Sección de Física
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Introduction to the Bosonic String Theory

Introducción a la Teoría de Cuerdas Bosónicas

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Paula Rodríguez Sánchez

Dirigido por:

José María Gómez Llorente

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Abstract

In this work we analyze the main aspects related to the appearance and development of bosonic string theory from an introductory point of view and with the knowledge obtained during the degree. First, we present the tools that modern physics uses to solve problems, the Principle of least action, and we work on some examples such as the action of the free particle with the aim of establishing the knowledge for the resolution of the movement of the relativistic particle, and study later the movement of the relativistic string. We carry out a brief review of the events that gave rise to this theory of bosonic strings as well as other theories that emerged with the aim of unifying the four forces: the gravity force, the electromagnetic force, the weak force and the strong force. We continue the study looking for symmetries and conserved quantities that will significantly reduce the complexity of the problem at hand. We carry out two types of quantization in our theory: canonical and light cone quantization, and we obtain the mass spectrum for bosonic strings. Finally, we discuss the current situation of string theory, the problems it has solved and the ones it intends to solve in the future.

Resumen

En la presente memoria analizamos los principales aspectos relacionados con la aparición y desarrollo de la teoría de cuerdas bosónicas desde un punto de vista introductorio y con los conocimientos obtenidos durante el grado. Primero presentaremos las herramientas que utiliza la física moderna para la resolución de problemas, el Prinicipio de mínima acción, y trabajaremos algunos ejemplos como el de la acción de la partícula libre con el objetivo de asentar los conocimientos para la resolución del movimiento de la partícula relativista, para posteriormente, estudiar el movimiento de la cuerda relativista. Llevaremos a cabo un breve repaso por los acontecimientos que dieron lugar a esta teoría de cuerdas bosónicas así como otras teorías que surgieron con el objetivo de unificar las cuatro fuerzas: la fuerza de la gravedad, la fuerza electromagnética, la fuerza débil y la fuerza fuerte. Continuamos el estudio buscando simetrías y cantidades conservadas que reducirán notablemente la complejidad del problema que nos ocupa. Llevaremos a cabo dos tipos de cuantización en nuestra teoría: la cuantización canónica y la del cono de luz, obtendremos el espectro de masas para cuerdas bosónicas. Finalmente, discutimos la situación de la teoría de cuerdas actualmente, los problemas que ha resuelto y los que pretende resolver en el futuro.

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Chapter 1

Principle of Least Action

En la física moderna, el procedimiento habitual al tratar de resolver un problema es desarrollar el Principio de Mínima Acción para dicho problema, es decir, minimizar la acción para obtener posteriormente las ecuaciones de Euler-Lagrange que darán la dinámica del sistema. Por ello, en primer lugar determinaremos, entre algunos ejemplos, la acción de una partícula libre relativista, las ecuaciones de Maxwell, las ecuaciones de la relatividad general de Einstein y, por último, la ecuación de Schrödinger.

First, we define the *Principle of Least Action.* This principle establishes that for each mechanical system there is a certain integral S, called *the action*, which takes its minimum value for the real movement, so that its variation, δS, is equal to zero[1].

1.1 Action for a free relativistic particle

Let us determine the action integral corresponding to a free material particle, i.e., to a particle not subjected to any external force. Let us observe for this that the integral cannot depend on the chosen reference system, that is, it must be invariant under Lorentz transformations. Hence it follows that it must depend on a scalar. Furthermore, it is clear that the integrand must be a first order differential, but the only scalar of this type that can be constructed for a free particle is the interval ds or $\alpha \, ds$, where α is a certain constant.

Therefore, for a relativistic free particle, the action must be of the form

$$S = -\alpha \int_a^b ds,$$

where $\int_a^b$ is an integral along the world line of the particle between the two events that represent its arrival at the initial position and at the final position at given instants t_1 and t_2, that is, between two given points of the universe; α is a certain constant that characterizes the particle and must be positive.

The integral, $\int_a^b ds$, reaches its maximum value along a world line, so by integrating along a curved world line, we can make the integral as small as we like. The integral, $\int_a^b ds$, taken with a positive sign cannot therefore have a minimum. On the other hand, if we take it with the opposite sign, it

[1]The Principle of Least Action states that the integral must be a minimum only for infinitesimal lengths of the integration path. For paths of any length, we can only say that S must have an extremum, not necessarily a minimum, [8].

evidently reaches its minimum value along a straight world line.

The action can be represented as an integral with respect to time

$$S = \int_{t_1}^{t_2} L dt,$$

where the coefficient L represents the Lagrange function of the given mechanical system.

Following the expression

$$dt' = \frac{ds}{c} = dt\sqrt{1 - \frac{v^2}{c^2}},$$

we get that

$$S = -\int_{t_1}^{t_2} \alpha c \sqrt{1 - \frac{v^2}{c^2}} dt,$$

where v is the velocity of the material particle. Therefore, the Lagrange function of the particle is

$$L = \alpha c \sqrt{1 - \frac{v^2}{c^2}}.$$

In classical mechanics each particle is characterized by its mass m. We will determine the relationship between α and m. This relation arise from the condition that, for $c \to \infty$, the expression obtained for L must be reduced to the classical expression $L = mv^2/2$. To carry out this step, we will expand L in series of powers of v/c. Ignoring higher-order terms, we find

$$L = -\alpha c \sqrt{1 - \frac{v^2}{c^2}} \simeq -\alpha c + \frac{\alpha v^2}{2c}.$$

The constant terms in the Lagrange function do not appear in the equations of motion. Ignoring the constant αc, we obtain $L = \frac{\alpha v^2}{2c}$, while in classical mechanics it is $L = \frac{mv^2}{2}$. Therefore, $\alpha = mc$.

To sum up, the action for a free material point is

$$S = -mc \int_a^b ds \tag{1.1}$$

and the Lagrange function is written

$$L = -mc^2 \sqrt{1 - \frac{v^2}{c^2}}.$$

1.2 Quadripotential of a field

The action corresponding to a particle moving in a electromagnetic field is given by the action for a free particle and a term that describes the interaction of the particles with the field. This last term must contain magnitudes that characterize the particle and magnitudes that characterize the field.

The properties of a particle with respect to its interaction with the electromagnetic field turn out to be determined by a single parameter, the charge of the particle, which can be a positive, negative or zero quantity[2]. The properties of the field, on the other hand, are characterized by a quadrivector

[2]Note that we are considering a classical theory and not a quantum one, and therefore, effects linked to the spin of the particles are not taken into account.

A_i, the quadripotential, whose components are functions of the coordinates and the time. These magnitudes appear in the action function as a term of the form

$$-\frac{e}{c}\int_a^b A_i dx^i, \tag{1.2}$$

where the functions A_i are fixed at points on the world line of the particle. The factor $1/c$ has been introduced for convenience.

Therefore, the action function corresponding to a charge in an electromagnetic field is

$$S = \int_a^b \left(-mc\, ds - \frac{e}{c} A_i dx^i\right).$$

The three spatial components of the quadrivector A_i form a three-dimensional vector $\mathbf{A}$ called the potential vector of the field. We will write the time component of the quadrivector A_i in the form $A_0 = \phi$. The quantity ϕ is called the scalar potential of the field. We have, $A_i = (\phi, \mathbf{A})$.

The action integral can, therefore, be written in the form

$$S = \int_a^b \left(-mc\, ds + \frac{e}{c}\mathbf{A} d\mathbf{r} - e\phi dt\right),$$

or, introducing the velocity of the particle $\mathbf{v} = \frac{d\mathbf{r}}{dt}$ and taking t as a variable of integration,

$$S = \int_{t_1}^{t_2} \left(-mc^2\sqrt{1-\frac{v^2}{c^2}} + \frac{e}{c}\mathbf{A}\mathbf{v} - e\phi\right) dt.$$

The integrand is precisely the Lagrange function of a charge in an electromagnetic field

$$L = -mc^2\sqrt{1-\frac{v^2}{c^2}} + \frac{e}{c}\mathbf{A}\mathbf{v} - e\phi.$$

This expression differs from the Lagrange function of a free particle in the terms $\frac{e}{c}\mathbf{A}\mathbf{v} - e\phi$ that describe the interaction of the charge with the field.

1.3 The action function for the electromagnetic field

The action function S for the electromagnetic field and their particles have three contributions

$$S = S_f + S_m + S_{mf},$$

where S_m is the part of the action that depends only on the properties of the particles, i.e., it is the action that corresponds to free particles. The action for a free particle is given by (1.1). For more than one particle, the total action is given by the sum of the actions corresponding to each of the particles,

$$S_m = -\sum mc \int ds.$$

The last contribution, S_{mf}, depends on the interaction between the particles and the field. For a system of particles we have the sum of (1.2),

$$S_{mf} = -\sum \frac{e}{c}\int A_k dx^k.$$

In each term of this sum, A_k is the potential of the field at the point in space and time where the corresponding particle is found. The sum $S_m + S_{mf}$ is the action of a charge on a field.

Last, S_f only depends on the properties of the field itself, i.e., S_f is the action that corresponds to a field when there are no charges. Until now, since we were only interested in the movement of charges in a given magnetic field, the magnitude S_f, which does not depend on the particles, had not to be taken into account, since this term cannot affect the movement of the particles. However, this term becomes necessary when trying to find equations to determine the field itself.

The action for the field, according to [8], is of the form

$$S_f = -\frac{1}{16\pi c}\int F_{ik}F^{ik}d\Omega, \quad \Omega = c\,dt\,dx\,dy\,dz,$$

where

$$F_{ik} = \frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k}.$$

Three-dimensionally is

$$S_f = \frac{1}{8\pi}\int (E^2 - H^2)dV dt.$$

This expression is obtained more detailed in Appendix A.1.

The action corresponding to the system formed by the field and the particles is

$$S = -\sum\int mcds - \sum\int \frac{e}{c}A_k dx^k - \frac{1}{16\pi c}\int F_{ik}F^{ik}d\Omega.$$

Note that the charges are not assumed to be small, as was done in deriving the equations of motion for a charge in a given field. Therefore, A_{ik} and F_{ik} correspond to the total field, that is, to the external field and the field produced by the particles themselves; A_k and F_{ik} now depend on the positions and velocities of the charges.

1.4 Motion of a particle in a gravitational field

The motion of a particle in a gravitational field has to be determined according to the Principle of least action in the same way. In a gravitational field the particle moves in such a way that its point of the universe runs through an extreme, as it is often said, a geodesic of space with four dimensions x_0, x_1, x_2, x_3. However, since space-time is not Galilean in the presence of the gravitational field, this line will not be a straight line and the actual motion of the particle in space is neither rectilinear nor uniform, [8].

We proceed to derive the equation of motion:

$$\frac{d^2x}{ds^2} + \Gamma^i_{kl}\frac{dx^k}{ds}\frac{dx^l}{ds} = 0.$$

From the principle of least action, $\delta S = -mc\,\delta\int ds = 0$, in a general curvilinear space-time characterized by the metric tensor g_{ik},

$$\delta ds^2 = 2ds\delta ds = \delta(g_{ik}dx^i dx^k) = dx^i dx^k\frac{\partial g_{ik}}{\partial x^l}\delta x^l + 2g_{ik}dx^i d\delta x^k.$$

Thus

$$\delta S = -mc\int\left\{\frac{1}{2}\frac{dx^i}{ds}\frac{dx^k}{ds}\frac{\partial g_{ik}}{\partial x^l}\delta x^l + g_{ik}\frac{dx^i}{ds}\frac{d\delta x^k}{ds}\right\}ds$$
$$= -mc\int\left\{\frac{1}{2}\frac{dx^i}{ds}\frac{dx^k}{ds}\frac{\partial g_{ik}}{\partial x^l}\delta x^l - \frac{d}{ds}\left(g_{ik}\frac{dx^i}{ds}\right)\delta x^k\right\}ds.$$

In the second term of the integral, we substitute the index k for the index l. Setting equal to zero the coefficient of arbitrary variation δx^l,

$$\frac{1}{2}u^iu^k\frac{\partial g_{ik}}{\partial x^l} - \frac{d}{ds}(g_{il}u^i) = \frac{1}{2}u^iu^k\frac{\partial g_{ik}}{\partial x^l} - g_{il}\frac{du^i}{ds} - u^iu^k\frac{\partial g_{il}}{\partial x^k} = 0.$$

Noting that the third term can be written in the form

$$-\frac{1}{2}u^iu^k\left(\frac{\partial g_{il}}{\partial x^k} + \frac{\partial g_{kl}}{\partial x^i}\right),$$

and introducing the Christoffel symbols,

$$\Gamma_{l,ik} = \Gamma_{i,lk} = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^l} + \frac{\partial g_{il}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^i}\right),$$

we get

$$g_{il}\frac{du^i}{ds} + \Gamma_{l,ik}u^iu^k = 0,$$

or

$$\frac{d^2x^i}{ds^2} + \Gamma^i{}_{kl}\frac{dx^2}{ds}\frac{dx^l}{ds} = 0.$$

1.5 The action function of the gravitational field

To find the equations that determine the gravitational field it is necessary, in the first place, to determine the action S_g of the field. The equations we are looking for are obtained by varying the sum of the actions of the field and the material particles.

As in the case of the electromagnetic field, the action S_g must be expressed in the form of a scalar integral

$$\int G\sqrt{-g}\,d\Omega$$

extended to all space and to all values of the time coordinate between two given values. This calculation is carried out in A.2.
We get

$$\delta S_g = -\frac{c^3}{16\pi k}\delta\int G\sqrt{-g}\,d\Omega = -\frac{c^3}{16\pi k}\delta\int R\sqrt{-g}\,d\Omega,$$

where k is a new universal positive constant and the magnitude G is

$$G = g^{ik}\left(\Gamma^m_{il}\Gamma^l_{km} - \Gamma^l_{ik}\Gamma^m_{lm}\right).$$

The components of the metric tensor are the quantities that determine the gravitational field. Therefore, the Principle of least action that corresponds to it, the quantities that are varied are the g_{ik}.

1.6 Gravitational field equations

Now, we proceed to deduce the equations of the gravitational field. These equations are obtained from the Principle of least action $\delta(S_m+S_g)=0$, where S_g and S_m are the actions of the gravitational field and matter, respectively.

For the variation of the action corresponding to matter we can write

$$\delta S_m = \frac{1}{2c}\int T_{ik}\delta g^{ik}\sqrt{-g}d\Omega,$$

where T_{ik} is the energy-momentum tensor of matter.

