

QUANTUM DECOHERENCE

Trabajo Final de Grado

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4 de Julio de 2022

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Acknowledgements

Este trabajo, como la culminación de mi carrera universitaria que es, ha tenido muchas personas involucradas en él en mayor o menor medida. En primer lugar quiero agradecer a Santi Brouard su labor de tutorización cercana, sencilla y clara, que despertó en mí el gusto por la decoherencia cuántica y me llevó a profundizar en sus implicaciones. Siento necesario agradecer también a mis padres Luis y Naya, que ya en aquel accidentado primer Enero universitario me mostraron su apoyo incondicional y me enseñaron que el valor de una persona proviene de un lugar mucho más profundo de lo que normalmente creemos. A mi hermana Blanca, que aún en la distancia me ha mostrado siempre su amor, y cuyos pequeños detalles y gran apoyo han sido un gran ejemplo para mí. Gracias también a Carmen, que ha sido una compañera inmejorable en este último curso y ha convertido un año agradable en un año maravilloso. Y por último, gracias también a la parte de mí que lo tenía claro desde el principio. No se equivocaba.

1 Abstract

A study of continuous measurement and quantum decoherence is presented in this work. Starting from the Schrödinger- von Neumann equation and developing it, a set of stochastic equations will be obtained such that they describe the behaviour of a system of few degrees of freedom affected by a bath of many degrees of freedom that is interacting with it. The bath, interpreted here as a measuring device, will make the system experiment quantum decoherence. The objective of this work is to analyze the processes of evolution and decoherence of several studied systems, where the Hamiltonian of the system and the measuring device play opposite roles in the evolution: while the first tries to make the system evolve, the second one tries to make it collapse. A further study will be made about the Master Equation of the system and the average final state of it, which is called asymptotic average state. By doing this analysis the reader shall get a deeper understanding of the concepts of quantum decoherence and continuous measurement.

2 Resumen

Este trabajo se centra en obtener las ecuaciones que describen sistemas cuánticos de pocos grados de libertad en interacción con un baño de muchos grados de libertad. Las ecuaciones expresan esta interacción con dos términos deterministas que presentan dependencia temporal y de una variable de ruido aleatoria, los cuales se añaden al término del Hamiltoniano del sistema. Se han elegido dos sistemas cuánticos, un oscilador armónico en el que se mide un observable proporcional a la energía del sistema y un sistema de dos niveles en el que se miden las componentes del spin, para ilustrar y estudiar el fenómeno de la decoherencia cuántica.

La decoherencia cuántica es el proceso por el cual un sistema cuántico pierde su coherencia y pasa a poder ser descrito únicamente en términos clásicos. Este proceso va asociado al concepto de medida continua, que es conveniente introducir antes de tratar la decoherencia en profundidad. Típicamente, el tipo de medida que se trata en física cuántica es aquella en la que el sistema se proyecta sobre uno de los autoestados de un observable dado. Esta clase de medida, denominada comúnmente medida de von Neumann, es en realidad solo un caso especial de los múltiples tipos de medidas posibles en física cuántica.

Mientras que una medida de von Neumann proporciona información completa sobre el sistema, es posible realizar otro tipo de medida que reduzca la incertidumbre inicial acerca del observable que está siendo medido sin llegar a eliminarla por completo. Suele referirse a esta clase de medidas como POVMs, acrónimo de *positive operator-valued measure*, y su interés yace en que pueden tomarse ciertos valores de parámetros del sistema de forma que éste siga evolucionando después de la medición. Una medida “fuerte” será aquella en la que la incertidumbre sea baja, mientras que una medida “débil” será aquella en la que la incertidumbre sea alta. Sabiendo esto es posible introducir las medidas continuas, que consisten en medidas en las cuales se está extrayendo continuamente información sobre el sistema.

Un sistema cuántico puede representarse a través de la matriz densidad, cuyos términos diagonales se denominan poblaciones y corresponden, en cierto modo, a la parte clásica del sistema, ya que equivalen a las probabilidades de hallar al sistema en el estado correspondiente. Los términos no diagonales se denominan coherencias, y dan cuenta de la parte cuántica del sistema.

El término decoherencia proviene del hecho de que, al proyectarse un sistema sobre un autoestado, las coherencias se hacen cero, de modo que el sistema pasa a poder ser descrito únicamente en términos clásicos a través de las poblaciones. Es decir, cuando se realiza una medida fuerte de un sistema, éste sufre un proceso de decoherencia por el cual sus coherencias, que dan cuenta de las propiedades cuánticas del sistema, se anulan. Mediante la realización de una medida débil es posible evitar la decoherencia y hacer que el sistema siga evolucionando después de la medida.

Los sistemas estudiados proporcionarán ejemplos de este tipo de mediciones, siendo posible alterar sus parámetros para realizar medidas más o menos fuertes y para dar mayor o menor fuerza al Hamiltoniano. De esta manera, es posible alcanzar puntos de equilibrio y estudiar las combinaciones que se deseen entre medida fuerte y débil para que el sistema siga evolucionando después de la medición o bien colapse completamente y se mantenga en el autoestado deseado.

Previamente a este estudio se realizará el desarrollo de las ecuaciones que describen estos sistemas, teniendo en cuenta las razones de ruido y las dimensiones de cada término, haciendo modificaciones si procede para optimizar la obtención de una solución.

Además, se buscará la obtención del estado asintótico promedio, que proporciona las poblaciones del sistema una vez las coherencias se han hecho cero. El hecho de hallar el estado asintótico promedio equivale a realizar un cálculo estadístico de las veces que el sistema colapsa a un autoestado u otro en relación al total. Este porcentaje es equivalente a las poblaciones de la matriz densidad, que son las componentes no nulas del estado asintótico promedio. El interés del estado asintótico promedio yace en que es posible alcanzar el resultado expuesto previamente de forma analítica, evitando así un cálculo estadístico farragoso. Además, el estado asintótico promedio hace evidente aquellos casos en los que los parámetros del sistema se han elegido de tal manera que pueden obtenerse resultados concluyentes, y aquellos en los que no. Este hecho se debe a que si se da el caso de que el Hamiltoniano no conmute con el observable que se está midiendo hará al sistema salir de cualquier estado en el que haya colapsado, de modo que la información que haya podido obtenerse en una medición no es válida, ya que el sistema ha evolucionado desde entonces. Ésto se refleja en el estado asintótico promedio como una distribución equitativa de probabilidades entre todos los posibles estados del sistema, lo cual no proporciona ninguna información de valor. En el caso contrario, cuando el Hamiltoniano conmuta con el observable que se está midiendo, los autoestados de dicho observable son también autoestados del Hamiltoniano, por lo que éste no es capaz de sacar al sistema de un autoestado al que haya colapsado; el sistema permanecerá en el estado en el que colapse siempre que sea autoestado del Hamiltoniano. En este caso la información que proporciona la medida sí que es de utilidad, ya que el sistema no evoluciona fuera del autoestado. El estado asintótico promedio describe este hecho mediante una distribución de probabilidades que no son necesariamente iguales para todos los estados posibles, sino que dependen de la configuración inicial del sistema.

3 Introduction

Resumen

Se comenzará dando un marco teórico para poder trabajar con los conceptos de medida continua y decoherencia. A continuación se realizará el estudio de dos sistemas diferentes, un sistema de dos niveles y un oscilador armónico. En ambos casos se obtendrán las ecuaciones que describen el comportamiento de dichos sistemas en interacción con un instrumento de medida, y a continuación se resolverán numéricamente y se analizarán los resultados obtenidos. Para completar el análisis se obtendrá el estado asintótico promedio, que proporcionará la estadística de posibles configuraciones del sistema.

Usually when measurement is first introduced to students of quantum mechanics, it is done without even mentioning the time that the measurement takes to be done; the measurements just "happens". This concept is enough for a first approach, but it becomes insufficient in two situations: the first one is when some aspect of the system is being continuously monitored. In this case it becomes obviously important to know what happens while the system is being measured. The second situation arises because nothing really happens instantaneously. Even the shortest one-shot measurement will take some time, and if such time is not small compared to the dynamics of the system, then it is relevant to understand the dynamics of the flow of information to the observer and the effect of the measurement on the system.

It then becomes necessary to introduce the concept of continuous measurement, which is a kind of measurement more general than the typical one-look at the system and that will be developed in the following sections.

Continuous measurement opens up the possibility of studying and even controlling to some point quantum decoherence, and it has become increasingly important in the last decades, due mainly to its application in fields like feedback control in quantum systems, metrology, quantum information and quantum computing [1-11]. Quantum decoherence is the process by which a system loses its quantum characteristics (the terms that describe its quantum properties in the density matrix go to zero) and it becomes possible to describe using classical probabilities. This phenomenon is bond

to continuous measurement through the importance of both of them in understanding the quantum to classical transition [12-18].

The structure of this work follows a simple scheme. Some concepts related to continuous measurement need to be introduced as a base on which further work shall be done; this shall be developed in section 5, where a general case of a Stochastic Schrödinger Equation will be developed.

After doing so, two systems are being analyzed: a two level system in section 6 and a harmonic oscillator in section 7. Both of them are in interaction with a bath of many degrees of freedom, which can be interpreted as a measuring device as it is used to extract information about the system. In each of the systems, a preliminary description of the treated system shall be done first in order to describe general aspects of it, as its Hamiltonian or the observable that will be measured. Then a quick look at some particular aspects of the equations of the system, as noise or dimensions, might become necessary, after which such equations will be projected and developed until they are gotten into a shape which eases their solution.

In both systems the solution is reached by a numerical procedure, and the results are exposed graphically and explained in order to get to the conclusions, which shall be summarized in section 8.

4 Objectives

Resumen

El objetivo de este trabajo es la obtención de un sistema de ecuaciones que describa el comportamiento de un sistema cuántico de pocos grados de libertad en interacción con un baño de muchos grados de libertad para su posterior resolución. Se realizará un análisis de los resultados centrándose en los conceptos de medida débil y fuerte de forma que el lector obtenga una comprensión más profunda de la decoherencia cuántica y del concepto de medida continua.

The obtainment of the equations that describe a quantum system of few degrees of freedom in interaction with a Markovian bath of many degrees of freedom is the first objective of this work. Such equation is usually called Stochastic Schrödinger Equation (SSE) or Stochastic Master Equation (SME), depending on if the system is described with the state vector $|\psi(t)\rangle$ or with the density operator $\rho(t)$. The equation obtained shall be projected onto the base of eigenstates of the measured observable in each case: in the two level system the observable S_x , which corresponds to the x-component of the spin, shall be measured, while in the harmonic oscillator the measure will be proportional to the energy of the system; the observable measured will be aa^\dagger .

The analysis of the obtained results will be done graphically. By changing parameters of the equations, it is possible to obtain weaker or stronger measurements. In particular, when the factor k grows, the system will tend to collapse quicker, and the noise will be more noticeable. On the other hand, the parameter ω (the frequency of the system) is related to the Hamiltonian, which tends to make the system evolve from any state it is collapsed on.

The evolution of the system is like an aggressive tango where both dancers, the Hamiltonian and the measuring device, try to prevail over their partner. This study will give the reader a further understanding of continuous measurement and of how quantum decoherence works.

5 Theoretical framework

Resumen

En este apartado se pretende establecer el marco teórico en el que se desarrollará posteriormente el trabajo. Se pretende obtener una ecuación estocástica que describa el comportamiento del sistema en contacto con un baño de muchos grados de libertad, para lo cual se partirá de la ecuación de Schrödinger- von Neumann. En el desarrollo que conducirá a la ecuación de Schrödinger estocástica se introducirán los conceptos de valor medio, medida continua, pureza, medida débil y POVM, a parte de dar la definición de matriz densidad y hacer un breve análisis de sus términos.

The objective of this section is to set the theoretical basis that will later allow to describe the evolution of quantum systems as they interact with a measuring instrument. For doing so, some important concepts must be introduced.

The process developed in this section has been extracted from Jacobs [19].

The starting point to obtain the equations required to study such behaviour shall be the *Schrödinger- von Neumann equation*

$$\partial_t \rho = -\frac{i}{\hbar} [H, \rho]. \quad (1)$$

Equation (1) can be obtained by differentiating the density operator and employing the Schrödinger equation $i\hbar\partial_t|\psi\rangle = H|\psi\rangle$. The use of the density operator allows to write more general evolution equations than those implied by state-vector dynamics. The density operator ρ is defined as the product

$$\rho = |\psi\rangle\langle\psi|. \quad (2)$$

In this case, the information content of the density operator is equivalent to that of the state vector (except for the overall phase, which is not of physical significance).

The state vector can represent states of *coherent* superposition. The power of the density operator lies in the fact that it can represent *incoherent* superpositions as well. For example, let $|\psi_\alpha\rangle$ be a set of states without any particular restrictions. Then the density operator

$$\rho = \sum_{\alpha} p_{\alpha} |\psi_{\alpha}\rangle\langle\psi_{\alpha}| \quad (3)$$

models the fact that it is not known which of the states $|\psi_\alpha\rangle$ the system is in, but it is known that it is in the state $|\psi_\alpha\rangle$ with probability p_α . Another way to say it: the state vector $|\psi\rangle$ represents a certain intrinsic uncertainty with respect to quantum observables; the density operator can represent uncertainty beyond the minimum required by quantum mechanics. Equivalently, the density operator can represent an ensemble of identical systems in possibly different states. A state of the form (2) is said to be a pure state. One that cannot be written in this form is said to be mixed, and can be written in the form (3).

5.1 Expectation values

Expectation values can be computed with respect to the density operator via the trace operation. The trace of an operator A is simply the sum over the diagonal matrix elements with respect to any complete, orthonormal set of states $|\beta\rangle$:

$$\text{Tr}[A] := \sum_{\beta} \langle\beta|A|\beta\rangle. \quad (4)$$

An important property of the trace is that the trace of a product is invariant under cyclic permutations of the product. For example, for three operators,

$$\text{Tr}[ABC] = \text{Tr}[BCA] = \text{Tr}[CAB]. \quad (5)$$

In the case of two operators, working in the position representation, the fact that $\int dx |x\rangle\langle x|$ is the identity operator can be used to see that

$$\begin{aligned}
\text{Tr}[AB] &= \int dx \langle x|AB|x\rangle \\
&= \int dx \int dx' \langle x|A|x'\rangle \langle x'|B|x\rangle \\
&= \int dx' \int dx \langle x'|B|x\rangle \langle x|A|x'\rangle \\
&= \int dx' \langle x'|BA|x'\rangle \\
&= \text{Tr}[BA].
\end{aligned} \tag{6}$$

The reader shall note that this argument assumes sufficiently 'nice' operators (it fails, for example, for $\text{Tr}[xp]$). Using this property, it is possible to write the expectation value with respect to a pure state as

$$\langle A \rangle = \langle \psi|A|\psi \rangle = \text{Tr}[A\rho]. \tag{7}$$

This argument extends to the more general form of the density operator (3).

5.2 The density matrix

The physical content of the density operator can be seen more clearly when computing the elements $\rho_{\alpha\alpha'}$ of the density matrix with respect to a complete, orthonormal basis. The density matrix elements are given by

$$\rho_{\alpha\alpha'} := \langle \alpha|\rho|\alpha' \rangle. \tag{8}$$

To analyse these matrix elements, the simple form of the density operator $\rho = |\psi\rangle\langle\psi|$ shall be assumed, though the arguments generalize easily to arbitrary density operators. The diagonal elements $\rho_{\alpha\alpha}$ are referred to as *populations*, and they give the probability of being in the state $|\alpha\rangle$:

$$\rho_{\alpha\alpha} = \langle \alpha|\rho|\alpha \rangle = |\langle \alpha|\psi \rangle|^2. \tag{9}$$

The off-diagonal elements $\rho_{\alpha\alpha'}$ (with $\alpha \neq \alpha'$) are referred to as *coherences*, since they give information about the relative phase of different components of the superposition. For example, if the state vector is written as a superposition with explicit phases,

$$|\psi\rangle = \sum_{\alpha} c_{\alpha} |\alpha\rangle = \sum_{\alpha} |c_{\alpha}| \exp(i\phi_{\alpha}) |\alpha\rangle, \tag{10}$$

then the coherences are

$$\rho_{\alpha\alpha'} = |c_{\alpha}c_{\alpha'}| \exp[i(\phi_{\alpha} - \phi_{\alpha'})]. \tag{11}$$

The reader shall notice that for a density operator not corresponding to a pure state, the coherences in general will be the sum of complex numbers corresponding to different states in the incoherent sum. The phases will not in general line up, so that while $|\rho_{\alpha\alpha}|^2 = \rho_{\alpha\alpha}\rho_{\alpha'\alpha'}$ for a pure state, for a generic mixed state it is expected $|\rho_{\alpha\alpha'}|^2 < \rho_{\alpha\alpha}\rho_{\alpha'\alpha'}$ ($\alpha \neq \alpha'$).

5.3 Purity

The difference between pure and mixed states can be formalized in another way. It is important to notice here that the diagonal elements of the density matrix form a probability distribution: the populations express probabilities. Proper normalization thus requires

$$\text{Tr}[\rho] = \sum_{\alpha} \rho_{\alpha\alpha} = 1. \tag{12}$$

It is possible to do the same computation for ρ^2 and purity will be defined as $\text{Tr}[\rho^2]$. For a pure state, the purity is simple to calculate:

$$\text{Tr}[\rho^2] = \text{Tr}[|\psi\rangle\langle\psi||\psi\rangle\langle\psi|] = \text{Tr}[\rho] = 1. \tag{13}$$

But for mixed states, $\text{Tr} [\rho^2] < 1$. For example, for the density operator in (3) purity will be

$$\text{Tr} [\rho^2] = \sum_{\alpha} p_{\alpha}^2, \quad (14)$$

if the states $|\psi_{\alpha}\rangle$ are assumed to be orthonormal.

For equal probability of being in N such states, $\text{Tr} [\rho^2] = 1/N$. Intuitively, then, it is possible to see that $\text{Tr} [\rho^2]$ drops to zero as the state becomes more mixed, that is, as it becomes an incoherent superposition of more and more orthogonal states.

5.4 Weak measurements and POVMs

In undergraduate courses the only kind of measurement that is usually discussed is one in which the system is projected onto one of the possible eigenstates of a given observable. If these eigenstates are written as $\{|n\rangle : n = 1, \dots, n_{\max}\}$, and the state of the system is $|\psi\rangle = \sum_n c_n |n\rangle$, the probability that the system is projected onto $|n\rangle$ is $|c_n|^2$. In fact, these kinds of measurements, which are often referred to as von Neumann measurements, represent only a special class of all the possible measurements that can be made on quantum systems. However, all measurements can be derived from von Neumann measurements.

One reason for which it is necessary to consider a larger class of measurements is so we can describe measurements that extract only partial information about an observable. A von Neumann measurement provides complete information: after the measurement is performed, it is known exactly what the value of the observable is, since the system is projected into an eigenstate. Naturally, however, there exist many measurements which, while reducing on average our uncertainty regarding the observable of interest, do not remove it completely.

