imposing any gauge condition. Therefore, we choose to work under a general perturbed FLWR metric of signature $(-,+,+,+)$ in comoving coordinates, with $dx^0 = d\eta = dt/a(\eta)$ and $dx^i = dx^i_p$, with subindex p denoting the physical distance, and taking natural units (N.U.) with $c \equiv 1$

$$ds^2 = a^2(\eta)\left\{-(1+2A)d\eta^2 - 2B_i d\eta dx^i + \left[(1+2D)\bar{g}_{ij} + 2E_{ij}\right]dx^i dx^j\right\}, \quad (2.1)$$

where $\bar{g}_{ij}$ is defined as the 3-space metric in an unperturbed universe (*i.e.* we define $ds^2 = \bar{g}_{ij}dx^i dx^j$ in $\mathbb{E}^3$). Here $A(\mathbf{r},\eta)$ and $D(\mathbf{r},\eta)$ are scalar metric perturbations while $B_i(\mathbf{r},\eta)$ and $E_{ij}(\mathbf{r},\eta)$ are vector and tensor metric perturbations respectively, which describe departures from homogeneity and isotropy. These can be be further decompose in other scalar, vector and tensor perturbations, which up to linear order will evolve independently of each other (12). However, for the purpose of this work no further decomposition is needed.

Along the paper we will use Greek indices such as μ, ν, ρ, σ running from 0 to 3, denoting spacetime variables, while Latin indices i, j, k, l will run from 1 to 3 labeling the spatial part of a four-tensor. Hence vector and tensor perturbations B_i, E_{ij} indices will be lowered and raised using $\bar{g}_{ij}$. Moreover, we will use a semicolon and a vertical bar for representing the covariant derivatives with respect to $g_{\mu\nu}$ (*e.g.* $\nabla_\mu A \equiv A_{;\mu}$) and $\bar{g}_{ij}$ (*e.g.* $\nabla_i A \equiv A_{|i}$), respectively. We will also adopt the Einstein summation convention for repeated indices and express partial derivatives as $\partial_i A \equiv \frac{\partial}{\partial i} A \equiv A_{,i}$.

2.2 General relativistic treatment of the ARF

As the starting point, the expression for the ARF found by Hernández Monteagudo *et al.* (2) is

$$\bar{z} + \delta z(\hat{\mathbf{n}}) = \frac{\int dr\, r^2 \bar{n}(r)\,(1+\delta_g(r,\hat{\mathbf{n}}))\, z_g(r,\hat{\mathbf{n}}) W(z_{obs} - z_g)}{\int dr\, r^2 \bar{n}(r)\,(1+\delta_g(r,\hat{\mathbf{n}}))\, W(z_{obs} - z_g)}, \tag{2.2}$$

where $\bar{z}$ refers to the redshift monopole, $n_g = \bar{n}(r)\,(1+\delta_g$ is the number density of galaxies at redshift z_g and $z_g(r,\hat{\mathbf{n}}) = z_H + z_{vlos}(r,\hat{\mathbf{n}}) + z_\phi(r,\hat{\mathbf{n}})$ following the notation adopted in that paper. There they define z_H as the redshift parameter of an homogeneous, isotropic universe ($1 + z_H = \frac{a_0}{a}$, with $a_0 \equiv a(\eta_0)$ at observer position), $z_{\rm vlos} = (1+z_H)\,\mathbf{v}(\eta,\hat{\mathbf{n}}) \cdot \hat{\mathbf{n}}/c$ as the redshift induced by the proper peculiar velocity $\mathbf{v}$ of the observed galaxy and z_ϕ as the redshift fluctuations of gravitational origin.

In order to take into account the relativistic effects into Eq. (2.2), we must express it in term of covariant quantities which transform under coordinate transformation as dictated by the space-time metric. The observed redshift can be trivially expressed as a function of the contraction of covariant 4-vectors[1] following its definition

$$1 + z_g = \frac{(k^\mu u_\mu)_g}{(k^\mu u_\mu)_o}, \tag{2.3}$$

where k^μ is the photon null momentum and u^μ its 4-velocity, and subscripts g and o refer to quantities at the source's redshift z_g, and at the observer's position respectively. With respect to the galaxy number density $n_g \equiv \bar{n}(r)\,(1+\delta_g(r,\hat{\mathbf{n}}))$, it can also be expressed in a covariant form using the relation[2]

$$dN(z,\hat{\mathbf{n}}) = n_p \sqrt{-g}\,\varepsilon_{\mu\nu\rho\sigma} u_g^\mu \frac{\partial x_g^\nu}{\partial z}\frac{\partial x_g^\rho}{\partial \theta}\frac{\partial x_g^\sigma}{\partial \phi} dz d\theta d\phi, \tag{2.4}$$

with $\sqrt{-g}$ the metric determinant and $\varepsilon_{\mu\nu\rho\sigma}$ is the *Levi-Civita symbol* (totally antisymmetric in its indices (*i.e.* $\varepsilon_{\mu\nu\rho\sigma} = \varepsilon_{[\mu\nu\rho\sigma]}$). This expression and the covariant form of n_g will be justified in Section 2.2.2.

We will follow a similar approach to the one proposed by J. Yoo, A. L. Fitz-

[1] As $k \cdot u$ is a scalar, it is independent of the frame we are working on.

[2] The proof for the covariant form of this expression will not be discussed here, but can be found at (13).

patrick and M. Zaldarriaga for the general relativistic effects on galaxy clustering (1). For a photon moving along a geodesic $x^\mu(\lambda)$ with an λ *affine parameter* along the geodesic, we can define its null-momentum by $k^\mu = \frac{dx^\mu}{d\lambda}$ with

$$\begin{cases} \mathrm{k}^\eta = \frac{\bar{\nu}}{a}(1+\delta\nu) \\ \mathrm{k}^i = -\frac{\bar{\nu}}{a}\left(e^i + \delta e^i\right), \end{cases} \tag{2.5}$$

in the observer's rest frame[3] Here we define $\bar{\nu}$ as the photon frequency and e^i as the photon propagation direction measured by the observer in a homogeneous universe ($\vec{e} \equiv \hat{\mathbf{n}}$); and $\delta\nu$, δe^i its respective dimensionless corrections as we expand the null vector to 1^{st} order in perturbations.

2.2.1 Corrections to the observed redshift

We can define the 4-velocity of a comoving observer as

$$u^\mu = \frac{dx^\mu}{\sqrt{-ds^2}} = \frac{dx^\mu}{d\tau} = \frac{1}{a}(1-A, \mathbf{v}), \tag{2.6}$$

for τ the proper time along the observer's worldline and $\mathbf{v}(\eta, \hat{\mathbf{n}}) << 1$ its relative 3-velocity respect to the observed galaxy (*i.e. the peculiar velocity*). This peculiar velocity is taken so that $v^i = 0$ in the rest frame of the observer. Hence, as it is known that $u^\mu u_\mu = -1$, in this rest frame

$$\begin{aligned} &u^\mu u_\mu = g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = g_{\eta\eta}\left(\frac{d\eta}{d\tau}\right)^2 = -a^2(1+2A)\left(\frac{d\eta}{d\tau}\right)^2 = -1 \quad \Rightarrow \\ &\Rightarrow \quad \frac{d\eta}{d\tau} = \frac{1}{a\sqrt{1+2A}} = \frac{1}{a}[1 - A + \mathcal{O}(2)] \end{aligned} \tag{2.7}$$

As $g_{\mu\nu}k^\mu u^\nu = k^\mu u_\mu = E \equiv \nu$, following the statement proposed at the beginning of this section we can find the observed redshift $z_g(r, \hat{\mathbf{n}})$ as

$$1 + z_g = \frac{(k^\mu u_\mu)_g}{(k^\mu u_\mu)_o} = \frac{(g_{\mu\nu}k^\mu u^\nu)_g}{(g_{\mu\nu}k^\mu u^\nu)_o}. \tag{2.8}$$

[3] All subsequent calculation will be performed in the observer's rest frame.

The scalar product $k \cdot u$ is given by

$$k^\mu u_\mu = g_{\mu\nu} k^\mu u^\nu = -a^2 \left\{ (1+2A)\frac{\bar\nu}{a}(1+\delta\nu)\frac{1}{a}(1-A) + B_i \frac{\bar\nu}{a}(1+\delta\nu)\frac{1}{a} v^i \right.$$
$$\left. - B_i \frac{\bar\nu}{a}\left(e^i + \delta e^i\right)\frac{1}{a}(1-A) - [(1+2D)\bar g_{ij} + 2E_{ij}]\frac{\bar\nu}{a}\left(e^i+\delta e^i\right)\frac{1}{a}v^i \right\} \tag{2.9}$$
$$= -\bar\nu\left[1 + \delta\nu + A + (v_i - B_i)\, e^i\right]$$

Given that $\bar\nu \propto \frac{1}{a}$, evaluated between the observer and the galaxy $\bar\nu_g/\bar\nu_o = a_o/a_g$. Thus we get

$$1 + z_g = \left(\frac{a_o}{a_g}\right)\left\{1 + \left[\delta\nu + A + (v_i - B_i)\, e^i\right]_o^g\right\}. \tag{2.10}$$

Here we have expressed the observed redshift in terms of null vector perturbations as well as metric perturbations. We need a relation between the null vector perturbations and the metric perturbations, so that the result can be expressed only in terms of the latter. Such relations can be found both by solving the null equation ($k_\mu k^\mu = 0$) and by solving the null geodesic equation for the photon path, specifically its temporal component (${k^0}_{;\mu} k^\mu = 0$).

