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Profit maximization in an inventory system with time-varying demand, partial backordering and discrete inventory cycle

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Abstract

In this paper, an inventory problem where the inventory cycle must be an integer multiple of a known basic period is considered. Furthermore, the demand rate in each basic period is a power time-dependent function. Shortages are allowed but, taking necessities or interests of the customers into account, only a fixed proportion of the demand during the stock-out period is satisfied with the arrival of the next replenishment. The costs related to the management of the inventory system are the ordering cost, the purchasing cost, the holding cost, the backordering cost and the lost sale cost. The problem is to determine the best inventory policy that maximizes the profit per unit time, which is the difference between the income obtained from the sales of the product and the sum of the previous costs. The modeling of the inventory problem leads to an integer nonlinear mathematical programming problem. To solve this problem, a new and efficient algorithm to calculate the optimal inventory cycle and the economic order quantity is proposed. Numerical examples are presented to illustrate how the algorithm works to determine the best inventory policies. A sensitivity analysis of the optimal policy with respect to some parameters of the inventory system is developed. Finally, conclusions and suggestions for future research lines are given.

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1 Introduction

The Inventory Theory is one of the most interesting fields of Operational Research. This area of knowledge has attracted significant attention from academics with the aim of improving managerial decision-making related to logistics and the commercial distribution of goods. The main purpose of inventory models is to determine appropriate stock levels to meet customer demand while minimizing inventory costs, providing effectiveness, efficiency and improved allocation of available resources. The first mathematical model developed for the control of the inventories, which is known as the economic order quantity model (EOQ), was proposed by Ford Whitman Harris at the beginning of the 20th century. Despite its simplicity, this model continues to be applied today in commerce and industry. Since the publication of that paper, a lot of additional research to analyze some variants of that model has been done by modifying some of its unrealistic hypotheses. For example, in the classical EOQ model, the demand rate is considered as constant. However, in some practical situations, customer demand may vary over time. Therefore, inventory models that consider time-varying demand can better represent the evolution of the inventory level. For this reason, it is not surprising that many researchers have studied different inventory models in which the demand rate varies over time. Banerjee and Sharma (2010) analyzed a deterministic inventory model for a product with seasonal demand which is a general function of time and price, considering that the shortage time cannot exceed the length of a seasonal interval. Sett et al. (2012) developed a two-warehouse inventory model with quadratically increasing time-dependent demand rate, whose solution is obtained by using an algorithmic procedure. Hsieh and Dye (2013) presented a production–inventory model with time-varying demand and finite replenishment rate, allowing a preservation technology cost as a decision variable in conjunction with production policy. Khanra et al. (2013) proposed an economic order quantity model over a finite time horizon for an item with a quadratic time-dependent demand by considering shortages in inventory under permissible delay in payments. Mishra (2013) developed the optimal inventory policy for a system whose demand rate is a quadratic-time function. Massonnet et al. (2014) adapted a cost balancing technique and applied it to continuous-review inventory models when demands and cost parameters are time-varying. Pervin et al. (2016) proposed an EPQ inventory model under trade credit financing by assuming that the demand rate is a linearly decreasing function of time and shortages are not allowed. Prasad and Mukherjee (2016) studied an inventory model for deteriorating items with stock and time-dependent demand. Benkherouf et al. (2017) proposed a finite horizon inventory control problem for two substitutable products with continuous time–varying demand, which are ordered jointly in each replenishment epoch. Pervin et al. (2018) developed a deterministic inventory control model where the demand rate and holding costs are totally time dependent and shortages are completely backlogged. Saha and Sen (2019) presented an inventory model with selling price and time dependent demand, constant holding cost and partial backlogging under the effect of inflation. Pervin et al. (2020) formulated an integrated vendor-buyer model with quadratic demand under inspection policy and preservation technology. Other works that consider time-dependent demand are, for example, Ghoreishi et al. (2015), Roy et al. (2020), Shah et al. (2020).

Demand properties are critical in determining optimal inventory management policies. In the conventional analysis of deterministic systems, it is assumed that the demand of products is uniform along the inventory cycle. This means that the products are demanded by consumers and removed from inventory at a constant rate per unit time. An approach to model the way in which items are extracted from inventory to satisfy customer demand is to consider demand patterns. One of the most versatile demand patterns is the power demand pattern, which allows the demand of the items to be adjusted to a wide variety of practical situations. This demand pattern allows us to model different ways of removing items from the inventory along the cycle. Thus, there may be a high percentage of requests to purchase items at the beginning of the period, or to keep the demand of the items constant throughout the inventory cycle, or also to have a more concentrated demand at the end of the period. In real markets, it is possible to identify a lot of products such that the customers' demand behavior is similar to the power pattern. In the literature on inventory models, there are several papers in which this type of demand is considered. Thus, Rajeswari and Vanjikkodi (2011) developed a deterministic inventory model with power pattern demand, constant deterioration rate and shortages completely backlogged. Mishra et al. (2012) presented an economic order quantity model for perishable items with power demand pattern under the influence of inflation and time-value of money. Mishra and Singh (2013) formulated a partial backlogging inventory model with time-dependent power demand pattern and quadratic deterioration rate. Rajeswari and Indrani (2015) analyzed an economic order quantity model under total cost minimization for linear time dependent deteriorating items with power demand pattern and partial backlogging. San-José et al. (2019a) developed an inventory model for items whose demand was the sum of a linear function with respect to the selling price and of a power-time function, assuming non-linear holding cost. San-José et al. (2020) developed an inventory model for products whose demand multiplicatively combines a potential function of time and a tri-exponential function of the selling-price where shortages are fully backordered. Other research papers that consider a power demand pattern are, for example, Adaraniwon and Omar (2019), Keshavarzfard et al. (2019a, b), San-José et al. (2019b); San-Jose et al. (2021).

When shortages occur in the inventory cycle, it is habitual that some customers are not willing to wait and go to buy items from other sellers, while other customers are willing to wait for the arrival of the next order. For this reason, several papers on inventory models have considered that only a part of the demand during the stock-out period is backordered. For example, Mishra and Singh (2010), Sicilia et al. (2012a), Taleizadeh (2018), Taleizadeh et al. (2020), Mallick et al. (2020), Gupta et al. (2020), Khan et al. (2020), Adak and Mahapatra (2020) studied several inventory models with partial backordering. San-José et al. (2017) obtained the optimal policy for an inventory system with power demand pattern, assuming partial lost sales. These hypotheses of power demand pattern and partial lost sales are also considered in other articles, such as San-José et al. (2018a, b).

In this paper, an inventory model for a single item is studied. It is assumed that there is a prescribed basic period of time in which the total demand is known. The distribution of this demand along this basic period follows a power demand pattern. The inventory cycle must be an integer multiple of that basic time-period. It is assumed that shortages are allowed and only a fixed fraction of demand is backordered. In the management of this inventory system, there are four significant costs: the holding cost, the shortage cost (includes the lost sale cost and the backordering cost), the purchasing cost and the setup cost for placing an order to replenish the inventory. A profit function is defined as the difference between income and the costs related to inventories. The objective is to determine the optimal inventory cycle and the economic order quantity that maximize the profit per unit time. Thus, the inventory problem

is formulated as an integer nonlinear mathematical program. To solve this problem, a new algorithm that determines the optimal inventory policy is developed.