First, it is worth noting that a von Neumann measurement can be described by using a set of projection operators $\{P_n = |n\rangle\langle n|\}$. Each of these operators describes what happens on one of the possible outcomes of the measurement: if the initial state of the system is $\rho = |\psi\rangle\langle\psi|$, then the n th possible outcome of the final state is given by

$$\rho_r = |n\rangle\langle n| = \frac{P_n \rho P_n}{\text{Tr} [P_n \rho P_n]}, \quad (15)$$

and this result is obtained with probability

$$P(n) = \text{Tr} [P_n \rho P_n] = c_n, \quad (16)$$

where c_n defines the superposition of the initial state $|\psi\rangle$ given above. It turns out that every possible measurement may be described in a similar way by generalizing the set of operators. Suppose a set of m_{\max} operators Ω_m is picked, the only restriction being that $\sum_{m=1}^{m_{\max}} \Omega_m^{\dagger} \Omega_m = I$, where I is the identity operator. Then it is in principle possible to design a measurement that has N possible outcomes,

$$\rho_r = \frac{\Omega_m \rho \Omega_m^{\dagger}}{\text{Tr} [\Omega_m \rho \Omega_m^{\dagger}]}, \quad (17)$$

with

$$P(m) = \text{Tr} [\Omega_m \rho \Omega_m^{\dagger}], \quad (18)$$

giving the probability of obtaining the m^{th} outcome. Each of these more general measurements may be implemented by performing a unitary interaction between the system and an auxiliary system, and then performing a von Neumann measurement on the auxiliary system. Thus all possible measurements may be derived from the basic postulates of unitary evolution and von Neumann measurement [20,21]. These 'generalized' measurements are often referred to as POVMs, where the acronym stands for 'positive operator-valued measure'.

This will now be put into practice to describe a measurement that provides partial information about an observable. In this case, instead of the measurement operators Ω_m being projectors onto a single eigenstate, they are chosen to be a weighted sum of projectors onto the eigenstates $|n\rangle$, each one peaked about a different value of the observable. It will be assumed now, for the sake of simplicity, that the eigenvalues n of the observable N take on all the integer values. In this case is chosen

$$\Omega_m = \frac{1}{\mathcal{N}} \sum_n \exp[-k(n-m)^2/4] |n\rangle\langle n|, \quad (19)$$

where \mathcal{N} is a normalization constant chosen so that $\sum_{m=-\infty}^{\infty} \Omega_m^t \Omega_m = I$. It has now been constructed a measurement that provides partial information about the observable N .

This is illustrated clearly by examining the case where at the beginning there is no information about the system. In such case the density matrix is completely mixed, so that $\rho \propto I$. After making the measurement and obtaining the result m , the state of the system is

$$\rho_r = \frac{\Omega_m \rho \Omega_m^t}{\text{Tr} [\Omega_m \rho \Omega_m^t]} = \frac{1}{\mathcal{N}} \sum_n \exp [-k(n - m)^2/2] |n\rangle\langle n|. \quad (20)$$

The final state is thus peaked about the eigenvalue m , but has a width given by $1/k^{1/2}$. The larger k , the less the final uncertainty regarding the value of the observable. Measurements for which k is large are often referred to as strong measurements, and conversely those for which k is small are weak measurements [22]. These are the kinds of measurements that will be needed in order to derive a continuous measurement in the next section.

5.5 A continuous measurement of an observable

A continuous measurement is one in which information is continually extracted from a system. Another way to say this is that when one is making such a measurement, the amount of information obtained goes to zero as the duration of the measurement goes to zero. To construct a measurement like this, time can be divided into a sequence of intervals of length Δt , so that a weak measurement is considered in each interval. To obtain a continuous measurement, the strength of each measurement is made proportional to the time interval, and then taken the limit in which the time intervals become infinitesimally short.

It is important to remark here that the measurement must be weak. If a strong measurement was done in each time interval, and the time intervals were becoming smaller, eventually the system would not be able to go out of any eigenstate once it had collapsed onto it. This block of the system on the state once Δt is small enough is called quantum Zeno effect.

To avoid the Zeno effect, the strength of the measurement is made proportional the time interval: $k \cdot \Delta t$ instead of k . By doing so when Δt tends to zero, the measurement becomes weaker. Under this conditions, the system has the possibility of evolving. It is, in summary, a very weak continuous measurement.

In what follows, the measured observable will be denoted by X (i.e. X is a Hermitian operator), and it will be assumed that it has a continuous spectrum of eigenvalues x . Eigenstates will be written as $|x\rangle$, so that $\langle x | x'\rangle = \delta(x - x')$. However, the equation that will be derived will be valid for measurements of any Hermitian operator.

Time is now divided into intervals of length Δt . In each time interval, it will be made a measurement described by the operators

$$A(\alpha) = \left(\frac{4k\Delta t}{\pi} \right)^{1/4} \int_{-\infty}^{\infty} \exp [-2k\Delta t(x - \alpha)^2] |x\rangle\langle x| dx. \quad (21)$$

Each operator $A(\alpha)$ is a Gaussian-weighted sum of projectors onto the eigenstates of X . Here α is a continuous index, so that there is a continuum of measurement results labelled by α . The fact that the coefficient of the exponential has $k\Delta t$ on it means that the Gaussian shall be shorter and wider the weaker the measurement is, and taller and thinner the stronger the measurement is. The inclusion of the product $k\Delta t$ in the Gaussian is called *weak measurement approximation* or *Bohr approximation*, and it implies that in perturbation theory the perturbation development is be cut in the first term. This is the first approximation necessary to get to the desired equations.

The thing needed to know now is the probability density $P(\alpha)$ of the measurement result α when Δt is small. To work this out it is first calculated the mean value of α . If the initial state is

$|\psi\rangle = \int \psi(x)|x\rangle dx$ then $P(\alpha) = \text{Tr} [A(\alpha)^\dagger A(\alpha)|\psi\rangle\langle\psi|]$, and

$$\begin{aligned}
\langle\alpha\rangle &= \int_{-\infty}^{\infty} \alpha P(\alpha) d\alpha \\
&= \int_{-\infty}^{\infty} \alpha \text{Tr} [A(\alpha)^\dagger A(\alpha)|\psi\rangle\langle\psi|] d\alpha \\
&= \sqrt{\frac{4k\Delta t}{\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \alpha |\psi(x)|^2 \exp[-4k\Delta t(x-\alpha)^2] dx d\alpha \\
&= \int_{-\infty}^{\infty} x |\psi(x)|^2 dx = \langle X \rangle.
\end{aligned} \tag{22}$$

To obtain $P(\alpha)$ it is now written

$$\begin{aligned}
P(\alpha) &= \text{Tr} [A(\alpha)^\dagger A(\alpha)|\psi\rangle\langle\psi|] \\
&= \left(\frac{4k\Delta t}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} |\psi(x)|^2 \exp[-4k\Delta t(x-\alpha)^2] dx.
\end{aligned} \tag{23}$$

If Δt is sufficiently small then the Gaussian is much wider than $\psi(x)$. This means it is possible to approximate $|\psi(x)|^2$ by a delta function, which must be centred at the expected position $\langle X \rangle$ so that $\langle\alpha\rangle = \langle X \rangle$ as calculated above. Therefore:

$$\begin{aligned}
P(\alpha) &\approx \left(\frac{4k\Delta t}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} \delta(x-\langle X \rangle) \exp[-4k\Delta t(x-\alpha)^2] dx \\
&= \left(\frac{4k\Delta t}{\pi}\right)^{1/2} \exp[-4k\Delta t(\alpha-\langle X \rangle)^2].
\end{aligned} \tag{24}$$

α can also be written as the stochastic quantity

$$\alpha_s = \langle X \rangle + \frac{\Delta W}{(8k)^{1/2}\Delta t}, \tag{25}$$

where ΔW is a zero-mean, Gaussian random variable with variance Δt . This alternate representation as a stochastic variable will be useful later. Since it will be clear from context, α will be used interchangeably with α_s in referring to the measurement results, although technically it would be necessary to distinguish between the index α and the stochastic variable α_s .

A continuous measurement results if a sequence of these measurements is made and the limit taken as $\Delta t \rightarrow 0$ (or equivalently, as $\Delta t \rightarrow dt$). As this limit is taken, more and more measurements are made in any finite time interval, but each is increasingly weak. By choosing the variance of the measurement result to scale as Δt , it has been ensured that it has been obtained a sensible continuum limit.

A stochastic equation of motion results due to the random nature of the measurements (a stochastic variable is one that fluctuates randomly over time). This equation of motion for the system can be derived under this continuous measurement by calculating the change induced in the quantum state by the single weak measurement in the time step Δt , to first order in Δt (Bohr approximation). Thus it will be computed the evolution when a measurement, represented by the operator $A(\alpha)$, is performed in each time step. This procedure gives

$$\begin{aligned}
|\psi(t+\Delta t)\rangle &\propto A(\alpha)|\psi(t)\rangle \\
&\propto \exp[-2k\Delta t(\alpha-X)^2] |\psi(t)\rangle \\
&\propto \exp\left(-2k\Delta t X^2 + X\left[4k\langle X \rangle\Delta t + (2k)^{1/2}\Delta W\right]\right) |\psi(t)\rangle.
\end{aligned} \tag{26}$$

The fact that $|\psi(t+\Delta t)\rangle$ is proportional to $A(\alpha)|\psi(t)\rangle$ and not to any time previous to t means that the future state of $|\psi\rangle$, which is $|\psi(t+\Delta t)\rangle$, is a function only of the state at time t . This fact configures a Markovian approximation, which supposes that the state of the system at $t+\Delta t$ depends only on the state of the system at t and not on any previous state of the system. The Markovian approximation is the second approximation done in order to get to the desired results, and it can be interpreted as that once part of the energy of the system has gone into the bath, it cannot go back to the system. If, for example, there was a decay where one photon was emitted, the Markovian approximation assumes that the photon does never come back, so that it cannot

undo the decay; the evolution of the atom does not depend then on what happened to the atom previously but on its actual state.

The exponential is now expanded to first order in Δt , which gives

$$|\psi(t + \Delta t)\rangle \propto \left\{ 1 - 2k\Delta t X^2 + X \left[4k\langle X \rangle \Delta t + (2k)^{1/2} \Delta W + kX(\Delta W)^2 \right] \right\} |\psi(t)\rangle. \quad (27)$$

Note that the second-order term in ΔW has been included in the power series expansion for the exponential. It is necessary to include this term because it turns out that in the limit in which $\Delta t \rightarrow 0$, $(\Delta W)^2 \rightarrow (dW)^2 = dt$.

Because of this, the $(\Delta W)^2$ term contributes to the final differential equation.

In order to take the limit as $\Delta t \rightarrow 0$, it is set $\Delta t = dt$, $\Delta W = dW$ and $(\Delta W)^2 = dt$, and the result is

$$|\psi(t + dt)\rangle \propto \left\{ 1 - [kX^2 - 4kX\langle X \rangle] dt + (2k)^{1/2} X dW \right\} |\psi(t)\rangle. \quad (28)$$

This equation does not preserve the norm $\langle \psi | \psi \rangle$ of the wave function, because before it was derived it threw away the normalization. An equation that does preserve the norm can easily be obtained by normalizing $|\psi(t + dt)\rangle$ and expanding the result to first order in dt (again, keeping terms to order dW^2). Writing $|\psi(t + dt)\rangle = |\psi(t)\rangle + d|\psi\rangle$, the resulting stochastic differential equation is given by

$$d|\psi\rangle = \left\{ -k(X - \langle X \rangle)^2 dt + (2k)^{1/2} (X - \langle X \rangle) dW \right\} |\psi(t)\rangle. \quad (29)$$

This is the sought equation: it describes the evolution of the state of a system in a time interval dt given that the observer obtains the measurement result

$$dy = \langle X \rangle dt + \frac{dW}{(8k)^{1/2}} \quad (30)$$

in that time interval.

The measurement result gives the expected value $\langle X \rangle$ plus a random component due to the width of $P(\alpha)$, and this is written as a differential since it corresponds to the information gained in the time interval dt . As the observer integrates $dy(t)$ the quantum state progressively collapses, and this integration is equivalent to solving (29) for the quantum-state evolution.

The stochastic Schrödinger equation (SSE) in equation (29) is usually described as giving the evolution conditioned upon the stream of measurement results. The state $|\psi\rangle$ evolves randomly, and $|\psi(t)\rangle$ is called the quantum trajectory. The set of measurement results $dy(t)$ is called the measurement record.

6 System 1: Two level system

Resumen

Una vez establecido el marco teórico, se procede a la obtención de las ecuaciones que describan el primer sistema: un sistema de dos niveles. Tras una breve introducción en la que se establecen las propiedades generales del sistema, se realizará un análisis del ruido relativo a la SSE y de sus dimensiones. Una vez conocidas las dimensiones de cada término se redimensionalizará la ecuación para hacerla adimensional, lo cual facilitará su posterior resolución. Tras proyectar la ecuación redimensionalizada en la base de autoestados del observable medido se procederá a la discusión de resultados. Entre la proyección de la ecuación y la discusión de resultados se realizó un trabajo que no ha sido incluido en este apartado sino en los éndaipces, que es la resolución de la ecuación mediante métodos numéricos utilizando un código de Python. Los resultados se presentarán gráficamente y se analizarán para lograr la comprensión de los procesos que se están observando y sus implicaciones. Por último se buscará obtener la ecuación maestra estocástica (SME) y el estado asintótico promedio de forma analítica.

6.1 Introduction

The first system to be analyzed is a two-level system, with the Hamiltonian

$$H = \omega S_z , \quad (31)$$

being ω the frequency of the system and S_z the observable related to the z-component of the spin. Initially the work will be developed in the base of eigenstates of S_z $\{|+\rangle, |-\rangle\}$ with S_z acting like: $S_z |\pm\rangle = \pm \frac{\hbar}{2} |\pm\rangle$. However, the measured observable shall be S_x , that describes the x-component of the spin of the system and that does not commute with the Hamiltonian of the system, as $[S_x, S_z] \neq 0$.

Equation (29) was obtained for the observable X , but as mentioned in the beginning of section 5.5, the equation will work for any Hermitian operator, i.e. S_x . It is possible to use equation (29) for describing the evolution of the system by just changing the observables:

$$d|\psi\rangle = \left\{ -\frac{i}{\hbar} H dt - k(S_x - \langle S_x \rangle)^2 dt + (2k)^{1/2} (S_x - \langle S_x \rangle) dW \right\} |\psi\rangle . \quad (32)$$

The first term of the equation, the one with the Hamiltonian on it, gives the effect of the Hamiltonian on the evolution of the system.. The second term (deterministic) expresses the effect of the measuring instrument over the system. The third term is an aleatory term due to noise.

6.2 Obtainment of the equations

6.2.1 Noise

The intention is now to analyze the noise term, specifically dW . For dW to be equivalent to noise two conditions must be imposed:

- $\langle dW \rangle = 0$
- $\langle (dW)^2 \rangle = \Delta t$

The reader shall notice that the second condition can be rewritten as:

$$Var(dW) = \Delta t ; \quad Var(dW) = \langle (dW)^2 \rangle - \langle dW \rangle^2 , \quad (33)$$

where for erasing the second term the first condition was used.

Let ξ be an arbitrary number generated randomly. By definition it will fulfill both $\langle \xi \rangle = 0$ and $\langle \xi^2 \rangle = 1$. This will allow to re-define dW as

$$dW = (\Delta t)^{1/2} \cdot \xi . \quad (34)$$

This definition of dW matches in fact with the two conditions established at the beginning of the section for any noise term:

$$i) \langle dW \rangle = \langle (\Delta t)^{1/2} \xi \rangle = (\Delta t)^{1/2} \langle \xi \rangle = 0 \quad (35)$$

$$ii) \langle (dW)^2 \rangle = \langle \Delta t \xi^2 = \Delta t \langle \xi^2 \rangle \rangle = \Delta t \quad (36)$$

The fact that both conditions (35) and (36) are fulfilled will allow to use definition (34) from now on in equation (32). This change is convenient as it allows to write the whole equation in terms of $d\tau$ instead of both $d\tau$ and dW , which makes much easier its resolution.

6.2.2 Dimensions

A check of the dimensions of equation (32) seems convenient at this point to make sure the changes done suit correctly. For doing so the three terms of the equation shall be analyzed separately, taking into account that the three of them must have the same dimensions.

First term: $-\frac{i}{\hbar} H dt$.

It is known that \hbar has dimensions of energy per time: $\hbar [=] E \cdot t$ thus:

$$-\frac{i}{\hbar} H dt \longrightarrow ad. \quad (37)$$

Second term: $-k(A - \langle A \rangle)^2 |\psi\rangle dt$

As the first term is adimensional, the second and third terms must be so as well. Obviously dt has dimensions of time so:

$$kA^2 dt [=] ad \longrightarrow kA^2 [=] t^{-1}. \quad (38)$$

Third term: $(2k)^{1/2}(A - \langle A \rangle) |\psi\rangle dW$

This term must be also adimensional. Knowing the dimensions of the second term it is possible to deduce:

$$k^{1/2} AdW [=] ad \longrightarrow dW [=] t^{1/2}. \quad (39)$$

This fact agrees with the characteristics that were set for the noise term in section 5.5, where $(dW)^2 = dt$.

6.2.3 Redimensionalizing the equation

A change in the dimensions of equation (32) is convenient now for further calculations. Knowing the dimensions of it, it is possible to introduce the proper constants so that the terms of the equation become adimensional, which again will make the calculations easier.