2.2.1.1 Null equation $k_\mu k^\mu = 0$

The first relation between null and metric perturbations can be easily obtained solving the null equation.

$$k_\mu k^\mu = g_{\mu\nu} k_\nu k_\mu = -a^2(\eta)(1+2A)(1+\delta\nu)(1+\delta\nu) - 2a^2(\eta)B_i(1+\delta\nu)(-e^i - \delta e^i)$$
$$+ a(\eta)^2\left[(1+2D)\bar g_{ij} + 2E_{ij}\right](-e^i-\delta e^i)(-e^j-\delta e^j) = -a(\eta)^2\Big[1 + 2A + 2\delta\nu$$
$$- 2B_i e^i - \bar g_{ij}(e^i\delta e^j + e^j \delta e^i) - (2D\bar g_{ij} + 2E_{ij} + \bar g_{ij})e^i e^j\Big] = 0 \tag{2.11}$$

As in an homogeneous and isotropic universe

$$\begin{cases} ds^2 = -a^2(\eta)d\eta^2 + a^2(\eta)\bar g_{ij} dx^i dx^j \\ k^\mu = \frac{\bar\nu}{a}\left(1, -e^i\right), \end{cases} \tag{2.12}$$

the null equation in the unperturbed background give us $k_\mu k^\mu = \bar\nu a\left(-1 + \bar g_{ij} e^i e^j\right) = 0 \Rightarrow \bar g_{ij} e^i e^j = 1$. Hence, as $\bar g_{ij}(e^i \delta e^j + e^j \delta e^i) = 2e^i \delta e_i$, we can rewrite Eq. (2.11) as

$$e^i \delta e_i = \delta\nu + A - B_i e^i - D - E_{ij} e^i e^j. \tag{2.13}$$

2.2.1.2 Temporal component of the geodesic equation $k^{\eta}{}_{;\mu}k^{\mu}=0$.

The other relation linking null and metric perturbations can be obtain by solving the temporal component of the geodesic equation. The first step in the calculation will be to obtain the Christoffel symbols of the perturbed metric. The Christoffel symbols are given by

$$\Gamma^{\mu}_{\nu\rho}=\frac{1}{2}g^{\mu\sigma}\left(g_{\nu\sigma,\rho}+g_{\rho\sigma,\nu}-g_{\nu\rho,\sigma}\right), \tag{2.14}$$

with commas denoting partial derivatives. Following this definition, for the perturbed FLWR considered in this work we obtain (for general coordinates)[4]

$$\Gamma^{\eta}_{\eta\eta}=\frac{\dot a}{a}+\dot A \tag{2.15}$$

$$\Gamma^{\eta}_{\eta i}=A_{,i}-\frac{\dot a}{a}B_i \tag{2.16}$$

$$\Gamma^{\eta}_{ij}=B_{(i|j)}+\frac{\dot a}{a}\bar g_{ij}+2\frac{\dot a}{a}\left(D\bar g_{ij}+E_{ij}-2\bar g_{ij}A\right)+\dot D\bar g_{ij}+\dot E_{ij} \tag{2.17}$$

$$\Gamma^{i}_{\eta\eta}=A^{|i}-\dot B^{i}-\frac{\dot a}{a}B^{i} \tag{2.18}$$

$$\Gamma^{i}_{\eta j}=\frac{1}{2}\left(B_j^{|i}-B^{i}_{|j}\right)+\frac{\dot a}{a}\delta^i_j+\dot D\delta^i_j+\dot E^i_j \tag{2.19}$$

$$\Gamma^{i}_{jk}=\overline{\Gamma}^{i}_{jk}+\frac{\dot a}{a}\bar g_{jk}B^{i}+D_{|k}\delta^i_j+D_{|j}\delta^i_k+2E^i_{(j|k)}-D^{|i}\bar g_{jk}-E_{jk}^{\ \ |i}, \tag{2.20}$$

where we have denoted dot variables as derivatives with respect to the conformal time η and $\bar\Gamma^i_{jk}$ as the Christoffel symbols based on the 3-space metric $\bar g_{ij}$. Also the subscript (ij) refers to the symmetric part of a tensor, *i.e.*

$$E_{(ij)}=\frac{1}{2}(E_{ij}+E_{ji})\Rightarrow E_{(j|k)}=\frac{1}{2}(E_{j|k}+E_{k|j}). \tag{2.21}$$

Once we have obtained the Christoffel symbols, we can solve the geodesic equation for $\mu=0$. However, in order to simplify calculations, as null geodesics are conformally invariant, we can apply a conformal transformation to the metric of the form

$$g_{\mu\nu}\rightarrow\hat g_{\mu\nu}=f(x^{\mu})g_{\mu\nu}, \tag{2.22}$$

for any analytic function f with non-zero first derivative everywhere in the considered manifold (14). Therefore, we can define the conformal transformation

$$g_{\mu\nu}\rightarrow\hat g_{\mu\nu}=\frac{1}{a(\eta)^2}g_{\mu\nu}, \tag{2.23}$$

[4] The step by step calculations are included in Annex A.

and so the geodesic equation remains invariant when described by the conformally transformed null vectors $\hat{k}^{\mu} = \frac{dx^{\mu}}{d\chi}$[5], with χ an affine parameter defined by the relation $\frac{d\lambda}{d\chi} = \mathcal{C}a^2$ (15). Hence, we can write the conformally transformed null vector as

$$\begin{cases} \hat{k}^{\eta} = \mathcal{C}\bar{\nu}a(1+\delta\nu) \\ \hat{k}^{i} = -\mathcal{C}\bar{\nu}a\left(e^{i}+\delta e^{i}\right), \end{cases} \tag{2.24}$$

If now we choose the normalization constant $\mathcal{C}$ so that $\mathcal{C}\bar{\nu}a = 1$ at the observer's position $x^{\mu}(\chi_0)$, we can write $\hat{k}^{\eta} = (1+\delta\nu, e^{i}-\delta e^{i})$ in the observer's rest frame. Therefore, given that the conformally transformed Christoffel symbols are[6]

$$\hat{\Gamma}^{\eta}_{\eta\eta} = \dot{A} \tag{2.25}$$
$$\hat{\Gamma}^{\eta}_{\eta i} = A_{,i} \tag{2.26}$$
$$\hat{\Gamma}^{\eta}_{ij} = B_{(i|j)} + \dot{D}\bar{g}_{ij} + \dot{E}_{ij} \tag{2.27}$$
$$\hat{\Gamma}^{i}_{\eta\eta} = A^{|i} - \dot{B}^{i} \tag{2.28}$$
$$\hat{\Gamma}^{i}_{\eta j} = \frac{1}{2}\left(B^{|i}_{j} - B^{i}_{|j}\right) + \dot{D}\delta^{i}_{j} + \dot{E}^{i}_{j} \tag{2.29}$$
$$\hat{\Gamma}^{i}_{jk} = \overline{\Gamma}^{i}_{jk} + D_{|k}\delta^{i}_{j} + D_{|j}\delta^{i}_{k} + 2E^{i}_{(j|k)} - D^{|i}\bar{g}_{jk} - E^{|i}_{jk} \tag{2.30}$$

we can calculate the geodesic equation for this new metric, simplifying considerably the calculation.

As we have stated, the temporal part of the geodesic equation is given by

$$k^{\eta}{}_{;\mu}k^{\mu} = k^{\eta}\nabla_{\mu}k^{\mu} = \frac{d^2\eta}{d\lambda^2} + \Gamma^{\eta}_{\sigma\rho}\frac{dx^{\sigma}}{d\lambda}\frac{dx^{\rho}}{d\lambda} = \frac{d^2\eta}{d\chi^2} + \overline{\Gamma}^{\eta}_{\sigma\rho}\frac{dx^{\sigma}}{d\chi}\frac{dx^{\rho}}{d\chi} = 0, \tag{2.31}$$

so for the conformally transformed metric

$$\begin{aligned} &\frac{d^2\eta}{d\chi^2} + \hat{\Gamma}^{\eta}_{\sigma\rho}\frac{dx^{\sigma}}{d\chi}\frac{dx^{\rho}}{d\chi} = \frac{d}{d\chi}\hat{k}^{\eta} + \hat{\Gamma}^{\eta}_{\eta\eta}\hat{k}^{\eta}\hat{k}^{\eta} + 2\hat{\Gamma}^{\eta}_{i\eta}\hat{k}^{i}\hat{k}^{\eta} + \hat{\Gamma}^{\eta}_{ij}\hat{k}^{i}\hat{k}^{j} \\ &= \frac{d}{d\chi}(1+\delta\nu) + \dot{A}(1+\delta\nu)(1+\delta v) + 2A_{,i}\left(-e^{i}-\delta e^{i}\right)(1+\delta\nu) \\ &+ \left[B_{(i|j)} + \dot{D}\bar{g}_{ij} + \dot{E}_{ij}\right]\left(-e^{i}-\delta e^{i}\right)\left(-e^{j}-\delta e^{j}\right) \\ &= \frac{d}{d\chi}\delta\nu + \dot{A} - 2A_{,i}e^{i} + \left[B_{(i|j)} + \dot{D}\bar{g}_{ij} + \dot{E}_{ij}\right]e^{i}e^{j} = 0. \end{aligned} \tag{2.32}$$

[5] All quantities Q related to the conformally transformed metric will be denoted by $\hat{Q}$.

[6] The calculation of the conformally transformed Christoffel symbols is equivalent to the one done in Annex A, without the terms involving a and its derivatives.

Note we can rewrite
$$\frac{dA}{d\chi} = \frac{dx^\mu}{d\chi}\frac{\partial}{\partial x^\mu}A = \dot{A} - e^i A_{,i}, \tag{2.33}$$
where the minus sign appears due to our definition of $\hat{k}^i$ in Eq. (2.24). Thus we have
$$\frac{d}{d\chi}(\delta\nu + 2A) = \dot{A} - \left[B_{i|j} + \dot{E}_{ij} + \bar{g}_{ij}\dot{D}\right] e^i e^j, \tag{2.34}$$
where $B_{(i|j)}e^i e^j = B_{i|j}e^i e^j$, as it is totally symmetric.

Having proved $\bar{g}_{ij}e^i e^j = 1$, we can substitute in our expression giving
$$\frac{d}{d\chi}(\delta\nu + 2A) = \left(\dot{A} - \dot{D}\right) - \left(B_{i|j} + \dot{E}_{ij}\right) e^i e^j. \tag{2.35}$$

Equations (2.13) and (2.35) gives us a relation between the null vector perturbations and the metric perturbations. Therefore, we can use the latter to express the result obtained in Eq. (2.10) in terms only of metric perturbations as
$$1+z_g = \left(\frac{a_o}{a_g}\right)\left\{1 + \left[(v_i - B_i)\, e^i - A\right]_o^g + \int_{\chi_0}^{\chi_g} d\chi \left[(\dot{A} - \dot{D}) - \left(B_{i|j} + \dot{E}_{ij}\right) e^i e^j\right]\right\} \tag{2.36}$$
Furthermore, we can relate the affine parameter χ along the geodesic to the comoving line of sight distance r since $r(\eta) = a(\eta_0)(\eta_0 - \eta) = \eta_0 - \eta$ and integrating k^η along the photon path gives
$$\int_{\chi_0}^{\chi} d\chi' \frac{d\eta}{d\chi'} = \eta - \eta_0 = \int_{\chi_0}^{\chi} d\chi'(1+\delta\nu) = \chi - \chi_o + \int_{\chi_0}^{\chi} d\chi' \delta\nu\,(\chi'), \tag{2.37}$$
we can see that $d/d\chi = -d/dr$. Hence we get to the final expression
$$1+z_g = \left(\frac{a_o}{a_g}\right)\left\{1 + \left[(v_i - B_i)\, e^i - A\right]_o^g - \int_{0}^{r_g} dr \left[(\dot{A} - \dot{D}) - \left(B_{i|j} + \dot{E}_{ij}\right) e^i e^j\right]\right\}, \tag{2.38}$$
where the integral is evaluated from the observer position at $r_0 = 0$ to the galaxy position r_g.