This new inventory model can be useful for items whose inventory replenishment must be ordered over a period of time that is a multiple of a basic prescribed period, and demand is sensitive to the effect of the time elapsed since the beginning of the basic period. For example, in a small supermarket, the replenishment of an item is done on the morning of each day (or every two days, or every three days, etc.) and, depending on the item, the manager decides how the demand required by customers is satisfied. For example, for cooked products, such as sweets, breads, cakes, etc., a larger quantity of items is demanded at the beginning than at the end of the basic period, because the customers prefer to buy these products when they believe that they have recently been put up for sale. Other items have a uniform demand rate along the inventory cycle. For instance, electrical goods, supplies, furniture, kitchen utensils and appliances, etc., have a more or less constant demand during the replenishment cycle. Lastly, there exist other products, such as cinema tickets, hamburgers, hot dogs, pizzas, which may have a lower demand at the beginning of the basic period and this increases during the day (basic period). Moreover, when shortages of the item occur, some customers are willing to wait for the arrival of the next replenishment, while other customers are not willing to wait and go to buy items from other sellers.

The main contribution of this paper is to obtain the optimal inventory policy that maximizes the total profit per unit time, assuming three interesting topics not dealt with jointly in the literature: (i) the inventory cycle must be a multiple of a known basic time period, (ii) customer demand follows a power demand pattern in each basic period, and (iii) only a fixed fraction of the demand during the stock-out period is satisfied with the arrival of the next replenishment. The consideration of all the previous assumptions makes the model more realistic, because they allow us to model a variety of real life situations.

The remainder of this paper is organized as follows. In Sect. 2, the basic assumptions and notation used throughout the work are introduced. In Sect. 3, the mathematical model is formulated. In Sect. 4, the necessary conditions to determine the optimal inventory policy are developed and an algorithmic procedure to obtain the optimal solution of the inventory problem is proposed. In Sect. 5, some numerical examples and a sensitivity analysis are given to illustrate the main results of the paper. Finally, Sect. 6 presents the conclusions and some suggestions for future research lines.

2 Hypothesis and notation

The mathematical model is developed under the following assumptions:

1. The inventory refers to a single item.
2. Shortages are allowed, but only a fixed fraction ρ (with $0 < \rho \leq 1$) of the demand during the stock-out period is satisfied with the arrival of the next replenishment.
3. The inventory replenishment is instantaneous.
4. The lead-time is insignificant.
5. During a basic time-period of length τ, $\lambda\tau$ units are demanded following a power demand pattern. That is, the demand rate along this basic period is $\lambda\delta\left(\frac{t}{\tau}\right)^{\delta-1}$, with $\delta > 0$ and $0 < t < \tau$. This demand pattern has been used by several authors in recent years (see, for example, Adaraniwon and Omar (2019), Keshavarzfard et al. (2019a, b), Mishra et al. (2012), San-José et al. (2017, 2018a, 2019a, 2020), Sicilia et al. (2012)).

6. The inventory is replenished when the number of backorders is equal to $-s$ units and the order quantity Q is always the same for every inventory cycle.

The notation used throughout the paper is presented in Table 1.

3 Mathematical formulation of the problem

Taking into account the assumptions admitted in the previous section, it follows that the net inventory level at time t, $I(t)$, is a periodic function of period $T = n\tau$, where τ is the length of the basic period and n is any positive integer. Moreover, $I(t)$ is a continuous function in the interval $(0, T)$. The inventory cycle is divided into n basic periods of length τ, so that in each basic period the demand has a power time-dependent pattern. In addition, on the first $(n - m)$ basic periods of each cycle, the net inventory level is positive and on the remaining m basic periods, the net inventory level is negative. Therefore, at the beginning of the inventory cycle, the maximum inventory level is $S = (n - m)\lambda\tau$. Next, the inventory decreases due to demand and, at time $t = t_1$, the inventory level is zero. In the period (t_1, T), the inventory level is negative and, at the end of this period, there are b shortages in the system. Since we have assumed that when shortage occurs only a fraction ρ of the demand is satisfied with the next replenishment, the number of units demanded in each cycle that are served late are $-s = \rho b = m\lambda\rho\tau$ and the amount of lost sales in each cycle is $m\lambda(1 - \rho)\tau$.

Taking into account the previous hypotheses, the inventory level $I(t)$ along the stock-in period is governed by the differential equation

$$\frac{dI(t)}{dt} = -\lambda\delta \left(\frac{t}{\tau} - (i - 1)\right)^{\delta-1}, t \in ((i - 1)\tau, i\tau), 1 \le i \le n - m$$

with $I(i\tau) = (n - m - i)\lambda\tau$, for $0 \le i \le n - m$. Therefore, the inventory level at any instant of time t during $[0, t_1]$ is given by

$$I(t) = (n - m - i + 1)\lambda\tau - \lambda\tau \left(\frac{t}{\tau} - i + 1\right)^{\delta}, t \in [(i - 1)\tau, i\tau), 1 \le i \le n - m \quad (1)$$

In the shortage period, the net stock level follows the differential equation

$$\frac{dI(t)}{dt} = -\lambda\delta\rho \left(\frac{t}{\tau} - (i - 1)\right)^{\delta-1}, t \in ((i - 1)\tau, i\tau), n - m < i \le n$$

with $I(i\tau) = (n - m - i)\lambda\rho\tau$, for $n - m \le i < n$. Consequently, the net inventory level in the interval $[t_1, T)$ is given by

$$I(t) = (n - m - i + 1)\lambda\rho\tau - \lambda\rho\tau \left(\frac{t}{\tau} - i + 1\right)^{\delta}, t \in [(i - 1)\tau, i\tau), n - m < i \le n \quad (2)$$

Figures 1 and 2 illustrate the behavior of the net inventory level for different demand pattern indexes.

The order quantity at the beginning of every inventory cycle is

$$Q = S - s = (n - (1 - \rho)m)\lambda\tau. \quad (3)$$

The aim is to maximize the profit function per unit time $P(m, n) = B(m, n)/n\tau$, where $B(m, n)$ is the profit

Table 1 Notation

Parameters	
τ	Basic time-period (> 0)
λ	Average demand per basic period (> 0)
δ	Index of the power demand pattern (> 0)
K	Cost of placing an order (> 0)
c	Unit purchasing cost (> 0)
p	Unit selling price ($p \geq c$)
h	Holding cost per unit and per unit time (> 0)
ρ	Fraction of demand in the stock-out period that is served with the next order ($0 < \rho \leq 1$)
ω	Shortage cost per backordered unit and per unit time (> 0)
π	Goodwill cost of a lost sale (≥ 0)
a_0	Auxiliary parameter, $a_0 = K/\tau$
a_1	Auxiliary parameter, $a_1 = (h + \rho\omega)\left(1/(\delta+1) - 1/2\right)\lambda\tau + (\pi + p - c)(1 - \rho)\lambda$
a_2	Auxiliary parameter, $a_2 = (h + \rho\omega)\lambda\tau/2$
Decision variables	
n	Number of basic periods included in the inventory cycle (> 0)
m	Number of basic periods included in the stock-out period ($0 \leq m \leq n$)
Other variables	
T	Inventory cycle, which is an integer multiple of the basic period, that is, $T = n\tau$
t_1	Length of the stock-in period, that is, $t_1 = (n - m)\tau$
t_2	Length of the stock-out period, that is, $t_2 = m\tau$
Q	Lot size per cycle (≥ 0)
S	Inventory level at the beginning of the inventory cycle, that is, $S = (n - m)\lambda\tau$
b	Maximum shortage per cycle, that is, $b = m\lambda\tau$
s	Minimum inventory level, that is, $s = -\rho b$
Functions	
$I(t)$	Inventory level at time t, with $0 \leq t < T$
$B(m,n)$	Total profit obtained during each inventory cycle
$P(m,n)$	Profit per unit time
$C(m,n)$	Total inventory cost per unit time

during each inventory cycle. It is calculated as the difference between the revenue per cycle and the sum of the ordering, the purchasing, the holding, the backordering and the lost sale costs.