- i. $(d|\psi\rangle)_{(1)} = -\frac{i}{\hbar} H dt |\psi\rangle = -\frac{i}{\hbar} \omega S_z dt |\psi\rangle$ As seen previously, \hbar has dimensions of $E \cdot t$, while $\omega \cdot S_z$ has dimensions of energy and trivially dt has dimensions of time. Two dimensionless quantities shall be introduced now:

$$\begin{aligned} \tilde{S}_z &= \frac{S_z}{\hbar} \\ \tau &= \omega \cdot t \end{aligned}$$

Substituting:

$$(d|\psi\rangle)_{(1)} = -i\tilde{S}_z \omega dt |\psi\rangle = -i\tilde{S}_z d\tau |\psi\rangle. \quad (40)$$

- ii. $(d|\psi\rangle)_{(2)} = -k(S_x - \langle S_x \rangle)^2 dt |\psi\rangle$ where S_z has the same dimensions as \hbar . The following quantities are also introduced:

$$\begin{aligned} \tilde{k} &= \hbar^2. \\ k' &= \frac{\tilde{k}}{\omega} \\ \tilde{S}_x &= \frac{S_x}{\hbar} \end{aligned}$$

Multiplying and dividing by ω and substituting the constants the result is the following:

$$\begin{aligned} (d|\psi\rangle)_{(2)} &= -k(S_x - \langle S_x \rangle)^2 dt |\psi\rangle \cdot \frac{\omega}{\omega} = -\frac{\tilde{k}}{\hbar^2 \omega} (S_x - \langle S_x \rangle)^2 d\tau |\psi\rangle = \\ &= -\frac{k'}{\hbar^2} (S_x - \langle S_x \rangle)^2 d\tau |\psi\rangle = -k' \left(\tilde{S}_x - \langle \tilde{S}_x \rangle \right)^2 d\tau |\psi\rangle \end{aligned} \quad (41)$$

iii. $(d|\psi\rangle)_{(3)} = (2k)^{1/2} (S_x - \langle S_x \rangle) |\psi\rangle dW$ Following the same procedure as before, this is, multiplying by \hbar/\hbar :

$$\begin{aligned} (d|\psi\rangle)_{(3)} &= 2^{1/2} k^{1/2} \frac{\hbar}{\hbar} (S_x - \langle S_x \rangle) dW |\psi\rangle = 2^{1/2} k^{1/2} \hbar \left(\tilde{S}_x - \langle \tilde{S}_x \rangle \right) dW |\psi\rangle \quad (42) \\ &= 2^{1/2} \tilde{k}^{1/2} \left(\tilde{S}_x - \langle \tilde{S}_x \rangle \right) dW |\psi\rangle . \end{aligned}$$

It is convenient now to multiply by $\omega^{1/2}/\omega^{1/2}$ in the search for simplicity. Taking into account that $dW = (dt)^{1/2} \xi$ (as seen in (34)) and also remembering that $\tau = \omega \cdot t$:

$$(d|\psi\rangle)_{(3)} = 2^{1/2} \tilde{k}^{1/2} \frac{\omega^{1/2}}{\omega^{1/2}} \left(\tilde{S}_x - \langle \tilde{S}_x \rangle \right) (dt)^{1/2} \xi |\psi\rangle = 2^{1/2} (k')^{1/2} \left(\tilde{S}_x - \langle \tilde{S}_x \rangle \right) (d\tau)^{1/2} \xi |\psi\rangle . \quad (43)$$

The dimensions of the three terms of the SSE have just been modified to make them adimensional. Joining them together it is possible to obtain the full redimensionalized SSE:

$$d|\psi\rangle = \left\{ -i\tilde{S}_z d\tau - k' \left(\tilde{S}_x - \langle \tilde{S}_x \rangle \right)^2 d\tau + 2^{1/2} (k')^{1/2} \left(\tilde{S}_x - \langle \tilde{S}_x \rangle \right) \xi (d\tau)^{1/2} \right\} |\psi\rangle . \quad (44)$$

6.2.4 Development of the equations

Until now, the only specification done was the definition of the measured observable as S_x . It is time now to introduce the information available about the system in the equation (44) in order to get a set of equations that defines the behaviour of the two level system that is the objective of study of this chapter. For getting such set of equations, a reasonable first step would be to project 44 in the base of eigenstates of S_z , which is, as the reader will remember, not the measured observable but part of the Hamiltonian. This means that the eigenstates of S_z are also eigenstates of the Hamiltonian, as S_z and H commute.

$$\begin{cases} d\langle +|\psi\rangle = \langle +|\{(44)\}|\psi\rangle \\ d\langle -|\psi\rangle = \langle -|\{(44)\}|\psi\rangle \end{cases} \quad (45)$$

where the notation (44) has been used to mark that the whole equation (44) is put between the bra and the ket. It is convenient to introduce the also the notation

$$d\langle +|\psi\rangle \equiv C_+(t) \quad (46)$$

$$d\langle -|\psi\rangle \equiv C_-(t) \quad (47)$$

for simplicity reasons. The reader shall notice that $|C_{\pm}|^2$ is an expression of the probability of the system to be in the states $|+\rangle$ or $|-\rangle$.

What has been obtained here is a system of two equations that reproduce how a two level system behaves. The procedure to calculate the projection over the first equation in (45) can be again divided in three terms:

$$dC_+ = -i d\tau \langle +|\tilde{S}_z|\psi\rangle - k' d\tau \langle +|\left(\tilde{S}_x - \langle \tilde{S}_x \rangle\right)^2|\psi\rangle + (2k')^{1/2} \xi (d\tau)^{1/2} \langle +|\left(\tilde{S}_x - \langle \tilde{S}_x \rangle\right)|\psi\rangle . \quad (48)$$

As (48) is written in the base of eigenstates of S_z , it is necessary to write S_x in terms of operators that act directly on the base of eigenstates of S_z .

$$S_x = \frac{1}{2}(S_+ + S_-) ,$$

where:

$$S_{\pm} |\pm\rangle = \hbar |\pm\rangle .$$

Taking this into account it is possible to proceed. The reader shall take into account that the calculations below only include the projections in each term but not the constants on them, which will be added later to the whole equation. This obviously does not mean that such quantities are canceled unless it is indicated so.

i.

$$\langle +|\tilde{S}_z|\psi\rangle = \frac{1}{\hbar} \langle +|\tilde{S}_z|\psi\rangle = \frac{\hbar}{2\hbar} \langle +|\psi\rangle = \frac{1}{2} C_+(t) \quad (49)$$

ii.

$$\langle + | (\tilde{S}_x - \langle \tilde{S}_x \rangle)^2 | \psi \rangle \quad (50)$$

$$(\tilde{S}_x - \langle \tilde{S}_x \rangle)^2 = \tilde{S}_x^2 + \langle \tilde{S}_x \rangle^2 - 2\tilde{S}_x \langle \tilde{S}_x \rangle = \frac{1}{4}(\tilde{S}_+ + \tilde{S}_-)^2 + \langle \tilde{S}_x \rangle - (\tilde{S}_+ + \tilde{S}_-) \langle \tilde{S}_x \rangle$$

$$\text{As } (\tilde{S}_+ + \tilde{S}_-)^2 = \tilde{S}_+^2 + \tilde{S}_-^2 + \tilde{S}_- \tilde{S}_+ + \tilde{S}_+ \tilde{S}_-:$$

$$\begin{aligned} & \langle + | \frac{1}{4}(\tilde{S}_+ + \tilde{S}_-)^2 | \psi \rangle + \langle + | \langle \tilde{S}_x^2 \rangle | \psi \rangle - \langle + | (\tilde{S}_+ + \tilde{S}_-) | \psi \rangle \langle \tilde{S}_x \rangle = \\ & = \frac{1}{4} \left[\langle + | \tilde{S}_+^2 | \psi \rangle + \langle + | \tilde{S}_-^2 | \psi \rangle + \langle + | \tilde{S}_- \tilde{S}_+ | \psi \rangle + \langle + | \tilde{S}_+ \tilde{S}_- | \psi \rangle \right] \\ & + \langle \tilde{S}_x \rangle^2 C_+ - \langle \tilde{S}_x \rangle \left[\langle + | \tilde{S}_+ | \psi \rangle + \langle + | \tilde{S}_- | \psi \rangle \right] = \\ & = \frac{1}{4} C_+ + \langle \tilde{S}_x \rangle^2 C_+ - \langle \tilde{S}_x \rangle \langle - | \psi \rangle = \frac{1}{4} C_+ + \langle \tilde{S}_x \rangle^2 C_+ - \langle \tilde{S}_x \rangle C_- \end{aligned}$$

iii.

$$\begin{aligned} \langle + | (\tilde{S}_x + \langle \tilde{S}_x \rangle) | \psi \rangle & = \langle + | \tilde{S}_x | \psi \rangle - \langle \tilde{S}_x \rangle \langle + | \psi \rangle = \langle + | \frac{1}{2}(\tilde{S}_+ - \tilde{S}_-) | \psi \rangle - \langle \tilde{S}_x \rangle C_+ = \\ & = \frac{1}{2} \langle + | \tilde{S}_- | \psi \rangle - \langle \tilde{S}_x \rangle C_+ = \frac{1}{2} C_- - \langle \tilde{S}_x \rangle C_+ \end{aligned}$$

Joining the results obtained for each term and adding again the constants on each of them:

$$dC_+ = -\frac{i}{2} C_+ d\tau - k' \left[\left(\frac{1}{4} + \langle \tilde{S}_x \rangle^2 \right) C_+ - \langle \tilde{S}_x \rangle C_- \right] d\tau + (2k')^{1/2} \xi \left[\frac{1}{2} C_- - \langle \tilde{S}_x \rangle C_+ \right] (d\tau)^{1/2} \quad (51)$$

The result of the second equation in (48) can be obtained by operating in a similar way, taking into account that $S_z |-\rangle = -\frac{\hbar}{2} |-\rangle$:

$$\begin{cases} dC_+ = -\frac{i}{2} C_+ d\tau - k' \left[\left(\frac{1}{4} + \langle \tilde{S}_x \rangle^2 \right) C_+ - \langle \tilde{S}_x \rangle C_- \right] d\tau + (2k')^{1/2} \xi \left[\frac{1}{2} C_- - \langle \tilde{S}_x \rangle C_+ \right] (d\tau)^{1/2} \\ dC_- = \frac{i}{2} C_- d\tau - k' \left[\left(\frac{1}{4} - \langle \tilde{S}_x \rangle^2 \right) C_- - \langle \tilde{S}_x \rangle C_+ \right] d\tau + (2k')^{1/2} \xi \left[\frac{1}{2} C_+ - \langle \tilde{S}_x \rangle C_- \right] (d\tau)^{1/2} \end{cases} \quad (52)$$

The upper (52) is a system of differential stochastic equations which solutions represent the process of decoherence of a two level system in interaction with a measuring device, and with S_x being the observable measured.

This system of equations has been solved numerically using a Python code, taking into account the redimensionalization done in 6.2.3, that makes the job much easier. The code created reproduces the behaviour of the system and shows both the evolution in time of C_+ and C_- and the evolution of $\langle S_x \rangle$, which will both be analyzed in the following section.

The factor 1/2 in the first part of both equations (the Hamiltonian part) at equations (52) was denoted as ω in the Python code, and it was used to modify the presence of the Hamiltonian in the evolution of the system, as it will be shown in the next section.

From now on, the elements on equations (52) will be labeled without any accent marks for clarity purposes, but the reader shall note that the redimensionalization done in section 6.2.3 is still valid and that the parameters have not changed.

6.3 Discussion of results

The behaviour of the system is intended to be studied now. For making a clear comparison between the possible cases, the discussion will be divided into the analysis of the case where $H = 0$ and the case where $H \neq 0$.

A quick look at the general behaviour of the system shall be gotten first:

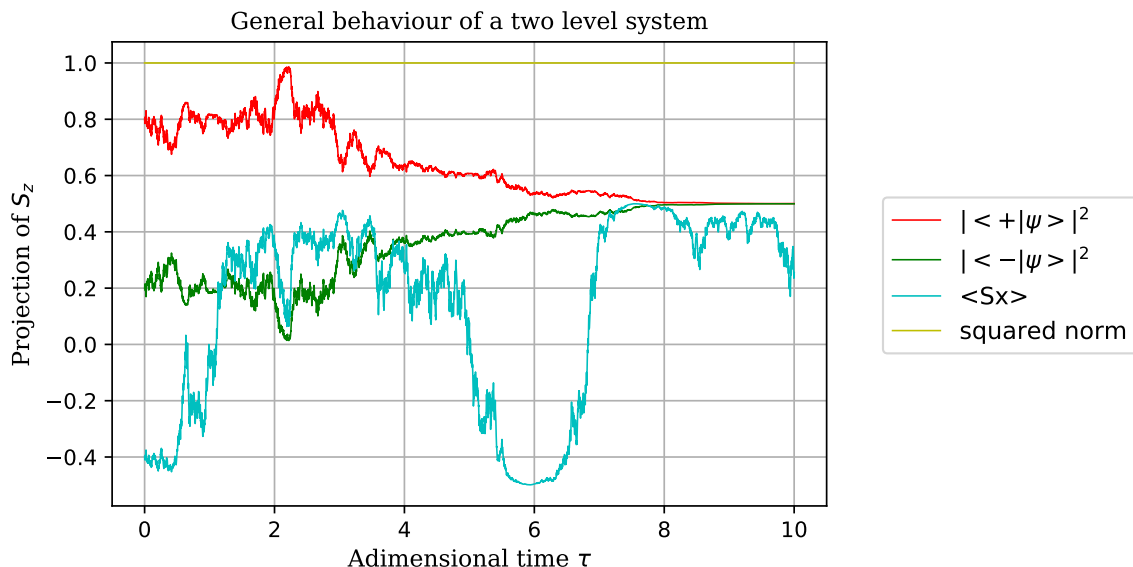


Figure 1: Projection of the state of the system in the base of S_z (red and green curves) and mean value of S_x (cyan curve) for the studies system with an initial state $C_+ = \sqrt{0.8}$, $C_- = -\sqrt{0.2}$

The red and green lines represent the squared module of the projection of $|+\rangle$ and $|-\rangle$ over the state of the system, respectively (C_+ and C_-), which represent the probabilities of the system of being in each respective state. It is possible to observe that, in the case shown, after some time fluctuating both lines get together at the value 0.5, which is, they end up having the same probability of happening. The system has the same probability of going into $|+\rangle$ than going into $|-\rangle$.

The cyan line represents $\langle S_x \rangle$, which fluctuates randomly due to the noise term.

The yellow line is the squared norm, which is represented here as a verification: if the calculus are correct, the squared norm will always be 1.

The initial state here is a lineal combination of C_+ and C_- , specifically $C_+ = \sqrt{0.8}$ and $C_- = -\sqrt{0.2}$.

The presence of both the Hamiltonian and the measuring device is quickly noted on the cyan curve, that represents the expectation value of S_x . The system was collapsed onto $\langle S_x \rangle \simeq 0.5$ for some time. At some point near $\tau = 3$, the Hamiltonian made the system evolve and leave $\langle S_x \rangle = 0.5$. A more detailed study of these processes will be done in the incoming sections.

6.3.1 $H = 0$

The fact that $H = 0$ implies that the energy of the spin is the same for the two possible values of the spin.

From now on in this section, the Hamiltonian will be considered to be zero.

Let there the state of the system be at some time $|\psi\rangle = |+\rangle_x$, where $S_x |+\rangle_x = \frac{\hbar}{2} |+\rangle_x$. This is equivalent to saying that the system is one of the eigenstates of S_x .

Obviously, $S_x |\psi\rangle = \frac{\hbar}{2} |\psi\rangle$.

This will also mean that $\langle S_x \rangle = \langle \psi | S_x | \psi \rangle = \langle + | S_x | + \rangle_x = \frac{\hbar}{2}$. What happens to the system then?

$$d|\psi\rangle = -k \left(S_x - \frac{\hbar}{2} \right)^2 |\psi\rangle dt + (2k)^{1/2} \left(S_x - \frac{\hbar}{2} \right) |\psi\rangle dW .$$

The reader shall note that the Hamiltonian term has been eliminated as $H = 0$. Now, as $S_x |\psi\rangle = \frac{\hbar}{2} |\psi\rangle$:

$$d|\psi\rangle = -k(0)^2 |\psi\rangle dt + (2k)^{1/2}(0) |\psi\rangle dW \longrightarrow d|\psi\rangle = 0 . \quad (53)$$

The result obtained means that, for any state of the system $|\psi(0)\rangle$, if the system evolves until reaching a state alike $|\psi\rangle \simeq |+\rangle_x$ at some time, then $d|\psi\rangle = 0$, which is the equivalent of saying that the system does not evolve anymore, as the differential variation of the system is 0.

Analogously, if $|\psi\rangle \simeq |-\rangle_x \rightarrow d|\psi\rangle = 0$. When the system gets close to one of the eigenstates of S_x , it will stay there. The eigenstates of S_x are the only values that make $d|\psi\rangle = 0$

In general, the measuring process has the shape:

$$\begin{aligned} (\hat{A} - \langle \hat{A} \rangle)^2 |\psi\rangle dt \\ (\hat{A} - \langle \hat{A} \rangle) |\psi\rangle dW \end{aligned}$$

where the first equation expresses a temporal dependence and the second one a noise dependence related to the measuring device. When the system gets to a eigenstate of \hat{A} , then for the reasons developed previously $d|\psi\rangle = 0$.

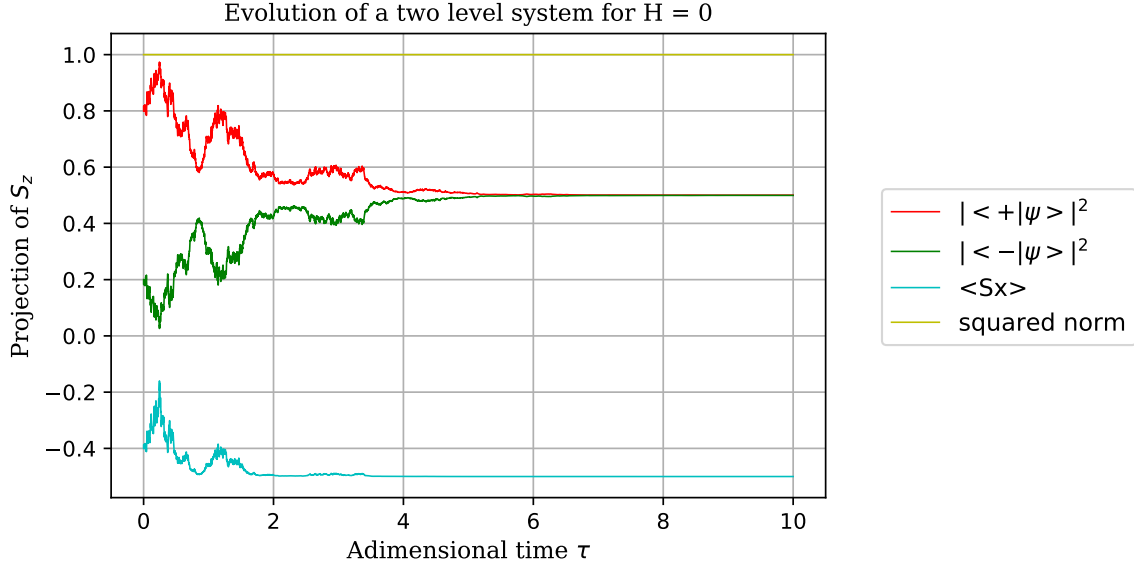


Figure 2: Projections in the base of S_z (red and green lines) and mean value of S_x (cyan line) for $k \gg \omega$ (a stronger presence of the measuring device), which is set by doing $H = 0$, for a two level system with initial state $C_+ = \sqrt{0.8}$, $C_- = -\sqrt{0.2}$

The upper graphic matches the conclusions obtained: the system gets eventually established at $\langle S_x \rangle = -0.5$, which is one of the eigenvalues of S_x . As $H = 0$, the system will stay there. This is an example of strong measurement: the measuring device has much more presence than the Hamiltonian in the system, which can be expressed as

$$k \gg \omega ,$$

where k is the factor that appear in the deterministic terms in equations (52) and ω is the factor used while coding to modify the strength of the Hamiltonian, and that corresponds to the frequency of the system as exposed in (31).