We can relate the expression found in Eq. (2.38) in terms of the notation followed in the original results of the ARF

$$1+z_g=\frac{a_o}{a_g}+\frac{a_o}{a_g}\left[v_i e^i\right]_o^g-\frac{a_o}{a_g}\left\{\left[B_i e^i+A\right]_o^g+\int_0^r dr\left[(\dot{A}-\dot{D})-\left(B_{i|j}+\dot{E}_{ij}\right)e^i e^j\right]\right\}$$
$$\equiv(1+z_H)+(1+z_H)\vec{v}\cdot\hat{\mathbf{n}}+z_\phi=1+z_H+z_{vlos}+z_\phi, \tag{2.39}$$

with z_ϕ referring to the terms of gravitational origin which were not calculated in the original paper for the ARF. Hence we have recall the expression $z_g(r,\hat{\mathbf{n}})=z_H+z_{vlos}(r,\hat{\mathbf{n}})+z_\phi(r,\hat{\mathbf{n}})$ identifying

$$\begin{cases} z_H=\frac{a_o}{a_g}-1 \\ z_{vlos}=\frac{a_o}{a_g}\left[v_i e^i\right]_o^g \\ z_\phi=-\left(\frac{a_o}{a_g}\right)\left\{\left[B_i e^i+A\right]_o^g+\int_0^r dr\left[(\dot{A}-\dot{D})-\left(B_{i|j}+\dot{E}_{ij}\right)e^i e^j\right]\right\} \end{cases} \tag{2.40}$$

Furthermore, in a homogeneous ad isotropic universe, $a(\eta_0)=1=a_0$. However, taking into account perturbations in the conformal time at the observer's position due to local gravitational potential effects

$$a_0=a(\eta_0+\delta\eta_0)\approx 1+\dot{a_0}\delta\eta_0=1+\mathcal{H}_0\delta\eta_0, \tag{2.41}$$

with $\mathcal{H}_0=\frac{\dot{a}(\eta_0)}{a(\eta_0)}$ the conformal hubble parameter. Including this perturbations to the former treatment we can express Eq. (2.36) to linear order as

$$1+z_g=\frac{1}{a_g}(1+\Delta z) \tag{2.42}$$

with

$$\Delta z=\left[(v_i-B_i)\,e^i-A\right]_o^g-\int_0^{r_g} dr\left[(\dot{A}-\dot{D})-\left(B_{i|j}+\dot{E}_{ij}\right)e^i e^j\right]+\mathcal{H}_0\delta\eta_0 \tag{2.43}$$

These are the relativistic corrections to $z_g(r,\hat{\mathbf{n}})$, which must be applied to Eq. (2.2), and which will also affect the window function $W\left(z_{obs}-z_g\right)$.

2.2.2 Corrections to the galaxy fluctuation field δ_g

The other term in Eq. (2.2) affected by the relativistic corrections is the galaxy number density n_g. This corrections will be gathered in what we called the galaxy fluctuation field δ_g. These corrections will be given by the convergence and angular and radial displacements caused by the gravitational lensing effect (16), which perturbs the observed luminosity distance $\mathcal{D}_L$, the corrections to the observed physical volume and the corrections due to the underlying matter distribution, given by the bias function.

2.2.2.1 Spatial components of the geodesic equation ${k^i}_{;\mu}k^\mu = 0$.

In Subsection 2.2.1.2 we have solved the geodesic equation for the photon path ($k^\nu_{;\mu}k^\mu = 0$) for $\nu = 0$ (temporal component). As a result, we obtained a relation between null vector and metric perturbations which was used to express Δz in terms only of the latter. If we solve it now for the spatial components $\nu = i$, it will give us the spatial displacements due to relativistic effects. Moreover, picking a set of coordinates such as the spherical coordinates $\{r, \theta, \phi\}$, we can account in this way for the angular displacements $\delta\theta$, $\delta\phi$ and the radial displacement δr. This spatial perturbations are responsible of the gravitational lensing effect.

As advanced knowledge on differential geometry is required at the finals steps of these calculations, in this subsection we will just derive the relations from the spatial component of the geodesic equation needed to express the spatial displacements in terms of the metric perturbations. The final expressions shown in Eqs. (2.49), (2.50), (2.51) are directly taken from the results obtain by Yoo *et al.* in (1), as well as Eqs. (2.56) and (2.60).

Again considering the geodesic equation for the conformally transformed metric $\hat{g}_{\mu\nu}$, we can calculate the geodesic equation for the spatial (general) components

$$\begin{aligned}
&\frac{d^2x^i}{d\chi^2} + \hat{\Gamma}^i_{\sigma\rho}\frac{dx^\sigma}{d\chi}\frac{dx^\rho}{d\chi} = \frac{d}{d\chi}\hat{k}^i + \hat{\Gamma}^i_{\eta\eta}\hat{k}^\eta\hat{k}^\eta + 2\hat{\Gamma}^i_{j\eta}\hat{k}^j\hat{k}^\eta + \hat{\Gamma}^i_{jk}\hat{k}^j\hat{k}^k \\
&= \frac{d}{d\chi}\left(-e^i - \delta e^i\right) + \left(A^{|i} - \dot{B}^i\right)(1+\delta\nu)(1+\delta\nu) + 2\left[\frac{1}{2}\left(B_j^{|i} - B^i_{|j}\right)\right. \\
&\left.+\dot{D}\delta^i_j + \dot{E}^i_j\right]\left(-e^j - \delta e^j\right)(1+\delta\nu) + \left(\overline{\Gamma}_{jk} + D_{|k}\delta^i_j + D_{|j}\delta^i_k + 2E^i_{(j|k)}\right. \\
&\left.-D^{|i}\bar{g}_{jk} - E_{jk}^{|i}\right)\left(-e^j - \delta e^j\right)\left(-e^k - \delta e^k\right) = \frac{d}{d\chi}\left(-e^i - \delta e^i\right) + A^{|i} \\
&- \dot{B}^i - \left(B_j^{|i} - B^i_{|j}\right)e^j - 2\dot{D}e^i - 2\dot{E}^i_je^j + D_{|k}e^ie^k + D_{|j}e^ie^j \\
&+ \left(2E^i_{(j|k)} - E_{jk}^{|i}\right)e^je^k - D^{|i} + \overline{\Gamma}^i_{jk}\left(e^je^k + e^k\delta e^j + e^j\delta e^k\right) = 0.
\end{aligned} \tag{2.44}$$

As in the observer's rest frame the photon propagation direction e^i is constant, and recalling Eq. (2.33)

$$\frac{d}{d\chi}\left(\delta e^i + B^i + 2De^i + 2E^i_j e^j\right) = A^{|i} - B_j^{|i} e^j - E_{jk}^{|i} e^j e^k - D^{|i} + \overline{\Gamma}^i_{jk}\left(B^k e^j + e^j e^k + 2e^j \delta e^k\right) + 2\left(\overline{\Gamma}^i_{kl} E^l_j - \overline{\Gamma}^l_{jk} E^i_l\right) e^j e^k. \tag{2.45}$$

In order to simplify this expression, we can assume we define the photon geodesic under the coordinate chart (U, x^μ) for an open subset U. Commonly, in general relativity (and differential geometry in general) we used what is called *standard formalism*, *i.e.* we express the vector and tensor components in terms of a *coordinate basis* $\{\partial_\mu\}$. However, we can choose another less restrictive basis on which to define tensors. Thus if we consider the set of vectors $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$ which form a local tetrad basis ($\mathbf{e_a} = e^i_a \partial_i$)[7] at $\chi = \chi_0$; for $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$ mutually orthogonal at χ_0, if they are parallelly propagated along the geodesic, they will remain orthogonal to each other along the geodesic, and so

$$\mathbf{g}(e_a, e_b) = g_{ab} = \bar{g}_{ij} e^i_a e^j_b = \delta_{ij} e^i_a e^j_b. \tag{2.46}$$

Hence the components of the connection (*i.e.* the Chrystoffel symbols) for the spatial part of the unperturbed metric along the geodesic are $\overline{\Gamma}^i_{jk} X^j = 0$ for any vector field X^j under this choice of local basis[8]. Therefore we get a final tensorial equation for the spatial perturbations

$$\frac{d}{d\chi}\left(\delta e^i + B^i + 2De^i + 2E^i_j e^j\right) = A^{|i} - B_j^{|i} e^j - E_{jk}^{|i} e^j e^k - D^{|i}, \tag{2.47}$$

valid only for our choice of local basis.

Using this relations, as $\delta e^i = -\frac{d}{d\chi}\delta x^i = \frac{d}{dr}\delta x^i$, the spatial perturbations can be obtained as

$$\delta x^i = -\int_{\chi_0}^{\chi} d\chi' \delta e^i = \int_0^{r_g} dr \delta e^i, \tag{2.48}$$

which if we consider spherical coordinates $\{r, \theta, \phi\}$ and recall the relations given

[7] The a index referring to a tetrad index, not indicative of the vector components. Notice that here we are only considering 3 of the 4 vectors which form the tetrad, as it is sufficient to characterize the 3-space metric $\bar{g}_{ij}$.

[8] For more information in tetrad formalism and local basis, see (17).

in Eqs. (2.13) and (2.47) gives rise to[9]

$$\delta\theta = -\int_0^{r_g} dr \left\{ \frac{\lfloor (B^i - B^i_o) + 2\left(E^{ij} - E^{ij}_o\right) e_j \rfloor e^\theta_i}{r_g} \right. \\ \left. + \left(\frac{r_g - r}{r r_g}\right) \partial_\theta \left(A - D - B_i e^i - E_{ij} e^i e^j\right) \right\} \tag{2.49}$$

$$\delta\phi = -\int_0^{r_g} dr \left\{ \frac{\lfloor (B^i - B^i_o) + 2\left(E^{ij} - E^{ij}_o\right) e_j \rfloor e^\phi_i}{r_g \sin\theta} \right. \\ \left. + \left(\frac{r_g - r}{r r_g \sin^2\theta}\right) \partial_\phi \left(A - D - B_i e^i - E_{ij} e^i e^j\right) \right\} \tag{2.50}$$

$$\delta r = \chi_o - \chi_s + e_i \delta x^i - \bar{r} = \delta\eta_o + \int_0^{r_g} dr \left(A - D - B_i e^i + E_{ij} e^i e^j\right), \tag{2.51}$$

with e^ϕ_i e^θ_i unit directional vectors defined as

$$\begin{aligned} e^\phi_i &= \partial_\theta \mathbf{e} \\ e^\theta_i &= \partial_\phi \mathbf{e}. \end{aligned} \tag{2.52}$$

2.2.2.2 Convergence κ and corrections to the luminosity distance $\delta\mathcal{D}_L$

Having the angular displacements $\delta\theta$ and $\delta\phi$, we can express the galaxy angular position as $\hat{\mathbf{s}} = [\theta+\delta\theta, (\phi+\delta\phi)\sin(\theta+\delta\theta)]$ (note that trivially $\hat{\mathbf{n}} \equiv \mathbf{e} = (\theta, \phi\sin(\theta))$ in spherical coordinates). Due to gravitational lensing, a source at high redshift can become magnified or de-magnified (18). The degree of magnification is related to the convergence κ, which describes the change in the solid angle as part of the distortion in a physical volume (not directly observable), and hence is given by the determinant of the deformation matrix

$$\left(\frac{\partial \hat{\mathbf{s}}}{\partial \mathbf{e}}\right) = \begin{pmatrix} 1 + \partial_\theta \delta\theta & \frac{(1+\partial_\theta\delta\theta)}{\phi\cos(\theta)} \\ (\phi+\delta\phi)(1+\partial_\theta\delta\theta)\cos(\theta+\delta\theta) & \frac{(\sin(\theta+\delta\theta)(1+\partial_\phi\delta\phi)}{\sin(\theta)} + \frac{(\phi+\delta\phi)(1+\partial_\theta\delta\theta)\cos(\theta+\delta\theta)}{\phi\cos(\theta)} \end{pmatrix} \tag{2.53}$$

which up to first order leads to

$$\left|\frac{\partial \hat{\mathbf{s}}}{\partial \mathbf{e}}\right| = \frac{\sin(\theta+\delta\theta)}{\sin(\theta)} \left[1 + \partial_\theta\delta\theta + \partial_\phi\delta\phi\right] = 1 + \left(\cot(\theta) + \partial_\theta\right)\delta\theta + \partial_\phi\delta\phi \equiv 1 - 2\kappa, \tag{2.54}$$

[9] The following results have been taken directly from the work by Yoo *et al.*, as stated at the beginning of this section.

where we have used the fact that up to first order $\sin(\theta+\delta\theta) = \sin(\theta)+\cos(\theta)\delta\theta$.