In the following paragraphs, the income and the costs are firstly calculated, and then the profit is determined as a function of the integer variables m and n.

- Revenue per cycle: $pQ = p\lambda\tau\left(n - (1 - \rho)m\right)$
- Ordering cost per cycle: K
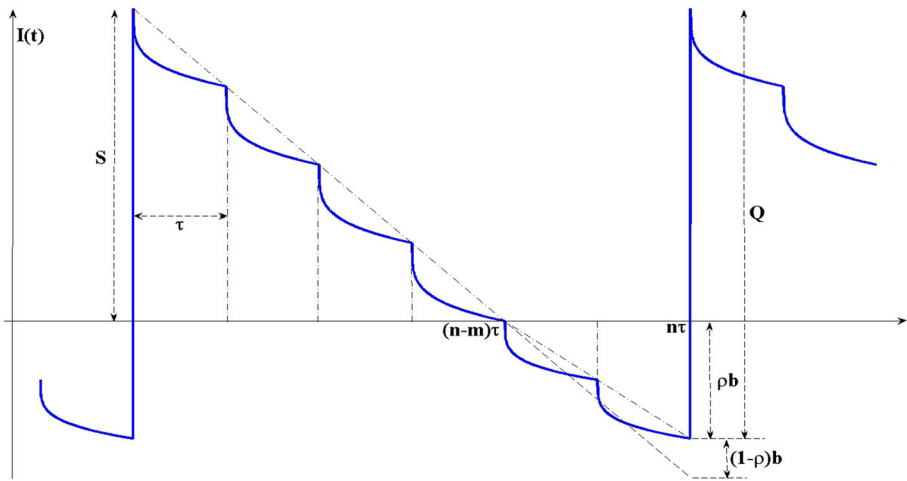- Purchasing cost per cycle: $cQ = c\lambda\tau\left(n - (1 - \rho)m\right)$

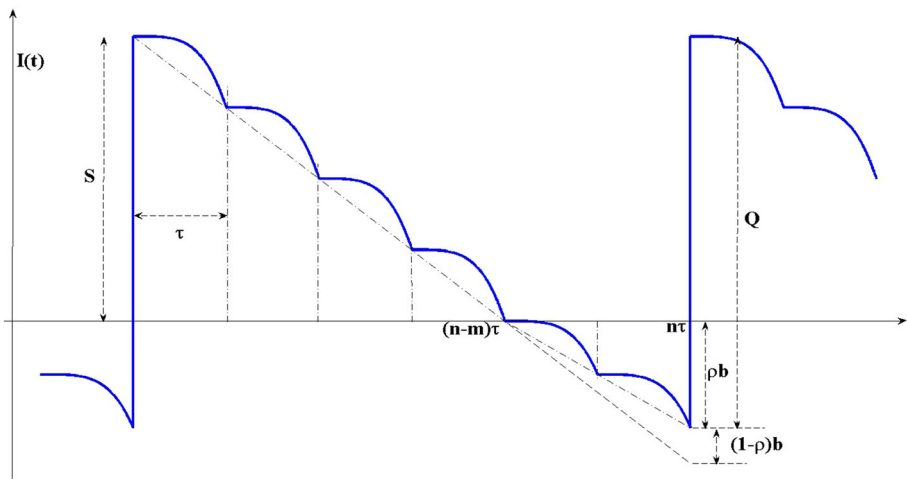**Fig. 1** Net inventory level when $\delta < 1$

Fig. 2 Net inventory level when $\delta > 1$

- Holding cost per cycle:

$$\int_0^{t_1} hI(t)dt = h \sum_{i=1}^{n-m} \int_{(i-1)\tau}^{i\tau} \left((n-m-i+1)\lambda\tau - \lambda\tau \left(\frac{t}{\tau} - i + 1 \right)^\delta \right) dt$$

$$= h\lambda\tau^2 \sum_{i=1}^{n-m} \left(n - m - i + 1 - \frac{1}{\delta+1} \right)$$

$$= h(n-m) \left(\frac{n-m+1}{2} - \frac{1}{\delta+1} \right) \lambda\tau^2. \tag{4}$$

- Backordering cost per cycle:

$$\int_{t_1}^{T} \omega\left(-I(t)\right) dt = \omega \sum_{i=n-m+1}^{n} \int_{(i-1)\tau}^{i\tau} \left(\lambda\rho\tau \left(\frac{t}{\tau} - i + 1\right)^{\delta} - (n - m - i + 1)\lambda\rho\tau\right) dt$$

$$= \omega\lambda\rho\tau^2 \sum_{i=n-m+1}^{n} \left(\frac{1}{\delta+1} + i - n + m - 1\right)$$

$$= \omega m \left(\frac{1}{\delta+1} + \frac{m-1}{2}\right) \lambda\rho\tau^2. \tag{5}$$

- Goodwill cost of lost sales per cycle:

$$\pi m\lambda(1-\rho)\tau. \tag{6}$$

Therefore, the total profit during each inventory cycle is given by

$$B(m,n) = (p-c)\lambda\tau\left(n - (1-\rho)m\right) - K - h(n-m)\left(\frac{n-m+1}{2} - \frac{1}{\delta+1}\right)\lambda\tau^2$$

$$- \omega m \left(\frac{1}{\delta+1} + \frac{m-1}{2}\right) \lambda\rho\tau^2 - \pi\lambda(1-\rho)\tau m$$

$$= (p-c)n\lambda\tau - K - h(n-m)\left(\frac{n-m+1}{2} - \frac{1}{\delta+1}\right)\lambda\tau^2$$

$$- \omega m \left(\frac{1}{\delta+1} + \frac{m-1}{2}\right) \lambda\rho\tau^2$$

$$- (\pi + p - c)m\lambda(1-\rho)\tau \tag{7}$$

Thus, the total profit per unit time is determined by

$$P(m,n) = \frac{B(m,n)}{T} = (p-c)\lambda - C(m,n) \tag{8}$$

where

$$C(m,n) = \frac{K}{n\tau} + h\left(1 - \frac{m}{n}\right)\left(\frac{n-m+1}{2} - \frac{1}{\delta+1}\right)\lambda\tau + \omega\frac{m}{n}\left(\frac{1}{\delta+1} + \frac{m-1}{2}\right)\lambda\rho\tau$$

$$+ (\pi + p - c)\frac{m}{n}\lambda(1-\rho) \tag{9}$$

So, taking into account (8), minimizing the function $C(m,n)$ is equivalent to maximizing $P(m,n)$. Therefore, the optimization problem addressed in this paper can be formulated as

$$\min_{(m,n)\in\Pi} C(m,n), \tag{10}$$

where $\Pi = \{(m,n) : n > 0, 0 \leq m \leq n \text{ and } m, n \in \mathbb{Z}\}$.