If the initial state of the system is changed, obviously the results might be different. For example, it is possible to write the initial state as the symmetric combination of eigenstates of S_x , which would be:

$$|\psi\rangle = \frac{1}{\sqrt{2}} |+\rangle_x + \frac{1}{\sqrt{2}} |-\rangle_x .$$

This initial state can be written in the base of eigenstates of S_z as $|\psi\rangle = |+\rangle$.

The coefficients that must be written as initial values in the code to reproduce this state are $C_+ = \sqrt{1.0}$ and $C_- = -\sqrt{0.0}$. The result is:

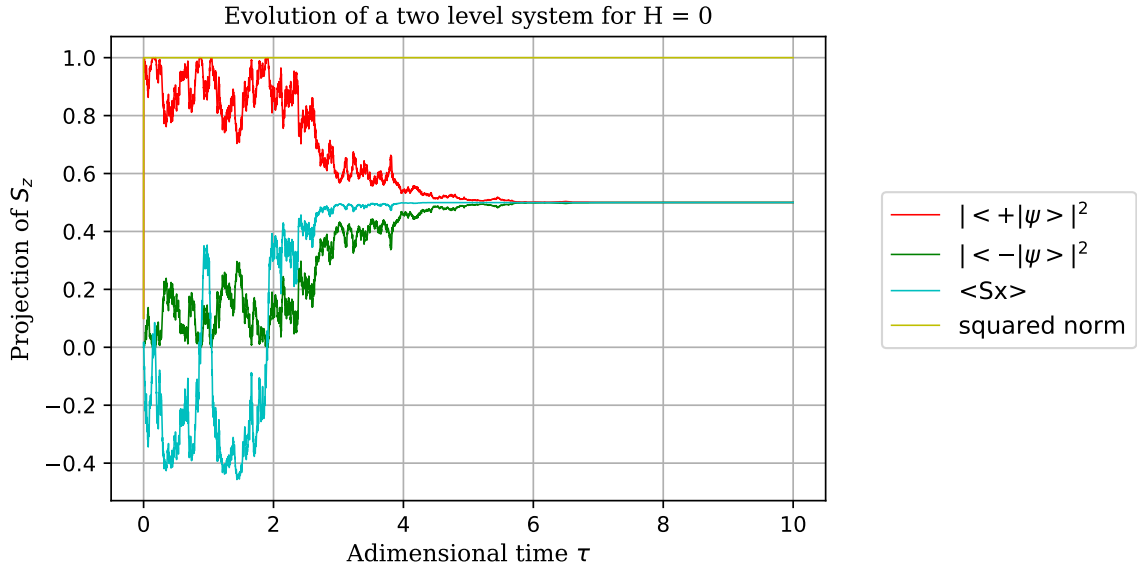


Figure 3: Evolution of a two level system for $H = 0$ with initial state $C_+ = 1$, which is equivalent to the symmetric combination of eigenstates of S_x

In this case, $\langle S_x \rangle$ started at 0, just as it was expected as both of its eigenvalues, 0.5 and -0.5 , had the same chance of happening. In other words, the initial state was a combination of $|+\rangle_x$ and $|-\rangle_x$ where both have the same weight.

In this specific run of the code, the system collapsed to $\langle S_x \rangle = 0.5$, but it is important to note that there is the same probability of the system collapsing onto $\langle S_x \rangle = -0.5$, as shown below.

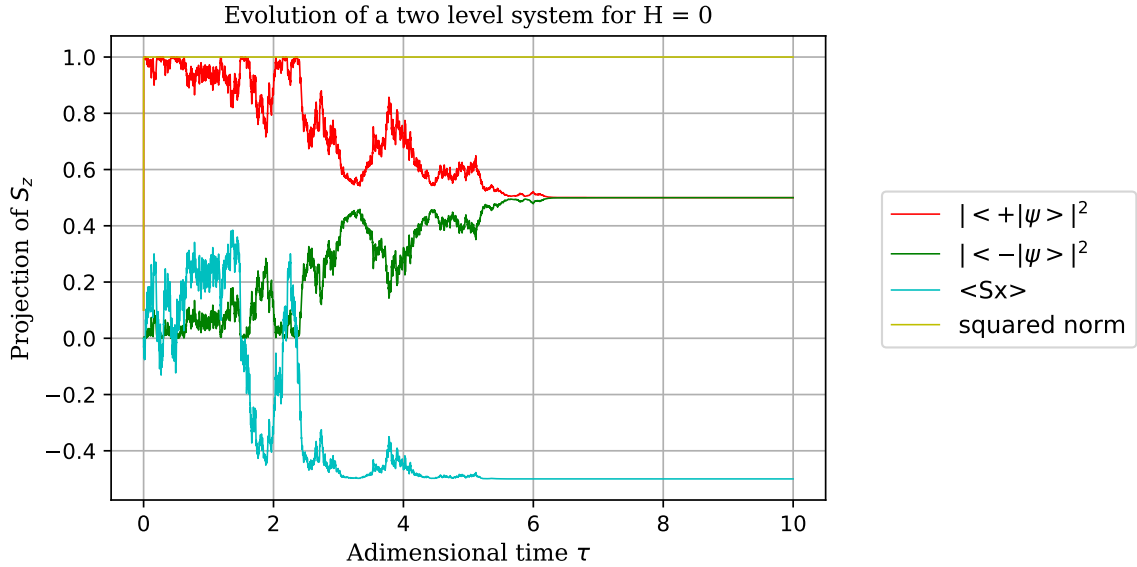


Figure 4: Evolution of a two level system for $H = 0$ with initial state $C_+ = 1$, with in this case the system collapsing to $\langle S_x \rangle = -0.5$

In both cases (Figures 3 and 4) the red line starts at 1.0 and the green one at 0, which is also obvious as the initial state was $|\psi\rangle = |+\rangle$. However, note that the system does not stay in the initial state, as it was not an eigenstate of S_x , so the process shown at (53) does not happen as the terms between the parenthesis do not get canceled between them.

As an additional note, it is possible to observe how the measuring time (the time until the system collapses totally) depends on k^{-1} . For a bigger k , for example $k = 5$ being previously $k = 1$, the system collapses much faster:

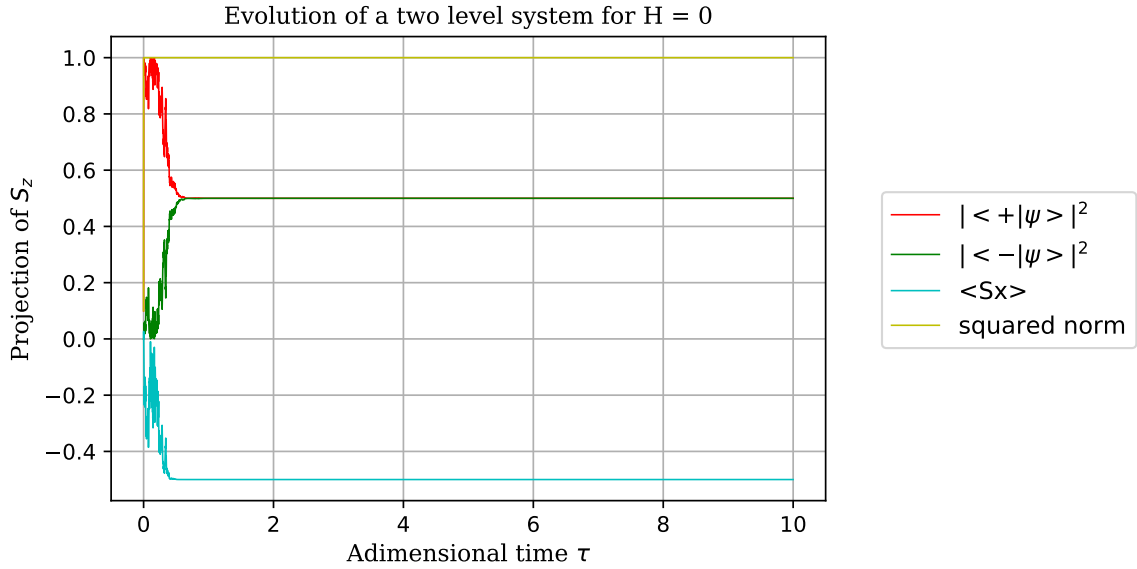


Figure 5: Quick collapse of the two level system for a faster measuring time, with $k=5$ and initial state $C_+ = 1$

It is convenient to have a look at what would happen if the initial state was $C_+ = \sqrt{0.5}$, $C_- = -\sqrt{0.5}$, that is

$$|\psi\rangle = \frac{1}{\sqrt{2}}|+\rangle - \frac{1}{\sqrt{2}}|-\rangle .$$

Such state of the system corresponds to exactly $|-\rangle_x$ in the base of eigenstates of S_x . The system is now expected not to evolve at all, as it is already at one of the eigenvalues of S_x at $t = 0$ (as shown in (53)).

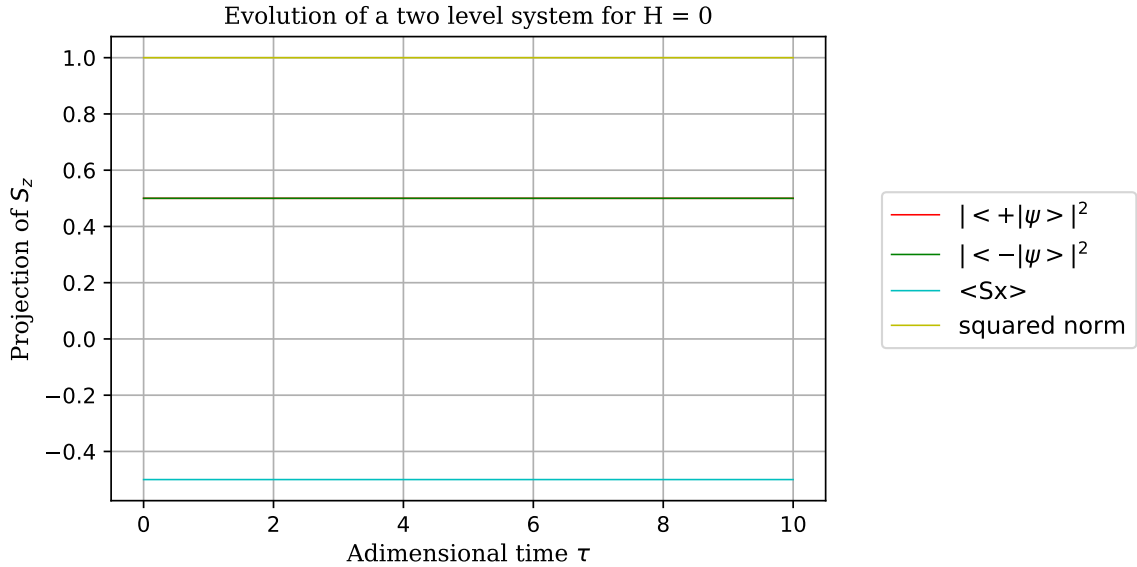


Figure 6: Evolution of a two level system for the initial state $|-\rangle_x$, this is, the anti-symmetric combination of eigenstates of S_x

Indeed, Figure 6 matches perfectly the results expected; the system does not evolve at all.

Obviously, for the initial state $C_+ = \sqrt{0.5}$, $C_- = \sqrt{0.5}$, which corresponds to $|+\rangle_x$ in the base of eigenstates of S_x , the system will stay collapsed at $\langle S_x \rangle = 0.5$.

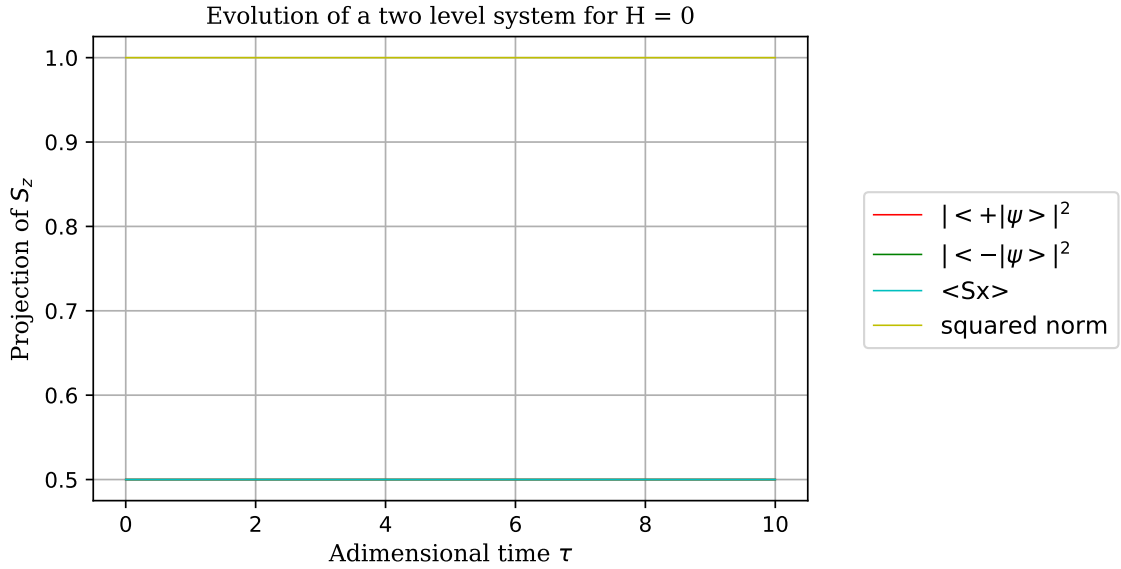


Figure 7: Evolution of a two level system for the initial state $|+\rangle_x$, this is, the symmetric combination of eigenstates of S_x

Some general conclusions can be obtained from this analysis.

First of all, it has become clear that for $H = 0$, if the system eventually gets to one of the eigenstates of the observable measured (\hat{S}_x in this case), it is not capable of further evolution and so it stays there indefinitely.

However, it is possible to extract a further conclusion. In a general case where $H \neq 0$, if the system gets into an eigenstate of the observable that is being measured \hat{A} , the Hamiltonian makes it evolve out of such eigenstate as it tries to get to the system to an eigenstate of H .

However, if $H \neq 0$, but the observable measured \hat{A} commutes with H , then each time the system got into a eigenstate of \hat{A} , H would not be capable of getting the system out of it and keep evolving, as the eigenstate of \hat{A} is also an eigenstate of H .

Therefore, a more general statement has been obtained from this analysis:

$$\text{If } [H, \hat{A}] = 0, \text{ the system will stay at any eigenstate of } \hat{A} \text{ he gets to.}$$

It is easy to see that this statement involves the previous one, as if $H = 0$, the obviously it is true that $[H, \hat{A}] = 0$.

A deeper understanding on this processes shall be obtained with the analysis done in the following section.

6.3.2 $H \neq 0$

The study detailed in the previous section made obvious that the presence of the measuring device has the effect of making the system collapse into one of the eigenstates of the observable measured.

The objective of this section is to analyze the effect of the Hamiltonian in this process. For doing so, it is mandatory to have a Hamiltonian capable of getting the system out of an eigenstate of the measured observable, so from now on on this section:

$$H \neq 0 \quad \text{and} \quad [H, \hat{A}] \neq 0 \quad (54)$$

To get started and make clear the influence of the Hamiltonian in the system, it might be good to get a look at the evolution of the system for a random combination of C_+ and C_- as initial state, with $k = 0$, which is, with no presence of the measuring device.

The initial state chosen is $C_+ = \sqrt{0.8}$, $C_- = -\sqrt{0.2}$, the same which was used in the previous section.

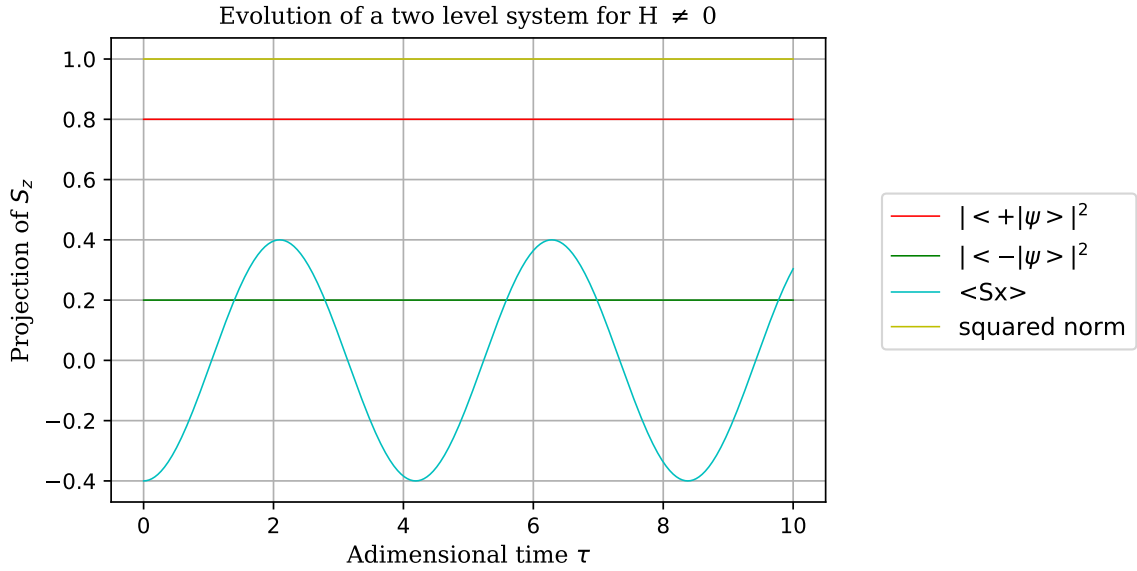


Figure 8: Evolution of a two level system for $k = 0$ (no presence of the measuring device) and initial system $C_+ = \sqrt{0.8}$, $C_- = -\sqrt{0.2}$, showing the projections in the base of S_z (red and green curves) and the mean value of S_x (cyan curve)

The effect of the Hamiltonian is to make the system oscillate. The eigenstates of the Hamiltonian are a lineal combination of the states of the observable being measured (\hat{S}_x in this case), so the Hamiltonian takes the system from one eigenstate of \hat{S}_x to another without letting it stay on them. In this case $\langle S_x \rangle$ is oscillating between $+0.4$ and -0.4 . For this to change, it is enough to modify the initial state.