Related to these perturbations, we will also have a perturbation in the luminosity distance. In cosmology, there are different ways of describing distances, as it was stated in Chapter (1). The luminosity distance $\mathcal{D}_L$ defines it for a given source with known luminosity in the rest frame of the observer. As the flux of source galaxies is affected by the defined angular displacements, for $D_L(z) = (1+z)r(z)$ the luminosity distance in an homogeneous universe (19) we can express $\mathcal{D}_L$ taking into account perturbations as

$$\mathcal{D}_L = D_L(z)\left(1+\delta\mathcal{D}_L\right), \tag{2.55}$$

with the expression for $\delta\mathcal{D}_L$ taken directly from the results obtained by Yoo *et al.* (1)

$$\begin{aligned}\delta\mathcal{D}_L = &\left[(v_i - B_i)\,e^i - A\right]_g - \frac{1+z_g}{Hr_g}\Delta z\\&+2\int_0^{r_g} dr\frac{A}{r_g} - \int_0^{r_g} dr\frac{r}{r_g}\left[(\dot{A}-\dot{D}) - \left(B_{i|j}+\dot{E}_{ij}\right)e^ie^j\right]\\&-\int_0^{r_g} dr\frac{(r_g-r)\,r}{2r_g}\delta\left(\hat{R}_{\mu\nu}\hat{k}^\mu\hat{k}^\nu\right) + \left(\mathcal{H}_0+\frac{1}{r_g}\right)\delta\eta_0,\end{aligned} \tag{2.56}$$

with

$$\begin{aligned}\delta\left(\hat{R}_{\mu\nu}\hat{k}^\mu\hat{k}^\nu\right) = &-k^2\left[A - \left(D+\frac{E}{3}\right) + \left(\frac{\dot{B}}{k}-\frac{\ddot{E}}{k^2}\right)\right] - 2\left(\ddot{D}+\frac{\ddot{E}}{3}\right)\\&+4\left(\dot{D}+\frac{\dot{E}}{3}\right)_{|i}e^i - \left[A+\left(D+\frac{E}{3}\right)+\left(\frac{\dot{B}}{k}-\frac{\ddot{E}}{k^2}\right)\right]_{|ij}e^ie^j\\&+\left(\ddot{E}^T_{ij}+k^2E^T_{ij}\right)e^ie^j\end{aligned} \tag{2.57}$$

where E and B are the scalar parts of E_{ij} and B_i respectively; and E^T_{ij} is the transverse part of E_{ij}[10].

2.2.2.3 Corrections to the physical volume occupied by the observed source galaxies.

If we consider now the volume occupied by the observed galaxy at $x^\mu(\chi_g) \equiv x^\mu_g$ as dV, we can express it in a covarinat way for a small interval of the observed

[10] More information on vector (and similarly for tensors) decomposition into transverse and longitudinal parts can be found at (20).

redshift dz and a solid angle $d\Omega$ as

$$dV = \sqrt{-g}\varepsilon_{\mu\nu\rho\sigma}u_g^\mu \frac{\partial x_g^\nu}{\partial z}\frac{\partial x_g^\rho}{\partial \theta}\frac{\partial x_g^\sigma}{\partial \phi}dzd\theta d\phi \equiv d\bar{V}(1+\delta V), \tag{2.58}$$

where $\sqrt{-g} = a^4(1+A+3D)$ is the metric determinant, $d\bar{V}$ refers to the volume in an homogeneous, isotropic universe and δV accounts for the perturbations to the physical volume due to the already defined redshift perturbation in Eq. (2.36) and the spatial corrections in Eqs. (2.49, 2.50, 2.51). As $d\bar{V}$ can be written in terms of dz and $d\Omega$ as

$$d\bar{V} = \frac{r^2 dz d\Omega}{H(1+z)^3}, \tag{2.59}$$

the expression for δV can be found after some tedious calculus and so we can express[11]

$$\begin{aligned} dV = {} & \frac{r^2 dz d\Omega}{H(1+z)^3}\Big(1 + A + 2D + \left(v^i - B^i\right)e_i + E_{ij}e^i e^j - (1+z)\partial_z \Delta z \\ & -2\frac{1+z}{Hr}\Delta z - \Delta z - 2\kappa + \frac{1+z}{H}\frac{dH}{dz}\Delta z + 2\frac{\delta r}{r}\Big) \equiv d\bar{V}(1+\delta V). \end{aligned} \tag{2.60}$$

Therefore, we can express the number of galaxies within the already define volume as

$$dN(z, \hat{\mathbf{n}}) = n_p d\bar{V} = n_g dV \quad \Rightarrow \quad n_g = n_p(1+\delta V), \tag{2.61}$$

where n_p is the physical number density and n_g the observed galaxy number density of the source galaxies. Here we have justified the expression introduced in Eq. (2.4) and hence demonstrate the covariant form of the galaxy number density. However, further corrections to n_g have to be considered.

2.2.2.4 Corrections due to selection effects.

As it was mentioned in Chapter (1), we must take into account selection effects which limit the galaxy sample up to a threshold apparent magnitude or, equivalently, a threshold flux $\mathcal{F}_{thr}$. Therefore the physical number density must be modified to take into account only galaxies with observed flux greater than $\mathcal{F}_{thr}$

$$n_p \rightarrow \int_{\mathcal{F}_{\rm thr}}^{\infty} \frac{n_p}{d\mathcal{F}} d\mathcal{F}. \tag{2.62}$$

Recalling the definition of the luminosity distance, we can define a luminosity threshold for the background as $L_{\rm thr} = 4\pi D_L^2(z)\mathcal{F}_{\rm thr}$, and assuming that the

[11] Again, as advanced knowledge on differential geometry was needed for this derivation, the result was directly taken following the work by Yoo *et al.* (1).

galaxy luminosity function follows a power law of the type $dn_p/dL \propto L^{-s}$ with $s \geq 1$, we can rewrite Eq. (2.62) as

$$\int_{\mathcal{F}_{\rm thr}}^{\infty} d\mathcal{F}\frac{dL}{d\mathcal{F}}\frac{dn_p}{dL} = 4\pi D_L^2(z)\mathcal{F}_{\rm thr}\int_{\mathcal{F}_{\rm thr}}^{\infty} d\mathcal{F}\frac{dn_p}{dL} = 4\pi D_L^2(z)\mathcal{F}_{\rm thr}\int_{L_{\rm thr}}^{\infty}\frac{dL}{4\pi D_L^2(z)}\mathcal{C}L^{-s}$$
$$= \mathcal{C}\frac{1}{s-1}L_{thr}^{1-s} \equiv n_p(L_{thr}), \tag{2.63}$$

where $\mathcal{C}$ is a constant. Here $n_p(L_{thr})$ defines the *cumulative* number density for galaxies with luminosity greater than L_{thr} without taking into account relativistic effects.

Incorporating the perturbations to the luminosity distance we get the corrected $\mathcal{D}_L$ defined in Eq. (2.55), so $L_{\rm thr} = 4\pi D_L^2(1+\delta\mathcal{D}_L)^2\mathcal{F}_{\rm thr}$. Hence $n_p(L_{thr})$ has to be modified as

$$n_p(L_{thr}) \rightarrow \frac{\mathcal{C}}{s-1}\left[4\pi D_L^2(1+\delta\mathcal{D}_L)^2\mathcal{F}_{\rm thr}\right]^{1-s} = \frac{\mathcal{C}}{s-1}\left[L_{thr}(1+2\delta\mathcal{D}_L)\right]^{1-s} \tag{2.64}$$

to linear order in perturbations. Expanding now the final expression

$$\frac{\mathcal{C}}{s-1}\left[L_{thr}(1+2\delta\mathcal{D}_L)\right]^{1-s} = \frac{\mathcal{C}}{s-1}L_{thr}^{1-s}\Big[1+(1-s)2\delta\mathcal{D}_L\Big]$$
$$\equiv n_p(L_{thr})\left[1+2(1-s)\delta\mathcal{D}_L\right]. \tag{2.65}$$

Defining the slope of the luminosity function in magnitude as $p = \frac{d\log_{10} n_p(M)}{dM}$, if we can express $M =$ constant $-2.5\log_{10}(L/L_0)$ (21), thus we have $p = 0.4(s-1)$ and so we can express the correction to the number density due to selection effects as $n_p(L_{\rm thr})(1-5p\delta\mathcal{D}_L)$, and so

$$n_g = n_p(1+\delta V)(1-5p\delta\mathcal{D}_L) \tag{2.66}$$

2.2.2.5 Corrections due to bias function.

We can also express corrections to the physical number density due to fluctuations in the matter number density. Galaxies tend to be formed on overdense regions of space, so for a mean number density $\bar{n}_p$, we can express $n_p = \bar{n}_p(1+\delta_n)$, with δ_n the galaxy density contrast field. As it was presented in Chapter 1, the bias function relates the galaxy density contrast with the matter density contrast δ_m. In (1), Yoo *et al.* define the bias on the linear bias approximation so that the physical number density of the observed galaxy is some function of the local matter density $n_p = f(\rho_m)$.