4 Solution to the inventory problem

As we have just indicated, our objective is to determine the optimal solution of the integer nonlinear program given by (10). Taking into account the structure of the feasible region Π of the problem (10), we begin by dividing this region Π into two subregions: $\Pi_0 = \{(0,n) : n > 0, n \in \mathbb{Z}\}$ and $\Pi_1 = \{(m,n) : 1 \leq m \leq n \text{ y } m, n \in \mathbb{Z}\}$.

We will analyze the behavior of the function $C(m, n)$ on each subregion. Firstly, in the next subsection let us study the function $C(m, n)$ on the sub-region Π_0.

4.1 Analysis of the objective function on the subregion Π_0

In this case, as $m = 0$, the problem (10) is reduced to

$$\min_{n>0, n \in \mathbb{Z}} C_0(n), \tag{11}$$

where

$$C_0(n) = \frac{K}{n\tau} + h \left(\frac{n+1}{2} - \frac{1}{\delta + 1} \right) \lambda\tau. \tag{12}$$

This problem represents the situation in which the system has no shortages.

The following result gives us the optimal number of basic periods that the inventory cycle must have.

Theorem 1 *An optimal solution of the problem* (11) *is given by*

$$n^*(0) = \left\lceil \frac{1}{2} \left(\sqrt{\frac{8K + h\lambda\tau^2}{h\lambda\tau^2}} - 1 \right) \right\rceil, \tag{13}$$

where $\lceil x \rceil$ is the least integer greater than or equal to x. The minimum cost is given by

$$C_0^i = C_0(n^*(0)) = \frac{K}{n^*(0)\tau} + \frac{h\lambda\tau}{2} n^*(0) + \frac{\delta - 1}{2(\delta + 1)} h\lambda\tau \tag{14}$$

Proof See "Appendix". □

Remark 1 Note that the continuous optimal inventory policy on the region $\Pi_0^c = \{(0, n) : n \in \mathbb{R}^+\}$ is given by

$$n_0^c = \sqrt{\frac{2K}{h\lambda\tau^2}} \tag{15}$$

with optimal value

$$C_0^c = \sqrt{2Kh\lambda} + \frac{\delta - 1}{2(\delta + 1)} h\lambda\tau. \tag{16}$$

4.2 Analysis of the objective function on the subregion Π_1

To study the function $C(m, n)$ on the subregion Π_1, we relax the conditions of integrality on the variables m and n, and then we analyze the behavior of this function on the continuous region $\Pi_1^c = \{(m, n) : m \geq 1, n > 0\}$.

Firstly, we assume that the variable m is fixed, with $m \geq 1$. Thus, we are considering the function of a continuous variable $C^m(n) = C(m, n)$. Note that $C^m(n)$ can be rewritten as

$$C^m(n) = \frac{A_1(m)}{n} + \frac{h}{2}\lambda\tau n + A_2(m), \tag{17}$$

where

$$A_1(m) = a_0 + a_1 m + a_2 m^2 \tag{18}$$

is a parabola that has the following parameters:

$$a_0 = \frac{K}{\tau}, \, a_1 = (h+\rho\omega)\left(\frac{1}{\delta+1} - \frac{1}{2}\right)\lambda\tau + (\pi+p-c)(1-\rho)\lambda \text{ and } a_2 = \frac{h+\rho\omega}{2}\lambda\tau, \quad (19)$$

and $A_2(m)$ is the linear and strictly negative function given by

$$A_2(m) = \left(\frac{\delta-1}{2(\delta+1)} - m\right)h\lambda\tau. \quad (20)$$

Thus, $A_1(m)$ is a strictly positive function, because it can also be expressed as

$$A_1(m) = \frac{2(\delta+1)K + \{2(\pi+p-c)(1-\rho)(1+\delta) + [\delta(m-1)+m+1](h+\rho\omega)\tau\}\lambda\tau m}{2(\delta+1)\tau}.$$

Therefore, the equation $a_0 + a_1 m + a_2 m^2 = 0$ has no real solutions. Hence, its discriminant $\Delta = a_1^2 - 4a_0 a_2$ is always negative.

A first result about the behavior of the function $C^m(n)$ is as follows:

Lemma 1 *The function $C^m(n)$ on the region Π_1^c is strictly convex and attains its minimum value at the point*

$$n^c(m) = \sqrt{\frac{2\left(a_0 + a_1 m + a_2 m^2\right)}{h\lambda\tau}} \quad (21)$$

with optimal value $C^m(n^c(m)) = C_1(m)$, where

$$C_1(m) = \sqrt{2h\lambda\tau\left(a_0 + a_1 m + a_2 m^2\right)} + [(\delta-1)/(2(\delta+1)) - m]h\lambda\tau. \quad (22)$$

Proof See "Appendix".   □

Next, we shall study some characteristics of the function $C_1(m)$ given by (22).

Lemma 2 *The function $C_1(m)$ is continuous and differentiable on the interval $[1, \infty)$. Furthermore, the sign of its derivative satisfies*

$$\text{sign}\left(C_1'(m)\right) = \text{sign}\left\{a_1^2 + 2\lambda\tau\left[\rho\omega\left(a_2 m^2 + a_1 m\right) - ha_0\right]\right\}, \quad (23)$$

where a_0, a_1 and a_2 are given by (19).

Proof See "Appendix".   □

This last result allows us to determine the minimum of the function $C_1(m)$ on the interval $[1, \infty)$.

Theorem 2 *Let Γ be a numerical value defined as $\Gamma = a_1^2 + 2\lambda\tau[\rho\omega(a_2 + a_1) - ha_0]$ and let $C_1(m)$ be the function defined by (22) for $m \geq 1$, with $a_0 = K/\tau$, $a_1 = (h + \rho\omega)(1/(\delta+1) - 1/2)\lambda\tau + (\pi+p-c)(1-\rho)\lambda$ and $a_2 = (h+\rho\omega)\lambda\tau/2$.*

1. *If $\Gamma \geq 0$, then the function $C_1(m)$ attains its minimum at the point $m = 1$.*
2. *Otherwise, the function $C_1(m)$ attains its minimum at the point*

$$m^1 = \frac{1}{2a_2}\left(\sqrt{\frac{\left(4a_0 a_2 - a_1^2\right)h}{\rho\omega}} - a_1\right). \quad (24)$$

Proof See "Appendix".   □

Note that, according to Lemma 1 and Theorem 2, the minimum of the function $C(m, n)$ on the set $\Pi_1^c = \{(m, n) : m \geq 1, n > 0\}$ is attained at the point (m^c, n^c) given by

$$(m^c, n^c) = \begin{cases} (1, n_1) & \text{if } \Gamma \geq 0 \\ (m^1, n^1) & \text{if } \Gamma < 0 \end{cases}$$

where $n_1 = n^c(1)$, $n^1 = n^c(m^1)$, the function $n^c(m)$ is given by (21) and the point m^1 is presented in (24). Hence

$$n_1 = \sqrt{\frac{2(a_0 + a_1 + a_2)}{h\lambda\tau}} \tag{25}$$

and

$$n^1 = \frac{1}{\lambda\tau}\sqrt{\frac{4a_0 a_2 - a_1^2}{h\rho\omega}} \tag{26}$$