Following the Principle of least action, $\delta S_m + \delta S_g = 0$, we get the expressions,

$$R_{ik} - \frac{1}{2}g_{ik}R = \frac{8\pi k}{c^4}T_{ik},$$

or in mixed components

$$R_i^k - \frac{1}{2}\delta^k_{\ i}R = \frac{8\pi k}{c^4}T^k_{\ i}.$$

These are precisely the gravitational field equations, the fundamental equations of the general theory of relativity, also known as the *Einstein equations*.

Contracting the last expression with respect to the indices i and k, we find:

$$R = -\frac{8\pi k}{c^4}T, \quad T = T_i^i.$$

Therefore, the field equations are

$$R_{ik} = \frac{8\pi k}{c^4}(T_{ik} - \frac{1}{2}g_{ik}T).$$

1.7 Schrödinger equation

In this section, we aim to find the Schrodinger equation for a particle in an external potential V. According to [4], we can write, for the Schrödinger equation as

$$L = \int d^3r\mathscr{L},$$

where the function $\mathscr{L}$ is called the *Lagrangian density*, is a real function of the fields ψ and ψ^*, time-derivatives $\dot{\psi}$ and $\dot{\psi}^*$ and also of the partial derivatives ($\partial_i\psi$ and $\partial_i\psi^*$, with $\partial_i = \partial_x, \partial_y, \partial_z$)

$$\mathscr{L} = \frac{i\hbar}{2}(\psi^*\dot{\psi} - \dot{\psi}^*\psi) - \frac{\hbar^2}{2m}\nabla\psi^*\nabla\psi - V(r)\psi^*\psi,$$

where $\psi(\mathbf{r})$ is a classical complex field and $\mathbf{r}$ can take all the possible values.

Note that the action S is the time integral of the Lagrangian. Because of how the Lagrangian density has been postulated it can be written as

$$S = \int_{t_1}^{t_2} dt \int d^3r\mathscr{L}(\psi, \dot{\psi}, \partial_i\psi).$$

The principle of least action does not change; S is an extremum when ψ corresponds to the movement of the field between times t_1 and t_2. So, we study now how S changes when the field varies by an amount $\delta\psi$ with respect to the path for which S is extreme. Stating that S is an extremum gives the Lagrange equations, which can be written in the form

$$\frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{\psi}} = \frac{\partial \mathscr{L}}{\partial \psi} - \sum_{i=x,y,z} \frac{\partial \mathscr{L}}{\partial(\partial_i \psi)}.$$

Now we make the generalization for the complex field case. Let be a Lagrangian L and a Lagrangian density $\mathscr{L}$ dependent on the complex fields ψ and $\dot{\psi}$. Since L must be real, $\mathscr{L}$ must depend on ψ^* and $\dot{\psi}^*$. Thus,

$$L = \int d^3k \mathscr{L}(\psi, \dot{\psi}, \partial_i\psi, \psi^*, \dot{\psi}^*, \partial_i\psi^*).$$

Following the same procedure that was followed for the previous case, we establish two Lagrange equations relative to ψ and ψ^*

$$\begin{aligned} \frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{\psi}} &= \frac{\partial \mathscr{L}}{\partial \psi} - \sum_i \partial_i \frac{\partial \mathscr{L}}{\partial(\partial_i \psi)} \\ \frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{\psi}^*} &= \frac{\partial \mathscr{L}}{\partial \psi^*} - \sum_i \partial_i \frac{\partial \mathscr{L}}{\partial(\partial_i \psi^*)}. \end{aligned} \tag{1.3}$$

Let's get the Schrödinger equation of the lagrangian $\mathscr{L}$. For this, we calculate the partial derivatives of the Lagrangian density that appear in the Lagrange equation

$$\frac{\partial \mathscr{L}}{\partial \dot{\psi}^*} = -\frac{i\hbar}{2}\psi, \quad \frac{\partial \mathscr{L}}{\partial \psi^*} = \frac{i\hbar}{2}\dot{\psi} - V(r)\psi$$
$$\frac{\partial \mathscr{L}}{\partial(\partial_j\psi)^*} = -\frac{\hbar^2}{2m}\partial_j\psi.$$

Substituting these expressions in (1.3), we get

$$-\frac{i\hbar}{2}\dot{\psi} = \frac{i\hbar}{2}\dot{\psi} - V(r)\psi - \sum \partial_i \left(\frac{-\hbar^2}{2m}\partial_i\psi\right)$$

which lead us to the Schrödinger equation of a particle of mass m in a potential V(r),

$$i\hbar\dot{\psi} - V(r)\psi + \frac{\hbar^2}{2m}\Delta\psi = 0.$$

Calculating $\partial\mathscr{L}/\partial\dot{\psi}$ leads us to the complex conjugate equation.

Chapter 2

An introduction to the Bosonic String Theory

En este capítulo desarrollaremos los acontecimientos que llevaron a la aparición y desarrollo de la Teoría de Cuerdas, veremos que aparece como solución a la interacción fuerte. Para entender por qué es tan importante hoy en día, veremos también los problemas que trata de resolver. Para ello introduciremos las Supercuerdas, la Teoría M, el Modelo Estándar y las Teorías de Gran Unificación. Al final de este capítulo desarrollaremos el Principio de mínima acción para una cuerda bosónica relativista, obteniendo así las acciones de Nambu-Goto y Polyakov.

2.1 Motivation for String Theory

Nowadays it is understood that physics can be described by four forces:

- ✧ *Gravity*: it was first described by Isaac Newton but was Albert Einstein who reformulated it in the theory of General Relativity. This is a classical theory of gravitation and has not been formulated as a quantum theory yet.
- ✧ *Electromagnetic force*: this force is well described by Maxwell's equation. Electromagnetism is formulated as a classical theory of electromagnetic fields. However, Maxwell theory is fully consistent with special relativity.
- ✧ *Weak force*: it is responsible for beta decay. The strength of this force is measured by the Fermi constant. Weak interactions are much weaker than electromagnetic interactions.
- ✧ *Strong force*: this force binds quarks, the constituent of neutrons, protons, pions and many other subnuclear particles.

Sting Theory pretends to be a theory that unifies these four forces in a single quantum mechanical framework. Currently there are quantization methods that allow to turn a classical theory into a quantum theory, i.e., a theory that can be calculated using the principles of quantum mechanics.

The quantum version of classical electrodynamics is *Quantum Electrodynamics* (QED). The theory of weak interactions is also a quantum field theory. The unified theory is the quantum electroweak theory. In the case of the strong color force the resulting theory is the *Quantum Chromodynamics* (QCD). This two theories together (electroweak and QCD) form the *Standard Model* of particles physics. However there is not real unification between this two forces even though some particles feel both types of forces. The Standard Model has two significant shortcomings: it does not include gravity and has 23 parameters that can not be calculated within its framework.

It is believed that the Standard Model is only a step towards the formulation of a complete theory of physics. It is also suspected that some unification of weak and strong forces into a *Grand Unified Theory* (GUT), will prove to be correct. With this unification, we obtain new particles not detected yet.

Another possibility is Supersymmetry, a more complete version of the Standard Model. A symmetry that relates bosons to fermions. This symmetry unifies matter and forces since matter particles are fermions and force carriers are bosons. However, since in the Standard Model this property does not appear, this supersymmetry must be spontaneously broken in nature if it exists.

On the other hand, the effects of the gravitational force are important on a macroscopic scale. Therefore gravity must be included into the particle physics framework. Nevertheless, there is a major problem when attempting to incorporate gravitational physics into the Standard Model. This is because the Standard Model is a quantum theory while Einstein's General Relativity is a classical theory. Gravity must be turned into a quantum theory. However many difficulties are encountered in the quantization procedure. It is observed that gravity goes wrong at scales shorter than the Planck length[1] because is not renormalizable, the resulting quantum theory is impractical or totally unpredictable, [14]. For processes at energies of the order of the Planck mass it is found that it is necessary more parameters to absorb the infinities that occur in the theory. In fact, we need and infinite number of parameters to renormalize the theory at these scales, [15]. Also, for each parameter that gets renormalized we must make a measurement. Therefore, it is necessary to formulate another theory.

2.2 A brief introduction to String Theory

String theory is a quantum theory that includes gravitation. For that reason, it is an excellent candidate for an unified theory of all four forces and all forms of matter in nature. So, it has become a focal point for physical and philosophical discussions.

String theory emerged in the late 1960s, as a attempt to describe the physics of strong interactions outside the framework of quantum field theory, in the called Dual Resonance Model. Later, around the mid-1970s it was reinterpreted as a quantum theory of gravity unified with the other forces, and its successive developments until reaching the superstring revolution in 1984[2].

In this theory each particle is identified as a particular vibrational mode of an elementary microscopic string. The modes of vibration of this fundamental string can be recognized as the different particles known. There are two types of string, open and closed strings. The first ones have two endpoints while the closed strings have no endpoints. We are going to study both types, with the difference of the boundary conditions.

Another division in string theories is between bosonic string and superstring theories. Bosonic strings live in 26 dimensions and all the modes of vibration represent bosons. These kind of theories are not realistic since the lack of fermions. However they are easier to work with than the Superstrings, which include fermions. Most of the important concepts in String Theory can be explained

[1] In 1899, Max Planck proposed a set of units to simplify the expression of physics laws. The Planck mass, length, and time are equivalent ways to describe the Planck scale. They are constructed from the three fundamental constants of physics: the speed of light in a vacuum, c, the gravitational constant, G,and the quantum of angular momentum, $\hbar$.

[2] More information of the beginnings of the String theory in [2].

with bosonic strings. On the other hand, Superstrings live in a ten dimensional spacetime and boson and fermions are related by supersymmetry. The Superstring theories can be formulated in more than these ten dimensions, not in less, although they are no longer consistent. They are called the *critical dimensions* of the theory. We will calculate the critical dimension of bosonic string theory in Chapter 5.

Due to the fact that Superstrings live in a ten dimensional spacetime, one dimension of time and nine of space, it is necessary that these extra spatial dimensions are small enough to not have been detected experimentally. If the extra dimensions are the size of the Planck length, l_p, they will remain beyond direct detection. Indeed, $l_p \sim 10^{-33}$cm, this distance is many orders of magnitude smaller than the smallest distance that has been explored with particle accelerators, 10^{-16}cm. Moreover, our three dimensional space emerges as a hypersurface embedded inside the nine-dimensional space. This hipersurface is called a *D-brane*. In this model, the gauge bosons and matter particles emerges from vibrations of open strings. The D-branes hold the endpoints of the open strings. The lowest vibrational modes could represent the particles in the Standard Model.

By the mid 1980s five other consistent string theories were established, which include fermions: Type I, Type IIA, Type IIB, Heterotic SO(32), Heterotic $E_8 \times E_8$. It turns out that these all five theories are assumed to be parts of a still unknown theory, which relates them through *dualities*. Moreover, another theory was discovered by taking a certain strong coupling limit of one of the Superstrings. This theory is eleven-dimensional and has been dubbed *M-theory*. It has now become clear that the five Superstrings are only facets of different limits of a single unique theory, the M-Theory[3].

Finally, it should be said that there has been no experimental verification of String theory. It is still at an early stage of development. Now it is time to ask if the Standard Model emerges from String Theory. It must be so since String Theory emerges as a unification theory of all interactions. However, it has not yet been possible to prove this in detail.

2.3 The Bosonic String Action

In spacetime, as a particle traces a line, a string traces a surface. The line traced out by the particle is called the *worldline* while the two dimensional surface traced out by a string in spacetime is called the *worldsheet*, as we can see in Figure 2.1. A closed string will trace out a tube and an open string will trace out a strip. Following with the analogy of the particle, the *proper length* of the worldline is defined as the proper time multiplied by c. This is the Lorentz invariant. For strings, it's defined the Lorentz invariant *proper area* of a worldsheet. The relativistic action is proportional to this proper area and it's called the Nambu-Goto action.

On the other hand, a $p-brane$ is defined as a p dimensional object moving through a D dimensional spacetime, where $D \geq p$. A $0-brane$ is a point particle, a $1-brane$ is a string and a $2-brane$ is a membrane, for example.

We want to generalize the notion of an action for a point particle, 0-brane, to the case of a p-brane. The generalization of the action for a point particle

$$S_0 = -m \int dS$$

[3]The term M-theory was introduced to refer the quantum theory in eleven dimensions whose leading low-energy effective action is eleven dimensional supergravity. More information on page 238 in [15].

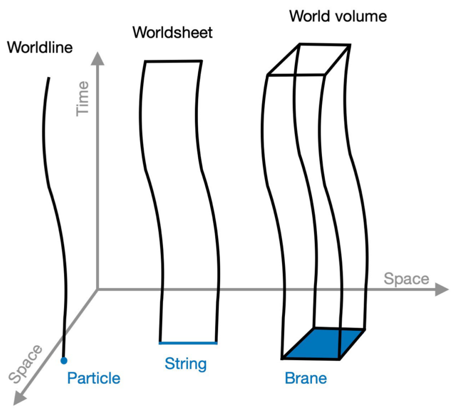


Figure 2.1: The traces of different objects according to their dimension.

to a p-brane in a D dimensional spacetime ($D \geq p$) is

$$S_p = -T_p \int d\mu_p,$$

where T_p is the tension of the p-brane (has units of mass/volume) and $d\mu_p$ is the $(p+1)$ dimensional volume element:

$$d\mu_p = \sqrt{-det(G_{\alpha\beta}(X))d^{p+1}\sigma}.$$

The induced metric of the worldsurface is $G_{\alpha\beta}$ and is given by:

$$G_{\alpha\beta} = \frac{\partial X^\mu}{\partial \sigma^\alpha}\frac{\partial X^\nu}{\partial \sigma^\beta}g_{\mu\nu}(X); \quad \alpha, \beta = 0, 1, \dots p,$$

where $\sigma^0 \equiv \tau$ and $\sigma^1, \sigma^2, \dots \sigma^p$ are the p spacelike coordinates for the $p+1$ dimensional world surface mapped out by the p-brane. This metric, $G_{\alpha\beta}$, arises from the embedding of the string into the D dimensional background spacetime. This embedding of the string is given by the functions, or fields, $X^\mu(\tau, \sigma)$, as we can see in Figure 2.2. From now on we will consider 1-branes.