However, as stated by Challinor and Lewis (22), they define a linear bias relation considering the definition of the matter density parameter $\Omega_m = \bar{\rho}_{m,0}/\rho_c$ with $\rho_c = \frac{8\pi G}{3H_0^2}$ and $\rho_m \propto a^{-1} \propto (1+z)$ for non-relativistic matter. Hence, they consider the linear relation $\bar{\rho}_m(z) = (3H_0^2/8\pi G)\,\Omega_m(1+z)$ at the observed redshift z. This is a background quantity but defined at the observed redshift $z = 1/a(1+\Delta z)$ rather than at its background value $\tilde{z}_H = 1/a$, so defining the bias over this quantity singles out the observational gauge known as *equal redshift gauge* or *zero-redshift perturbation gauge* ($\Delta z = 0$) (23). Therefore, as the bias is related to the structure formation itself and not to the way we observe it, it seems incorrect to define it for an observational gauge, so this choice may not be the ideal.

Gauges in cosmology

Before continuing with our proposal for the bias, we will introduce a brief explanation of gauges in cosmology. From considering a perturbation approach, some gauge degrees of freedom arises in the Lagrangian of a relativistic theory of gravity, as relativistic gravity is a constrained system and thus there exist constrained equations relating variables only algebraically (24). These degrees of freedom can be eliminated by working under a specific gauge imposing some gauge conditions. In this way, we could consider *e.g.* the synchronous gauge taking $A = 0$ in the general FLWR metric considered in Eq. 2.1 or the uniform-density gauge for $\delta = 0$.

However, different gauges could be preferred depending on the problem at hand, so fixing a gauge condition could make us lose some advantages of other gauges in a particular problem. Following that approach, Bardeen in 1980 (25) propose a *gauge ready* method, which allow us to work without imposing any gauge condition at early stages of the calculation, and therefore the use of various gauge conditions in different situations.

As was introduced in Chapter 1, we have been following this gauge-ready method and working without imposing any gauge condition yet. According to Bardeen "Since the background 3-space is homogeneous and isotropic, the perturbation in all physical quantities must in fact be gauge invariant under purely spatial gauge transformations" (26). Following this approach, we can express scalar-type perturbations in a spatially gauge-invariant form but without fixing the temporal gauge condition, and so we can implement this temporal gauge condition later depending on the situation.

For our proposal for the bias, we can write the gauge-invariant combination[12]

$$\delta_v \equiv \delta - \frac{\dot{\bar{n}}_p}{\bar{n}_p}\frac{v}{k}, \tag{2.67}$$

where k refers to the wavenumber of the observed photons and v is the scalar part of the peculiar velocity $v^i = v^i_s + v^i_v$, as $v^i_s = v^{,i}$ and $\nabla \cdot \mathbf{v}_v = \mathbf{0}$ (28). Moreover, $\bar{n}_p$ represents the average physical number density of observed galaxies. For a relativistic treatment of the bias, we will expect relativistic corrections arising from the Newtonian definition. However, if we consider an orthogonal, comoving and synchronous gauge, the relativistic Poisson equation coincides with the classical description, and hence no relativistic corrections are needed (29). Therefore, the bias (as it is classically defined) is only meaningful on an orthogonal, synchronous ($A = 0$) and comoving ($v/k = 0$) gauge. In such a gauge, if we consider Eq. (2.67) for the galaxy density contrast δ_n

$$\delta_v = \delta_n^{syn} = b\delta_m^{\mathrm{syn}} \tag{2.68}$$

with the linear bias parameter b and δ_m^{syn} the matter density contrast in this gauge. As δ_v is gauge-invariant , for any other gauge (*e.g.* the Newtonian-longitudinal gauge) we have

$$\delta_v = \delta_n - \frac{\dot{\bar{n}}_p}{\bar{n}_p}\frac{v}{k} = \delta_n^{syn} = b\delta_m^{\mathrm{syn}}, \tag{2.69}$$

and so as our ansatz for the bias we propose

$$\delta_n = b\delta_m^{\mathrm{syn}} + \frac{\dot{\bar{n}}_p}{\bar{n}_p}\frac{v}{k}. \tag{2.70}$$

After considering all these correction, we can finally write the galaxy number density as

$$n_g = \bar{n}_p(1+\delta V)(1-5p\delta\mathcal{D}_L)(1+\delta_n) \equiv \bar{n}_p(1+\delta_g), \tag{2.71}$$

so up to linear order we get

$$\begin{aligned}\delta_g = b\delta_m^{syn} + \frac{\dot{\bar{n}}_p}{\bar{n}_p}\frac{v}{k} - \left(2\frac{1+z_H}{Hr} - \frac{1+z_H}{H}\left.\frac{dH}{dz}\right|_{z=z'} + 1 + (1+z_H)\partial_z\right)\Delta z \\ + A + 2D + 2\frac{\delta r}{r} - 5p\delta\mathcal{D}_L - 2\kappa + \left(v^i - B^i\right)e_i + E_{ij}e^i e^j.\end{aligned} \tag{2.72}$$

[12]Gauge-invariant implies it remains unchanged after a gauge transformation, *i.e.* after a coordinate transformation of the form $x^\mu \to \tilde{x}^\mu = x^\mu + \xi^\mu$. More information on gauge transformations can be found at (27).

3 Results

Abstract:
Tomando las expresiones obtenidas en el capítulo anterior, se escribe la expresión final para las ARF teniendo en cuenta las correcciones relativistas, resultado fundamental de nuestro trabajo. Teniendo esta ecuación en su expresión final, se procede a expresar también en los gauges *synchronous* y Newtoniano, con el fin de mostrar la utilidad del método *gauge-ready*.

Having applied the relativistic corrections to both z_g and δ_g, we can rewrite Eq. (2.2) in a general form for as

$$\bar{z} + \delta z(\hat{\mathbf{n}}) = \frac{\int dr\, r^2 \bar{n}_p(r)\,(1+\delta_g)\,\frac{1}{a_g}(1+\Delta z - a_g) W(z_{obs} - \frac{1}{a_g}(1+\Delta z - a_g))}{\int dr\, r^2 \bar{n}_p(r)\,(1+\delta_g))\, W(z_{obs} - \frac{1}{a_g}(1+\Delta z - a_g))}. \tag{3.1}$$

Given that $1+\tilde{z}_H = \frac{1}{a_g} \Rightarrow \tilde{z}_H = \frac{1}{a_g}(1-a_g)$, with $\tilde{z}_H$ the gauge-dependent redshift parameter in a homogeneous and isotropic universe, we can substitute it in the expression.[1] Furthermore, if we define the normalised functional

$$\mathcal{F}[Y] = \frac{\int dr r^2 \bar{n}_p(r) W\left(z_{obs} - \tilde{z}_H\right) Y(r)}{\int dr r^2 \bar{n}_p(r) W\left(z_{obs} - \tilde{z}_H\right)} = \frac{1}{\mathcal{N}} \int dr r^2 \bar{n}_p(r) W\left(z_{obs} - \tilde{z}_H\right) Y(r) \tag{3.2}$$

with the normalisation constant $\mathcal{N}$ referring to the average number of galaxies under the considered Gaussian shell W, we can express Eq. (3.1) in an easier way, and expand it to first order in perturbation theory[2]

$$\begin{aligned} \bar{z} + \delta z(\hat{\mathbf{n}}) &= \mathcal{F}\left[\tilde{z}_H\right] + \mathcal{F}\left[\delta_g\left(\tilde{z}_H - \mathcal{F}\left[\tilde{z}_H\right]\right)\right] \\ &+ \mathcal{F}\left[\frac{\Delta z}{a_g}\left(1 - \left.\frac{d\ln W}{dz}\right|_{z=z'}\left(\tilde{z}_H - \mathcal{F}\left[\tilde{z}_H\right]\right)\right)\right] + \mathcal{O}\left(2^{nd}\right), \end{aligned} \tag{3.3}$$

where $\frac{d\ln W}{dz}$ is evaluated at $z' = z_{obs} - \tilde{z}_H$. This is a general expression for the ARF without having fixed a gauge condition yet, and hence in gauge-ready form.

[1] The tilde in $\tilde{z}_H$ is used to differentiate it from the already defined $z_H = a_0/a$, as we have seen $a_0 = a(\eta_0 + \delta\eta_0)$ (see Eq. (2.41)).

[2] Calculations are included in Annex B.

This is the main result of our work, and will be discussed and particularized for some common gauges in the next section.

3.1 Particularization for specific gauges

As was stated at the end of the last section, we have found an expression for the ARF in a gauge-ready form. If we identify $\mathcal{F}[\tilde{z}_H]$ as the redshift monopole $\bar{z}$ in Eq. (3.3), we get that we can write the ARF to linear order in perturbations as

$$\delta z(\hat{\mathbf{n}}) = \mathcal{F}\left[\delta_g\left(\tilde{z}_H - \mathcal{F}\left[\tilde{z}_H\right]\right)\right] + \mathcal{F}\left[\frac{\Delta z}{a_g}\left(1 - \left.\frac{d\ln W}{dz}\right|_{z=z'}\left(\tilde{z}_H - \mathcal{F}\left[\tilde{z}_H\right]\right)\right)\right]. \quad (3.4)$$

Being the principal advantage of expressing Eq. (3.4) in a gauge-ready form, we can characterize it for different gauges. Two of the most common gauges in cosmology when working on linear perturbation theory are the synchronous gauge and the conformal Newtonian (or longitudinal) gauge. Here we will present our solutions for these two gauges. As $\mathcal{F}[\tilde{z_H}]$ and the window function $W(z_{obs} - \tilde{z}_H)$ are gauge-invariant, when particularizing for the two gauges we will only account for modifications in the defintions of Δz and δ_g.

3.1.1 Synchronous gauge

First introduced by Lifshitz in 1946 (30) is defined by the conditions $A = 0$ and $B_i = 0$[3]. Hence, the general FLWR metric in Eq. (2.1) is modified so that

$$ds^2 = a^2(\eta)\left\{-d\eta^2 + \left(\bar{g}_{ij} + h_{ij}\right)dx^i dx^j\right\}, \quad (3.5)$$

with $h_{ij} = 2(D\bar{g}_{ij} + E_{ij})$. Therefore, the corrections to the observed redshift are modified too as

$$\Delta z = \int_0^{r_g} dr\left(\dot{D} + \dot{E}_{ij}e^i e^j\right) + \mathcal{H}_0\delta\eta_0. \quad (3.6)$$

Following the classical notation adopted in Eq. (3.5)

$$h_{ij}e^i e^j = 2(D\bar{g}_{ij}e^i e^j + E_{ij}e^i e^j) = 2(D + E_{ij}e^i e^j), \quad (3.7)$$

where we used again $\bar{g}_{ij}e^i e^j = 1$. As e^i is constant along the geodesic we get

$$\Delta z = \frac{1}{2}\int_0^{r_g} dr\, \dot{h}_{ij}e^i e^j + \mathcal{H}_0\delta\eta_0. \quad (3.8)$$

[3]This, in fact, implies it is also a comoving gauge, so we also have $v/k = 0$. See (28) for more information.