Therefore, the minimum of the function $C(m, n)$ in Π_1^c is

$$C_1^c = C(m^c, n^c) = \begin{cases} C(1, n_1) = \sqrt{2h\lambda\tau(a_0 + a_1 + a_2)} - \dfrac{\delta + 3}{2(\delta + 1)}h\lambda\tau & \text{if } \Gamma \geq 0 \\ C(m^1, n^1) = \dfrac{\sqrt{h\rho\omega(4a_0 a_2 - a_1^2)} + ha_1}{h + \rho\omega} + \dfrac{\delta - 1}{2(\delta + 1)}h\lambda\tau & \text{if } \Gamma < 0 \end{cases} \tag{27}$$

Corollary 1 *Assume that $m \geq 1$ is fixed. An optimal solution of the problem*

$$\min_{n \geq m, n \in \mathbb{Z}} C^m(n), \tag{28}$$

where

$$C^m(n) = \frac{A_1(m)}{n} + \frac{h}{2}\lambda\tau n + A_2(m) \tag{29}$$

is given by

$$n^*(m) = \left\lceil \frac{1}{2}\left(\sqrt{\frac{8A_1(m) + h\lambda\tau}{h\lambda\tau}} - 1\right) \right\rceil \tag{30}$$

Proof See "Appendix". □

Then, taking into account all the previous results, we propose an algorithmic procedure to determine the optimal solution for the inventory problem formulated in (10).

Note that the minimum inventory cost on the region $\Pi_0^c \cup \Pi_1^c$ is given by $C^c = \min\{C_0^c, C_1^c\}$, where C_0^c and C_1^c are given by (16) and (27), respectively.

5 Numerical examples

In this section, we present some numerical examples, along with a sensitivity analysis to illustrate the applicability of the main results of the paper.

Example 1 Consider an inventory system with the hypotheses assumed in this paper, in which the values of the parameters are as follows: the known basic period is $\tau = 1$ week and the average demand is $\lambda = 40$ units per week. During that basic time-period, we assume that demands toward the beginning of the week are larger that the demands at the end of the week.

Algorithm 1

Step 1	From (13), determine $n^*(0)$ and, from (14), compute C_0^i.
	From (19), calculate a_0, a_1 and a_2.
	Take $(m^*, n^*) = (0, n^*(0))$ as an initial solution, with minimum cost
	$\quad C(m^*, n^*) = C_0^i$.
Step 2	Calculate $\Gamma = a_1^2 + 2\lambda\tau\left[\rho\omega\left(a_2 + a_1\right) - ha_0\right]$.
	Determine the value C_1^c by using (27).
Step 3	If $C(m^*, n^*) > C_1^c$, then:
	$\quad$ If $\Gamma \geq 0$, then go to step 4.
	$\quad\quad$ Else, put $m = 1$ and go to step 5.
	$\quad$ End_If.
	$\quad$ Else, go to step 6.
	$\quad\quad$ End_If.
Step 4	From (30), calculate the point $n^*(1)$.
	$\quad$ If $C(1, n^*(1)) < C(m^*, n^*)$, then:
	$\quad\quad$ Take $(m^*, n^*) = (1, n^*(1))$, with cost $C(m^*, n^*) = C(1, n^*(1))$.
	$\quad$ End_If.
	$\quad\quad$ Go to step 6.
Step 5	Repeat
	$\quad$ Calculate the point $n^*(m)$ by using the formula (30).
	$\quad$ If $C(m, n^*(m)) < C(m^*, n^*)$, then:
	$\quad\quad$ Take $(m^*, n^*) = (m, n^*(m))$, with cost $C(m^*, n^*) = C(m, n^*(m))$.
	$\quad$ End If.
	$\quad$ $m = m + 1$.
	$\quad$ Compute $C_1(m)$ by using (22).
	Until $C_1(m) \geq C(m^*, n^*)$.
Step 6	The optimal inventory policy is (m^*, n^*), the minimum inventory cost is
	$\quad C^* = C(m^*, n^*)$ and the maximum profit per unit time is $P^* = (p - c)\lambda - C^*$.
	Stop.

Thus, we consider that the demand follows a power pattern with index $\delta = 0.5$. The ordering cost is $K = \$600$ per replenishment. The unit purchasing cost is $c = \$8$ and the selling price is $p = \$18$ per unit. The holding cost is $h = \$1$ per unit and per week. The fraction of demand in the stock-out period that is served with the next replenishment is $\rho = 0.9$. The shortage cost per backordered unit and per week is $\omega = \$10$, while the goodwill cost of a lost sale is $\pi = \$2$. With these input parameters, we apply Algorithm 1 to obtain the optimal policy. Firstly, from (13), we compute $n^*(0) = 5$. Thus, we take $(m^*, n^*) = (0, 5)$ as the initial solution, with cost $C(m^*, n^*) = C_0^i = 213.333$. Since $\Gamma = 191708$, from (27), it follows that $C_1^c = 223.839$. Therefore, from step 3, we see that the optimal inventory policy is $(m^*, n^*) = (0, 5)$, with minimum cost $C^* = C(0, 5) = \$213.333$. Consequently, the optimal inventory cycle is $T^* = 5$ weeks, there is no shortage (that is, $b^* = 0$), the economic lot size is $Q^* = 200$ units and the maximum inventory profit per week is $P^* = \$186.667$.

Example 2 Now, Let us consider a product in which a larger portion of demand occurs at the end of the week. Thus, we assume the following parameters of the model: $\tau = 1$, $\lambda = 10$, $\delta = 10$, $K = \$5$, $c = \$10$, $p = \$15$, $h = \$2$, $\rho = 1$, $\omega = \$2.5$ and $\pi = \$2$. Firstly, we have $n^*(0) = 1$. Then, the initial solution is $(m^*, n^*) = (0, 1)$, with cost $C(m^*, n^*) = 23.1818$. Taking into account that $\Gamma = 343.440$, from (27) we obtain $C_1^c = 7.25107$. From step 4, we get $n^*(1) = 1$ and $C(1, 1) = 7.27273$. Since $C(m^*, n^*) > C(1, 1)$, we update the solution at $(m^*, n^*) = (1, 1)$, with cost $C(m^*, n^*) = 7.27273$. From step 6 of the algorithm, it follows that the optimal inventory policy is $(m^*, n^*) = (1, 1)$, with cost $C^* = \$7.27273$.