If, for example, the initial state is set as $|x\rangle_x$, which would be $C_+ = \sqrt{0.5}$, $C_- = \sqrt{0.5}$, then:

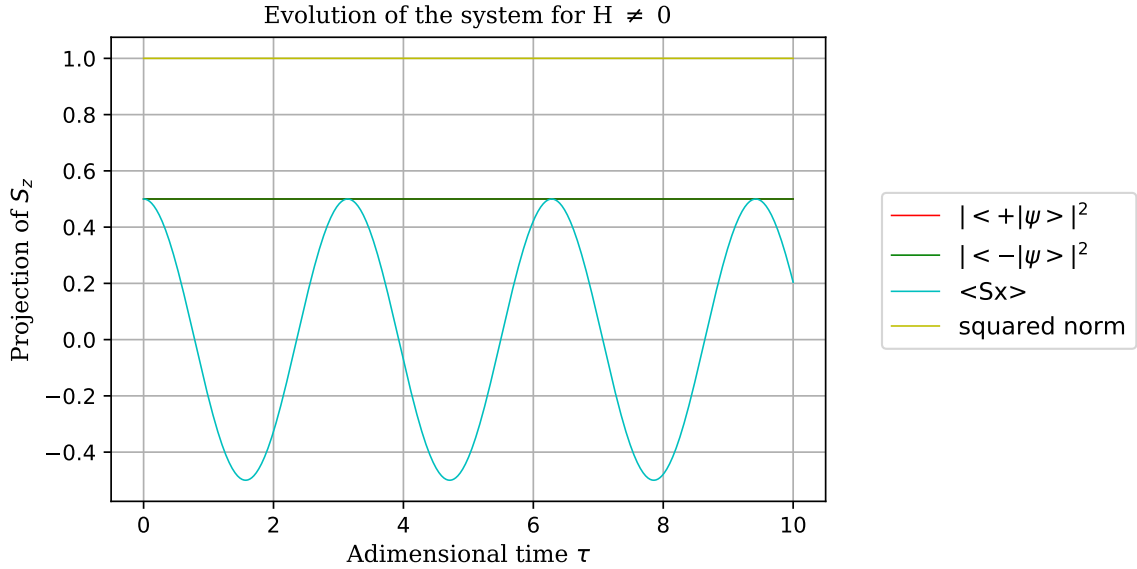


Figure 9: Evolution of a two level system for $k = 0$ and initial state the symmetric combination of eigenstates of S_z , which is equivalent to $|+\rangle_x$

As expected, now the system starts at $\langle S_x \rangle = 0.5$ and the Hamiltonian makes it keep evolving from 0.5 to -0.5 .

Again, the Hamiltonian is not letting the system stay at $|+\rangle_x$ neither at $|-\rangle_x$ but forcing it to keep evolving periodically.

Taking this into account it is possible to make an approach to the case where $H \neq 0$ and $k \neq 0$ at the same time.

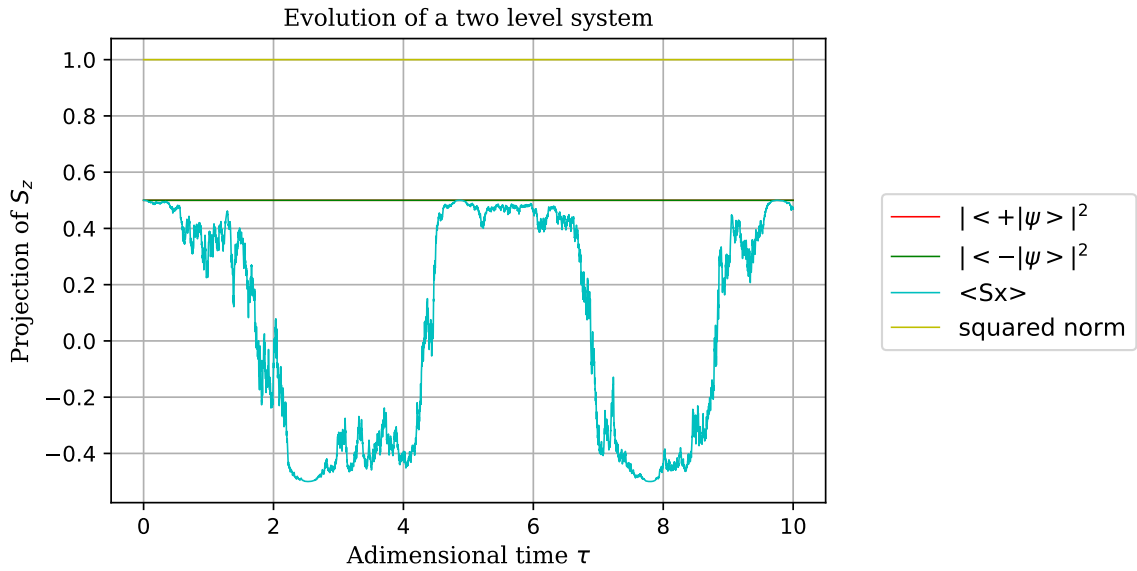


Figure 10: Evolution of a two level system with presence of both the Hamiltonian and the measuring device, being the initial state of the system $|+\rangle_x$

Figure 10 shows how the system experiments a mixture of the effects of the measuring device, which tries to make it stay at one of the eigenvalues of S_x , and the Hamiltonian, that tries to take it out from the eigenvalues of S_x and make it keep evolving.

This confrontation will be more tilted to one side or the other as far as weight is given to the Hamiltonian term in the equations or to k (the measuring terms). In a case where

$$k \gg \omega$$

the measure will have clear predominance, as shown in Figure 11.

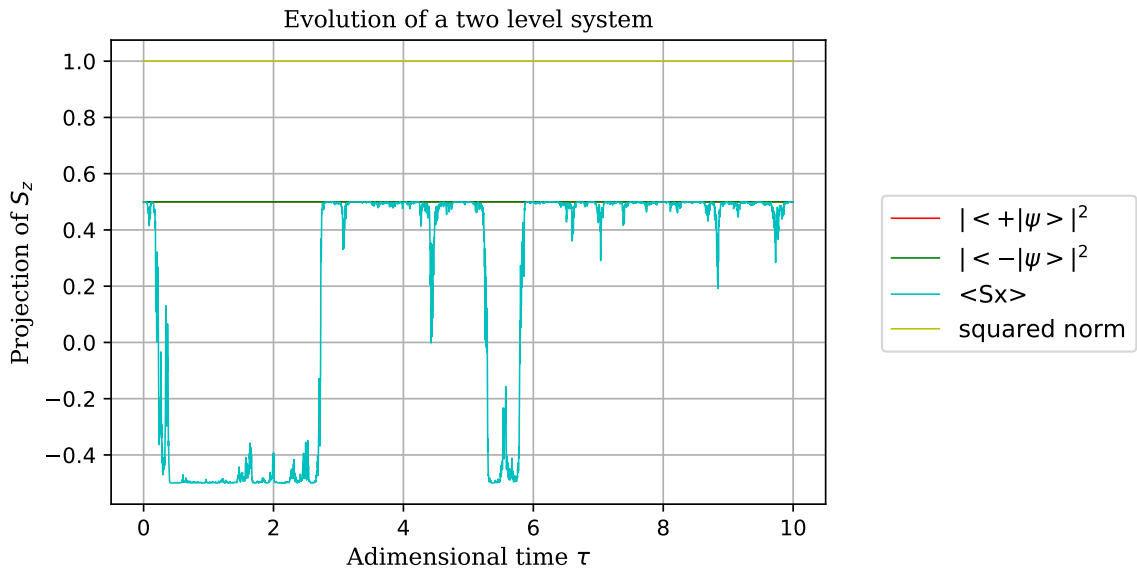


Figure 11: Evolution of a two level system for $k \gg \omega$, being the initial state of the system $|+\rangle_x$

Still it is possible to observe the effect of the Hamiltonian, which from time to time gets to take the system out of the eigenvalue it is fixed on . However the measure makes it collapse quickly just after.

In the opposite case, where

$$k \ll \omega ,$$

the system will oscillate from one value to the other with small perturbations in the oscillation due to the effect of the measure. This effect is shown in Figure 12.

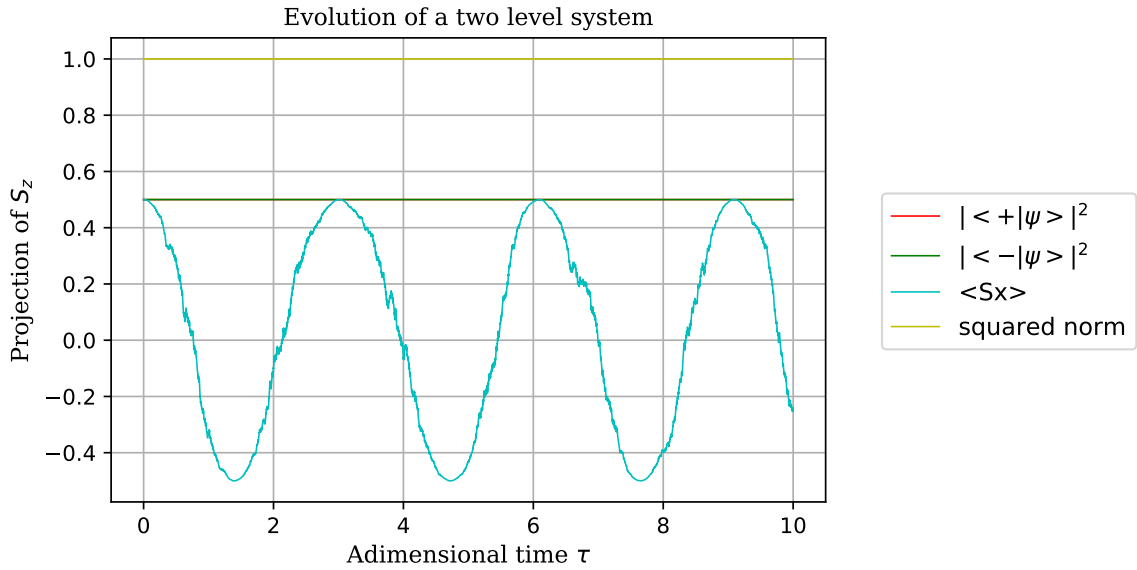


Figure 12: Evolution of a two level system for $k \ll \omega$, being the initial state of the system $|+\rangle_x$

This is an example of weak measurement: only partial information is obtained from the system, which allows it not to collapse and to keep evolving. This kind of measure does not usually provide useful information about the system, as it cannot be guaranteed that the system will be at the state measured after some time.

In order to see better this phenomenon the difference between H and k can be made smaller, but keeping $H > k$. The system will then oscillate with bigger perturbations and eventually collapse into one eigenvalue and maybe stay close to it for some time, but the system will keep its evolution after a certain time.

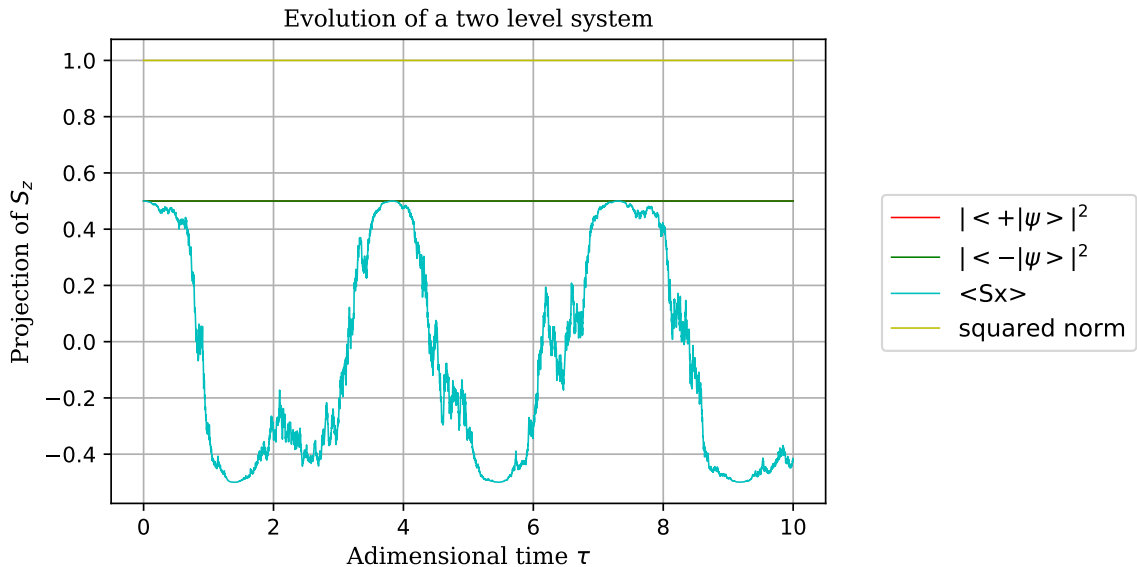


Figure 13: Evolution of a two level system for $k \sim \omega$. The Hamiltonian has a slightly stronger presence than the measuring device and the initial state of the system is $|+\rangle_x$

As a summary, it can be said that the measuring device constantly tries to make the system lose coherence while the Hamiltonian constantly tries to make it preserve coherence.

This can also be interpreted as the Hamiltonian "breaking" the measurement, meaning that the system is weakly measured and some time later the state of the system is not the measured one

anymore, because the Hamiltonian made the system evolve.

This will happen always that the conditions previously exposed are fulfilled: $[H, \hat{A}] \neq 0$ and $H \neq 0$.

6.4 Asymptotic Average State

The state of the system can be expressed using the density matrix ρ instead of $|\psi\rangle$. The elements of the density matrix have the shape

$$\rho_{\alpha\alpha'} = \langle \alpha | \rho | \alpha' \rangle . \quad (55)$$

By definition $\rho = |\psi\rangle\langle\psi|$, its elements can be divided into populations and coherences, as mentioned previously on section (5.2).

- The populations (diagonal terms) are related to the real part of the system and are equivalent to the probabilities of finding the system in the state $|\alpha\rangle$.

$$\rho_{\alpha\alpha} = \langle \alpha | \rho | \alpha \rangle = |\langle \alpha | \psi \rangle|^2 . \quad (56)$$

- The coherences (non-diagonal terms) are the the non-diagonal terms of the density matrix; the product between a state and its complex conjugate.

$$\rho_{\alpha\alpha'} = |c_\alpha c_{\alpha'}| e^{i(\phi_\alpha - \phi_{\alpha'})} . \quad (57)$$

The coherences are the the crossed terms in the density matrix; the product between a state and its complex conjugate.

In a quantum system, both populations and coherences are non-zero. However, in the process of decoherence, when a system collapses into one of the eigenstates of the observable that is being measured, the coherences get to zero and the density matrix becomes diagonal.

As mentioned previously, the populations (the only non-zero elements in the density matrix once the system has lost its coherence) represent the probability of the system of getting into one specific state.

In order to obtain these values one could make an statistical study, running the Python code multiple times and then calculating what percentage of the total the system collapsed to one system or another.

There is, however, another method to obtain the populations, that will also be a verification of the fact that populations are indeed the probability of finding the system in the corresponding state.

This analytical method is focused on obtaining the *asymptotic average state*, which is a state of the system composed by the states of it reached when the system collapses. The procedure to obtain it is detailed in the following sections.

6.4.1 Development of the SSE

The SSE (32) can be described as giving the evolution of the state conditioned by the measurement results.

This development has been extracted from Jacobs [19].

It is possible to write this SSE in terms of the density operator ρ instead of $|\phi\rangle$. Remembering that all terms must be kept proportional to dW^2 (as exposed in (36)) and defining $\rho(t+dt) \equiv \rho(t) + d\rho$, it is possible to obtain:

$$\begin{aligned} d\rho &= (d|\psi\rangle)\langle\psi| + |\psi\rangle(d\langle\psi|) + (d|\psi\rangle)(d\langle\psi|) \\ &= \frac{i}{\hbar} [H, \rho] - k [S_x [S_x, \rho]] dt + (2k)^{1/2} (S_x \rho - \rho S_x - 2 \langle S_x \rangle \rho) dW . \end{aligned} \quad (58)$$

This is usually referred to as a stochastic master equation (*SME*), which (as well as $|\psi(t)\rangle$) defines a quantum trajectory $\rho(t)$. The advantage of using a SME over using a SSE is that a the state vector $|\psi\rangle$ only gives uncertainty with respect to quantum observables, while the density operator represents, apart from this quantum uncertainty, uncertainty over the results of quantum

trajectories, which can be seen as a classical uncertainty that could be due to, for example, a mistake on the measuring device at the moment of writing down the result of the measurement. All previous developments around the SSE have been done assuming efficient measurements, where this classical uncertainty is considered zero. The reason for this approximation is simplicity: if the efficient measurement approximation is valid, the SSE allows to solve the evolution of the system with $N+1$ differential equations, being N the number of levels of the system (plus 1 because of the ground level), while the density operator is a $N \times N$ matrix, which gives places to $(N+1)^2$ differential equations that must be solved to obtain the evolution of the system, which is obviously harder to solve numerically.

The density operator at time t gives the observer's state of knowledge of the system always that the measurement record $y(t)$ has been obtained until time t , where the measurement result is given by

$$dy = \langle S_x \rangle dt + \frac{dW}{(8k)^{1/2}} . \quad (59)$$

Since the observer has access to dy but not to dW , to calculate $\rho(t)$ it is necessary to calculate dW at each time step:

$$dW = (8k)^{1/2} (dy - \langle S_x \rangle dt) . \quad (60)$$

By substituting this expression in the SME (6.4.1), it is possible to write the evolution of the system directly in terms of the measurement record, which is the natural thing to do from the point of view of the observer. This is

$$d\rho = \frac{i}{\hbar} [H, \rho] - k [S_x [S_x, \rho]] dt + 4k (S_x \rho + \rho S_x - 2 \langle S_x \rangle \rho) (dy - \langle S_x \rangle dt) . \quad (61)$$

If the observer makes a continuous measurement, but throws away the information regarding the measurement results, the observer must average over the different possible results. Since ρ and dW are statistically independent, $\langle \rho dW \rangle = 0$, where $\langle \rho dW \rangle$ denotes the average previously mentioned. The result is then given by making zero all terms proportional to ρdW in equation (6.4.1)

$$\frac{d\rho}{dt} = \frac{i}{\hbar} [H, \rho] - k [S_x [S_x, \rho]] , \quad (62)$$

where the density operator here represents the state averaged over all possible measurement results. Equation (62) is a Master Equation (ME) (non-stochastic). The method used above to derive the stochastic Schrödinger equation is an extension of a method initially developed by Caves and Milburn to derive the non-stochastic master equation (62) [23].

6.4.2 Obtainment of the asymptotic average state

The intention is now to obtain the asymptotic average state ρ_{asympt} . For doing so, a base shall be chosen and the evolution of all the elements of the density matrix ρ will be written in that base. The base of eigenstates of S_x is a good choice for this.

Two options appear here, set by the previous discussion of results: $H = 0$ and $H \neq 0$. Now, for $H \neq 0$ in the case treated here the Hamiltonian does not commute with the measured observable, as seen before, which produces that the system keeps jumping from one state to another without staying in any of them.

The asymptotic average state represents the average of results when the system has ceased to evolve. For $H \neq 0$ this average will not be determined by the initial state or other properties of the system but for the noise term, which produces the stochastic behaviour of the system. Thus the asymptotic average state for $H \neq 0$ will be just an equal distribution of probabilities between the possible outcomes of the measurement.