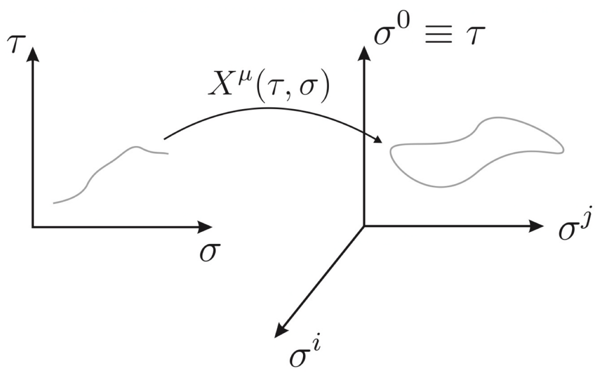


Figure 2.2: Embedding of a string into a background spacetime, [15].

The fields $X^\mu(\tau, \sigma)$, which are parameterized by the worldsheet coordinates gives how the string propagates and oscillates through the background spacetime. The propagation of the particles throught spacetime is defined with the fields $X^\mu(\tau)$, parameterized by the worldline coordinate τ. Note that if σ is periodic, the embedding gives a closed string in the spacetime.

Now, assuming that the background space time is Minkowskian:

$$G_{00} = \frac{\partial X^\mu}{\partial \tau}\frac{\partial X^\nu}{\partial \tau}\eta_{\mu\nu} \equiv \dot{X}^2$$
$$G_{11} = \frac{\partial X^\mu}{\partial \sigma}\frac{\partial X^\nu}{\partial \sigma}\eta_{\mu\nu} \equiv X'^2$$
$$G_{10} = G_{01} = \frac{\partial X^\mu}{\partial \tau}\frac{\partial X^\nu}{\partial \sigma}\eta_{\mu\nu} \equiv \dot{X}X'$$

This gives us

$$G_{\alpha\beta} = \begin{pmatrix} \dot{X}^2 & \dot{X}X' \\ \dot{X}X' & X'^2 \end{pmatrix}$$

and

$$det(G_{\alpha\beta}) = \dot{X}^2 X'^2 - (\dot{X}X')^2.$$

So the previous action reduces to:

$$S_{NG} = -T\int d\tau d\sigma \sqrt{(\dot{X}X')^2 - (\dot{X}^2)(X'^2)}$$

this is know as the *Nambu-Goto action.* This action can be interpreted as providing the area of the worldsheet mapped out by the string in spacetime.

In order to get rid of the square root an auxiliary field can be introduced $h_{\alpha\beta}(\tau,\sigma)$, this is another metric living on the worldsheet. The resulting action is given by:

$$S_\sigma = -\frac{T}{2}\int d\tau d\sigma \sqrt{-h}\, h^{\alpha\beta}\frac{\partial X^\mu}{\partial \alpha}\frac{\partial X^\nu}{\partial \beta}g_{\mu\nu}, \quad \text{where } h \equiv det(h_{\alpha\beta}),$$

and is called the *Polyakov action* or the string sigma-model. Note that this action holds for a general background.

Theorem 2.3.1. *The Polyakov action S_σ is equivalent to the Nambu-Goto action S_{NG}*[4]*.*

[4]Proof on page 22 in [15].

Chapter 3

Symmetries and Field Equations of the Bosonic String

En física las cantidades conservadas, debido a las propiedades de simetría, vía el teorema de Noether, reducen la complejidad del problema que nos ocupa. Nuestro objetivo a continuación es buscar las simetrías globales y locales de nuestra teoría. Luego, con esta información obtendremos las ecuaciones de campo para la acción de Polyakov, previamente obtenidas, para la cuerda bosónica cerrada y abierta, aplicando las condiciones de contorno. Después, haremos una expansión en términos de los modos normales.

3.1 Global Symmetries of the Bosonic String Theory

In this chapter we will discuss the invariance of this theory under the Poincaré group, but first we will define global and local transformations and their consequences. We begin by describing a *global transformation* in some spacetime, this is a transformation whose parameter/s do not depend on where in the spacetime the transformation is being performed. A *local transformation* in some spacetime does, however, depend on where the transformation is begin performed in the spacetime. Note that invariance of a theory under global transformations leads to conserved currents and charges via Noether's theorem[1], while invariance under local transformations, or gauge transformations, is a sign of redundant degrees of freedom in the theory. First, we will discuss the Poincaré transformations which are global transformations.

Poincaré Transformations

These are global transformations of the form

$$\begin{aligned} \delta X^{\mu}(\tau,\sigma) &= a^{\mu}{}_{\nu} X^{\nu}(\tau,\sigma) + b^{\mu}, \\ \delta h_{\alpha\beta}(\tau,\sigma) &= 0, \end{aligned} \tag{3.1}$$

where fields $X^{\mu}(\tau,\sigma)$ are defined on the worldsheet and $a^{\mu}{}_{\nu}$ is antisymmetric, $a_{\mu\nu} = -a_{\nu\mu}$ and correspond to spatial rotations and boost. We can see that the transformations do not depend on the worldsheet coordinates, σ and τ. Now, we will show that the $a^{\mu}{}_{\nu}$ generate the Lorentz transformations.

Its known that the speed of light is the same in all inertial frames. Thus, if (t, X_i) is the spacetime position of a light ray in one inertial frame and (t', X'_i) in another, then the relation between the two

[1] Noether's theorem states that if the Lagrangian of a system has a continuous symmetry, then there exists a conserved quantity by the system, and vice versa.

is given by

$$\eta_{\mu\nu}X^\mu X^\nu = -c^2t^2 + X_iX^i = -c^2t'^2 + X'_iX'^i = \eta_{\mu\nu}X'^\mu X'^\nu$$

The linear transformations, denoted by Λ, which preserve this relation are called Lorentz transformations

$$X'^\mu = \Lambda^\mu{}_\nu X^\nu.$$

Infinitesimally, the above transformation is given by

$$\Lambda^\mu{}_\nu = \delta^\nu_\mu + a^\mu{}_\nu \tag{3.2}$$

The infinitesimal form of the Lorentz transformation says that

$$X'^\mu = X^\mu + a^\mu{}_\nu X^\nu,$$

which implies that the variation of X, under the Lorentz transformation, is given by

$$\delta X^\mu = a^\mu{}_\nu X^\nu.$$

Now, if we impose that under a Lorentz transformation the spacetime interval vanishes

$$\delta\,|_{L.T.}\,(\eta_{\mu\nu}X^\mu X^\nu) = 0,$$

then we have that

$$\delta\,|_{L.T.}\,(\eta_{\mu\nu}X^\mu X^\nu) = 2\eta_{\mu\nu}(\delta X^\mu)X^\nu = 2\eta_{\mu\nu}(a^\mu{}_k X^k)X^\nu = 2a_{k\nu}X^kX^\nu = 0.$$

The most general solution to this is to have $a_{\mu\nu} = -a_{\nu\mu}$, and thus we have that $a^\mu{}_\nu$ defined in (3.1) is equivalent to (3.2).

The next step is to ask if our Polyakov action is invariant under the Poincaré Transformation[2]. In order to see this, we consider

$$\delta S_\sigma = -T\int d\tau d\sigma\sqrt{-h}\,h^{\alpha\beta}\partial_\alpha(\delta X^\mu)\partial_\beta X^\nu g_{\mu\nu},$$

where we have used the fact that $h^{\alpha\beta}$ is invariant under the transformation, $\delta h^{\alpha\beta} = 0$, and symmetry of the metric in its indices. Plugging the infinitesimal Poincaré transformation (that includes translations), $\delta X^\mu = a^\mu{}_k X^k + b^\mu$, we get that

$$\delta S_\sigma = -T\int d\tau d\sigma\sqrt{-h}\,h^{\alpha\beta}\partial_\alpha(a^\mu{}_k X^k + b^\mu)\partial_\beta X^\nu g_{\mu\nu}.$$

This can be simplified by noting that $a^\mu{}_k$ and b^k are spacetime independent and thus we can drop the b^k term and pull the $a^\mu{}_k$ out of the parentesis

$$\delta S_\sigma = -T\int d\tau d\sigma\sqrt{-h}\,h^{\alpha\beta}a^\mu{}_k\partial_\alpha X^k\partial_\beta X^\nu g_{\mu\nu}.$$

Now, using the metric $g_{\mu\nu}$ to lower the upper index on $a_m u_k$,

$$\delta S_\sigma = -T\int d\tau d\sigma\sqrt{-h}\ \underbrace{[a_{\nu k}]}_{\text{antisym}}\ \underbrace{[h^{\alpha\beta}\partial_\alpha X^k\partial_\beta X^\nu]}_{\text{symmetric}}.$$

This is the the integral of the product of an antisymmetric part with a symmetric part, and thus equal to zero. We have proved that the variation in the Polyakov action is zero, $\delta S_\sigma = 0$, under a Poincaré transformation. So, this action is invariant under these transformations.

[2]This means to be invariant under space and time translations and under Lorentz transformations.

3.2 Local Symmetries of the Bosonic String Theory

Local symmetries, also called *Gauge symmetries*, lead to a reduction in the number of degrees of freedom. The next question to ask is what local symmetries does our bosonic string theory has. These are transformations whose parameters depend on the worldsheet coordinates, we will see two of them:

1. Reparametrization invariance (diffeomorphisms[3]): This is a local symmetry for the worldsheet. The Polyakov action is invariant under the change of the parameter σ to $\sigma' = f(\sigma)$ since the fields $X^\mu(\tau, \sigma)$ transform as scalars while the auxiliary field $h^{\alpha\beta}(\tau, \sigma)$ transforms as a 2-tensor. Thus, our bosonic string is invariant under reparametrizations.

2. Weyl Symmetry: these are transformations that change the scale of the metric,
$$h_{\alpha\beta}(\tau, \sigma) \to h'_{\alpha\beta}(\tau, \sigma) = e^{2\phi(\sigma)} h_{\alpha\beta}. \tag{3.3}$$
Under a Weyl transformation the variation of $X^\mu(\tau, \sigma)$ is zero, $\delta X^\mu(\tau, \sigma) = 0$. This is a local transformation since the parameter $\phi(\sigma)$ depends on the worldsheet coordinates. Since our theory is invariant under Weyl transformations this implies that the stress-energy tensor associated with this theory is traceless, $h^{\alpha\beta} T_{\alpha\beta} = 0$. This is proved in Appendix B.1 as well as the invariance of our theory under Weyl transformations.

Since this theory has local, or gauge, symmetries, the theory has a redundancy in its degrees of freedom, we can use these symmetries to cope with these redundancies, a procedure which is known as gauge fixing. In order to simplify the bosonic string theory, we will show that if our theory is invariant under these two particular transformations, we can fix a gauge so that our metric, $h_{\alpha\beta}$, becomes flat.

First, note that since the metric is symmetric:
$$h_{\alpha\beta} = \begin{pmatrix} h_{00} & h_{01} \\ h_{10} & h_{11} \end{pmatrix},$$
there are only three independent components, $h_{00}(X)$, $h_{11}(X)$, and $h_{10}(X) = h_{01}(X)$. Now, a diffeomorphism, or reparametrization, allows us to change two of the independent components by using two coordinate transformations to fix $h_{10}(X) = 0 = h_{01}(X)$ and $h_{00}(X) = \pm h_{11}(X)$ (the $\pm$ depends on the signature of the metric).

Because our theory is diffeomorphism invariant, we can see that the two dimensional metric $h_{\alpha\beta}(X)$ is of the form $h(X)\eta_{\alpha\beta}$. Now, using a Weyl transformation, we get that $h_{\alpha\beta}(X) = \eta_{\alpha\beta}$. If our theory is invariant under diffeomorphisms and Weyl transformations[4], then the two-dimensional intrinsic metric, $h_{\alpha\beta}(X)$, can be gauged into the two-dimensional flat metric,
$$h_{\alpha\beta}(X) = \eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$
Note that this is only valid locally since we cannot, in general, extend $h_{\alpha\beta}(X)$ to the whole worldsheet, since gauge symmetries are local symmetries.

The Polyakov action, in terms of the gauge fixed flat metric, becomes:
$$S_\sigma = \frac{T}{2} \int d\tau d\sigma ((\dot{X})^2 - (X')^2),$$
where $\dot{X} \equiv dX^\mu/d\tau$ and $X' \equiv dX^\mu/d\sigma$.

[3] A *diffeomorphism* is an invertible function that maps one differentiable manifold to another such that the function and its inverse are differentiable. In this case, the transformations and their inverses are differentiable.

[4] Their combinations are called *conformal transformations*.

3.3 Field Equations for the Polyakov Action

Once we have seen the symmetries through the Euler-Lagrange equations of the string action, we obtain the equations of our field in particular. To do this, we make use of the Principle of least action and expand to first order, thereby we obtain our Euler-Lagrange equations.

First, we will assume that our worldsheet topology allows for the gauge fixed locally defined flat metric $h^{\alpha\beta}$ to be extended globally. The field equations for the fields $X^\mu(\tau,\sigma)$ on the worldsheet come from setting the variation of S_σ with respect to $X^\mu \to X^\mu + \delta X^\mu$ equal to zero. This leads to

$$\delta S_\sigma = \frac{T}{2}\int d\tau d\sigma (2\dot{X}\Delta\dot{X} - 2X'\delta X').$$

Integrating both terms by parts, we deduce the equations that give the dynamics

$$T\int d\tau d\sigma[(-\partial_\tau^2 + \partial_\sigma^2)X^\mu]\delta X^\mu + T\int d\sigma \dot{X}^\mu \delta X^\mu \mid_{\partial\tau}$$
$$- \left[T\int d\tau X'\delta X^\mu \mid_{\sigma=\pi} + T\int d\tau X'\delta X^\mu \mid_{\sigma=0}\right].$$

We set the variation of X at the boundary of τ to be zero, i.e. $\delta X^\mu \mid_{\partial\tau} = 0$, and are left with the field equations for $X^\mu(\tau,\sigma)$ for the Polyakov action,

$$(-\partial_\tau^2 + \partial_\sigma^2)X^\mu - T\int d\tau\, [X'\delta X^\mu \mid_{\sigma=\pi} + X'\delta X^\mu \mid_{\sigma=0}]$$

The σ boundary terms tell us what type of strings we have, either closed or open strings:

- **Closed Strings**: We take σ to have a periodic boundary condition,

$$X^\mu(\tau,\sigma+n) = X^\mu(\tau,\sigma), \tag{3.4}$$

which implies that the boundary terms appearing in the variation of S_σ vanish since if $X^\mu(\tau,\sigma+n) = X^\mu(\tau,\sigma)$ then $\delta X(\tau,\sigma=0) = \delta X(\tau,\sigma+n)$. Thus, we have the following field equations for the closed string

$$(\partial_\tau^2 - \partial_\sigma^2)X^\mu(\tau,\sigma) = 0, \tag{3.5}$$

The fields satisfy the wave equation in two dimensions with the boundary conditions (3.4).