Table 2 Application of Algorithm 1 to Example 3. The optimal inventory policy is shown in bold

m	$C_1(m)$	$n^*(m)$	$C(m, n^*(m))$	m^*	n^*	$C(m^*, n^*)$	Γ	C_1^c
0				0	5	226.667	−39139.6	184.215
1		6	196.889	1	6	196.889		
2	185.072	6	185.778	**2**	**6**	**185.778**		
3	185.999							

Table 3 Application of Algorithm 1 to Example 4. The optimal inventory policy is shown in bold

m	$C_1(m)$	$n^*(m)$	$C(m, n^*(m))$	m^*	n^*	$C(m^*, n^*)$	Γ	C_1^c
0				0	2	336.667	−59136	221.490
1		2	234.667	1	2	234.667		
2	226.359	3	228.000	**2**	**3**	**228.000**		
3	258.622							

Consequently, the optimal inventory cycle is $T^* = 1$ week, no units are stored (i.e., $S^* = 0$), the economic lot size is $Q^* = 10$ units and the maximum profit per week is $P^* = \$42.7273$.

Example 3 Consider the same parameters as in Example 1, but modifying the values of δ, c, ω and π to $\delta = 2$, $c = \$12.25$, $\omega = \$2$ and $\pi = \$0.25$. Applying Algorithm 1, the result of the solution procedure is shown in Table 2. Thus, the optimal inventory policy is $(m^*, n^*) = (2, 6)$, with cost $C^* = \$185.778$. Therefore, the optimal inventory cycle is $T^* = 6$ weeks, the stock-out period is 2 weeks, the inventory level at the beginning of the inventory cycle is $S^* = 160$ units, the minimum inventory level is -72, the lost sales per cycle are 8 units, the economic lot size is $Q^* = 232$ units and the maximum profit per week is $P^* = \$44.2222$.

Example 4 Assume the same parameters as in Example 3, but modifying the values of τ, h and ρ to $\tau = 2$, $h = \$4$ and $\rho = 0.95$. Table 3 shows the results obtained by applying the algorithm developed in the previous section. Consequently, the optimal inventory policy is $(m^*, n^*) = (2, 3)$, with cost $C^* = C(2, 3) = \$228$. Therefore, the optimal inventory cycle is $T^* = 6$ weeks, the stock-out period is 4 weeks, the maximum inventory level is $S^* = 80$, the minimum inventory level is -152, the lost sales per cycle are 8 units, the economic lot size is $Q^* = 232$ units and the maximum profit per week is $P^* = \$2$.

Example 5 Now, let us consider the following parameters of the model: $\tau = 1$, $\lambda = 10$, $\delta = 20$, $K = \$20$, $c = \$50$, $p = \$75$, $h = \$10$, $\rho = 1$, $\omega = \$1$ and $\pi = \$5$. Executing Algorithm 1, the result of the solution procedure is shown in Table 4. Thus, we conclude that $(m^*, n^*) = (2, 2)$. The optimal inventory policy consists of ordering each $T^* = 2$ weeks for a batch of lot size $Q^* = 20$ units, with profit $P^* = \$234.524$. Note that, in this case, both $n^*(1)$ and $n^*(2)$ take their smallest value, that is, 1 and 2, respectively.

5.1 Sensitivity analysis

In this subsection, we study how the optimal policy of the inventory system varies when some values of the input parameters of the system are modified. To do this, initially consider

Table 4 Application of Algorithm 1 to Example 5. The optimal inventory policy is shown in bold

m	$C_1(m)$	$n^*(m)$	$C(m, n^*(m))$	m^*	n^*	$C(m^*, n^*)$	Γ	C_1^c
0				0	1	115.238	−1418.99	12.6090
1		1	20.4762	1	1	20.4762		
2	12.8544	2	15.4762	**2**	**2**	**15.4762**		
3	15.6875							

an inventory system with the assumptions described in Sect. 2 and the following input data: $\lambda = 48$, $K = \$600$, $c = \$13$, $p = \$18$, $h = \$1$, $\omega = \$2$ and $\pi = \$0$.

In order to analyze the effect of the fraction ρ of demand that is served with the next replenishment, the basic time period τ and the index δ of the power demand on the optimal policy, we provide a table showing the behavior of m^*, n^*, Q^*, s^* and C^* as functions of ρ, τ and δ. More specifically, Table 5 presents the results when $\rho \in \{0.05, 0.10, 0.75, 0.90, 0.95, 1\}$, $\tau \in \{0.25, 0.5, 1, 2, 3\}$ and $\delta \in \{0.0625, 0.5, 1, 2\}$. These computational results present certain insights about the behavior of the inventory system studied here. Thus, we can make the following observations:

1. With fixed ρ and τ, the minimum inventory level s^* decreases, while the optimal number m^* of basic periods contained in the stock-out period and the optimal ratio m^*/n^* increase as the power demand pattern index δ increases. The optimal number $n^* - m^*$ of basic periods contained in the stock-in period is practically insensitive to changes in the parameter δ. The minimum cost increases weakly as the parameter δ increases.
2. With fixed δ and τ, the minimum inventory level s^* and the minimum inventory cost C^* decrease, while the optimal profit per unit time P^* increases as the parameter ρ increases. Moreover, if the value of ρ is increasing, then the optimal number m^* of basic periods contained in the stock-out period increases when $\delta < 1$, while m^* decreases when $\delta > 1$.
3. With fixed δ and ρ, the optimal number m^* of basic periods contained in the stock-out period and the optimal number n^* of basic periods contained in the inventory cycle decrease as the basic time period τ increases.
4. In general, the minimum inventory cost is more sensitive to changes in the fraction ρ of demand that is served with the next replenishment than to changes in the parameters δ and τ.
5. In general, the optimal policy (m^*, n^*) is more sensitive to changes in the basic time period τ than to changes in the parameters δ and ρ.

5.2 Management insights

In this subsection, some comments or suggestions are proposed for inventory systems managers that could help to improve the effectiveness and efficiency of the inventory control practices. The sensitivity analysis reveals that the length τ of the basic time period has the greatest impact on the total profit per unit time among the parameters τ, ρ and δ. That impact is more representative when the percentage ρ of backlogged demand is small. Hence, decision makers should consider a basic period that is as large as possible in order to obtain a greater benefit.

When τ and ρ are small values, the profit per unit time drops when the demand pattern index δ increases. This is due to the amount held in stock rising when δ increases. In this case,