However, the behaviour for $H = 0$ is different, and will be developed and studied below. After doing $H = 0$, equation (62) will have the shape:

$$d\rho = -k [S_x, [S_x, \rho]] = -k (S_x^2 \rho - 2S_x \rho S_x - \rho S_x^2) . \quad (63)$$

At $t = 0$, ρ has the shape:

$$\rho(t) = |\psi_0\rangle \langle \psi_0| , \quad (64)$$

where

$$|\psi_0\rangle = \alpha |+\rangle_x + \beta |-\rangle_x . \quad (65)$$

However, at a later time t :

$$\rho(t) = \begin{pmatrix} {}_x \langle + | \rho | + \rangle_x & -{}_x \langle + | \rho | - \rangle_x \\ {}_x \langle - | \rho | + \rangle_x & {}_x \langle - | \rho | - \rangle_x \end{pmatrix} \quad (66)$$

Projecting each element of the matrix in equation (63):

$$\begin{cases} d_x \langle + | \rho | + \rangle_x = 0 \\ d_x \langle + | \rho | - \rangle_x = -k4 \frac{\hbar^2}{4} {}_x \langle + | \rho | - \rangle_x \\ d_x \langle - | \rho | + \rangle_x = -k4 \frac{\hbar^2}{4} {}_x \langle - | \rho | + \rangle_x \\ d_x \langle - | \rho | - \rangle_x = 0 \end{cases} \quad (67)$$

It is known that in order to obtain ρ_{asympt} it is necessary to take $d\rho = 0$. Therefore, equaling to 0 each equation in (67):

$$\begin{cases} d_x \langle + | \rho | + \rangle_x = 0 \\ -k4 \frac{\hbar^2}{4} {}_x \langle + | \rho | - \rangle_x = 0 \implies {}_x \langle + | \rho | - \rangle_x = 0 \\ -k4 \frac{\hbar^2}{4} {}_x \langle - | \rho | + \rangle_x = 0 \implies {}_x \langle - | \rho | + \rangle_x = 0 \\ d_x \langle - | \rho | - \rangle_x = 0 \end{cases} \quad (68)$$

The first and fourth equations in (68) provide the information that ${}_x \langle + | \rho | + \rangle_x$ neither ${}_x \langle - | \rho | - \rangle_x$ (the diagonal terms) change in time. As the diagonal terms are equivalent to the probabilities of finding the system in $|+\rangle$ and $|-\rangle$ respectively, the fact that they do not change in time means they are constant, which means they will keep their initial values.

On the other side, the non-diagonal terms, that correspond to equation 2 and 3 in (68), are 0. Therefore the shape of the asymptotic average state will be:

$$\rho_{asympt} = \begin{bmatrix} {}_x \langle + | \rho | + \rangle_x & 0 \\ 0 & {}_x \langle - | \rho | - \rangle_x \end{bmatrix} \quad (69)$$

ρ_{asympt} defines the state of the system after it has lost its coherence, and shows the percentage of probability of the system of being collapsed in one state or the other. The calculations done in this section are the equivalent of doing a statistical average of a high enough number of results as the ones obtained in section 6.3, and then doing the calculations of the percentage of times over the total the system ended up in one eigenstate or another. Both methods are equally valid.

For obtaining such probabilities it is only necessary to take into account the shape (69) and to know the initial state of the system. The coefficients corresponding to the terms on the initial state will determine the sought probabilities.

7 System 2: Harmonic Oscillator

Resumen

La estructura de este apartado es idéntica a la del anterior. Comienza con una introducción en la que se describen las propiedades generales del sistema que se va a estudiar (un oscilador armónico) y algunas características de los observables involucrados en el estudio que serán de utilidad posteriormente. En el caso de este sistema los apartados de ruido, dimensiones y redimensionalización de la ecuación (presentes en el sistema anterior) se han obviado, ya que son idénticos al caso previo. La obtención de las ecuaciones pasa por proyectar la SSE en la base de autoestados del Hamiltoniano, tras lo cual se procede a la discusión de resultados. Al igual que en el sistema de dos niveles, se termina obteniendo el estado asintótico promedio para este sistema.

7.1 Introduction

The study made for the previous system can be extended to other systems, which will make possible to establish comparisons and to find the differences between the behaviour of the systems. In this second part the objective is to analyze the behaviour of a harmonic oscillator by measuring its energy. The Hamiltonian of this system will be

$$H = \hbar\omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right) + b (\hat{a} + \hat{a}^\dagger) , \quad (70)$$

where \hat{a} and \hat{a}^\dagger are the operators creation and annihilation respectively; acting together they are proportional to the energy of the system and $\hat{a}\hat{a}^\dagger$ is the observable that will be measured in this system. \hbar is again the Planck constant, ω the frequency of the system and b a real constant.

The first part of the equation is related to the energy of the system. The observable $\hat{a}\hat{a}^\dagger$ is proportional to the energy of the system, and can be denoted as the *number operator* $\hat{N} = \hat{a}\hat{a}^\dagger$. The number operator act as:

$$\hat{N} |n\rangle = \hat{a}\hat{a}^\dagger |n\rangle = n |n\rangle , \quad (71)$$

where $\{|n\rangle\}$, $n = 0, 1, 2, \dots$ is the base of eigenstates of the Hamiltonian of the harmonic oscillator and n the eigenvalue associated to each eigenvector. The reader shall note that (71) is fulfilled due to:

$$\hat{a} |n\rangle = \sqrt{n} |n-1\rangle \quad (72)$$

$$\hat{a}^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle . \quad (73)$$

Denoting H_0 as the first term of the Hamiltonian, $H_0 = \hbar\omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right)$:

$$H_0 |n\rangle = \hbar\omega \left(\hat{N} + \frac{1}{2} \right) |n\rangle = \hbar\omega \left(n + \frac{1}{2} \right) |n\rangle , \quad (74)$$

where $E = \hbar\omega \left(n + \frac{1}{2} \right)$ is typically the energy of a harmonic oscillator.

The second term in the Hamiltonian is proportional to the position \hat{X} , and is interpreted as the effect of the application of an external field which is proportional to position.

The reason of the addition of this term to the Hamiltonian is simple: $\hat{a}\hat{a}^\dagger$ is the observable that will be measured in this case, but $[H, \hat{a}\hat{a}^\dagger] = [H, \hat{N}] = 0$. The fact that the number operator and the Hamiltonian commute means that, when analyzing the behaviour of the system, the same thing as in the section 6.3.1 will happen: as the eigenstates of the observable measured are also eigenstates of the Hamiltonian, the Hamiltonian is not capable of taking the system out of a state once it has collapsed onto it.

For avoiding this, the term $b(\hat{a} + \hat{a}^\dagger)$ is added, so that now $[H, \hat{N}] \neq 0$ and the Hamiltonian can take the system out of the eigenstates of \hat{N} when the system collapses to them.

Therefore, the cases studied for this system will be those where $b = 0$ and $b \neq 0$, which is equivalent to saying $[H, \hat{N}] = 0$ and $[H, \hat{N}] \neq 0$ respectively.

7.2 Obtainment of the equations

Before getting to analyze the system it is necessary to obtain first the equations that describe it. The starting point will be equation (32), as the procedure to get to it is identical independently of the system.

The observable which is being measured in this case is \hat{N} , so equation (32) will now have the shape

$$d|\psi\rangle = -\frac{i}{\hbar}H|\psi\rangle dt - k\left(\hat{N} - \langle\hat{N}\rangle\right)^2|\psi\rangle dt + (2k)^{1/2}\left(\hat{N} - \langle\hat{N}\rangle\right)|\psi\rangle dW . \quad (75)$$

The process of redimensionalization is the same as showed in section 6.2.3. After so, it would be obtained an equation alike

$$d|\psi\rangle = \left\{-iHd\tau - k'\left(\tilde{N} - \langle\tilde{N}\rangle\right)^2 d\tau + (2k')^{1/2}\left(\tilde{N} - \langle\tilde{N}\rangle\right)\xi(d\tau)^{1/2}\right\}|\psi\rangle , \quad (76)$$

where the accents express redimensionalization. Knowing this, they will be skipped from now on for clearness reasons, without this meaning a change in the dimensions of the redimensionalized quantities.

The objective is to obtain a set of equations that describes the behaviour of the system and that are possible to code in order to obtain numerical solutions. For doing so, the base of eigenstates of $H\{|n\rangle\}$ will be projected on equation (76).

$$d\langle n|\psi\rangle = -id\tau\langle n|H|\psi\rangle - k(\langle n|N|\psi\rangle - \langle N\rangle\langle n|\psi\rangle)^2 d\tau + (2k)^{1/2}(\langle n|N|\psi\rangle - \langle N\rangle\langle n|\psi\rangle)\xi(d\tau)^{1/2} \quad (77)$$

A notation will be introduced here for simplicity:

$$C_n = \langle n|\psi\rangle . \quad (78)$$

C_n is then the projection of the state of the system on the eigenstate $|n\rangle$, and $|C_n|^2$ will express the probability of finding the system on such state. The reader shall note that, as C_n is proportional to $|\psi\rangle$, which defines the evolution of the system, then C_n presents time dependence.

$$C_n = C_n(t) \quad (79)$$

The terms in equation (77) will now be analyzed separately.

i. $-id\tau\langle n|H|\psi\rangle$

The form of the Hamiltonian (70) shall be introduced here:

$$\langle n|H|\psi\rangle = \hbar\omega\langle n|N|\psi\rangle + \frac{\hbar\omega}{2}\langle n|\psi\rangle + b\langle n|a|\psi\rangle + b\langle n|a^\dagger|\psi\rangle , \quad (80)$$

where the operators \hat{N} , \hat{a} and \hat{a}^\dagger act as (71), (72) and (73) respectively. As in this case a projection is being done, it is important to also take into account:

$$\langle n|a = \langle n+1|\sqrt{n+1} \quad (81)$$

$$\langle n|a^\dagger = \langle n-1|\sqrt{n} \quad (82)$$

$$\langle n|H|\psi\rangle = \hbar\omega \cdot n\langle n|\psi\rangle + \frac{\hbar\omega}{2}\langle n|\psi\rangle + b\sqrt{n+1}\langle n+1|\psi\rangle + b\sqrt{n}\langle n-1|\psi\rangle . \quad (83)$$

Using the notation introduced in (78):

$$\langle n|H|\psi\rangle = \hbar\omega C_n\left(n + \frac{1}{2}\right) + b(\sqrt{n+1}C_{n+1} + \sqrt{n}C_{n-1}) . \quad (84)$$

The first term of equation (77) will then have the shape

$$-id\tau\left\{\hbar\omega C_n\left(n + \frac{1}{2}\right) + b(\sqrt{n+1}C_{n+1} + \sqrt{n}C_{n-1})\right\} . \quad (85)$$

ii. $-k(\langle n|N|\psi\rangle - \langle N\rangle\langle n|\psi\rangle)^2 d\tau$ For this case it is important to remember that $\langle N\rangle$ is actually a number (even though dependent of time) so it can be taken out of the bra-ket projection.

$$-kd\tau(\langle n|N|\psi\rangle - \langle N\rangle\langle n|\psi\rangle)^2 = -k d\tau C_n(n - \langle N\rangle)^2 . \quad (86)$$

- iii. $(2k)^{1/2} (\langle n|N|\psi\rangle - \langle N\rangle \langle n|\psi\rangle) \xi(d\tau)^{1/2}$ The third and last term is similar to the previous one:

$$(2k)^{1/2} \xi(d\tau)^{1/2} (\langle n|N|\psi\rangle - \langle N\rangle \langle n|\psi\rangle) = (2k)^{1/2} \xi(d\tau)^{1/2} C_n (n - \langle N\rangle) . \quad (87)$$

The equation obtained when putting the three terms together will be the result of the projection of the base of eigenstates of H on the redimensionalized SSE.

$$dC_n = -id\tau \left\{ \hbar\omega \left(n + \frac{1}{2} \right) C_n + b (\sqrt{n+1} C_{n+1} + \sqrt{n} C_{n-1}) \right\} - k d\tau C_n (n - \langle N\rangle)^2 + (2k)^{1/2} \xi(d\tau)^{1/2} (n - \langle N\rangle) C_n . \quad (88)$$

The reader shall note that, as $n = 0, 1, 2, \dots$, (88) is actually a set of infinite equations. This is obviously impossible to code or represent, so when coding it was important to look for convergence: for a given b , the system must not evolve over a certain n .

7.3 Discussion of results

The procedure for solving (88) is similar as the process followed in the study of the previous system. In this case, (88) will be projected on the base of eigenstates of $\{|n\rangle\}$. As there exist a relation between an arbitrary C_n and C_{n-1} and C_{n+1} , the equation for the first level dC_0 was programmed manually, while the other one were done in an iterative way.

It is important to keep an eye on convergence here. The fact that the equation for any C_n contains a term for C_{n+1} would be a convergence problem, so it is important to make sure that the number of levels N chosen for the system is big enough for the value of b chosen such that the population of the last level C_N is empty and the system does not grow over this level. For making sure this condition is fulfilled, the Python code included an alert that would spawn any time the population of the last level grew over 0.001, so that the number of levels can be raised. As an extra check, the evolution of the last level will be drawn in black in the correspondent graph so that it can be easily seen that it does not raise over 0.

Three different graphs can be of interest in the case of this system: the expectation value of the number of energy quanta $\langle N\rangle$, the evolution of the energy levels $C_n(t)$ and the evolution of the coherences $C_i C_j^*$.

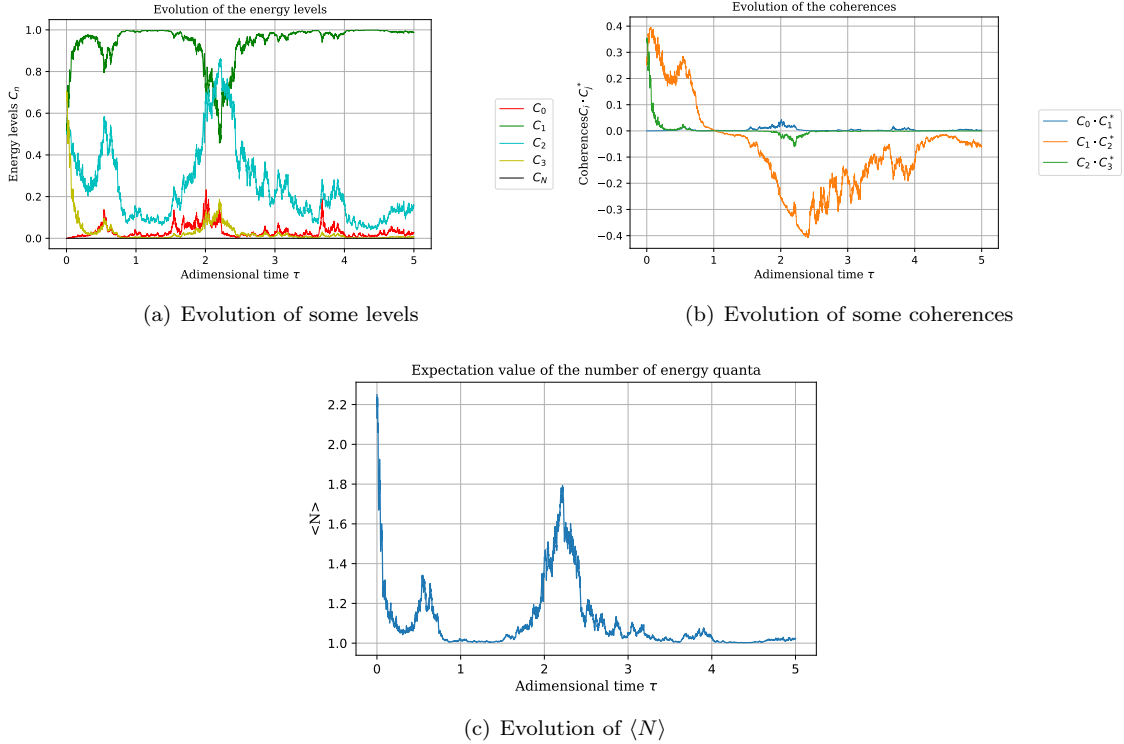


Figure 14: Evolution of some levels (C_0 , C_1 , C_2 and C_3), some coherences ($C_0 \cdot C_1^*$, $C_1 \cdot C_2^*$ and $C_2 \cdot C_3^*$) and the mean value of N for a harmonic oscillator of initial state $C_1 = 0.5$, $C_2 = 0.5$, $C_3 = \sqrt{0.5}$

The parameters chosen here are as initial state where $C_1 = 0.5$, $C_2 = 0.5$, $C_3 = \sqrt{0.5}$ while all other C_n are zero, with both k and w being 1 and giving b a low value to guarantee convergence ($b = 0.2$). The system was given 10 levels here, but only the lower ones were represented for clarity reasons. It is important to take into account while coding that in general C_n is an imaginary number. In this case, the expectation value of N stayed at $\langle N \rangle = 1$ most of the time, which means that the system stayed mostly in the first level (not level 0, but level 1). This fact can be corroborated easily by just looking at Figure 14(a), where C_1 stays most of the time at 1. As the state of the system is normalized, this fact means that the probability of finding the system in the first level is almost 1. However, there are some noticeable changes in the evolution of the system. Approximately between $\tau = 2$ and $\tau = 3$, $\langle N \rangle$ increases, as well as the probability of finding the system in the level 2, which can be noticed in the peak suffered in the curve that describes the evolution of C_2 . More or less for that value of τ it is noticeable a change in the coherences, whose values get further from zero as the system stops being collapsed at one single level and the state vector begins to have presence of more than just one level. The coherences will be non-zero when the system is evolving and zero when the system collapses to one of the eigenstates of N . This represents the general behaviour of the system. Once understood, it might be convenient to go on and have a look at the cases of interest.