- **Open Strings** (*Neumann Boundary Conditions*): We set the derivative of X^μ, by σ, at the σ boundary to vanish, i.e.

$$\partial_\sigma X^\mu(\tau,\sigma=0) = \partial_\sigma X^\mu(\tau,\sigma=n) = 0. \tag{3.6}$$

Under these boundary conditions the boundary terms over σ also vanish and the field equations become

$$(\partial_\tau^2 - \partial_\sigma^2)X^\mu(\tau,\sigma) = 0,$$

with the boundary conditions (3.6). Note that the Neumann boundary conditions preserve Poincaré invariance since

$$\partial_\sigma(X'^\mu) \mid_{\sigma=0,n} = \partial_\sigma(a^\mu{}_\nu X^\nu + b^\mu) \mid_{\sigma=0,n} = a^\mu{}_\nu \partial_\sigma X^\nu \mid_{\sigma=0,n} = 0.$$

- **Open Strings** (*Dirichlet Boundary Conditions*): We set the value of X^μ to a constant at the σ boundary,

$$X^\mu(\tau,\sigma=0) = X_0^\mu, \tag{3.7}$$
$$X^\mu(\tau,\sigma=n) = X_n^\mu \tag{3.8}$$

where X_0^μ and X_n^μ are constants. This also makes the σ boundary terms vanish and so the field equations are

$$(\partial_\tau^2 - \partial_\sigma^2) X^\mu(\tau, \sigma) = 0,$$

with the boundary conditions (3.7) and (3.8). The Dirichlet boundary conditions do not preserve Poincaré invariance, this does not invalidate the theory. We have that the extremes are in the branes, therefore we must include the dynamics of the brane

$$(X'^\mu)\mid_{\sigma=0,n} = (a^\mu{}_\nu X^\nu + b^\mu)\mid_{\sigma=0,n} = a^\mu{}_\nu X^\nu_{0,n} + b^\mu \neq X^\mu_{0,n}.$$

Thus, under a Poincaré transformation the ends of the string actually change.

Finally, note that the resulting field equations for the open and closed strings are equivalent, but with different boundary conditions. In addition, we must impose the field equations, which result from setting the variation of S_σ with respect to $h^{\alpha\beta}$ equals to zero. This field equations are given by

$$0 = T_{\alpha\beta} = \partial_\alpha X \cdot \partial_\beta X - \frac{1}{2} h_{\alpha\beta}\, h^{\gamma\delta} \partial_\gamma X \cdot \partial_\delta X,$$

and gauge fixing $h^{\alpha\beta}$ to be flat, we get that the fields equations transform into

$$0 = T_{00} = T_{11} = \frac{1}{2}(\dot{X}^2 + X'^2) \tag{3.9}$$

$$0 = T_{01} = T_{10} = \dot{X} \cdot X'. \tag{3.10}$$

We have seen that gauge symmetry gives redundant degrees of freedom that give rise to constraints. The expression (3.5) and the constraints (3.9) and (3.10) give the dynamics of the fields.

3.4 Expansion in normal modes

As in any classical string, the solution is expressed in normal modes. The oscillations of this string will represent particles. It will be a string that moves and also oscillates so, it will be expressed as the sum of a translation and an oscillation.

First, we will assume that we can extend the gauge fixed flat metric to a global flat metric, $h_{\alpha\beta} \to \eta_{\alpha\beta}$, on the worldsheet. Now, we will solve the system of equations by introducing light-cone coordinates for the worldsheet

$$\sigma^\pm = (\tau \pm \sigma),$$

which implies that $\tau = \frac{1}{2}(\sigma^+ + \sigma^-)$ and $\sigma = \frac{1}{2}(\sigma^+ - \sigma^-)$. The derivatives, in terms of light-cone coordinates, become

$$\partial_+ = \frac{1}{2}(\partial_\tau + \partial_\sigma),$$

$$\partial_- = \frac{1}{2}(\partial_\tau - \partial_\sigma),$$

and since the metric transforms as

$$\eta'_{\alpha'\beta'} = \frac{\partial \sigma^\gamma}{\partial \sigma^{\alpha'}} \frac{\partial \sigma^\delta}{\partial \sigma^{\beta'}} \eta_{\gamma\alpha}$$

the metric, in terms of light-cone coordinates, is given by

$$\eta_{\alpha\beta}|_{l-c\ c} = -\frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and so,

$$\eta^{\alpha\beta}|_{l-c\ c} = -2\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

In terms of the light-cone coordinates, the field equations (3.5) become

$$\partial_+\partial_- X^\mu = 0.$$

The most general solution to the field equations for $X^\mu(\sigma^+,\sigma^-)$ is given by a linear combination of two arbitrary functions whose arguments depend only on one of the light-cone coordinates, $X^\mu(\sigma^+,\sigma^-) = X^\mu_R(\sigma^-) + X^\mu_L(\sigma^+)$.

Now that we have the general form of the solution to the field equation in terms of the worldsheet light-cone coordinates, we want to map this solution to the usual worldsheet coordinates, τ and σ. Since $\sigma^- = \tau - \sigma$ and $\sigma^+ = \tau + \sigma$, the arbitrary functions $X^\mu_R(\sigma^-)$ and $X^\mu_L(\sigma^+)$ can be thought of as left and right moving waves which propagate through space

$$X^\mu = \underbrace{X^\mu_R(\tau-\sigma)}_{\text{right mover}} + \underbrace{X^\mu_L(\tau+\sigma)}_{\text{left mover}}.$$

Now, we apply the boundary conditions for the closed and open string:

- **Closed String**: Applying the conditions $X^\mu(\tau,\sigma+n) = X^\mu(\tau,\sigma)$ gives the particular solution (mode expansion) for the left and right movers as

$$X^\mu_R = \frac{1}{2}x^\mu + \frac{1}{2}l_s^2(\tau-\sigma)p^\mu + \frac{i}{2}l_s\sum_{n\neq 0}\frac{1}{n}\alpha^\mu_n e^{-2in(\tau-\sigma)}$$
$$X^\mu_L = \frac{1}{2}x^\mu + \frac{1}{2}l_s^2(\tau+\sigma)p^\mu + \frac{i}{2}l_s\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}^\mu_n e^{-2in(\tau+\sigma)}$$

  where x^μ, the center of mass of the string, is a constant, p^μ, the total momentum of the string, is a constant, l_s, the string length is also a constant. We can see that $X^\mu = X^\mu_R + X^\mu_L$ satisfies the boundary conditions.

Now that we have solved the field equations for the closed string boundary conditions we study the solutions obeying the open string boundary conditions, both Neumann and Dirichlet.

- **Open String** (*Neumann Boundary Conditions*): Recall that the Neumann boundary conditions imply that the derivative, with regard to σ, of the field at the σ boundary is zero, $\partial_\sigma X^\mu(\tau,\sigma)|_{\sigma=0,\pi} = 0$ (we have set $n=\pi$ for the boundary conditions). The general solution to the field equations is given by [5]

$$X^\mu(\sigma,\tau) = a_0 + a_1\sigma + a_2\tau + a_3\sigma\tau + \sum_{k\neq 0}(b^\mu_k e^{ik\sigma} + \tilde{b}^\mu_k e^{-ik\sigma})e^{-ik\tau},$$

  where a_i $i=(1,3)$, b_k and $\tilde{b}_k$ are constants and the only constraint on k in the summation is that it cannot equal to zero. Now, when we apply the Neumann boundary conditions we get

[5] For a hint of how to get to this, see Problem 4.2, [15].

the specific solution, here we have introduced new constants to resemble the closed string mode expansion,

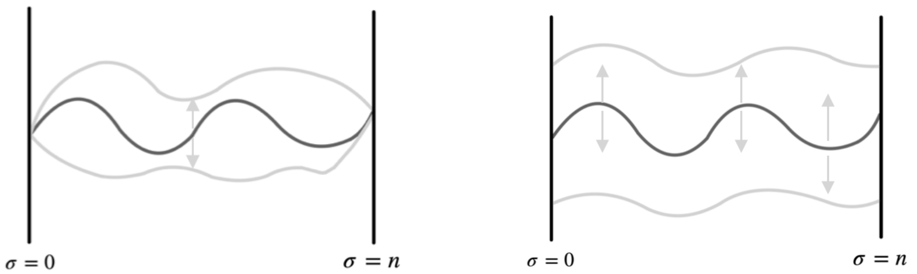
$$X^\mu(\sigma,\tau) = \underbrace{x^\mu + l_s \tau p^\mu}_{\text{c.o.m. motion of the string}} \underbrace{+ l_s \sum_{m\neq 0} \frac{1}{m}\alpha^\mu_m e^{-im\tau}\cos(m\sigma)}_{\text{oscillation of string}},$$

where c.o.m. is center of mass.

✧ **Open String** (*Dirichlet Boundary Conditions*): The Dirichlet boundary conditions say that at the σ boundary the field assumes the value of a constant, $X^\mu(\tau,\sigma=0) = X^\mu_0$ and $X(\tau,\sigma=\pi) = X^\mu_\pi$. The solution to the field equation obeying the Dirichlet boundary conditions is given by,

$$X^\mu(\tau,\sigma) = \underbrace{x^\mu_0 + \frac{\sigma}{\pi}(x^\mu_\pi - x^\mu_0)}_{\text{c.o.m motion of the string}} \underbrace{+ \sum_{m\neq 0} \frac{1}{m}\alpha^\mu_m e^{-im\tau}\sin(m\sigma)}_{\text{oscillation of string}}.$$

We can see, following the study of non relativistic strings, that Dirichlet boundary conditions arise if string endpoints are attached to some physical objects. In this case of relativistic strings, the endpoints must lie in objects that are characterized by the number of spatial dimensions that they have. This objects are the D-branes.



(a) Dirichlet Boundary Conditions (b) Neumann Boundary Conditions

When the open strings endpoints have free boundary conditions along all spatial directions, we have a space-filling D-brane. Since open strings endpoints can be anywhere on the D-brane, open strings endpoints are completely free, [16]. When analyzing the motion of open string points two important properties emerges: the endpoints move with the speed of light and the endpoints move transversely to the string.

Now, we need to impose the constraints (3.9) and (3.10). In the worldsheet lightcone coordinates $\sigma^\pm$, these become,

$$(\partial_+ X)^2 = (\partial_- X)^2 = 0.$$

These equations give constraints on the momenta p^μ and the Fourier modes α^μ_n and $\tilde{\alpha}^\mu_n$.

More generally, the periodic solution can be expanded into Fourier modes as

$$X^\mu_L(\sigma^+) = \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu \sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}^\mu_n e^{-in\sigma^+}, \tag{3.11}$$

$$X^\mu_R(\sigma^+) = \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu \sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha^\mu_n e^{-in\sigma^-}, \tag{3.12}$$

Since X^μ is real, it requires that the coefficients of the Fourier modes, α_n^μ and $\tilde{\alpha}_n^\mu$, obey

$$\alpha_n^\mu = (\alpha_n^\mu)^* \quad \tilde{\alpha}_n^\mu = (\tilde{\alpha}_n^\mu)^*$$

So, the constraint (3.9) can be written as

$$\begin{aligned}(\partial_- X)^2 &= \frac{\alpha'}{2}\sum_{m,p}\alpha_m \cdot \alpha_p e^{-i(m+p)\sigma^-} = \frac{\alpha'}{2}\sum_{m,n}\alpha_m \cdot \alpha_{n-m} e^{-in\sigma^-} \\ &\equiv \alpha' \sum_n L_n e^{-in\sigma^-} = 0\end{aligned}$$

where we have defined the sum of oscillator modes,

$$L_n = \frac{1}{2}\sum_m \alpha_{n-m} \cdot \alpha_m,$$

with the zero mode defined to be

$$\alpha_0^\mu \equiv \sqrt{\frac{\alpha'}{2}} p^\mu.$$

We can do the same for the left-moving modes, the sum of operator modes is

$$\tilde{L}_n = \frac{1}{2}\sum_m \tilde{\alpha}_{n-m} \cdot \tilde{\alpha}_m,$$

with

$$\tilde{\alpha}_0^\mu \equiv \sqrt{\frac{\alpha'}{2}} p^\mu.$$

We see that $\alpha_0^\mu = \tilde{\alpha}_0^\mu$. The L_n and $\tilde{L}_n$ are the Fourier modes of the constraints. Any classical solution of the string of the form (3.11) must further obey the infinite number of constraints,

$$L_n = \tilde{L}_n = 0, \quad n \in \mathbb{Z}. \tag{3.13}$$

Chapter 4

Canonical Quantization

En este capítulo procederemos a realizar una cuantización canónica de la teoría. Para ello buscaremos cantidades conservadas y escribiremos primero el hamiltoniano y el tensor energía-impulso para cuerdas cerradas y abiertas en la teoría clásica. Obtendremos así la expresión de masa clásica. Para cuantificar la teoría introduciremos las álgebras de Witt y Virasoro; y cambiaremos los paréntesis de Poisson a conmutadores y los generadores L_m a operadores. La elección del orden normal introducirá una constante arbitraria y términos de corrección en la fórmula de masa. Por último caracterizaremos los estados físicos en términos de los generadores de Virasoro. Esto nos va a permitir calcular representaciones del álgebra de Virasoro que se relacionarán con representaciones del grupo de Lorentz con el objetivo de ver a qué tipo de partícula corresponde nuestro estado físico.

Now, our aim is to analyze the conserved currents and charges. Due to Emmy Noether's theorem, we know that associated with any global symmetry of a system, the worldsheet in our case, there exists a conserved current, j^μ, and a conserved charge, Q,

$$\begin{aligned}
&\partial_\alpha j^\alpha = 0, \\
&\frac{d}{d\tau} Q = \frac{d}{d\tau}\left(\int d\sigma j^0\right) = 0.
\end{aligned}$$

We will use this result to construct an algorithm for finding these currents and charges for any symmetry.