With respect to δ_g, it will result in

$$\delta_g = b\delta_m - \left(2\frac{1+z_H}{Hr} - \frac{1+z_H}{H}\frac{dH}{dz}\bigg|_{z=z'} + 1 + (1+z_H)\partial_z\right)$$
$$\left(\frac{1}{2}\int_0^{r_g} dr\,\dot{h}_{ij}e^ie^j + \mathcal{H}_0\delta\eta_0\right) + 2D + 2\frac{\delta r}{r} - 5p\delta\mathcal{D}_L - 2\kappa + E_{ij}e^ie^j. \tag{3.9}$$

Without loss of generality, we can take E_{ij} to be traceless, and hence identify $D \equiv h \equiv h_{ii}$, *i.e.* with the trace part of the synchronous perturbation h_{ij}, which characterizes the scalar mode of the metric perturbations in this gauge (31). Hence we can rewrite Eq. (3.9) in this notation as

$$\delta_g = b\delta_m - \frac{1}{2}\left(2\frac{1+z_H}{Hr} - \frac{1+z_H}{H}\frac{dH}{dz}\bigg|_{z=z'} + 1 + (1+z_H)\partial_z\right)$$
$$\left(\frac{1}{2}\int_0^{r_g} dr\,\dot{h}_{ij}e^ie^j + \mathcal{H}_0\delta\eta_0\right) + h + \frac{1}{2}h_{ij}e^ie^j + 2\frac{\delta r}{r} - 5p\delta\mathcal{D}_L - 2\kappa. \tag{3.10}$$

Note here that we should also characterize the expressions for the spatial displacements δr, $\delta\theta$, $\delta\phi$, the corrections to the luminosity distance $\delta\mathcal{D}_L$ and the convergence κ, which appear implicitly in the definition of δ_g. However, as these results are presented only as examples of the advantages of the gauge-ready method in particularizing for different gauges, we will avoid to show them as it will not add any more information.

3.1.2 Conformal-Newtonian gauge

Also known as longitudinal gauge, it was advocated by Mukhanov, Feldman and Brandenberger in 1992 (32). It is a particularily simple metric since tensor and vector perturbation modes are not considered (*i.e.* we impose the gauge conditions $B_i = 0$ and $E_{ij=0}$). Hence we get a metric applicable only for the scalar mode of the metric perturbations and characterized by two scalar fields $\psi \equiv A$ (which corresponds to the gravitational potential in the Newtonian limit, and thus the name) and $\phi \equiv -D$

$$ds^2 = a^2(\eta)\left\{-(1+2\psi)d\eta^2 + (1-2\phi)\bar{g}_{ij}dx^idx^j\right\}. \tag{3.11}$$

In this way we get a diagonal metric tensor $g_{\mu\nu}$ which describes the structure of the Universe (inferred by the scalar perturbations only). It is easier to express the results for Δz and δ_g in this gauge

$$\Delta z = \psi_o - \psi_g - \int_0^{r_g} dr(\dot{\psi} + \dot{\phi}) + \mathcal{H}_0\delta\eta_0. \tag{3.12}$$

$$
\begin{aligned}
\delta_g = b\delta_m^{syn} + \frac{\dot{\bar{n}}_p}{\bar{n}_p}\frac{v_N}{k} - \left(2\frac{1+z_H}{Hr} - \frac{1+z_H}{H}\frac{dH}{dz}\bigg|_{z=z'} + 1 + (1+z_H)\,\partial_z\right) \\
\left(\psi_o - \psi_g - \int_0^{r_g} dr(\dot{\psi} + \dot{\phi}) + \mathcal{H}_0\delta\eta_0\right) + \psi - 2\phi + 2\frac{\delta r}{r} \\
- 5p\delta\mathcal{D}_L - 2\kappa + v^i e_i.
\end{aligned}
\tag{3.13}
$$

with v_N the Newtonian velocity. Again for this gauge we should also characterize the expressions for δr, $\delta\theta$, $\delta\phi$, $\delta\mathcal{D}_L$ and κ. Note here we can identify some terms, like the usual Sachs Wolfe effect $(\psi_0 - \psi_g)$ accounting for the difference in the gravitational potential at photon emission and reception, or the integrated Sachs-Wolfe effect $(\int_0^{r_g} dr(\dot{\psi} + \dot{\phi}))$, which traces the net photon energy gain/loss as light crosses time evolving gravitational potentials (33).

4 Conclusions

Starting from a Newtonian definition of the ARF, we have taken into account the relativistic effects under a *gauge-ready* approach. Up to linear order in perturbations, we have derived the corrected expressions for the observed redshift and number density of galaxies in a generic form, thus yielding a expression for the ARF in terms of quantities that are invariant under coordinate transformations. Once these expressions have been obtained, we could express the ARF to linear order in Eq. (3.4), with a fairly similar form to the one obtained originally by Hernández-Monteagudo *et al.* but accounting for the new effects arising from the relativistic treatment. Finally, in order to obtain a more physically intuitive form of the expressions and show the advantages of the *gauge-ready* method, the final Eq. (3.4) have been formulated in the synchronous and Newtonian gauges.

As a purely analytical work, no conclusions about the amount of information ARF can reveal can be obviously extracted from our results. Further numerical work must be performed in order to estimate the amount of cosmological information encoded in the ARF in comparison with the ADF in regard of the general relativistic corrections. However, here we have succeeded in writing the ARF in a fully relativistic way, which can be (and has been) particularized for different gauges.

Some of that work have already been performed in the internship which took place between March and May of this year at IAC, supervised by Carlos Hernández-Monteagudo. Under a different perspective, working directly in the Newtonian-gauge following a similar approach from Lewis and Challinor (22), we were able to obtain the C_ℓ for the ARF in some simple cases, equivalent to the ones obtained by Hernández-Monteagudo *et al.*. In this way, we were able to check the validity of our results (see Figures 4.1 and 4.2). Further work on the matter is being developed to obtain the C_ℓ including all corrections, and hence reveal all information ARF has to offer as a cosmological probe. We expect this work to be submitted to a Q1 refereed journal in the upcoming months.

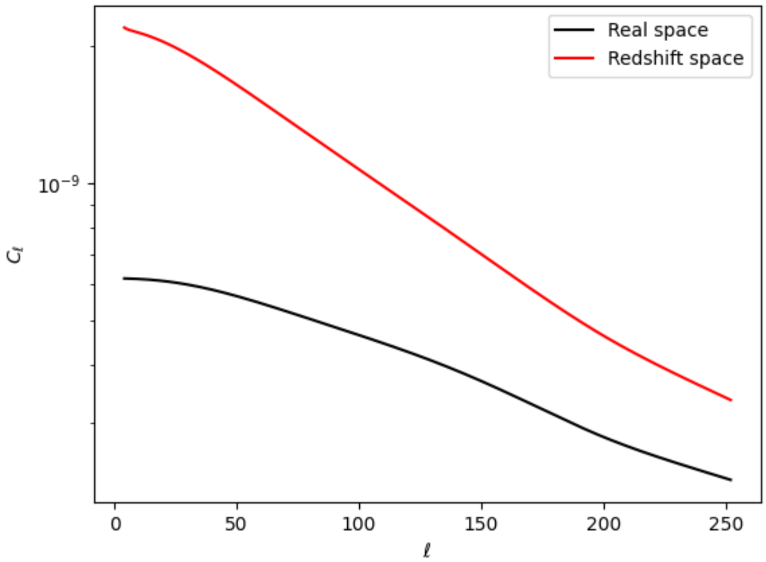

Figure 4.1: Angular power spectra from ARF in real and redshift space for $z_{obs} = 1$, $\sigma_z = 0.01$ and $b = 1$, obtained modifying the cosmological code `CAMB sources` integrated in `CAMB` (34). This was the main result of our work at the IAC, showing the correspondence with the classical results displayed in Figure 4.2.

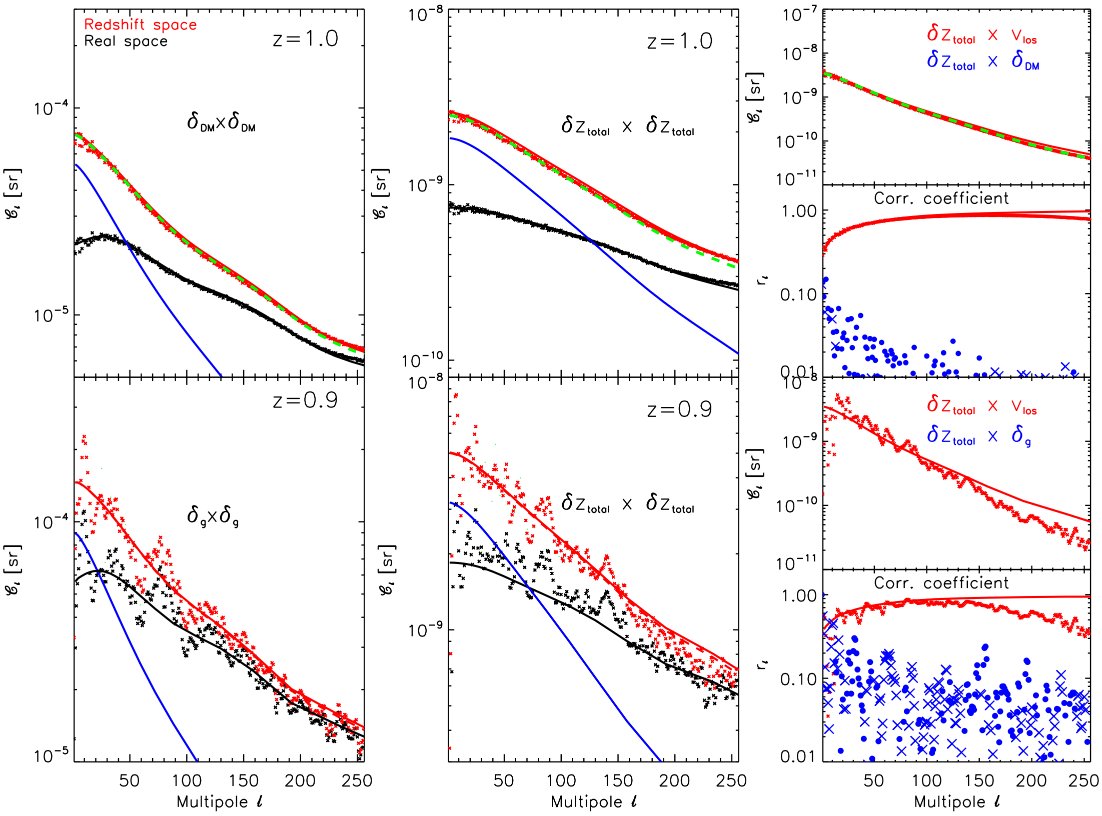

Figure 4.2: Comparison of the angular power spectra from ADF (left panels) and from ARF (middle panels) obtained in (2) by Hernández-Monteagudo *et al.*. For the purpose of showing the correspondance with our results, we should focus on the top middle panel, which displays the angular power spectra from ARF in real (solid black) and redshift (solid red) space for $z_{obs} = 1$, $\sigma_z = 0.01$ and $b = 1$.