7.3.1 $b = 0$

The discussion of results of the two level system led to a conclusion that is convenient to introduce here: if $[H, \hat{A}] = 0$, being \hat{A} the observable that is being measured, then once the system collapses onto an eigenstate of \hat{A} it will stay on it. However, if $[H, \hat{A}] \neq 0$, the Hamiltonian will make the system evolve out of any eigenstate it collapses onto. In this case, the first term of the Hamiltonian $\omega (a^\dagger a + \frac{1}{2})$ commutes with the observable being measured $N = aa^\dagger$, but the second term $b (a + a^\dagger)$ does not. This means that, for this system, the difference in the behaviour of the system will be noticed for $b = 0$ and $b \neq 0$, which are the cases for which H commutes and not commutes with N respectively. For $b = 0$ the quantities represented graphically look this way:

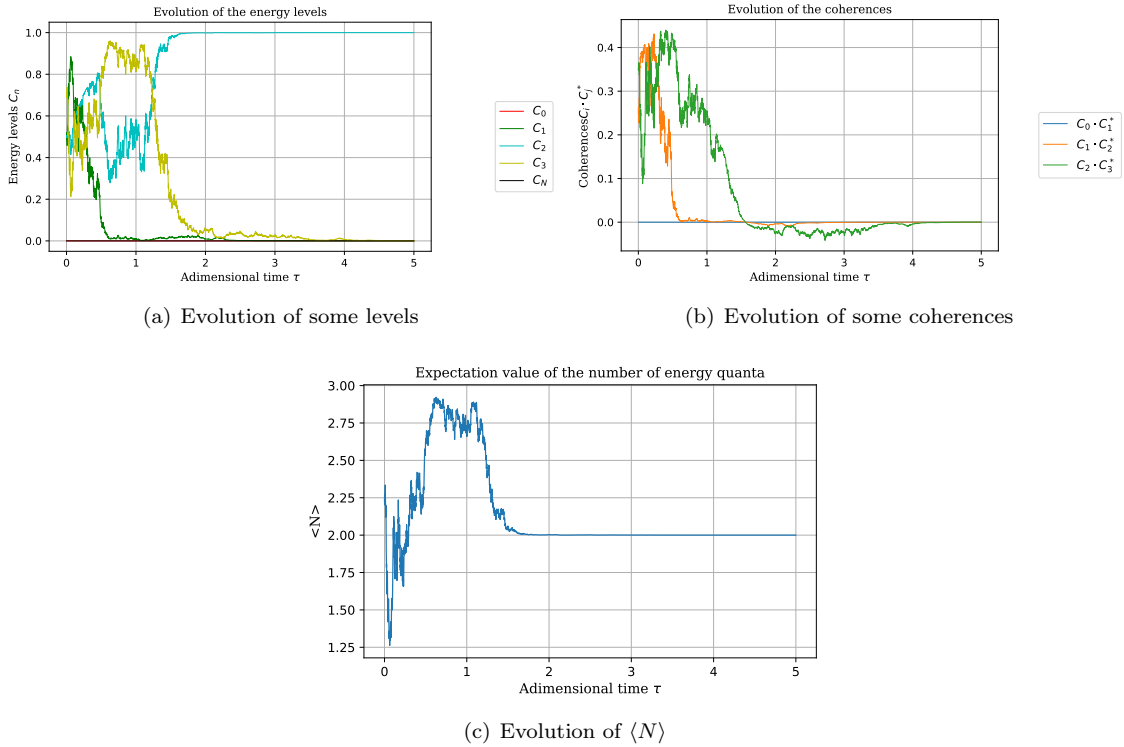


Figure 15: Evolution of some levels (C_0 , C_1 , C_2 and C_3), some coherences ($C_0 \cdot C_1^*$, $C_1 \cdot C_2^*$ and $C_2 \cdot C_3^*$) and the mean value of N for a harmonic oscillator of initial state $C_1 = 0.5$, $C_2 = 0.5$, $C_3 = \sqrt{0.5}$ and $b=0$, which implies that $[H, N] = 0$

where the initial state was the same as in the previous more general case. As expected, it is noticeable here how the system eventually collapses onto one eigenstate of N (in this case C_2) and

stays there. C_2 is now eigenstate not only of N but of H as well since $b = 0$, so the Hamiltonian is not capable of taking the system out of one of its own eigenstates to make it continue evolving. It is possible to observe a process of decoherence here. In Figure 15(b), initially the coherences were non-zero. At the time when all C_n 's go to zero while C_2 grows until 1, approximately for $\tau = 1.5$, the coherences of the system begin to decrease. By the time the system stays completely at C_2 , the coherences are zero and the expectation value stays at 2.0. The fact that some coherences stay for some time under 0 is of no particular relevance.

For a longer time, it is possible to check that the system does not grow its coherences back; it stays in the eigenstate it has collapsed onto indefinitely.

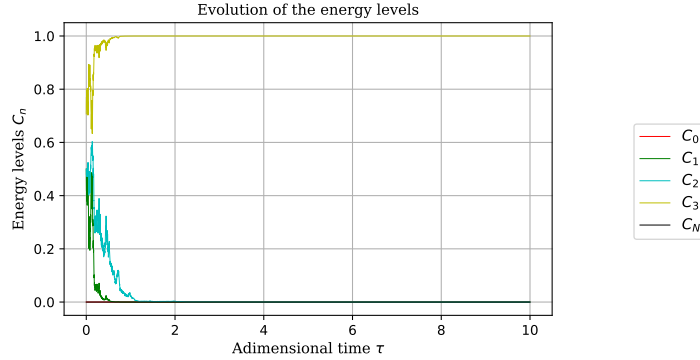


Figure 16: Evolution of a harmonic oscillator showing the levels C_0 , C_1 , C_2 and C_3 , which makes obvious that once the system collapses it does not evolve out of the state it has collapsed onto.

7.3.2 $b \neq 0$

Previously it has been studied the case where $b = 0$, which makes $[H, N] = 0$. In the case where $b \neq 0$, the Hamiltonian H and the number operator N do not commute, so the eigenstates of N are not eigenstates of H . This fact can be noticed in the following:

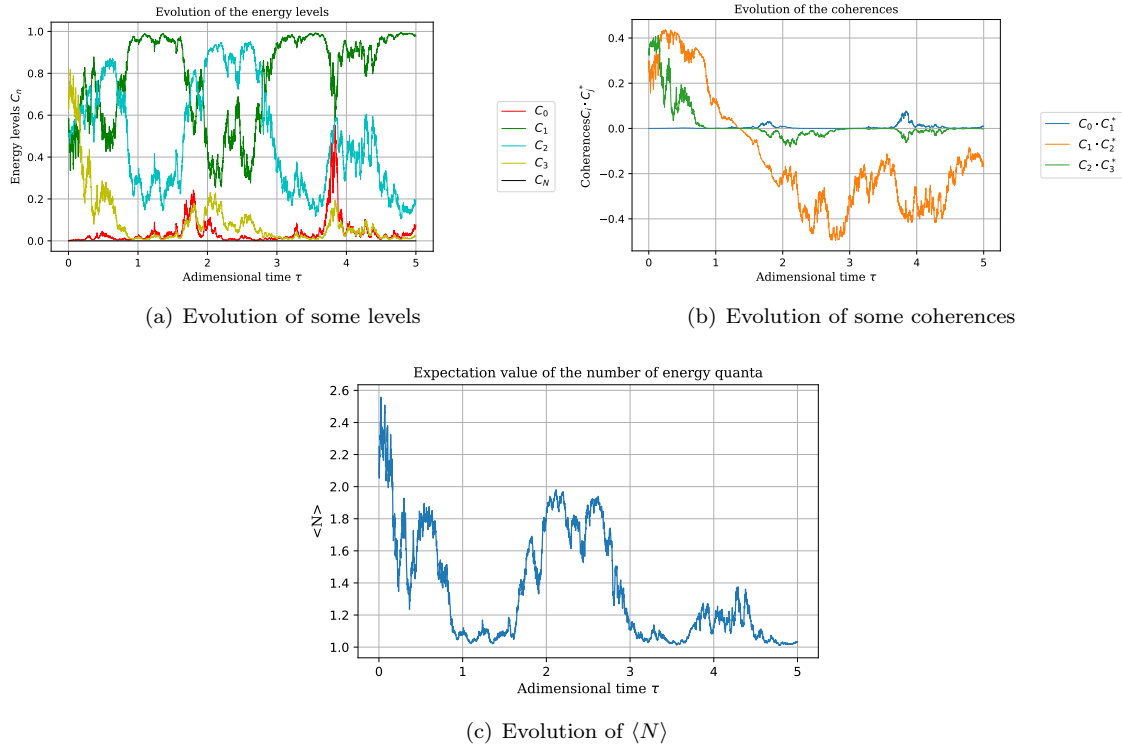


Figure 17: Evolution of some levels (C_0 , C_1 , C_2 and C_3), some coherences ($C_0 \cdot C_1^*$, $C_1 \cdot C_2^*$ and $C_2 \cdot C_3^*$) and the mean value of N for a harmonic oscillator of initial state $C_1 = 0.5$, $C_2 = 0.5$, $C_3 = \sqrt{0.5}$ and $b \neq 0$, which implies that $[H, N] \neq 0$

where b was given again the value $b = 0.2$. The system has a clearly different behaviour now. At some points, like at $\tau = 1$, it kind-of collapses

to one eigenstate (C_1 in this case) but it does not stay there but evolves out of it. This can be seen in Figure 17(c), for the expectation value of N does not stay at any particular value but jumps from one to another. Even for the moments when the system looks collapsed entirely onto one state, the coherences are not entirely zero and the expectation value of N keeps having small oscillations.

For a change of parameters, for example a higher k , it is possible to see a similar behavior as in the two level system.

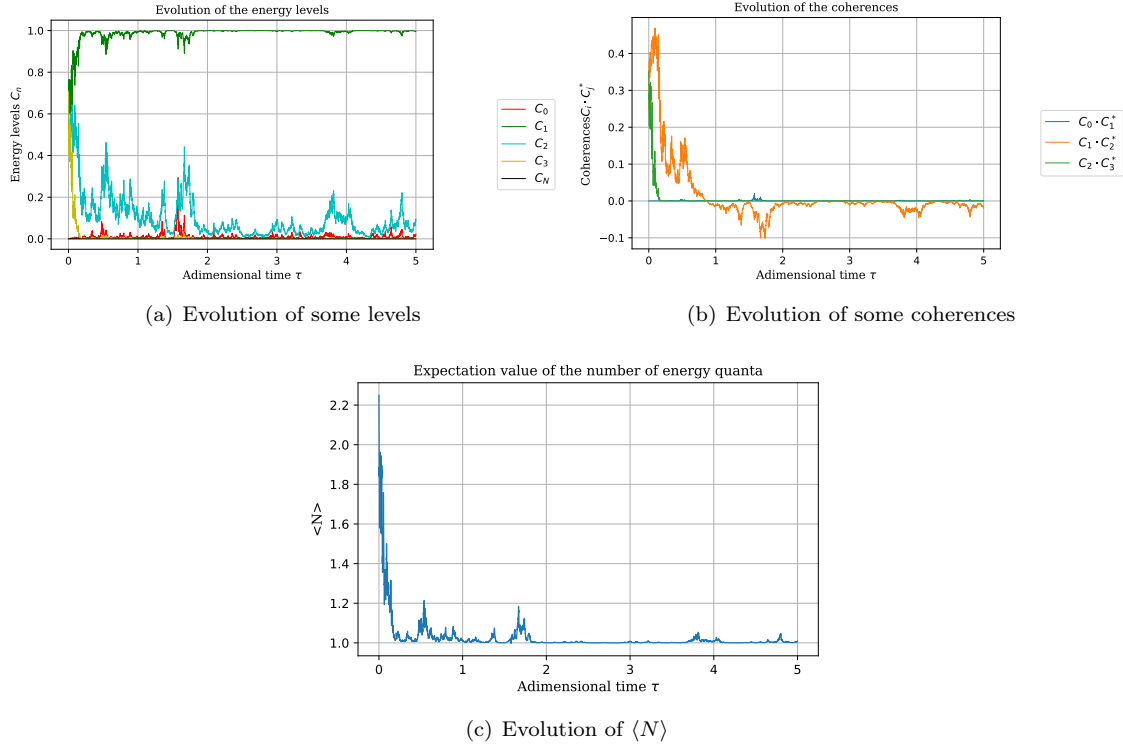


Figure 18: Evolution of some levels (C_0 , C_1 , C_2 and C_3), some coherences ($C_0 \cdot C_1^*$, $C_1 \cdot C_2^*$ and $C_2 \cdot C_3^*$) and the mean value of N for a harmonic oscillator of initial state $C_1 = 0.5$, $C_2 = 0.5$, $C_3 = \sqrt{0.5}$ and $b \neq 0$, which implies that $[H, N] \neq 0$. The measuring instrument has a stronger presence in this case (higher k)

A stronger k makes the system collapse quickly and the Hamiltonian can barely make it evolve. The peaks the coherences draw over or under zero are due to the moments where the Hamiltonian gets to take the system out of the level it has collapsed to.

In the same way, for a stronger Hamiltonian:

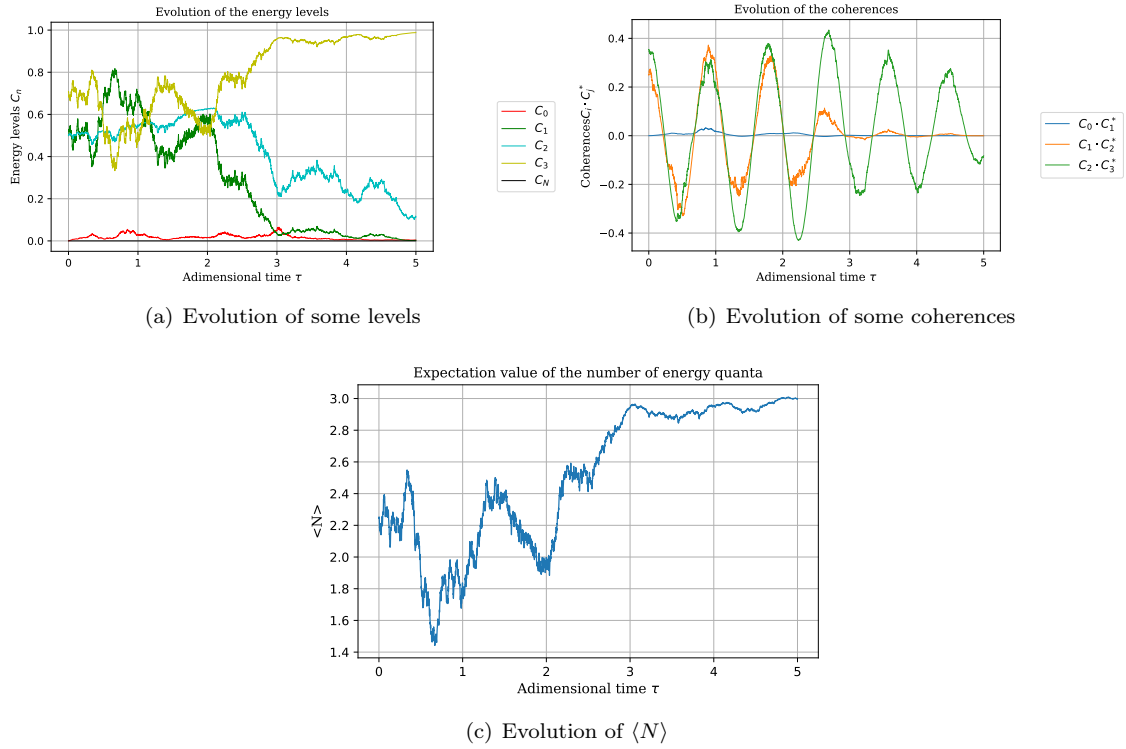


Figure 19: Evolution of some levels (C_0 , C_1 , C_2 and C_3), some coherences ($C_0 \cdot C_1^*$, $C_1 \cdot C_2^*$ and $C_2 \cdot C_3^*$) and the mean value of N for a harmonic oscillator of initial state $C_1 = 0.5$, $C_2 = 0.5$, $C_3 = \sqrt{0.5}$ and $b \neq 0$, which implies that $[H, N] \neq 0$. The Hamiltonian has a stronger presence in this case (higher ω)

The presence of the system in the levels is much more mixed than in the previous cases, and it can be seen that only after some time the system begins to have more weight in one level.

7.4 Asymptotic Average State

The concept of asymptotic average state developed in section 6.4 can be applied to a harmonic oscillator as well. The procedure is the same: starting from equation (62), where in this case N is the observable that is being measured

$$\dot{\rho} = \frac{1}{i\hbar} [H, \rho] - k [N, [N, \rho]] . \quad (89)$$

it is possible to obtain the asymptotic average state by projecting equation (89) on the base of eigenstates of H . Two cases that could be treated appear here: the case for $b = 0$ and the case for $b \neq 0$. The second case, however, where $b \neq 0$, does not provide any special information about the system due to the fact that, as the Hamiltonian does not commute with the number operator, the asymptotic average state will provide every state possible with the same fraction, equivalent to the cocient $\frac{1}{N+1}$, being N the number of levels. The reason for adding one is the inclusion of level zero. There is no information of interest that can be extracted from this conclusion, so for the exposed reasons the author finds more interesting to develop the case where $b = 0$.

It is convenient to develop equation (89) in order to ease the later projection.

$$\dot{\rho} = \frac{1}{i\hbar} (H\rho - \rho H) - k (N^2\rho - 2N\rho N + \rho N^2) , \quad (90)$$

where, for $b = 0$,

$$H = \hbar\omega \left(a^\dagger a + \frac{1}{2} \right) .$$

Each projection will have the shape

$$\dot{\rho}_{nm} = \langle n | \dot{\rho} | m \rangle , \quad (91)$$

which, as there is a number N of levels, will give place to $(N + 1)^2$ equations. However, as the density matrix is symmetric, some elements will be the complex conjugate of others and there will be need to solve just $\frac{(N+1) \cdot N}{2}$ equations. The value of N is an election of the reader; here some of the first combinations for equation (91) will be solved in order to show the procedure and the general scheme of the results that can be obtained.

$$\begin{aligned} \dot{\rho}_{00} &= \langle 0 | \dot{\rho} | 0 \rangle = \\ & \frac{1}{i\hbar} \hbar\omega \left(\langle 0 | a^\dagger a \rho | 0 \rangle + \langle 0 | 0 \rangle \frac{1}{2} - \langle 0 | \rho a^\dagger a | 0 \rangle - \langle 0 | 0 \rangle \frac{1}{2} \right) \\ & -k \left(\langle 0 | N^2 \rho | 0 \rangle - 2 \langle 0 | N \rho N | 0 \rangle + \langle 0 | \rho N^2 | 0 \rangle \right) = \frac{\omega}{i} \left(\frac{1}{2} - \frac{1}{2} \right) = 0, \end{aligned} \quad (92)$$

where the relations (71), (72), (73), (81), (82) have been used to solve the behavior of N , a^\dagger and a over the eigenstates.

$$\begin{aligned} \dot{\rho}_{01} &= \langle 0 | \dot{\rho} | 1 \rangle = \\ & \frac{1}{i\hbar} \hbar\omega \left(\langle 0 | a^\dagger a \rho | 1 \rangle + \langle 0 | 1 \rangle \frac{1}{2} - \langle 0 | \rho a^\dagger a | 1 \rangle - \langle 0 | 1 \rangle \frac{1}{2} \right) \\ & -k \left(\langle 0 | N^2 \rho | 1 \rangle - 2 \langle 0 | N \rho N | 1 \rangle + \langle 0 | \rho N^2 | 1 \rangle \right) = -k \langle 0 | \rho | 1 \rangle = -k \rho_{01} \end{aligned} \quad (93)$$

The equations for $\dot{\rho}_{11}$, $\dot{\rho}_{12}$, $\dot{\rho}_{02}$ and $\dot{\rho}_{22}$ have been solved and included in Appendix III (10.3). The result of all six projections must now be equal to zero in order to obtain the asymptotic average state.

$$\dot{\rho}_{00} = 0 = 0 \quad (94)$$

$$\dot{\rho}_{01} = -k \rho_{01} = 0 \implies \rho_{01} = 0 \quad (95)$$

$$\dot{\rho}_{11} = 0 = 0 \quad (96)$$

$$\dot{\rho}_{12} = -\left(\frac{\omega}{i} + k\right) \rho_{12} = 0 \implies \rho_{12} = 0 \quad (97)$$

$$\dot{\rho}_{02} = -2\left(\frac{\omega}{i} + 2k\right) \rho_{02} = 0 \implies \rho_{02} = 0 \quad (98)$$

$$\dot{\rho}_{22} = 0 = 0 \quad (99)$$

It is possible to observe that the projections on all diagonal terms $\dot{\rho}_{ii}$ are zero. This means that the change in time of those elements is zero, this is

$$\rho_{00}^{asympt} = \rho_{00}(0). \quad (100)$$

The populations of the density matrix stay as they were in the beginning, which means that the probabilities of finding the system in one state or another do not change since its initial configuration. On the other side, the non-diagonal elements ρ_{ij} have as result $\rho_{ij} = 0$. It is important to note that this conclusion can be reached because, as k and ω are both real there is no way for the elements between the parenthesis to cancel each other. The fact that the coherences of the density matrix are zero in the asymptotic average state means, as in the case of the previous system, that the system eventually suffers a process of decoherence that makes it collapse into one of the eigenstates of N , making the coherences to be zero.