4.1 Conserved Quantities

In order to verify that the charge, defined as the spatial integral of the zeroth component of the current, is conserved we consider the following

$$\frac{d}{d\tau}\left(\int d\sigma j^0\right) = \int d\sigma \frac{d}{d\tau}(j^0) = -\int d\sigma \partial_\sigma j^\sigma = -j^\sigma \mid_{\sigma=0}^{\pi} = 0,$$

where the substitution in the second equality comes from the current being conserved and the third equality follows from Stoke's theorem. So, the charge is indeed conserved.

Now we ask how to actually construct a current from a given symmetry. Let ϕ be a field, then if there exists a global symmetry for the theory under this transformation, $f : \phi \to \phi + \delta\phi$, where $\delta\phi = \varepsilon f(\phi)$ and ε is infinitesimal, the variation in the action δS is equal to zero (the equations of motion do not change). To construct the corresponding current to this transformation we proceed as follows: First, consider ε to be a local parameter, its derivative with respect to spacetime coordinates does not vanish. Now, since ε is infinitesimal, the only contributing part to the variation of S will be linear in ε. Then, the transformation $\delta\phi = \varepsilon f(\phi)$ leads to a variation in the action S which is given by

$$\delta S = \int d\tau d\sigma (\partial_\alpha \varepsilon) j^\alpha.$$

Integrating this by parts gives

$$\delta S = -\int d\tau d\sigma \varepsilon (\partial_\alpha j^\alpha).$$

If this transformation is a symmetry then this variation vanishes for all ε and thus $\partial_\alpha j^\alpha = 0$. We shown that the current is conserved. To construct this current we plug the variation of the transformation, $\varepsilon f(\phi)$, into the variation of the action and then the current will be given by all the terms which multiply the $\partial_\alpha \varepsilon$ term.

4.2 The Hamiltonian and Energy-Momentum Tensor

The Hamiltonian that generates the Worldsheet time evolution is defined by

$$H = \int_{\sigma=0}^{\pi} d\sigma (\dot{X}_\mu P^\mu - \mathcal{L}),$$

where P^μ is the canonical momentum and $\mathcal{L}$ is the Lagrangian. In the case of the bosonic string theory, we have that the canonical momentum is given by $P^\mu = T\dot{X}^\mu$ while the Lagrangian is $\mathcal{L} = \frac{1}{2}(\dot{X}^2 \cdot X'^2)$. So, the bosonic string Hamiltonian is

$$H = T\int_{\sigma=0}^{\pi} d\sigma (\dot{X}^2 - \frac{1}{2}(\dot{X}^2 \cdot X'^2)) = \frac{T}{2}\int_{\sigma=0}^{\pi} d\sigma (\dot{X}^2 + X'^2),$$

where $\dot{X}^2 \equiv \dot{X}_\mu \dot{X}^\mu$. To express the Hamiltonian in terms of an open or closed string we need to expand the above in terms of the mode expansions for the fields $X^\mu(\tau,\sigma)$. So, for a **closed string** the Hamiltonian becomes

$$H = \sum_{n=-\infty}^{\infty} (\alpha_{-n} \cdot \alpha_n + \tilde{\alpha}_{-n} \cdot \tilde{\alpha_n}),$$

we have defined $\alpha_0^\mu = \tilde{\alpha}_0^\mu = \frac{1}{2} l_s p^\mu$. While for **open strings** we have that

$$H = \frac{1}{2}\sum_{n=-\infty}^{\infty} \alpha_{-n} \cdot \alpha_n,$$

with $\alpha_0^\mu = \tilde{\alpha}_0^\mu = l_s p^\mu$.

Note that H is conserved, $\frac{d}{d\tau}(H) = 0$, since neither α or $\tilde{\alpha}$ depend on τ. Also, these results only hold in the classical theory, when we quantize the theory we will have order ambiguities.

Now, we will see the mode expansion for the stress-energy tensor in terms of a closed string theory (the open string version follows analogously). The components of the stress-energy tensor are given by

$$T_{--} = (\partial_- X_R^\mu)^2; \quad T_{++} = (\partial_+ X_L^\mu)^2; \quad T_{-+} = T_{+-} = 0.$$

For a closed string, we can plug in the mode expansions for X_R^μ and X_L^μ to get

$$T_{--} = 2l_s^2 \sum_{m=-\infty}^{\infty} L_m e^{-2im(\tau-\sigma)}$$

$$T_{++} = 2l_s^2 \sum_{m=-\infty}^{\infty} \tilde{L}_m e^{-2im(\tau-\sigma)},$$

where we have defined
$$L_m = \frac{1}{2}\sum_{n=-\infty}^{\infty} \alpha_{m-n}\cdot\alpha_n, \tag{4.1}$$
$$\tilde{L}_m = \frac{1}{2}\sum_{n=-\infty}^{\infty} \tilde{\alpha}_{m-n}\cdot\tilde{\alpha}_n$$

These expressions only hold for the classical case and must be modified in the quantization. We can write the Hamiltonian in terms of the newly defined quantities L_m and $\tilde{L}_m$; for a closed string we have that
$$H = 2(L_0 + \tilde{L}_0),$$
while for an open string
$$H = L_0.$$

We can use the expression for the Hamiltonian and the stress-energy tensor in terms of the modes to derive a mass formula for both the classical open and closed string theories.

4.3 Classical Mass Formula for a Bosonic String

Recalling the mass-energy relation
$$M^2 = -p^\mu p_\mu.$$
We have for our bosonic string theory that
$$p^\mu = \int_{\sigma=0}^{\pi} d\sigma P^\mu = T\int_{\sigma=0}^{\pi} d\sigma \dot{X}^\mu = \begin{cases} \frac{2\alpha_0^\mu}{l_s}, & \text{for a closed string,} \\ \frac{\alpha_0^\mu}{l_s}, & \text{for an open string.} \end{cases}$$
and so
$$p^\mu p_\mu = \begin{cases} \frac{2\alpha_0^2}{\alpha'}, & \text{for a closed string,} \\ \frac{\alpha_0^2}{2\alpha'} & \text{for an open string,} \end{cases}$$
where $\alpha' = l_s^2/2$.
In the case of an open string, we have the following mass formula,
$$M^2 = \frac{1}{\alpha'}\sum_{n=1}^{\infty} \alpha_{-n}\cdot\alpha_n.$$

For the closed string we have to take into account both left and right movers and thus we must use both the condition (3.13). So, the closed string mass formula is given by
$$M^2 = \frac{2}{\alpha'}\sum_{n=1}^{\infty}(\alpha_{-n}\cdot\alpha_n + \tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n).$$

These are the mass-shell conditions for open and closed strings and they express the mass corresponding to a certain classical string state. These expressions are only valid classically.

4.4 Witt Algebra

Let $\{L_m\}$ be a set of elements with the multiplication given by
$$\{L_m, L_n\}_{P.B.} = i(m-n)L_{m+n} \tag{4.2}$$

where $\{\cdot,\cdot\}_{P.B.}$ is the Poisson bracket. This set with the operation forms an algebra called the *Witt algebra* or the classical Virasoro algebra. Note that the elements L_m defined before satisfy (4.2), so they form the algebra. Now we will study the physical meaning of the $\{L_m\}$.

We gauge fixed the metric $h_{\alpha\beta}$ to the flat metric $\eta_{\alpha\beta}$. However, this does not completely gauge fix the diffeomorphism and Weyl symmetries. For instance, consider the transformations given by

$$\begin{aligned}\delta_D\eta^{\alpha\beta} &= -(\partial^\alpha\xi^\beta + \partial^\beta\xi^\alpha),\\ \delta_W\eta^{\alpha\beta} &= \Lambda\eta^{\alpha\beta},\end{aligned}$$

where ξ^α is an infinitesimal parameter of reparametrization, Λ is an infinitesimal parameter for Weyl rescaling, $\delta_D\eta^{\alpha\beta}$ gives the variation of the metric under reparametrization and $\delta_W\eta^{\alpha\beta}$ gives the variation under a Weyl rescaling. Combining these two transformations we get

$$(\delta_D + \delta_W)\eta^{\alpha\beta} = (-\partial^\alpha\xi^\beta - \partial^\beta\xi^\alpha + \Lambda\eta^{\alpha\beta}). \tag{4.3}$$

Finding the most general solution for ξ and Λ such that (4.3) is zero, we get additional symmetries for the system. To solve (4.3) for ξ and Λ we will use the light-cone coordinates:

$$\begin{aligned}\xi^\pm &= \xi^0 \pm \xi^1\\ \sigma^\pm &= \tau + \sigma.\end{aligned}$$

So, in terms of the light-cone coordinates, the equation (4.3) becomes

$$\partial^\alpha\xi^\beta + \partial^\beta\xi^\alpha = \Lambda\eta^{\alpha\beta}.$$

In order to solve this, we consider:

1. $\alpha = \beta = +$: Noting that $\eta^{++} = 0$. Thus, we have to solve

   $$\partial^+\xi^+ + \partial^+\xi^+ = \Lambda\eta^{++} \Rightarrow 2\partial^+\xi^+ = 0 \Rightarrow \partial^+\xi^+ = 0.$$

   The solution is given by some arbitrary function whose argument is only a function of σ^-, we denote it as $\xi^-(\sigma^-)$.

2. $\alpha = \beta = -$: In this case $\eta^{--} = 0$, so

   $$\partial^-\xi^- + \partial^-\xi^- = \Lambda\eta^{--} \Rightarrow 2\partial^-\xi^- = 0 \Rightarrow \partial^-\xi^- = 0.$$

   which has as its solution some arbitrary function of σ^+. This solution is denoted by $\xi^+(\sigma^+)$.

3. $\alpha = +$, $\beta = -$: We have that

   $$\partial^+\xi^- + \partial^-\xi^+ = \Lambda\eta^{-+} \Rightarrow \partial^+\xi^- + \partial^-\xi^+ = 2\Lambda.$$

So, we have found another set of gauge transformations that let our bosonic string theory invariant. These are the local transformations which satisfy:

$$\begin{aligned}\delta\sigma^+ &= \xi^+(\sigma^+),\\ \delta\sigma^- &= \xi^-(\sigma^-),\\ \Lambda \quad &= \partial^-\xi^+ + \partial + \xi^-\end{aligned}$$

The infinitesimal generators for the transformations $\delta\sigma^\pm = \xi^\pm$ are given by

$$V^\pm = \frac{1}{2}\xi^\pm(\sigma^\pm)\frac{\partial}{\partial\sigma^\pm}$$

and a complete basis for these transformations is given by

$$\xi_n^{\pm}(\sigma^{\pm}) = e^{2in\sigma^{\pm}}, \quad n \in \mathbb{Z}.$$

In the case of the open string the infinitesimal generators are given by

$$V_n^{\pm} = e^{in\sigma^+}\frac{\partial}{\partial\sigma^+} + e^{in\sigma^-}\frac{\partial}{\partial\sigma^-}, \quad n \in \mathbb{Z}.$$

In the next section we will look at the quantized theory.

4.5 Canonical Quantization

First, we will quantize the bosonic string theory in terms of canonical quantization and later we will study the light-cone gauge quantization of the theory.

In the canonical quantization procedure, we quantize the theory by changing Poisson brackets to commutators,

$$\{\cdot,\cdot\}_{P.B.} \to i[\cdot,\cdot],$$

and we promote the field X^μ to an operator in our corresponding Hilbert space. This is equivalent to promoting the modes α, the constant x^μ and the total momentum p^μ to operators. In particular, for the modes α_m^μ , we have that (dropping the i factor)

$$\begin{aligned}
[\hat{\alpha}_m^\mu, \hat{\alpha}_n^\nu] &= m\eta^{\mu\nu}\delta_{m,-n} \\
[\hat{\tilde{\alpha}}_m^\mu, \hat{\tilde{\alpha}}_n^\nu] &= m\eta^{\mu\nu}\delta_{m,-n}, \\
[\hat{\alpha}_m^\mu, \hat{\tilde{\alpha}}_n^\nu] &= 0.
\end{aligned}$$

Defining this new operators $\hat{a}^\mu \equiv \frac{1}{\sqrt{m}}\hat{\alpha}_m^\mu$ and $\hat{a}^{\mu\dagger} \equiv \frac{1}{\sqrt{m}}\hat{\alpha}_{-m}^{\mu\dagger}$ for $m > 0$, they satisfy

$$[\hat{a}_m^\mu, \hat{a}_n^{\nu\dagger}] = [\hat{\tilde{a}}_m^\mu, \hat{\tilde{a}}_n^{\nu\dagger}] = \eta^{\alpha\beta}\delta_{m,n}, \quad \text{for } m, n > 0.$$

For $\mu = \nu = 0$, we obtain a negative sign due to the signature of the metric,

$$[\hat{a}_m^0, \hat{a}_n^{0\dagger}] = \eta^{00}\delta_{m,n} = -\delta_{m,n}.$$

This negative sign in the commutators leads to the prediction of negative norm physical states, or *ghost states*. We will discuss this in Chapter 5.

Next, we introduce the ground state, denoted by $|0\rangle$, as the state which is annihilated by all of the lowering operators $\hat{a}_m^\mu$,

$$\hat{a}_m^\mu |0\rangle = 0, \quad \text{for } m > 0.$$

In the other hand, physical states are states that are constructed by acting on the ground state with the raising operators $\hat{a^{\mu\dagger}}_m$,

$$|\phi\rangle = \hat{a}_{m_1}^{\mu_1\dagger}\hat{a}_{m_2}^{\mu_2\dagger}\cdots\hat{a}_{m_n}^{\mu_n\dagger}|0; k^\mu\rangle,$$

which are eigenstates of the momentum operator $\hat{p}^\mu$, $\hat{p}^\mu|\phi\rangle = k^\mu|\phi\rangle$. Note that this is first quantization, and all of these states, including the ground state, are one-particle states.

In order to prove the claim of negative norm states, we consider the state $|\psi\rangle = \hat{a}_m^{0\dagger}|0; k^\mu\rangle$, for $m > 0$, and calculate

$$\|\psi\|^2 = \langle 0|\hat{a}_m^0\hat{a}_m^{0\dagger}|0\rangle = \langle 0|[\hat{a}_m^0, \hat{a}_m^{0\dagger}]|0\rangle = -\langle 0|0\rangle.$$

If we set $\langle 0|0\rangle$ to be positive then we will get some negative norm states, but if we define $\langle 0|0\rangle$ to be negative then there will be other states with negative norm (in particular any state not of the same form as $|\psi\rangle$). These negative norm states are a issue because they are unphysical. In Chapter 5, we will see that these negative norm states can be removed but that will put a constraint on the number of dimensions of the background spacetime in which our theory is defined.