Table 5 Effects of the parameters p, τ and δ on the optimal inventory policy

ρ	τ	$\delta = 0.0625$						$\delta = 0.5$						$\delta = 1$						$\delta = 2$					
		m^*	n^*	Q^*	s^*	C^*	P^*	m^*	n^*	Q^*	s^*	C^*	P^*	m^*	n^*	Q^*	s^*	C^*	P^*	m^*	n^*	Q^*	s^*	C^*	P^*
0.05	0.25	5	25	243	−3	234.071	5.92941	7	27	244.2	−4.2	236.548	3.45185	8	28	244.8	−4.8	237.943	2.05714	10	30	246	−6	239.267	0.733333
	0.50	0	10	240	0	229.412	10.5882	3	13	243.6	−3.6	235.077	4.92308	4	14	244.8	−4.8	237.944	2.05714	5	15	246	−6	240.533	−0.533333
	1	0	5	240	0	218.824	21.1765	1	6	242.4	−2.4	231.867	8.13333	2	7	244.8	−4.8	241.2	−1.2	3	8	247.2	−7.2	242.9	−2.9
	2	0	3	288	0	201.647	38.3529	0	3	288	0	228	12	0	3	288	0	228	12	2	4	201.6	−9.6	249	−9
	3	0	2	288	0	180.471	59.5294	0	2	288	0	220	20	0	2	288	0	220	20	2	4	302.4	−14.4	254	−14
0.10	0.25	6	26	247.2	−7.2	232.295	7.70498	8	28	249.6	−9.6	234.571	5.42857	8	28	249.6	−9.6	235.886	4.11429	9	29	250.8	−10.8	237.159	2.84138
	0.50	2	12	244.8	−4.8	228.329	11.6706	4	13	247.2	−7.2	233.231	6.76923	4	14	249.6	−9.6	235.886	4.11429	5	15	252	−12	238.4	1.6
	1	0	5	240	0	218.824	21.1765	1	6	244.8	−4.8	230.4	9.6	2	7	249.6	−9.6	239.2	0.8	3	8	254.4	−14.4	240.8	−0.8
	2	0	3	288	0	201.647	38.3529	0	3	288	0	228	12	0	3	288	0	228	12	2	4	211.2	−19.2	247	−7
	3	0	2	240	0	180.471	59.5294	0	2	240	0	220	20	1	3	252	−14.4	239.467	0.533333	1	3	252	−14.4	253.8667	−13.8667
0.75	0.25	8	26	288	−72	206.471	33.5294	8	25	276	−72	207.2	32.8	8	25	276	−72	207.6	32.4	8	25	276	−72	208	32
	0.50	4	13	288	−72	205.249	34.7511	4	13	288	−72	206.769	33.2308	4	13	288	−72	207.692	32.3077	4	13	288	−72	208.615	31.3846
	1	2	7	312	−72	203.092	36.9076	2	6	264	−72	206.667	33.3333	2	6	264	−72	208	32	2	6	264	−72	209.333	30.6667
	2	1	4	360	−72	200.118	39.8824	1	3	264	−72	205.333	34.6667	1	3	264	−72	208	32	1	3	264	−72	210.667	29.3333
	3	0	2	288	0	180.471	59.5294	3	3	396	−108	214.667	25.3333	3	3	396	−108	218.667	21.3333	1	2	252	−108	214	26
0.90	0.25	8	25	290.4	−86.4	200.137	39.8626	8	25	290.4	−86.4	200.48	39.52	8	25	290.4	−86.4	200.688	39.312	8	25	290.4	−86.4	200.896	39.104
	0.50	4	13	302.4	−86.4	199.580	40.4199	4	13	302.4	−86.4	200.492	39.5077	4	12	278.4	−86.4	200.8	39.2	4	12	278.4	−86.4	201.067	38.9333
	1	2	6	326.4	−86.4	198.736	41.2639	2	6	278.4	−86.4	200.267	39.7333	2	6	278.4	−86.4	200.8	39.2	2	6	278.4	−86.4	201.3333	38.6667
	2	1	4	374.4	−86.4	197.894	42.1059	1	3	278.4	−86.4	199.733	40.2667	1	3	278.4	−86.4	200.8	39.2	1	3	278.4	−86.4	201.867	38.1333
	3	0	2	288	0	180.471	59.5294	3	3	417.6	−129.6	212.267	27.7333	2	2	273.6	−129.6	212.8	27.2	1	2	273.6	−129.6	203.2	36.8
0.95	0.25	8	25	295.2	−91.2	198.003	41.9972	8	25	295.2	−91.2	198.24	41.76	8	25	295.2	−91.2	198.384	41.616	8	24	283.2	−91.2	198.467	51.533
	0.50	4	13	307.2	−91.2	197.690	42.3095	4	12	283.2	−91.2	198.267	41.7333	4	12	283.2	−91.2	198.4	41.6	4	12	283.2	−91.2	198.533	41.4667
	1	2	7	331.2	−91.2	197.248	42.7160	2	6	283.2	−91.2	198.133	41.867	2	6	283.2	−91.2	198.4	41.6	2	6	283.2	−91.2	198.667	41.3333
	2	1	3	283.2	−91.2	196.988	43.0118	1	3	283.2	−91.2	197.867	42.1333	1	3	283.2	−91.2	198.4	41.6	1	3	283.2	−91.2	198.933	41.0667

Table 5 continued

ρ	τ	δ = 0.0625						δ = 0.5						δ = 1						δ = 2					
		m^*	n^*	Q^*	s^*	C^*	P^*	m^*	n^*	Q^*	s^*	C^*	P^*	m^*	n^*	Q^*	s^*	C^*	P^*	m^*	n^*	Q^*	s^*	C^*	P^*
	3	0	2	288	0	180.471	59.5294	1	3	424.8	−136.8	211.467	28.5333	1	2	280.8	−136.8	210.4	29.6	1	2	280.8	−136.8	199.6	40.4
1	0.25	8	25	300	−96	195.868	44.1318	8	24	288	−96	196	44	8	24	288	−96	196	44	8	24	288	−96	196	44
	0.50	4	13	312	−96	195.801	44.1991	4	12	288	−96	196	44	4	12	288	−96	196	44	4	12	288	−96	196	44
	1	2	7	336	−96	195.832	44.1681	2	6	288	−96	196	44	2	6	288	−96	196	44	2	6	288	−96	196	44
	2	1	3	288	−96	196	44	1	3	288	−96	196	44	1	3	288	−96	196	44	1	3	288	−96	196	44
	3	0	2	288	0	180.471	59.5294	1	3	432	−144	210.667	29.3333	1	2	288	−144	208	32	1	2	288	−144	196	44

the manager should apply some marketing policies to incentivize demand at the beginning of the inventory cycle and thus reduce the amount stored in inventory.

Notice that when the parameters τ and δ are fixed, the profit per unit time increases as the proportion ρ of backlogged shortages increases. Thus, the manager should boost the number of customers who are willing to wait to the next replenishment to receive the product. This may be done by implementing policies that augment this backlogging (e.g., by offering a small discount in the selling price of this or other products to those customers who decide to wait for the arrival of the new batch of items ordered).

6 Conclusions

Inventory management models aim to improve the effectiveness and efficiency of business operations related to inventory control. It allows an appropriate design schedule to be made for the sale and distribution of products. The main contribution of this paper is the detailed description of the optimal inventory policies obtained for an inventory system in which customers' demand follows a power pattern during a known basic time-period. That power pattern represents the temporal distribution of customer demand throughout the basic period. In the model, we have assumed the existence of an inventory of products that must be periodically replenished. Shortages are allowed, but only a fixed fraction of the demand during the stock-out period is satisfied with the arrival of the next replenishment. The inventory cycle must be an integer multiple of the basic time-period. Moreover, the stock-out period must also be an integer multiple of the basic period. In the analysis of the problem, five significant costs have been considered in the management of the inventory system: the ordering cost, the purchasing cost, the holding cost, the backordering cost and the lost sale cost. The aim is to determine the optimal inventory policy that maximizes the total profit per unit time. This inventory problem is formulated as an integer nonlinear mathematical program.

Some theoretical results that help to characterize the optimal solutions have been presented. To solve the inventory problem, a new algorithm that determines the optimal inventory policy (the optimal inventory cycle and the optimal stock-out period) is developed. The minimum total cost and the maximum profit per unit time are also obtained.