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Annex A: Christoffel symbols for the perturbed FLWR metric

The Christoffel symbols are given by

$$\Gamma^{\mu}_{\rho\sigma} = \frac{1}{2} g^{\rho\nu} \left(g_{\nu p,\sigma} + g_{\sigma\nu,\rho} - g_{\rho\sigma,\nu} \right). \tag{4.1}$$

First of all, we can obtain the inverse metric tensor components using $g^{\mu\nu} g_{\nu\rho} = \delta^{\mu}_{\rho}$ to first order

$$g^{\eta\eta} = -a^{-2}(1+2A), \quad g^{\eta i} = -a^{-2} B^{i}, \quad g^{ij} = a^{-2}\Big(\bar{g}^{ij}(1-2D) - 2E^{ij}\Big). \tag{4.2}$$

Hence, for the FLWR metric defined in (2.1) we can obtain the general expression for $\Gamma^{\mu}_{\rho\sigma}$ to linear order in a covariant form (in terms only of tensors):

- $\mu = \eta$, $\rho = \eta$, $\sigma = \eta$

$$\begin{aligned}
&\Gamma^{\eta}_{\eta\eta} = \frac{1}{2} g^{\eta\eta} \left(g_{\eta\eta,\eta} \right) + \frac{1}{2} g^{\eta i} \left(g_{i\eta,\eta} + g_{\eta i.\eta}, - g_{\eta\eta,i} \right) = \frac{1}{2} \Bigg\{ -\frac{1}{a^{2}}(1+2A) \\
&\Big(-2a\dot{a}(1+2A) - 2a^{2}\dot{A}\Big) - \frac{1}{a^{2}} B^{i} \Big(-4a\dot{a}B_{i} - 2a^{2}\dot{B}_{i} + a^{2} 2A_{,i}\Big)\Bigg\} \\
&= \frac{\dot{a}}{a} + \frac{\dot{A}}{(1+2A)} + 2\frac{\dot{a}}{a} B^{i} B_{i} + \dot{B}_{i} B_{i} - B^{i} A_{,i} = \frac{\dot{a}}{a} + \dot{A}(1-2A) \Rightarrow \\
&\Rightarrow \Gamma^{\eta}_{\eta\eta} = \frac{\dot{a}}{a} + \dot{A}.
\end{aligned} \tag{4.3}$$

- $\mu = \eta,\ \rho = \eta,\ \sigma = i$

$$\begin{aligned}
&\Gamma^{\eta}_{\eta i} = \frac{1}{2}g^{\eta\eta}\left(g_{\eta\eta,i}\right) + \frac{1}{2}g^{\eta j}\left(g_{ij,\eta} + g_{\eta j,i}, -g_{\eta i,j}\right) = \frac{1}{2}\Bigg\{-\frac{1}{a^2}(1+2A) \\
&(-a^2 2A_{,i}) - \frac{1}{a^2}B^j\Bigg[2a\dot{a}[(1+2D)\bar{g}_{ij} + 2E_{ij}] + a^2(2\dot{D}\bar{g}_{ij} + 2\dot{E}_{ij}) \\
&- a^2(B_{j,i} - B_{i,j})\Bigg]\Bigg\} = A_{,i}(1-2A) - \frac{\dot{a}}{a}B^j\bar{g}_{ij} \Rightarrow \\
&\Rightarrow \Gamma^{\eta}_{\eta i} = A_{,i} - \frac{\dot{a}}{a}B_i.
\end{aligned} \tag{4.4}$$

- $\mu = \eta,\ \rho = i,\ \sigma = j$

$$\begin{aligned}
&\Gamma^{\eta}_{ij} = \frac{1}{2}g^{\eta\eta}\left(g_{i\eta,j} + g_{\eta j,i} - g_{ij,\eta}\right) + \frac{1}{2}g^{\eta k}\left(g_{kj,i} + g_{ik,j}, -g_{ij,k}\right) = \\
&\frac{1}{2}\Bigg\{-\frac{1}{a^2}(1+2A)\Big[-a^2(B_{j,i} + B_{i,j}) - 2a\dot{a}\left[(1+2D)\bar{g}_{ij} + 2E_{ij}\right] \\
&- a^2\left(2\dot{D}\bar{g}_{ij} + 2\dot{E}_{ij}\right)\Big] - \frac{1}{a^2}B^k a^2\Big[2D_{,i}\bar{g}_{kj} + (1+2D)\bar{g}_{kj,i} \\
&+2E_{kj,i} + 2D_{,j}\bar{g}_{ik} + (1+2D)\bar{g}_{ik,j} + 2E_{ik,j} - 2D_{,k}\bar{g}_{ij} \\
&- (1+2D)\bar{g}_{ij,k} - 2E_{ij,k}\Big]\Bigg\} = B_{(i,j)} + \frac{\dot{a}}{a}\bar{g}_{ij} + 2\frac{\dot{a}}{a}\left(D\bar{g}_{j_j} + E_{ij}\right) \\
&- 2\frac{\dot{a}}{a}\bar{g}_{ij}A + \dot{D}\bar{g}_{ij} + \dot{E}_{ij} - \frac{1}{2}B^k\left(\bar{g}_{kj,i} + \bar{g}_{ik,j} - \bar{g}_{ij,k}\right).
\end{aligned} \tag{4.5}$$

As we can rewrite

$$\frac{1}{2}B^k\left(\bar{g}_{kj,i} + \bar{g}_{ik,j} - \bar{g}_{ij,k}\right) = \frac{1}{2}\bar{g}^{lk}B_l\left(\bar{g}_{kj,i} + \bar{g}_{ik,j} - \bar{g}_{ij,k}\right) = \overline{\Gamma}^l_{ij}B_l \tag{4.6}$$

and $B_{i|j} = B_{i,j} - \overline{\Gamma}^l_{ij}B_l$, then

$$B_{(i,j)} = \frac{1}{2}(B_{i,j} + B_{j,i}) = \frac{1}{2}(B_{i|j} + \overline{\Gamma}^l_{ij}B_l + B_{j|i} + \overline{\Gamma}^l_{ji}B_l) = B_{(i|j)} + \overline{\Gamma}^l_{ij}B_l, \tag{4.7}$$

as the Christoffel symbols are symmetric in their lower indices ($\overline{\Gamma}^l_{ij}B_l = \overline{\Gamma}^l_{ji}B_l$). Hence we have

$$\Gamma^{\eta}_{ij} = B_{(i|j)} + \frac{\dot{a}}{a}\bar{g}_{ij} + 2\frac{\dot{a}}{a}\left(D\bar{g}_{ij} + E_{ij} - \bar{g}_{ij}A\right) + \dot{D}\bar{g}_{ij} + \dot{E}_{ij}. \tag{4.8}$$

- $\mu = i,\ \rho = \eta,\ \sigma = \eta$

$$\Gamma^i_{\eta\eta} = \frac{1}{2}g^{i\eta}\left(g_{\eta\eta,\eta}\right) + \frac{1}{2}g^{ij}\left(g_{\eta j,\eta} + g_{j\eta,\eta}, -g_{\eta\eta,j}\right) = \frac{1}{2}\Bigg\{-\frac{1}{a^2}B^i$$
$$\left[-2a\dot{a}(1+2A) - 2a^2\dot{A}\right] + \frac{1}{a^2}\left[\bar{g}^{ij}(1-2D) - 2E^{ij}\right]\left[-4a\dot{a}B_j\right.$$
$$\left. - 2a^2\dot{B}_j + 2a^2A_{,j}\right]\Bigg\} = \frac{1}{2}\left(2\frac{\dot{a}}{a}B^i - 4\frac{\dot{a}}{a}B^i - 2\dot{B}^i + 2A^{,i}\right), \tag{4.9}$$

which as A is a scalar, $A^{,i} = A^{|i}$ and so it results in

$$\Gamma^i_{\eta\eta} = A^{|i} - \dot{B}^i - \frac{\dot{a}}{a}B^i. \tag{4.10}$$

- $\mu = i,\ \rho = \eta,\ \sigma = j$

$$\Gamma^i_{\eta j} = \frac{1}{2}g^{i\eta}\left(g_{\eta\eta,j}\right) + \frac{1}{2}g^{ik}\left(g_{\eta k,j} + g_{jk,\eta} - g_{\eta j,k}\right) = \frac{1}{2}\Bigg\{-\frac{1}{a^2}B^i(-a^2 2A_{,j})$$
$$+ \left[\frac{1}{a^2}\bar{g}^{ik}(1-2D) - 2E^{ik}\right]\left[-a^2B_{k,j} + 2a\dot{a}\left[(1+2D)\bar{g}_{jk} + 2E_{jk}\right]\right.$$
$$\left. + a^2\left(2\dot{D}\bar{g}_{jk} + 2\dot{E}_{jk}\right) + a^2B_{j,k}\right]\Bigg\} = \frac{1}{2}\Bigg\{-B^i_{,j} + 2\frac{\dot{a}}{a}\left[(1+2D)\delta^i_j + 2E^i_j\right]$$
$$+2\left(\dot{D}\delta^i_j + 2\dot{E}^i_j\right) + B_j^{,i} + 2\frac{\dot{a}}{a}\left(-2D\delta^i_j - 2E^i_j\right)\Bigg\} = \frac{1}{2}\left(B_j^{,i} - B^i_{,j}\right) + \frac{\dot{a}}{a}\delta^i_j$$
$$+ \left(\dot{D}\delta^i_j + \dot{E}^i_j\right). \tag{4.11}$$

As

$$B_j^{,i} - B^i_{,j} = \bar{g}^{il}\left(B_{j,l} - B_{l,j}\right) = \bar{g}^{il}\left(B_{j|l} + \overline{\Gamma}^m_{jl}B_m - B_{l|j} - \overline{\Gamma}^m_{lj}B_m\right)$$
$$= B_j^{|i} - B^i_{|j} \tag{4.12}$$

we finally can write $\Gamma^i_{\eta j}$ as

$$\Gamma^i_{\eta j} = \frac{1}{2}\left(B_j^{|i} - B^i_{|j}\right) + \frac{\dot{a}}{a}\delta^i_j + \dot{D}\delta^i_j + \dot{E}^i_j. \tag{4.13}$$

- $\mu = i,\ \rho = j,\ \sigma = k$

$$\Gamma^i_{jk} = \frac{1}{2}g^{i\eta}\left(g_{j\eta,k} + g_{\eta k,j} - g_{jk,\eta}\right) + \frac{1}{2}g^{il}\left(g_{jl,k} + g_{lk,j}, -g_{jk,l}\right). \tag{4.14}$$