4.6 Virasoro Algebra

In the previous section we saw that when we quantize the bosonic string theory the modes α become operators. This implies that the generators L_m will also become operators, since they are constructed from the α. However, we must be careful because we cannot just say that $\hat{L}_m$ is given by (4.1), but we must normal order[1] the operators, because they are non-commuting operators. We will see when this ordering introduces ambiguities. Thus, we define L_m to be

$$\hat{L}_m = \frac{1}{2}\sum_{n=-\infty}^{\infty} :\hat{\alpha}_{m-n}\cdot\hat{\alpha}_n: .$$

For $m = 0$, $\hat{L}_0$ is given by

$$\hat{L}_0 = \frac{1}{2}\hat{\alpha_0^2} + \sum_{n=1}^{\infty}\hat{\alpha}_{-n}\cdot\hat{\alpha}_n.$$

We get the commutation relations for the operators $\hat{L}_m$ in terms of the commutation relations for the operators $\hat{\alpha}$

$$[\hat{L}_m, \hat{L}_n] = (m-n)\hat{L}_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m,-n},$$

where c is called the *central charge*. We will see that in the bosonic string theory, c is equal to the dimension of the spacetime where the theory lives, and it must take the value $c = 26$ in order to no have negative norm states. Also, note that for $m = -1, 0, 1$ the c term drops out, we get that the set $\{\hat{L}_{-1}, \hat{L}_0, \hat{L}_1\}$ along with the relations

$$[\hat{L}_m, \hat{L}_n] = (m-n)\hat{L}_{m+n},$$

becomes an algebra which is isomorphic to $SL(2,\mathbb{R})$[2].

4.7 Physical States

Classically we have seen that $L_0 = 0$ but when we quantize the theory we cannot say that $\hat{L}_0 = 0$, this is because when we quantize the theory we have to normal order the operator $\hat{L}_0$ and so we could have some arbitrary constant due to this normal ordering. Thus, after quantizing we get that for an open string, the vanishing of the L_0 constraint transforms to

$$(\hat{L}_0 - a)\,|\phi\rangle = 0,$$

where a is a constant. This is called the *mass-shell condition* for the open string. For a closed string, we have that

$$(\hat{L}_0 - a)\,|\psi\rangle = 0, \tag{4.4}$$
$$(\hat{\tilde{L}}_0 - a)\,|\psi\rangle = 0, \tag{4.5}$$

where $\hat{\tilde{L}}$ is the operator corresponding to the classical generator $\tilde{L}$.

[1]Normal ordering is defined to be

$$:\alpha_i\cdot\alpha_j := \begin{cases} \alpha_i\cdot\alpha_j \text{ when } i\le j, \\ \alpha_j\cdot\alpha_i \text{ when } i > j. \end{cases}$$

[2]$SL(2,\mathbb{R})$ is the group of all linear transformations of $\mathbb{R}^2$ that preserve oriented area.

Normal ordering also adds correction terms to the mass formula. For an open string theory, the mass formula becomes

$$\alpha' M^2 = \frac{1}{\alpha'}\sum_{n=1}^{\infty} :\hat{\alpha}_{-n}\cdot\hat{\alpha}_n: -a = \hat{N} - a,$$

where the number operator $\hat{N}$ is

$$\hat{N} = \sum_{n=1}^{\infty} :\hat{\alpha}_{-n}\cdot\hat{\alpha}_n := \sum_{n=1}^{\infty} n : \hat{a}_n^\dagger \cdot \hat{a}_n : .$$

We can use the number operator to compute the mass spectrum, for example,

$$\begin{aligned}
\alpha' M^2 &= -a, &&\text{ground state } n = 0\\
\alpha' M^2 &= -a + 1, &&\text{first excited state } n = 1\\
\alpha' M^2 &= -a + 2, &&\text{second excited state } n = 2.\\
&\vdots
\end{aligned}$$

For a closed string we have the mass formula

$$\frac{4}{\alpha'}M = \sum_{n=1}^{\infty} :\hat{\alpha}_{-n}\cdot\hat{\alpha}_n: -a = \sum_{n=1}^{\infty} \hat{\tilde{\alpha}}_{-n}\cdot\hat{\tilde{\alpha}}_n : -a, \tag{4.6}$$

or

$$\hat{N} - a = \hat{\tilde{N}} - a,$$

where $\hat{N}$ is the number operator for right movers and $\hat{\tilde{N}}$ is the number operator for left movers. Also, note that if we subtract the left moving physical state condition, (4.4), from the right moving physical state condition, (4.5), we get that $\hat{N} = \hat{\tilde{N}}$.

Virasoro Generators and Physical States

We have shown that, classically, $L_m = 0$ for all m which does not hold for $\hat{L}_0$. If $L_m |\phi\rangle = 0$ for all $m \neq 0$ then we would have that, if we take n in such a way that $n + m \neq 0$,

$$[\hat{L}_m, \hat{L}_n]\,|\phi\rangle = 0.$$

Plugging this in the commutation relations we get

$$(m-n)\hat{L}_{m+n}\,|\phi\rangle + \frac{c}{12}m(m^2-1)\delta_{m,-n}\,|\phi\rangle = 0$$

and since the first term vanishes, because we are assuming $\hat{L}_m |\phi\rangle = 0$ for all $m \neq 0$, we see that it must be that either $m = -1$, $m = 0$ or $m = 1$, for $c \neq 0$. Thus, we must restrict our Virasoro algebra to only $\{\hat{L}_{-1}, \hat{L}_0, \hat{L}_1\}$. Instead of doing this, we will only impose that $L_m |\phi\rangle = 0 = \langle\phi|\,\hat{L}_m^\dagger$ for $m > 0$. Physical states are then characterized by

$$\hat{L}_{m>0}\,|\phi\rangle = 0 = \langle\phi|\,\hat{L}_{m>0}^\dagger \tag{4.7}$$

and the mass-shell condition

$$(\hat{L}_0 - a)\,|\phi\rangle = 0.$$

Equivalently, one could replace (4.7) by

$$\langle\phi|\,\hat{L}_m = 0 = \langle\phi|\,\hat{L}_m^\dagger, \quad \text{for all } m < 0.$$

Classically, the Lorentz generators or charges, $Q^{\mu\nu}$, are given by the spatial integral of the time component of the current corresponding to the Lorentz transformations, $j^{\mu\nu}_{\alpha}$, i.e.,

$$Q^{\mu\nu} = \int_{\sigma=0}^{\pi} d\sigma j^{\mu\nu}_{0}.$$

This can be rewrite as

$$Q^{\mu\nu} = T\int_{\sigma=0}^{\pi} d\sigma (X^{\mu}\dot{X}^{\nu} - X^{\nu}\dot{X}^{\mu}),$$

which can be expanded into modes as

$$Q^{\mu\nu} = (x^{\mu}p\nu - x^{\nu}p^{\mu}) - 1\sum_{m=1}^{\infty}\frac{1}{m}(\alpha^{\mu}_{-m}\alpha^{\nu}_{m} - \alpha^{\nu}_{-m}\alpha^{\mu}_{m}).$$

We can quantize the expression since this does not have any normal ordering ambiguities,

$$\hat{Q}^{\mu\nu} = (\hat{x}^{\mu}\hat{p}^{\nu} - \hat{x}^{\nu}\hat{p}^{\mu}) - 1\sum_{m=1}^{\infty}\frac{1}{m}(\hat{\alpha}^{\mu}_{-m}\hat{\alpha}^{\nu}_{m} - \hat{\alpha}^{\nu}_{-m}\hat{\alpha}^{\mu}_{m}).$$

Now we have an operator expression for the Lorentz generators. Computing the commutator $[\hat{L}_m, \hat{Q}^{\mu\nu}]$ we see it is equal to zero. This implies that physical states, defined in terms of the Virasoro generators, appear in complete Lorentz multiplets. This allow us to calculate representations of the Virasoro algebra and be able to relate them to representations of the Lorentz group with the aim of seeing what type of particle our physical state corresponds to.

Chapter 5

Removing Ghost States and Light-Cone Quantization

En este capítulo construiremos el espacio de Fock donde se desarrolla nuestra teoría. Sin embargo, hay un problema con este espacio, no tiene norma positiva. Para deshacernos de los estados físicos normales negativos, los llamados estados fantasma, introduciremos los estados espurios. Por último, proponemos cuantizar la teoría de cuerdas bosónica siguiendo la cuantización del cono de luz en lugar de la cuantización canónica realizada anteriormente. Obtendremos la condición de concha de masa para cuerdas bosónicas abiertas, así como el espectro de masas para cuerdas bosónicas abiertas y cerradas.

The physical Hilbert space for the theory is given by taking the Fock space[1] generated by vectors $|\psi\rangle$ of the form

$$|\psi\rangle = \hat{a}_{m_1}^{\mu_1 \dagger} \hat{a}_{m_2}^{\mu_2 \dagger} \cdots \hat{a}_{m_n}^{\mu_n \dagger} |0; k^\mu\rangle \,, \tag{5.1}$$

and then modding out by the subspace formed from the constraint $\hat{L}_{m>0} |\psi\rangle = 0$. More precisely, we had that the physical states $|\phi\rangle$ were defined as $|\phi\rangle = \hat{a}_{m_1}^{\mu_1 \dagger} \hat{a}_{m_2}^{\mu_2 \dagger} \cdots \hat{a}_{m_n}^{\mu_n \dagger} |0; k^\mu\rangle$, which obeyed the following constraints

$$(\hat{L}_0 - a) |\phi\rangle = 0, \tag{5.2}$$

$$\hat{L}_{m>0} |\phi\rangle = 0. \tag{5.3}$$

However we saw that some of the states defined in (5.1) may have negative norm. We will see that one can get rid of these negative norm physical states by fixing the constant a, appearing in (5.2), and by also fixing the central charge c of the Virasoro algebra.

We will also get a constraint on the allowed dimensions for the background. We will study physical states of zero norm which satisfy the physical state conditions. To do this, we must introduce *spurious states*.

5.1 Spurious States

Definition 5.1.1. *A state $|\psi\rangle$ is said to be spurious if it satisfies the mass-shell condition,*

$$(\hat{L}_0 - a) |\phi\rangle = 0$$

and is orthogonal to all other physical states,

$$\langle \phi | \psi \rangle = 0, \text{ for all physical states } |\phi\rangle \,.$$

[1] More information of Fock spaces in [1].

In general, a spurious state can be written as

$$|\psi\rangle = \sum_{n=1}^{\infty} \hat{L}_{-n} |\chi_n\rangle \,, \tag{5.4}$$

where $|\chi_n\rangle$ is some state which satisfies the mass-shell condition given by

$$(\hat{L}_0 - a + n) |\chi_n\rangle = 0.$$

Now, since any $\hat{L}_{-n}$, for $n \geq 1$, can be written as a combination of $\hat{L}_{-1}$ and $\hat{L}_{-2}$ the general expression for a spurious state (5.4) can be simplified to

$$|\psi\rangle = \hat{L}_{-1} |\chi_1\rangle + \hat{L}_{-2} |\chi_2\rangle$$

where $|\chi_1\rangle$ and $|\chi_2\rangle$ are called level 1 and level 2 states, respectively, and they satisfy the mass-shell conditions $(\hat{L}_0 - a + 1) |\chi_1\rangle = 0$ and $(\hat{L}_0 - a + 2) |\chi_2\rangle = 0$.

To see that a spurious state $|\psi\rangle$ is orthogonal to any physical state $|\phi\rangle$ consider

$$\langle\phi|\psi\rangle = \sum_{n=1}^{\infty} \langle\phi| \hat{L}_{-n} |\chi_n\rangle = \sum_{n=1}^{\infty} (\langle\chi_n| \hat{L}_n |\phi\rangle)^* = \sum_{n=1}^{\infty} (\langle\chi_n| 0 |\phi\rangle)^* = 0,$$

where the second equality follows from the fact that $\hat{L}_{-n}^\dagger = \hat{L}_n$ and the third equality follows from the fact that since $|\phi\rangle$ is a physical state it is annihilated by all $\hat{L}_{n>0}$.

Since a spurious state $|\psi\rangle$ is perpendicular to all physical states, in particular, if $|\psi\rangle$ is both physical and spurious state then it must be perpendicular to itself,

$$\| \, |\psi\rangle \, \|^2 = \langle\psi|\psi\rangle = 0.$$

Thus, we have constructed physical states whose norm is zero, these are the states we need to study in order to get rid of the negative norm physical states in our bosonic string theory[2].

5.2 Removing Ghost States

Now, we will study physical spurious states in order to determine the values of a and c that project out the negative norm physical states, called *ghost states*. In order to find the corresponding a value we should start with the spurious state,

$$|\psi\rangle = \hat{L}_{-1} |\chi_1\rangle \,,$$

with $|\chi_1\rangle$ satisfaying $(\hat{L}_0 - a + 1) |\chi_1\rangle = 0$. Now, if $|\psi\rangle$ is physical then it must satisfy the mass-shell condition for physical states,

$$(\hat{L}_0 - a) |\psi\rangle = 0,$$

along with the condition

$$\hat{L}_{m>0} |\psi\rangle = 0.$$

[2]Since the set of spurious-physical states is at the boundary of physical and non physical states, there is ambiguity, since this boundary must have a "contamination" of these two kind of states. If we show that this boundary only contain physical states, we can state that there is no elements with negative norm, these are, non physical states. Furthermore, for the theory to be consistent we must impose values for a and c.

So, if $\hat{L}_{m>0}|\psi\rangle = 0$ then this holds for, in particular, the operator $\hat{L}_1$, i.e. $\hat{L}_1|\psi\rangle = 0$ which implies that $0 = \hat{L}_1(\hat{L}_{-1}|\chi_1\rangle) = ([\hat{L}_1, \hat{L}_{-1}] + \hat{L}_{-1}\hat{L}_1)|\chi_1\rangle = [\hat{L}_1, \hat{L}_{-1}]|\chi_1\rangle = 2\hat{L}_0|\chi_1\rangle = 2(a-1)|\chi_1\rangle$. Since $2(a-1)|\chi_1\rangle$ must be zero (since $|\psi\rangle$ is physical), we have $a = 1$. This tells us that $a = 1$ if $|\psi\rangle$ is a physical spurious state. This is part of the boundary between positive and negative norm physical states.