To illustrate the application of the algorithmic procedure, several numerical examples are provided. Furthermore, we have presented computational results to analyze the sensitivity of the optimal inventory policy when some parameters of the model are modified. As expected, the optimal number of basic periods contained in the stock-in period and the optimal number of basic periods contained in the stock-out interval decrease as the basic time period increases. Moreover, the minimum inventory level decreases as the fraction of the backordered demand increases.

From the point of view of corporate decision making, incorporating a power demand pattern may have interesting implications for management and inventory control within organizations. The results obtained have a direct impact on the optimal policy that organizations should adopt to reduce their operating costs and to maximize the profit related to inventory management. The consideration of time-varying demand, partial backordering and discrete inventory cycle is a more realistic assumption and contributes to the existing literature on inventory models.

Future research lines in this subject could be the following: (i) to assume in the inventory system that the replenishment is non-instantaneous, which implies that a finite production rate must be considered in the model; (ii) to incorporate in the model the possibility that products

may suffer some deterioration over time; (iii) to consider discounts in purchasing costs; (iv) to develop the inventory system with price-dependent demand rate; (v) to analyze the case of non-linear holding cost; (vi) to study the inventory system under stochastic demand and (vii) to consider multiple products and a limited storage capacity.

Appendix

Proof of Theorem 1 The problem (11) is similar to the problem studied by García-Laguna et al. (2010). Thus, applying analogous arguments to those used there, it is easy to check that $n^*(0)$ is an optimal solution to the problem (11). □

Proof of Lemma 1 The result is immediate. Since $A_1(m) > 0$, the function $C^m(n)$ has the same structure as the objective function of Wilson's basic EOQ model. Therefore, it attains its minimum value at

$$n^*(m) = \sqrt{\frac{2A_1(m)}{h\lambda\tau}} = \sqrt{\frac{2\left(a_0 + a_1 m + a_2 m^2\right)}{h\lambda\tau}}$$

with the optimal value

$$C^m(n^*(m)) = \sqrt{2h\lambda\tau A_1(m)} + A_2(m)$$
$$= \sqrt{2h\lambda\tau\left(a_0 + a_1 m + a_2 m^2\right)} + \left[(\delta - 1)/(2(\delta + 1)) - m\right]h\lambda\tau.$$

□

Proof of Lemma 2 From (22), it is clear that the function $C_1(m)$ is continuous and differentiable on the interval $[1, \infty)$. The first derivative of this function is

$$C_1'(m) = \left(\frac{a_1 + 2a_2 m}{\sqrt{2h\lambda\tau\left(a_0 + a_1 m + a_2 m^2\right)}} - 1\right)h\lambda\tau \qquad (31)$$

Taking into account that, if $m \geq 1$, then

$$a_1 + 2a_2 m = (h + \rho\omega)\left(\frac{1}{\delta + 1} + \frac{2m - 1}{2}\right)\lambda\tau + (\pi + p - c)(1 - \rho)\lambda > 0, \qquad (32)$$

it follows that

$$\left(\frac{a_1 + 2a_2 m}{\sqrt{2h\lambda\tau\left(a_0 + a_1 m + a_2 m^2\right)}} - 1 \right)$$

$$= \left(\frac{a_1 + 2a_2 m - \sqrt{2h\lambda\tau\left(a_0 + a_1 m + a_2 m^2\right)}}{\sqrt{2h\lambda\tau\left(a_0 + a_1 m + a_2 m^2\right)}} \right)$$

$$= \left(\frac{(a_1 + 2a_2 m)^2 - 2h\lambda\tau\left(a_0 + a_1 m + a_2 m^2\right)}{\sqrt{2h\lambda\tau\left(a_0 + a_1 m + a_2 m^2\right)}\left(a_1 + 2a_2 m + \sqrt{2h\lambda\tau\left(a_0 + a_1 m + a_2 m^2\right)}\right)} \right)$$

$$= \left(\frac{a_1^2 + 2a_2 m^2(2a_2 - h\lambda\tau) + 2a_1 m(2a_2 - h\lambda\tau) - 2h\lambda\tau a_0}{\sqrt{2h\lambda\tau\left(a_0 + a_1 m + a_2 m^2\right)}\left(a_1 + 2a_2 m + \sqrt{2h\lambda\tau\left(a_0 + a_1 m + a_2 m^2\right)}\right)} \right)$$

$$= \left(\frac{a_1^2 + 2\lambda\tau\left[\rho\omega(a_2 m^2 + a_1 m) - ha_0\right]}{\sqrt{2h\lambda\tau\left(a_0 + a_1 m + a_2 m^2\right)}\left(a_1 + 2a_2 m + \sqrt{2h\lambda\tau\left(a_0 + a_1 m + a_2 m^2\right)}\right)} \right)$$

and, therefore, sign $\left(C_1'(m)\right)$ =sign $\left\{a_1^2 + 2\lambda\tau\left[\rho\omega\left(a_2 m^2 + a_1 m\right) - ha_0\right]\right\}$. □

Proof of Theorem 2 From (23), it follows that the behavior of the function $C_1(m)$ can be determined by studying the parabola $p(m) = a_1^2 + 2\lambda\tau\left[\rho\omega\left(a_2 m^2 + a_1 m\right) - ha_0\right]$. For this, we begin by calculating the first derivative of the function $p(m)$, that is, $p'(m) = 2\rho\omega\lambda\tau\left(2a_2 m + a_1\right)$. From (32), we deduce that $p(m)$ is a strictly increasing function for $m \geq 1$. Therefore, to obtain the minimum of $C_1(m)$, we must evaluate the function $p(m)$ at the point $m = 1$. Since $p(1) = a_1^2 + 2\lambda\tau\left[\rho\omega\left(a_2 + a_1\right) - ha_0\right] = \Gamma$, we can distinguish two situations:

1. If $\Gamma \geq 0$, then $p(m) > 0$ for all $m > 1$ and, therefore, the function $C_1(m)$ attains its minimum at the point $m = 1$.
2. If $\Gamma < 0$, then the parabola $p(m)$ intersects the abscissa axis at the point $m^1 > 1$ given in (24). Moreover, $p(m) < 0$ for all $m \in (1, m^1)$ and $p(m) > 0$ for all $m \in (m^1, \infty)$. Consequently, the function $C_1(m)$ attains its minimum at m^1. □

Proof of Corollary 1 We proceed analogously to García-Laguna et al. (2010). Thus, it is easy to deduce that $n^*(m)$ is an optimal solution of the problem $\min_{n \in \mathbb{Z}} \{C^m(n)\}$. Therefore, we only need to show that $n^*(m) \geq m$. This is equivalent to proving that

$$\sqrt{\frac{1}{4} + \frac{2A_1(m)}{h\lambda\tau}} - \frac{1}{2} > m - 1.$$

After some simple algebraic manipulations, we see that this inequality is equivalent to the condition $2A_1(m) - h\lambda\tau m(m - 1) > 0$. This is true because $f(m)$ satisfies the following properties: (i) $f(m) = 2A_1(m) - h\lambda\tau m(m - 1)$ is a quadratic function strictly increasing, since

$$f'(m) = \lambda\left(\frac{2(h + \rho\omega)\tau}{\delta + 1} + 2(\pi + p - c)(1 - \rho) + (2m - 1)\rho\omega\tau\right)$$

is positive for all $m \geq 1$, and (ii) $f(1) = 2A_1(1) > 0$. □

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