At this step, in order to simplify calculations, we can define $C_{ij} = D\bar{g}_{ij} + E_{ij}$, and thus

$$
\begin{aligned}
\Gamma^i_{jk} &= \frac{1}{2}\Bigg\{ -\frac{1}{a^2}B^i \left[-a^2\left(B_{j,k}+B_{k,j}\right) - 2a\dot{a}\left(\bar{g}_{jk}+2C_{jk}\right) - a^2 2\dot{C}_{jk} \right] \\
&+ \frac{1}{a^2}\left(g^{il}-2C^{il}\right)a^2\left[2C_{jl,k}+\bar{g}_{jl,k}+2C_{lk,j}+\bar{g}_{lk,j}-2C_{jk,l}-\bar{g}_{jk,l}\right]\Bigg\} \\
&= \frac{1}{2}\Bigg[2\frac{\dot{a}}{a}B^i\bar{g}_{jk} + 2\left(C_{j,k}^{\ i} + C^i_{k,j} - C_{jk}^{\ ,i}\right) + \bar{g}^{il}\left(\bar{g}_{jl,k}+\bar{g}_{lk,j}-\bar{g}_{jk,l}\right) \\
&- 2C^{il}\left(\bar{g}_{jl,k}+\bar{g}_{lk,j}-\bar{g}_{jk,l}\right) = \frac{\dot{a}}{a}B^i\bar{g}_{jk} + C_{j,k}^{\ i} + C^i_{k,j} - C_{jk}^{\ ,i} + \overline{\Gamma}^i_{jk} \\
&- C^i_m\bar{g}^{ml}\left(\bar{g}_{jl,k}+\bar{g}_{jl,k}-\bar{g}_{jl,k}\right)\Bigg].
\end{aligned}
\tag{4.15}
$$

As we can express

$$
\begin{aligned}
& C_{j,k}^{\ i} + C^i_{k,j} - C_{jk}^{\ ,i} = g^{il}\left(C_{jl,k}+C_{lk,j}-C_{jk,l}\right) = \\
& \bar{g}^{il}\left(C_{jl|k} + \overline{\Gamma}^m_{kj}C_{ml} + \overline{\Gamma}^m_{kl}C_{jm} + C_{lk|j} + \overline{\Gamma}^m_{jk}C_{ml} + \overline{\Gamma}^m_{jl}C_{km} - C_{jk|l}\right. \\
& \left. -\overline{\Gamma}^m_{lj}C_{km} - \overline{\Gamma}^m_{lk}C_{jm}\right) = C^i_{j|k} + C^i_{k|j} + 2\overline{\Gamma}^m_{jk}C_m^{\ i} - C_{jk}^{\ |i} \\
& = 2C^i_{(j|k)} + 2\overline{\Gamma}^m_{jk}C_m^{\ i} - C_{jk}^{\ |i}
\end{aligned}
\tag{4.16}
$$

and also taking into account that C_{ij} is symmetric and

$$
C^i_m\bar{g}^{ml}\left(\bar{g}_{jl,k}+\bar{g}_{lk,j}-\bar{g}_{jk,l}\right) = 2C^i_m\overline{\Gamma}^m_{jk},
\tag{4.17}
$$

we can finally express Γ^i_{jk} to first order as

$$
\overline{\Gamma}^i_{jk} = \frac{\dot{a}}{a}B^i\bar{g}_{jk} + \overline{\Gamma}^i_{jk} + 2C^i_{(j|k)} - C_{jk}{}^{|i},
\tag{4.18}
$$

which if we further decompose C_{ij} into E_{ij} and D we recover the expression

$$
\Gamma^i_{jk} = \overline{\Gamma}^i_{jk} + \frac{\dot{a}}{a}\bar{g}_{jk}B^i + D_{|k}\delta^i_j + D_{|j}\delta^i_k + 2E^i_{(j|k)} - D^{|i}\bar{g}_{jk} - E_{jk}^{\ |i}.
\tag{4.19}
$$

Annex B: Expansion of the relativistic expression for the ARF

As was introduced in Eq.(2.2), the expression for the ARF given by Hernández-Monteagudo *et al.* (2) is

$$\bar{z} + \delta z(\hat{\mathbf{n}}) = \frac{\int dr\, r^2 \bar{n}_p(r)\,(1+\delta_g(r,\hat{\mathbf{n}}))\, z_g(r,\hat{\mathbf{n}}) W(z_{obs} - z_g)}{\int dr\, r^2 \bar{n}_p(r)\,(1+\delta_g(r,\hat{\mathbf{n}}))\, W(z_{obs} - z_g)}. \tag{4.20}$$

Now this can be expanded to first order in perturbations accounting for the definition of z_g as

$$1 + z_g = \frac{1}{a_g}(1+\Delta z) \Rightarrow z_g = \frac{1}{a_g}(1+\Delta z - a_g), \tag{4.21}$$

and so

$$\bar{z} + \delta z(\hat{\mathbf{n}}) = \frac{\int dr\, r^2 \bar{n}_p(r)\,(1+\delta_g)\, \frac{1}{a_g}(1+\Delta z - a_g) W(z_{obs} - \frac{1}{a_g}(1+\Delta z - a_g))}{\int dr\, r^2 \bar{n}_p(r)\,(1+\delta_g)\, W(z_{obs} - \frac{1}{a_g}(1+\Delta z - a_g))}. \tag{4.22}$$

Now we can expand the window function to first order as

$$\begin{aligned} W(z_{obs} - \frac{1}{a_g}(1+\Delta z - a_g)) &= W\left(z_{obs} - \frac{1}{a_g}(1-a_g)\right) - \left.\frac{dW}{dz}\right|_{z=z'} \frac{\Delta z}{a_g} \\ + \mathcal{O}(2^{nd}) &= W(z_{obs} - \tilde{z}_H)\left(1 - \left.\frac{d\ln W}{dz}\right|_{z=z'} \frac{\Delta z}{a_g}\right), \end{aligned} \tag{4.23}$$

where we have use the definition of $\tilde{z}_H$ as the background Hubble parameter $(1 + \tilde{z}_H = \frac{1}{a_g})$ and $\frac{d\ln W}{dz}$ evaluated at $z' = z_{obs} - \tilde{z}_H$. Now for the numerator of Eq.(4.22)

$$\begin{aligned} &\int dr\, r^2 \bar{n}_p(r)\,(1+\delta_g)\left(\tilde{z}_H + \frac{\Delta z}{a_g}\right) W(z_{obs} - \tilde{z}_H)\left(1 - \left.\frac{d\ln W}{dz}\right|_{z=z'} \frac{\Delta z}{a_g}\right) \\ &= \int dr\, r^2 \bar{n}_p \tilde{z}_H W(z_{obs} - \tilde{z}_H) + \int dr\, r^2 \bar{n}_p \delta_g \tilde{z}_H W(z_{obs} - \tilde{z}_H) \\ &+ \int dr\, r^2 \bar{n}_p \frac{\Delta z}{a_g} W(z_{obs} - \tilde{z}_H) \\ &- \int dr\, r^2 \bar{n}_p \tilde{z}_H W(z_{obs} - \tilde{z}_H)\left(1 - \left.\frac{d\ln W}{dz}\right|_{z'} \frac{\Delta z}{a_g}\right) \end{aligned} \tag{4.24}$$

and for the denominator

$$
\begin{aligned}
&\int dr\, r^2 \bar{n}_p(r)\,(1+\delta_g)\, W(z_{obs}-\tilde{z}_H)\left(1-\left.\frac{d\ln W}{dz}\right|_{z'}\frac{\Delta z}{a_g}\right)\\
&=\int dr\, r^2 \bar{n}_p W(z_{obs}-\tilde{z}_H)+\int dr\, r^2 \bar{n}_p \delta_g W(z_{obs}-\tilde{z}_H)\\
&-\int dr\, r^2 \bar{n}_p W(z_{obs}-\tilde{z}_H)\left.\frac{d\ln W}{dz}\right|_{z'}\frac{\Delta z}{a_g}.
\end{aligned}
\tag{4.25}
$$

Now in terms of the normalised functionals $\mathcal{F}[Y]$ defined in Eq. (3.2), as we can rewrite Eq. (4.26) identifying $\mathcal{N}=\int dr\, r^2 \bar{n}_p W(z_{obs}-\tilde{z}_H)$,

$$
\begin{aligned}
&\int dr\, r^2 \bar{n}_p(r)\,(1+\delta_g)\, W(z_{obs}-\tilde{z}_H)\left(1-\left.\frac{d\ln W}{dz}\right|_{z'}\frac{\Delta z}{a_g}\right)\\
&=\mathcal{N}\left[1+\frac{1}{\mathcal{N}}\left(\int dr\, r^2 \bar{n}_p \delta_g W(z_{obs}-\tilde{z}_H)\right.\right.\\
&\left.\left.-\int dr\, r^2 \bar{n}_p W(z_{obs}-\tilde{z}_H)\left.\frac{d\ln W}{dz}\right|_{z'}\frac{\Delta z}{a_g}\right)\right],
\end{aligned}
\tag{4.26}
$$

we then have

$$
\begin{aligned}
\bar{z}+\delta z &= \frac{\mathcal{F}(\tilde{z}_H)+\mathcal{F}(\tilde{z}_H\delta_g)+\mathcal{F}\left(\frac{\Delta z}{a_g}\right)-\mathcal{F}\left(\tilde{z}_H \left.\frac{d\ln W}{dz}\right|_{z'}\frac{\Delta z}{a_g}\right)}{1+\mathcal{F}(\delta_g)-\mathcal{F}\left(\left.\frac{d\ln W}{dz}\right|_{z=z'}\frac{\Delta z}{a_g}\right)}\\
&=\left\{\mathcal{F}(\tilde{z}_H)+\mathcal{F}(\tilde{z}_H\delta_g)+\mathcal{F}\left[\frac{\Delta z}{a_g}\left(1-\left.\frac{d\ln W}{dz}\right|_{z'}\right)\right]\right\}\left[1-\mathcal{F}(\delta_g)\right.\\
&\left.+\mathcal{F}\left(\left.\frac{d\ln W}{dz}\right|_{z=z'}\frac{\Delta z}{a_g}\right)\right]=\mathcal{F}(\tilde{z}_H)+\mathcal{F}(\tilde{z}_H\delta_g)-\mathcal{F}(\tilde{z}_H)\mathcal{F}(\delta_g)\\
&+\mathcal{F}(\tilde{z}_H)\mathcal{F}\left(\left.\frac{d\ln W}{dz}\right|_{z=z'}\frac{\Delta z}{a_g}\right)+\mathcal{F}\left[\frac{\Delta z}{a_g}\left(1-\left.\frac{d\ln W}{dz}\right|_{z'}\right)\tilde{z}_H\right]\\
&=\mathcal{F}(\tilde{z}_H)+\mathcal{F}\left[\delta_g(\tilde{z}_H-\mathcal{F}(\tilde{z}_H))\right]+\mathcal{F}\left\{\frac{\Delta z}{a_g}\left[1-\left.\frac{d\ln W}{dz}\right|_{z'}(\tilde{z}_H-\mathcal{F}(\tilde{z}_H)\right]\right\}.
\end{aligned}
\tag{4.27}
$$

Hence we have obtained the expression of Eq. (3.4) expanding Eq. (2.2) to first order in perturbations.