In order to determine the appropriate value of c for spurious physical states we need to look at a level 2 spurious state. These states are given by

$$|\psi\rangle = (\hat{L}_{-2} + \gamma\hat{L}_{-1}\hat{L}_{-1})|\chi_2\rangle$$

where γ is a constant, that will be fixed to insure that $|\psi\rangle$ has a zero norm and $|\chi_2\rangle$ obeys the relations

$$\begin{aligned}(\hat{L}_0 - a + 2)|\chi_2\rangle &= 0,\\ \hat{L}_{m>0}|\chi_2\rangle &= 0.\end{aligned}$$

Now, if $|\psi\rangle$ is to be physical, and thus have zero norm, then it must satisfy $\hat{L}_{m>0}|\psi\rangle = 0$ and, in particular, $\hat{L}_1|\psi\rangle = 0$. This implies that $\gamma = \frac{3}{2}$, (obtained in [15]).
With the previous results, any general level 2 physical spurious state is of the form

$$|\psi\rangle = (\hat{L}_{-2} + \frac{3}{2}\hat{L}_{-1}\hat{L}_{-1})|\chi_2\rangle$$

Now, since $\hat{L}_{m>0}|\psi\rangle = 0$, we have that, in particular, $\hat{L}_2|\psi\rangle = 0$ which yields

$$\begin{aligned}
&\hat{L}_2\left(\hat{L}_{-2} + \frac{3}{2}\hat{L}_{-1}\hat{L}_{-1}\right)|\chi_2\rangle = 0;\\
&\left([\hat{L}_2, \hat{L}_{-2} + \frac{3}{2}\hat{L}_{-1}\hat{L}_{-1}] + (\hat{L}_{-2} + \frac{3}{2}\hat{L}_{-1}\hat{L}_{-1})\hat{L}_2\right)|\chi_2\rangle = 0;\\
&\left([\hat{L}_2, \hat{L}_{-2} + \frac{3}{2}\hat{L}_{-1}\hat{L}_{-1}]\right)|\chi_2\rangle = 0;\\
&\left(13\hat{L}_0 + 9\hat{L}_{-1}\hat{L}_1 + \frac{c}{2}\right)|\chi_2\rangle = 0;\\
&\left(-13 + \frac{c}{2}\right)|\chi_2\rangle = 0 \Rightarrow c = 26.
\end{aligned}$$

Finally, we have obtained $a = 1$, $\gamma = 3/2$ and $c = 26$ in order to project out the negative norm physical states. Note that since the central charge c is equivalent to the dimension of the background space-time for our bosonic string theory, then our theory is only physically acceptable for 26 dimensions, this is called the *critical dimension*.

Next, we want to quantize the theory in a different manner, it will no longer have negative norm physical states, but it will not be Lorentz invariant. However, this can be fixed constraining the constants a and c.

5.3 Light-Cone Gauge Quantization

There are different ways to quantize, we propose to quantize the bosonic string theory in another way, instead of the canonical quantization previously carried out. When we impose that our theory be Lorentz invariant we will see that this forces $a = 1$ and $c = 26$. We get that, even after we choose a gauge such that the space-time metric $h^{\alpha\beta}$ becomes Minkowskian, the bosonic string theory still has residual diffeomorphism symmetries. In terms of the worldsheet light-cone coordinates σ^+ and σ^-, this residual symmetry corresponds to being able to reparameterize these coordinates as

$$\sigma^{\pm} \to \sigma'^{\pm} = \xi^{\pm}(\sigma^{\pm})$$

without changing the theory, the bosonic string action is invariant under these reparametrizations. So, we have the possibility of making an additional gauge choice, and if we choose a particular non-covariant gauge, the light-cone gauge, it is then possible to describe a Fock space which has not ghost states. To proceed, we will first define the light-cone coordinates for the background spacetime in which our bosonic string is moving.

In general, the light-cone coordinates for a spacetime can be defined by taking linear combinations of the temporal coordinate along with another transverse, or spacelike, coordinate. In our case, we will pick the $D-1$ spacetime coordinate. Thus, the light-cone coordinates for the background spacetime, X^+ and X^-, are defined as

$$X^+ \equiv \frac{1}{\sqrt{2}}(X^0 + X^{D-1})$$
$$X^- \equiv \frac{1}{\sqrt{2}}(X^0 - X^{D-1})$$

So, the spacetime coordinates become the set $\{X^-, X^+, X^i\}_{i=1}^{D-2}$.

In this light-cone coordinate system the inner product of two vectors V and W is given by

$$V \cdot W = -V^+W^- - V^-W^+ + \sum_{i=1}^{D-2} V^iW^i.$$

While raising and lowering of indices goes as

$$V_+ = -V^-; \quad V_- = -V^+; \quad V_i = V^i, \quad i = 1, \dots, D-2.$$

Note that since we are treating two coordinates of spacetime differently from the rest, namely X^0 and X^{D-1}, we have lost manifest Lorentz invariance and so our Lorentz symmetry $SO(1, D-1)$ becomes $SO(D-2)$.

We know that the residual gauge symmetry corresponds to being able to reparameterize $\sigma^\pm$ as $\sigma^\pm \to \xi^\pm(\sigma^\pm)$ without changing the theory. This implies that we can reparameterize τ and σ, since they are given by linear combinations of σ^+ and σ^-, as

$$\tau \to \tilde{\tau} = \frac{1}{2}(\tilde{\sigma}^+ + \tilde{\sigma}^-) = \frac{1}{2}(\xi^+(\sigma^+) + \xi^-(\sigma^-)),$$
$$\sigma \to \tilde{\sigma} = \frac{1}{2}(\tilde{\sigma}^+ - \tilde{\sigma}^-) = \frac{1}{2}(\xi^+(\sigma^+) - \xi^-(\sigma^-))$$

From the form of $\tilde{\tau}$, we can see that it is a solution to the massless wave equation,

$$\partial_+\partial_-\tilde{\tau} = (\partial_\tau^2\partial_\sigma^2)\tilde{\tau} = 0. \tag{5.5}$$

We can take a particular $\tilde{\tau}$ that satisfy (5.5) and simplifies the theory. However, once we pick $\tilde{\tau}$ then we have fixed $\tilde{\sigma}$.

In [15] we find the following result: one can express the bosonic string theory in terms of transverse oscillators only and so a (critical) string only has transverse oscillations, just as massless particles only have transverse polarizations.

5.3.1 Mass-Shell Condition for Open Bosonic Strings

The mass-shell condition in light-cone coordinates is given by

$$-p^{\mu}p_{\mu} = M^2 = 2p^+p^- - \sum_{i=1}^{D-2} p^i p^i$$

Also, for $n = 0$ we have that, by expanding P^{μ} in modes (calculation done in [15]),

$$M^2 = \frac{2}{l_s^2}(N - a).$$

So, in the light-cone gauge we have that the mass-shell formula for an open string is given by

$$M^2 = \frac{2}{l_s^2} \sum_{i=1}^{D-2} \sum_{n=-\infty}^{\infty} : \alpha_{-n}^i \alpha_n^i : -a = \frac{2}{l_s^2}(N - a).$$

5.3.2 Mass Spectrum for Open Bosonic Strings

Note that in the light-cone gauge all of the excitations are generated by transverse oscillators (α_n^i). Before, in the canonical quantization, we had to include all the oscillators in the spectrum which lead to negative norm states. Now, the commutator of the transverse oscillations no longer have the negative value coming from the 00 component of the metric and so we do not have negative norm states in the light-cone gauge quantization.

The first excited state, $\alpha_{-1}^i |0; k^{\mu}\rangle$, belongs to a $(D-2)$-component vector representation of the rotation group $SO(D-2)$[3] in the transverse space. Lorentz invariance implies that physical states form a representation of $SO(D-1)$ for massive states and $SO(D-2)$ for massless states. Since $\alpha_{-1}^i |0; k^{\mu}\rangle$ belongs to a representation of $SO(D-2)$ it must correspond to a massless state.

Lets consider the result of acting on the first excited state by the squared mass operator,

$$M^2(\alpha_{-1}^i |0; k^{\mu}\rangle) = \frac{2}{l_s^2}(N - a)(\alpha_{-1}^i |0; k^{\mu}\rangle) = \frac{2}{l_s^2}(1 - a)(\alpha_{-1}^i |0; k^{\mu}\rangle).$$

And so, in order to have an eigenvalue of 0 for the mass operator, and thus to be in agreement with Lorentz invariance, we must impose that $a = 1$.

Now, that we have a value for a, we want to determine the space-time dimension D (or c). Let us try to calculate the normal ordering constant directly. Recall that the normal ordering constant a arose when we had to normal order the expression

$$\frac{1}{2} \sum_{i=1}^{D-2} \sum_{m=-\infty}^{\infty} \alpha_{-m}^i \alpha_m^i.$$

So, we get

$$\frac{1}{2} \sum_{i=1}^{D-2} \sum_{m=-\infty}^{\infty} \alpha_{-m}^i \alpha_m^i = \sum_{i=1}^{D-2} \sum_{m=-\infty}^{\infty} : \alpha_{-m}^i \alpha_m^i : + \frac{(D-2)}{2} \sum_{m=1}^{\infty} m, \tag{5.6}$$

[3] $SO(n)$ is a normal subgroup, called the special orthogonal group in dimension n. It is also called the *rotation group*.

since $[\alpha^i_m, \alpha^i_{-m}] = m\delta_{ij}$. The second sum on the right hand side is divergent and we will use Riemann ζ-function regularization to take care of this problem. So, first consider the sum

$$\zeta(s) = \sum_{m=1}^{\infty} m^{-s} \quad s \in \mathbb{C}, \quad Re(s) > 1.$$

The zeta function, $\zeta(s)$, has a unique analytic continuation to $s = -1$, for which it takes the value $\zeta(-1) = -1/12$. Thus, plugging this into (5.6), the second term becomes

$$\frac{1}{2}\sum_{i=1}^{D-2}\sum_{m=-\infty}^{\infty} : \alpha^i_{-m}\alpha^i_m : -\frac{(D-2)}{24}.$$

Now, inserting the normal ordering constant back in, as in (5.6) with n = 0,

$$\frac{1}{2}\sum_{i=1}^{D-2}\sum_{m=-\infty}^{\infty} : \alpha^i_{-m}\alpha^i_m : -a,$$

which gives

$$\frac{D-2}{24} = a,$$

and since we already know $a = 1$, for Lorentz invariance, we get that $D = 26$.

5.3.3 Analysis of the Mass Spectrum

Finally, we will analyze the spectrum of a single free bosonic string.

Open Strings

Lets see the physical states of the open string for first few mass levels:

- For $N = 0$, the ground state $|0; k^\mu\rangle$ has mass imaginary mass, $\alpha' M^2 = -1$, where $\alpha' = l_s^2/2$. It is a *tachyon*. The ground state is unstable due to the imaginary mass. One can think that these particles travel faster than the speed of light but this is not the right interpretation. Suppose a field in space-time, whose quanta will give rise to this particle. The mass squared of the particle is the quadratic term in the action. The negative mass-squared is telling us that we are expanding around a maximum of the potential for the tachyon field. However, once we introduce fermions, supersymmetry eliminates the closed and open string tachyon. Therefore a tachyon is an unstable vacuum.

- For $N = 1$, the first excited state. There is a vector boson $\alpha^i_{-1}|0; k^\mu\rangle$ which, due to Lorentz invariance, is massless. This state gives a vector representation of $SO(24)$. According to [14], these group decompose into three irreducible representations:

  $$\text{traceless symmetric} \oplus \text{anti-symmetric} \oplus \text{singlet}.$$

  To each of these modes, there is associated a massless field in spacetime such that the string oscillation can be identified with a quantum of these fields. The fields are $G_{\mu\nu}(X)$, $B_{\mu\nu}(X)$ and $\Phi(X)$. These three massless fields are common to all string theories and the scalar field is called the *dilaton*.

- For $N = 2$ the states are $\alpha^i_{-2}|0; k^\mu\rangle$ and $\alpha^i_{-1}\alpha^j_{-1}|0; k^\mu\rangle$ with $\alpha' M^2 = 1$. These have 24 and $24 \times 25/2$ states, respectively. Thus, the total number of states is 324. This would be a huge mass state. We have that for higher states, even greater masses.

Closed String

For the closed string one must take into account there are both left-moving and right-moving modes. The mass of the states in the closed string spectrum is given by (4.6),

$$\alpha' M^2 = 4(N-1).$$

The physical states of the closed string at the first two mass levels are:

- For $N=0$, the ground state $|0;k^\mu\rangle$ has mass $\alpha' M^2 = -4$ and is again a *tachyon*.
- For $N=1$, there is a set of $24^2 = 576$ states of the form $|\Omega^{ij}\rangle = \alpha^i_{-1}\tilde{\alpha}^j_{-1}|0;k^\mu\rangle$, corresponding to the tensor product of two massless vectors, one left mover and one right mover. The part of $|\Omega^{ij}\rangle$ that is symmetric transforms under $SO(24)$ as a massless spin-2 particle, the *graviton*.

An infinite spectrum of masses comes out but they increase in mass and energy, so they are very difficult to produce.

Note that all of the states for open and closed strings either fall into multiplets of $SO(24)$ or $SO(25)$. This depends whether the state is massless or massive, because for a massive state it can Lorentz transform to a state of the form

$$|E, \underbrace{0,0,\dots,0}_{\text{25 times}}\rangle\,.$$

Now, the set of all transformations that leave this state unchanged is the set of rotations in 25 dimensions, i.e. the Little group[4] is given by SO(25), and so the massive state corresponds to some representation of the rotation group, SO(25). While for a massless state the Lorentz transformation transforms into the state of the form

$$|E, E, \underbrace{0,0,\dots,0}_{\text{24 times}}\rangle\,,$$

and so massless states have a Little group given by $SO(24)$. Thus, the massless states corresponds to some representation of $SO(24)$.

As we have already mentioned, the main difference between the bosonic string theory and the superstring theory is the addition of fermionic modes on its worldsheet. The resulting worldsheet theory is supersymmetric.