As in the previous case, the populations of ρ^{asympt} give the probabilities of the system to collapse onto the eigenstate correspondent to each population, which is also given by the initial state of the system. In this case of lose of coherence the system can be described in classical terms.

8 Conclusions

Resumen

La primera conclusión que puede obtenerse es la importancia de la conmutabilidad del Hamiltoniano con el observable que se está midiendo. Si conmutan, el Hamiltoniano no será capaz de sacar al sistema de un estado al que colapse, si no conmutan el Hamiltoniano sacará al sistema de cualquier autoestado al que colapse y lo hará seguir evolucionando. Por otro lado, el instrumento de medida tiende a hacer al sistema colapsar; el que tenga mayor presencia entre

ambos, Hamiltoniano e instrumento de medida, determinará el comportamiento general del sistema. Por último, cuando el sistema colapsa a un estado las coherencias de la matriz densidad se hacen cero, haciendo que el sistema pueda ser descrito en términos probabilísticos clásicos a través de las poblaciones. La medida continua y la medida débil son las herramientas de las que se dispone para evitar este proceso de decoherencia.

Some conclusions can be drawn after the study over the systems done before. First of all, it is clear that the commutation of the Hamiltonian with the operator that is being measured is crucial for understanding the behavior of the system. If H commutes with the measured observable, then the system will stay indefinitely at any state it has collapsed onto. If H does not commute with the measured observable, the Hamiltonian will do the work of getting the system out of any state it collapses onto and make it keep evolving.

This fact gives place to a second conclusion: it is possible to observe a fight between the Hamiltonian and the many degrees of freedom bath the system is wrapped in, which in this work has been interpreted as a measuring device. While (if not commuting with the measured observable) the Hamiltonian makes the system evolve periodically over all possible states, the measuring device makes the system collapse onto one eigenstate and stay there. Whoever of the two is the strongest will determine the behaviour of the system.

The third conclusion to get to is the fact that, when the system collapses, it suffers a process of decoherence where the coherences of the density matrix go to zero, such that the asymptotic average state will have the probabilities of finding the system on each state in its diagonal and zero in all non-diagonal terms. Such a system, having lost its coherence, can be described in classical terms of probabilities. The concepts of continuous measurement and weak measurement are the tool that can be used to obtain partial information about the system without forcing it to collapse to one eigenstate.

9 References

- [1]: V.P. Belavkin, in *Information, Complexity and Control in Quantum Physics*, edited by A. Blaquiére, S. Diner and G. Lochak, Springer-Verlag, New York, (1987).
- [2]: P. Bushev, D. Rotter, A. Wilson, et al., *Phys. Rev. Lett.* 96 043003 (2006).
- [3]: J. Geremia, J.K. Stockton and H. Mabuchi, *Phys. Rev. Lett.* 94 203002 (2005).
- [4]: C. D’Helon and M.R. James, *Phys. Rev. A* 73, 053803 (2006).
- [5]: D.T. Pope, H.M. Wiseman and N.K. Langford, *Phys. Rev. A* 70, 043812 (2004).
- [6]: K. Jacobs, *Quant. Informat. Comp.* 7 127 (2007).
- [7]: P. Rabl, V. Steixner and P. Zoller, eprint: quant-ph/0506185 (2005).
- [8]: T. Bhattacharya, S. Habib and K. Jacobs, *Phys. Rev. A* 67 042103 (2003).
- [9]: C. Ahn, A.C. Doherty and A.J. Landahl, *Phys. Rev. A* 65 042301 (2002).
- [10]: R. van Handel and H. Mabuchi, eprint: quant-ph/0511221 (2006).
- [11]: M.J. Everitt, T.D. Clark, P.B. Stiffell, et al., *New J. Phys.* 7 64 (2005).
- [12]: W.H. Zurek :Decoherence and the Transition from Quantum to Classical- Revisited, *Los Alamos Science* 27, (2002)
- [13]: T. Bhattacharya, S. Habib and K. Jacobs, *Phys. Rev. A* 67 042103 (2003).
- [14]: H. Carmichael: *An Open Systems Approach to Quantum Optics*, Lectures presented at the Université Libre de Bruxelles, Springer Verlag, Oct 8 to Nov 4, (1991)
- [15]: S. Ghose, P. Alsing, I. Deutsch, et al., *Phys. Rev. A* 69 052116 (2004).
- [16]: S. Ghose, P.M. Alsing, B.C. Sanders, et al., *Phys. Rev. A* 72 014102 (2005).
- [17]: A. Venugopalan: *Measurement in Quantum Mechanics: Decoherence and the Pointer Basis*, InTech Open (2012)
- [18]: S. Habib, K. Jacobs and K. Shizume, *Phys. Rev. Lett.* 96 010403 (2006).
- [19]: K. Jacobs, D.A. Steck: *A Straightforward Introduction to Continuous Quantum Measurement*, *Contemporary Phys.* vol. 47, n^o 5, 279-303, Sept-Oct (2006)
- [20]: M.A. Nielsen and I.L. Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press, Cambridge, (2000).
- [21]: B. Schumacher, *Phys. Rev. A* 54 2614 (1996).
- [22]: C.A. Fuchs and K. Jacobs, *Phys. Rev. A* 63 062305 (2001).
- [23]: C.M. Caves and J.G. Milburn, *Phys. Rev. A* 36 5543 (1987).

10 Appendixes

10.1 Appendix I: Python code for the first system

```
8
9 import numpy as np
10 import pylab as plt
11
12
13 n=10000
14 w=1.0
15 Cmas = np.array([np.sqrt(0.5)+0.j])
16 Cmenos = np.array([np.sqrt(0.5)+0.j])
17 dtau=0.001
18 k_prim = 0.8
19
20 for i in range(n):
21
22     rand=np.random.normal(0,1)
23     vmsx = 0.5 * (np.conjugate(Cmas[-1])*Cmenos[-1] + np.conjugate(Cmenos[-1])*Cmas[-1])
24     vmsx2=vmsx**2.0
25
26     eq1_mas = -1j * Cmas[-1] * dtau *w/2
27     eq2_mas = -k_prim * ( (0.25 + vmsx2) * Cmas[-1] - vmsx * Cmenos[-1]) * dtau
28     eq3_mas = np.sqrt(2.0*k_prim) * rand * (0.5*Cmenos[-1] - vmsx*Cmas[-1]) * np.sqrt(dtau)
29
30     eq1_menos = 1j * Cmenos[-1] * dtau *w/2
31     eq2_menos = -k_prim * ( (0.25 + vmsx2) * Cmenos[-1] - vmsx * Cmas[-1]) * dtau
32     eq3_menos = np.sqrt(2.0*k_prim) * rand * (0.5*Cmas[-1] - vmsx*Cmenos[-1]) * np.sqrt(dtau)
33
34     Cmas_mod = Cmas[-1] + eq1_mas + eq2_mas + eq3_mas
35     Cmenos_mod = Cmenos[-1] + eq1_menos + eq2_menos + eq3_menos
36     norma=np.sqrt((abs(Cmas_mod))**2.0+(abs(Cmenos_mod))**2.0)
37     Cmas_mod=Cmas_mod/norma
38     Cmenos_mod=Cmenos_mod/norma
39     Cmas = np.append(Cmas, Cmas_mod)
40     Cmenos = np.append(Cmenos, Cmenos_mod)
41
42 tiempo = dtau*np.arange(0,n+1,1)
43
44 plt.plot(tiempo, np.abs(Cmas)**2.0, 'r-', label = '|<+|r'$\psi$ >|^2')
45 plt.plot(tiempo, np.abs(Cmenos)**2.0, 'g-', label = '|<-|r'$\psi$ >|^2')
46 plt.plot(tiempo, (0.5 * (np.conjugate(Cmas)*Cmenos + np.conjugate(Cmenos)*Cmas)).real, 'c-', label = '<Sx>')
47 plt.plot(tiempo, (np.abs(Cmas))**2.0+(np.abs(Cmenos))**2.0, 'y-', label = 'norm^2')
48 plt.title('Evolution of a two level system')
49 plt.xlabel('Adimensional time' r' $\tau$')
50 plt.ylabel('Projection of ' r'$S_z$')
51 plt.legend(bbox_to_anchor = (1.25, 0.5), loc = 'center right', fontsize = 11.2, bbox_transform = plt.gcf().transFigure)
52 plt.grid()
53 plt.savefig('similares.pdf', bbox_inches = 'tight')
54
```

Figure 20: Python code for the two level system

10.2 Appendix II: Python code for the second system

```

1 import numpy as np
2 import pylab as plt
3
4
5
6 Niveles = 10
7 w = 1.0
8 k = 1.0
9 b = .2
10 dtau = 0.001
11
12 tiempo = np.arange(0,5,dtau)
13
14 vmN = np.zeros(len(tiempo), dtype = complex)
15 psi = np.zeros(Niveles,dtype=complex)
16 ev = np.zeros(Niveles, len(tiempo)), dtype = complex)
17
18 #Denotamos un estado inicial:
19 psi[1], psi[2], psi[3] = 0.5+0.j, 0.5+0.j, np.sqrt(0.5)+0.j
20
21 norma = np.sum(abs(psi)**2)
22 psi = psi/np.sqrt(norma)
23 ev[:,0]=psi
24
25 #Calculamos el valor medio de N = suma de todos los |C_n|^2 * n
26 indices = np.array([])+0j
27 valor = 0
28 for idx, x in enumerate(psi):
29     valor = valor + idx * abs(x)**2
30 vmN[0] = valor
31 print('Valor medio',valor)
32
33 for i in range(len(tiempo)-1):
34     rand = np.random.normal(0,1)
35     dC0 = -1j*dtau * (w*psi[0]/2 + b*psi[1]) - k*dtau*psi[0]*vmN[0]**2 + (2*k*dtau)**0.5 *rand*vmN[0]*psi[0]
36     psi[0] = psi[0] + dC0
37
38     for n in range(1,Niveles-1):
39         eq1 = -1j*dtau* (w* (n+0.5) *psi[n] + b*(np.sqrt(n+1)*psi[n+1] + np.sqrt(n)*psi[n-1]))
40         eq2 = -k*dtau*psi[n] * (n-vmN[i])**2
41         eq3 = (2*k*dtau)**0.5 *rand*psi[n]* (n-vmN[i])
42
43         psi[n] = psi[n] + eq1 + eq2 + eq3
44
45     norma = np.sum(abs(psi)**2)
46     psi = psi/np.sqrt(norma) #Normalizamos el estado obtenido
47
48     indices = np.array([])+0j
49     valor = 0
50     for idx, x in enumerate(psi):
51         valor = valor + idx * abs(x)**2
52
53     vmN[i+1] = valor
54     ev[:,i+1] = psi #Calculamos el valor medio de N para este paso de tiempo
55     if ev[-1,i+1] > 0.001: #Guardamos el estado correspondiente a cada paso de tiempo en las columnas de la matriz ev
56         print('La población del último nivel es mayor a 0.001, debe aumentarse el número de niveles')
57

```

Figure 21: Python code for the harmonic oscillator (I)

```

58 #print(vmN[-1])
59 #print(psi)
60 print('Norma', norma)
61
62 plt.plot(tiempo, np.real(vmN), linewidth=1.0)
63 plt.xlabel('Adimensional time ' r'$\tau$', fontname='serif', fontsize=11)
64 plt.ylabel('<N>', fontname='serif', fontsize=11)
65 plt.title('Expectation value of the number of energy quanta', fontname='serif', fontsize=11)
66 plt.grid()
67 plt.tight_layout()
68 plt.savefig('1_b.pdf', bbox_inches = 'tight')
69 plt.show()
70
71 plt.plot(tiempo, abs(ev[0,:]), 'r-', label='$C_0$', linewidth=0.75)
72 plt.plot(tiempo, abs(ev[1,:]), 'g-', label='$C_1$', linewidth=0.75)
73 plt.plot(tiempo, abs(ev[2,:]), 'c-', label='$C_2$', linewidth=0.75)
74 plt.plot(tiempo, abs(ev[3,:]), 'y-', label='$C_3$', linewidth=0.75)
75 plt.plot(tiempo, abs(ev[-1,:]), 'k-', label='$C_N$', linewidth=0.75) #La línea negra debe mantenerse a 0 para asegurar la convergencia
76 plt.xlabel('Adimensional time ' r'$\tau$', fontname='serif', fontsize=11)
77 plt.ylabel('Energy levels ' r'$C_n$', fontname='serif', fontsize=11)
78 plt.title('Evolution of the energy levels', fontname='serif', fontsize=11)
79 plt.legend(bbox_to_anchor = (1.25, 0.5), loc = 'center right', fontsize = 11.2, bbox_transform = plt.gcf().transFigure)
80 plt.grid()
81 plt.tight_layout()
82 plt.savefig('2_b.pdf', bbox_inches = 'tight')
83 plt.show()
84
85 plt.plot(tiempo, np.real(ev[0,:]*np.conjugate(ev[1,:])), label=r'$C_0 \cdot C_1$', linewidth=0.75)
86 plt.plot(tiempo, np.real(ev[1,:]*np.conjugate(ev[2,:])), label=r'$C_1 \cdot C_2$', linewidth=0.75)
87 plt.plot(tiempo, np.real(ev[2,:]*np.conjugate(ev[3,:])), label=r'$C_2 \cdot C_3$', linewidth=0.75)
88 plt.xlabel('Adimensional time ' r'$\tau$', fontname='serif', fontsize=11)
89 plt.ylabel('Coherences ' r'$C_i \cdot C_j$', fontname='serif', fontsize=11)
90 plt.title('Evolution of the coherences', fontname='serif', fontsize=11)
91 plt.legend(bbox_to_anchor = (1.25, 0.5), loc = 'center right', fontsize = 11.2, bbox_transform = plt.gcf().transFigure)
92 plt.grid()
93 plt.tight_layout()
94 plt.savefig('3_b.pdf', bbox_inches = 'tight')
95 plt.show()
96
97

```

Figure 22: Python code for the harmonic oscillator (II)

10.3 Appendix III: Solving of some more projections

$$\begin{aligned}
\dot{\rho}_{11} &= \langle 1 | \dot{\rho} | 1 \rangle = \frac{1}{i\hbar} \hbar\omega \left(\langle 1 | a^\dagger a \rho | 1 \rangle + \langle 1 | 1 \rangle \frac{1}{2} - \langle 1 | \rho a^\dagger a | 1 \rangle - \langle 1 | 1 \rangle \frac{1}{2} \right) \\
&\quad - k \left(\langle 1 | N^2 \rho | 1 \rangle - 2 \langle 1 | N \rho N | 1 \rangle + \langle 1 | \rho N^2 | 1 \rangle \right) = \\
&= \frac{\omega}{i} \left(\rho_{11} + \frac{1}{2} - \rho_{11} - \frac{1}{2} \right) - k (\rho_{11} - 2\rho_{11} + \rho_{11}) = 0
\end{aligned} \tag{101}$$

$$\begin{aligned}
\dot{\rho}_{12} &= \langle 1 | \dot{\rho} | 2 \rangle = \frac{1}{i\hbar} \hbar\omega \left(\langle 1 | a^\dagger a \rho | 2 \rangle + \langle 1 | 2 \rangle \frac{1}{2} - \langle 1 | \rho a^\dagger a | 2 \rangle - \langle 1 | 2 \rangle \frac{1}{2} \right) \\
&\quad - k \left(\langle 1 | N^2 \rho | 2 \rangle - 2 \langle 1 | N \rho N | 2 \rangle + \langle 1 | \rho N^2 | 2 \rangle \right) = \\
&= \frac{\omega}{i} (\rho_{12} - 2\rho_{12}) - k (\rho_{12} - 4\rho_{12} + 4\rho_{12}) = \frac{\omega}{i} (\rho_{12} - 2\rho_{12}) - k\rho_{12} = - \left(\frac{\omega}{i} + k \right) \rho_{12}
\end{aligned} \tag{102}$$

$$\begin{aligned}
\dot{\rho}_{02} &= \langle 0 | \dot{\rho} | 2 \rangle = \frac{1}{i\hbar} \hbar\omega \left(\langle 0 | a^\dagger a \rho | 2 \rangle + \langle 0 | 2 \rangle \frac{1}{2} - \langle 0 | \rho a^\dagger a | 2 \rangle - \langle 0 | 2 \rangle \frac{1}{2} \right) \\
&\quad - k \left(\langle 0 | N^2 \rho | 2 \rangle - 2 \langle 0 | N \rho N | 2 \rangle + \langle 0 | \rho N^2 | 2 \rangle \right) = \\
&= -\frac{\omega}{i} 2\rho_{02} - k4\rho_{02} = -2 \left(\frac{\omega}{i} + 2k \right) \rho_{02}
\end{aligned} \tag{103}$$

$$\begin{aligned}
\dot{\rho}_{22} &= \langle 2 | \dot{\rho} | 2 \rangle = \frac{1}{i\hbar} \hbar\omega \left(\langle 2 | a^\dagger a \rho | 2 \rangle + \langle 2 | 2 \rangle \frac{1}{2} - \langle 2 | \rho a^\dagger a | 2 \rangle - \langle 2 | 2 \rangle \frac{1}{2} \right) \\
&\quad - k \left(\langle 2 | N^2 \rho | 2 \rangle - 2 \langle 2 | N \rho N | 2 \rangle + \langle 2 | \rho N^2 | 2 \rangle \right) = \\
&= \frac{\omega}{i} (2\rho_{22} - 2\rho_{22}) - k (4\rho_{22} - 8\rho_{22} + 4\rho_{22}) = 0
\end{aligned} \tag{104}$$