While the bosonic string is unique, there are a number of discrete choices that one can make when adding fermions to the worldsheet. This gives different perturbative superstrings theories. Although later developments reveal that they are all part of the same framework which goes by the name of *M theory*, as we mentioned in Section 2. So, whether we add fermions in both the left-moving and right-moving sectors of the string, or whether we choose the fermions to move only in one direction, usually taken the right-moving, this leads to two different classes of string theory: Type II, strings have both left and right-moving worldsheet fermions and Heterotic, strings have just right-moving fermions. In both cases, there is then one further discrete choice that we can make. This leaves us with four superstring theories: Type IIA, Type IIB, Heterotic $SO(32)$ and Heterotic $E_8 \times E_8$. It is sometimes said that there are five perturbative superstring theories in ten dimensions. The remaining theory is called Type I and includes open strings moving in flat ten dimensional space as well as closed strings, [14].

[4] Given a point $x \in X$ we set $G_x := \{g \in X | gx = x\}$. G_x is called the stabilizer or *little group* of x, [11].

Chapter 6

Conclusions

Throughout this study we have seen that String Theory is a theory of quantum gravity and provides new and surprising methods to understand aspects of quantum gauge theories. Also, it provides new results in mathematics, the most well know is *mirror symmetry*[1], a relation between topologically different Calabi-Yau manifolds[2], [14].

String Theory has also begun to address some of the deeper questions of quantum gravity, in particular the quantum mechanics of black holes, the collapsed remnants of large stars or the massive central cores of many galaxies, [10]. In 1976, Stephen Hawking proposed that black holes are associated with information loss, [6]. Physically, we can associate information with pure states in quantum mechanics. What Hawking found was that pure quantum states evolved into mixed states[3]. The implication is that perhaps a quantum theory of gravity would drastically alter quantum theory to allow for non-unitary evolution[4]. However, string theory is a fully quantum theory so evolution is unitary and using this theory it is possible to count the microscopic states of a black hole and compare this to the result obtained using the laws of black hole mechanics[5]. It was found that there is an exact agreement using this two methods, [15].

String Theory and the different dimensions that arise from it and in the other theories that we have seen has also led to physical and philosophical debate. With the intention of explaining why we live in the world in which we live, the anthropic principle (weak and strong) postulated by Brandon Carter in 1974 arises, [3]. The anthropic principle refers to the position that human occupies in the universe, is based on the reflection on how delicate the necessary conditions are for there to be life in the universe and the fact that life could not have appeared if any of the constants of nature had a slightly different value. Stephen Hawking states that the anthropic principle consists in stating that we see the universe the way it is because we exist [9]. An essential hypothesis of the anthropic principle is the existence of an enormous number of universes, each with different physical laws. The collection of all these universes is called the multiverse[6].

Now, to finish the study, we will review some problems that have appeared and some that have been solved in String Theory over the years. In the mid-1980s, when String Theory became a hot

[1]Mirror symmetry is an example of a general phenomenon known as duality, which occurs when two different physical systems are isomorphic in a non-trivial way, [7].

[2]A Calabi-Yau manifold is a compact, complex, Kähler manifold which has $SU(d)$ holonomy. An equivalent statement is that a Calabi-Yau manifold admits a Ricci-flat metric. This information is in more detail in [5].

[3]In ordinary quantum physics, it is not possible for a pure quantum state to evolve into a mixed state. This is because of the unitary nature of time evolution.

[4]This is a disadvantage because non-unitary transformations do not preserve probabilities.

[5]Black hole mechanics state that entropy is proportional to area, $S = A/4G$.

[6]More information about multiverse on page 212 in [13].

subject, some of the problems and questions that came up were as follows, [12]:

- Hundreds of Calabi-Yau manifolds were known. Now we know that there are many thousands. So, which one of them, if any, is the right one?
- Why are there four other consistent Superstring theories? Are some of them inconsistent, or else, could some of them somehow be equivalent?
- What ensures that the vacuum energy density or dark energy is sufficiently small, namely of order $10 - 120$ in Planck units? In 1985 it was generally believed to be zero, so the idea was to look for a symmetry principle that would enforce this. Nowadays, we know that the vacuum energy density is not exactly zero.
- What does String Theory have to say about cosmology?

Later it was shown that there is just one theory, moreover, new objects, called branes, arise non perturbatively, where p is an integer that represents the number of spatial dimensions of the brane. Stable branes carry conserved charges and satisfy generalized Dirac quantization conditions. There are also unstable branes that do not carry conserved charges. The main categories of stable branes are D-branes, M-branes, and NS5-branes. It has been seen that D-branes are characterized by Dirichlet boundary conditions for open strings[7].

Introducing Supersymmetry we are able to eliminate the tachyon, corresponding to the ground state, which would be unstable (due to the negative mass of the particle). This theory makes use of supersymmetry that relates the properties of bosons to fermions, As we have already mentioned in Chapter 2. For example, the analogous particle to the graviton (boson) is be the gravitino (fermion). Because fermions are included in the theory, all matter is represented. Also the dimensions of space-time are reduced from 26 to 10. However, Superstring theory is still under development and has problems to solve as well.

As we have been able to observe throughout this study, String Theory still lacks much development, although it has been seen that it is a great solution to the quantization of Einstein's theory, in addition to the importance of addressing some of the most deeper questions of the quantum mechanics of black holes.

[7] More information about the problems that arise in [12].

Appendix A

A.1 The action function for the electromagnetic field

In order to establish the form of the action that corresponds to the electromagnetic field, we will start from the following property of magnetic fields: the electromagnetic field obeys the Principle of Superposition. This principle states that if one charge produces a certain field and another charge produces a second field, the field produced by the two charges results from the simple composition of the fields produced by each of them individually. According to this, the sum of any number of such fields must be a field that can exist in nature, that is, it must satisfy the equations of the field. We know that for linear homogeneous differential equations the sum of any number of solutions is also a solution. Then the field equations must be homogeneous linear differential equations.

From this discussion we have that the integrand corresponding to the action S_f must be a quadratic expression in the components of the field. Only in this case the equations will be linear, since the equations of the field are obtained by varying the action.

The potentials cannot appear in the expression of the action S_f, since they are not uniquely determined. Therefore, S_f must be the integral of a certain function of the tensor of the electromagnetic field F_{ik}. But the action must be a scalar and therefore an integral of a certain scalar. This can only be the product $F_{ik}F^{ik}$.

Therefore, S_f must be of the form:

$$S_f = a \int \int F_{ik}F^{ik} dV dt, \quad dV = dxdydz,$$

where the integral extends over all space and all time between two given instants; a is some constant and

$$F_{ik} = \frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k}.$$

In the integrand there appears $F_{ik}F^{ik} = 2(H^2 - E^2)$. The field $\mathbf{E}$ contains the derivative $\frac{\partial \mathbf{A}}{\partial t}$, but it is easy to see that $(\partial \mathbf{A}/\partial t)^2$ must appear in the action with a positive sign (and, therefore, $\mathbf{E}^2$ must also have a positive sign). Indeed, if $(\partial \mathbf{A}/\partial t)^2$ appears in S_f with a negative sign, a sufficiently rapid change in potential with time (in the time interval considered) could always make S_f a negative quantity of arbitrarily large absolute value. Under these conditions, S_f could not have a minimum, contrary to what is required by the principle of least action. Therefore, it must be negative.

The numerical value of a depends on the choice of units to measure the field. Note that once a value for a and the units for the field measurement have been chosen, the units for the field measurement, the units for all other electromagnetic quantities are determined. In what follows we will use the so-called Gaussian system of units; in this system a is of zero dimension and equal to $-\frac{1}{16\pi}$.

So, the action for the field is of the form

$$S_f = -\frac{1}{16\pi c}\int F_{ik}F^{ik}d\Omega, \quad \Omega = c\,dt\,dx\,dy\,dz,$$

three-dimensionally,

$$S_f = \frac{1}{8\pi}\int (E^2 - H^2)dV dt.$$

The Lagrange function of the field is

$$L_f = \frac{1}{8\pi}\int (E^2 - H^2)dV.$$

A.2 The action function of the gravitational field

To determine this scalar we will assume that the equations of the gravitational field must contain derivatives of the "potentials" of order not higher than the second (in a similar way to what happens in the case of the electromagnetic field). Since the equations of the field are obtained by varying the action, it is necessary that the integrand G contains derivatives of the g_{ik} of order not higher than the first; therefore G must contain only the tensor g_{ik} and the quantities Γ^i_{kl}.

However, it is impossible to construct a scalar from the quantities g_{ik} and Γ^i_{kl} alone. It goes without saying that, by a suitable choice of coordinate system, we can always reduce all Γ^i_{kl} quantities at a given point to zero. There is, however, the scalar R (the curvature of four-dimensional space), which, although it contains, in addition to the g_{ik} and its first derivatives, the second derivatives of the g_{ik}, is linear with respect to them. Thanks to this character, the invariant integral $\int R\sqrt{-g}\,d\Omega$ can be transformed, through Gauss's theorem, into the integral of an expression that does not contain second derivatives.

Indeed, $\int R\sqrt{-g}\,d\Omega$ can be represented in the form:

$$\int R\sqrt{-g}\,d\Omega = \int G\sqrt{-g}\,d\Omega + \int \frac{\partial(\sqrt{-g}\omega^i)}{\partial x^i}d\Omega,$$

where G contains only the tensor g_{ik} and its first derivatives, and the integrand of the second integral has the form of the divergence of some magnitude w^i. According to Gauss's theorem, this last integral can be transformed into an integral extended to a hypersurface surrounding the quadvolume in which the integration of the other two integrals takes place. When the action is varied, the variation of the second term of the second member vanishes, since in the principle of least action the variations of the field in the limits of the integration region are equal to zero. Consequently, we can write:

$$\int R\sqrt{-g}\,d\Omega = \int G\sqrt{-g}\,d\Omega.$$

The first member is a scalar; therefore, the expression that appears on the second member is also a scalar.

The magnitude G fulfills the previously imposed condition, since it contains only the g_{ik} and their first derivatives. We can therefore write

$$\delta S_g = -\frac{c^3}{16\pi k}\delta\int G\sqrt{-g}\,d\Omega = -\frac{c^3}{16\pi k}\delta\int R\sqrt{-g}\,d\Omega,$$

where k is a new universal constant. As in the case of the action of the electromagnetic field, it can be seen that the constant k must be positive. The constant k is called the constant of gravitation.

The dimensions of the action are $\mathrm{g\,cm^2 s^{-1}}$; it can be considered that all the coordinates have the dimensions of a length and that the g_{ik} are magnitudes of null dimension, so the dimensions of R are cm^{-2}. It turns out, then, that the dimensions of k are $\mathrm{cm^3 g^{-1} s^{-2}}$. Its numerical value is $k = 6.67 \times 10^{-8} \mathrm{cm^3 g^{-1} s^{-2}}$. Note that we could have set k equal to unity. However, this would have fixed the value of the unit of mass. We can calculate the magnitude

$$G = g^{ik} \left(\Gamma^m_{il} \Gamma^l_{km} - \Gamma^l_{ik} \Gamma^m_{lm} \right).$$

The components of the metric tensor are the quantities that determine the gravitational field. Therefore, the Principle of least action that corresponds to it, the quantities that are varied are the g_{ik}. However, we cannot now say that in a really existing field the action integral presents a minimum (and not just an extremum) with respect to all possible variations of the g_{ik}. This is linked to the fact that not every change in the g_{ik} corresponds to a change in the space-time metric, that is, to a real change in the gravitational field. The components g_{ik} also change in a simple transformation of coordinates linked merely with the passage from one system to another in the same space-time. Each of these coordinate transformations is, roughly speaking, a set of four independent functions (one per coordinate). In order to exclude such changes in the g_{ik}, changes that are not associated with a change in the metric, we can impose four supplementary conditions and require that these conditions be fulfilled in the variation process. Thus, when a gravitational field is applied, the Principle of Least Action states only that it is possible to impose additional conditions on the g_{ik} such that, when fulfilled, the action presents a minimum with respect to the variations of the g_{ik}.

Appendix B

B.1 Bosonic String Theory invariance under Weyl transformation

Lets show that our bosonic string theory is invariant under Weyl transformations. First, we preceed transforming $\sqrt{-h}$ and $\sqrt{-h}\ h^{\alpha\beta}$.

$$\sqrt{-h'} = \sqrt{-det(h'_{\alpha\beta})} = e^{2(2\phi(\sigma))/2}\sqrt{-det(h_{\alpha\beta})} = e^{2\phi(\sigma)}\sqrt{-h}$$

Expanding (3.3) in ϕ yields that

$$h'^{\alpha\beta} = e^{-2\phi}h^{\alpha\beta} = (1-2\phi+\cdots)h^{\alpha\beta}$$

Thus, the variation infinitesimally of $h^{\alpha\beta}$ is given by

$$\delta h^{\alpha\beta} = -2\phi h^{\alpha\beta}.$$

Now, for $\sqrt{-h}\ h^{\alpha\beta}$, we have

$$\sqrt{-h'}h'^{\alpha\beta} = \sqrt{-h}e^{2\phi(\sigma)}e^{-2\phi(\sigma)}h^{\alpha\beta} = \sqrt{-h}h^{\alpha\beta}.$$

Thus, under a Weyl transformation S_σ is invariant, which implies that the variation of S_σ under a Weyl transformation vanishes. This says that our bosonic string theory is invariant under Weyl transformations. We will now show that since our theory is invariant under Weyl transformations this implies that the stress-energy tensor associated with this theory is traceless, $h^{\alpha\beta}T_{\alpha\beta} = 0$, , where the stree-energy tensor is given by

$$T_{\alpha\beta} = -\frac{2}{T}\frac{1}{\sqrt{h}}\frac{\delta S_\sigma}{\delta h_{\alpha\beta}}. \tag{B.1}$$

This implies that under a generic transformation of the field $h^{\alpha\beta}$, the variation of S_σ can be written as

$$\delta S_\sigma \equiv \int \frac{\delta S_\sigma}{\delta h^{\alpha\beta}}\delta h^{\alpha\beta} = -\frac{T}{2}\int d\tau d\sigma\sqrt{-h}\ \delta h^{\alpha\beta}T_{\alpha\beta}.$$

Restricting to a Weyl transformation,

$$\begin{aligned} S_\sigma &= -\frac{T}{2}\int d\tau d\sigma\sqrt{-h}\ \delta h^{\alpha\beta}T_{\alpha\beta} \\ &= -\frac{T}{2}\int d\tau d\sigma\sqrt{-h}(-2\phi)h^{\alpha\beta}T_{\alpha\beta} \end{aligned}$$

This is equal to zero since there is not variation in S_σ under a Weyl transformation. Since $\sqrt{-h}$ and ϕ are arbitrary,

$$h^{\alpha\beta}T_{\alpha\beta} = 0.$$

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