C_0-semigroups of m-isometries on Hilbert spaces

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Abstract

Let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup on a separable Hilbert space H. We show that $T(t)$ is an m-isometry for any t if and only if the mapping $t \in \mathbb{R}^+ \to \|T(t)x\|^2$ for each $x \in H$ is a polynomial of degree at most m. This property is used to study m-isometric right translation semigroup on weighted L^p-spaces. We also provide alternative characterizations of the above property by imposing conditions on the infinitesimal generator operator and on the cogenerator operator of $\{T(t)\}_{t\geq 0}$. Moreover, we prove that a non-unitary 2-isometry T on a Hilbert space satisfying the kernel condition, that is,

$$T^*T(KerT^*) \subset KerT^* \,,$$

can be embedded into a C_0-semigroup if and only if $dim(KerT^*) = \infty$.

1 Introduction

Let H be a complex Hilbert space and $B(H)$ denote the C^*-algebra of all bounded linear operators on H.

A one-parameter family $\{T(t)\}_{t\geq 0}$ of bounded linear operators from H into H is a C_0-*semigroup* if:

1. $T(0) = I$.
2. $T(s+t) = T(t)T(s)$ for every $t, s \geq 0$.
3. $\lim\limits_{t\to 0^+} T(t)x = x$ for every $x \in H$, in the strong operator topology.

The linear operator A defined as

$$Ax := \lim_{t\to 0^+} \frac{T(t)x - x}{t},$$

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for every

$$x \in D(A) := \left\{ x \in H \quad : \quad \lim_{t\to 0^+} \frac{T(t)x - x}{t} \text{ exists } \right\}$$

is called the *infinitesimal generator* of the semigroup $\{T(t)\}_{t\geq 0}$. It is well-known that A is a closed and densely defined linear operator.

If 1 is in the resolvent set of A, $\rho(A)$, then Cayley transform of A defined as $V := (A+I)(A-I)^{-1}$ is a bounded linear operator, since $V = I + 2(A-I)^{-1}$. The operator V is called the *cogenerator* of the C_0-semigroup $\{T(t)\}_{t\geq 0}$.

For a positive integer m, an operator $T \in B(H)$ is called an *m-isometry* if

$$\sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \|T^k x\|^2 = 0 \ ,$$

for any $x \in H$.

It is said that T is a *strict m-isometry* if T is an m-isometry but it is not an $(m-1)$-isometry.

Remark 1.1.

1. For $m \geq 2$, the strict m-isometries are not power bounded. If T is an m-isometry, then $\|T^n x\|^2$ is a polynomial of degree at most $m-1$, for every x, [6, Theorem 2.1]. In particular, $\|T^n\| = O(n)$ for 3-isometries and $\|T^n\| = O(n^{\frac{1}{2}})$ for 2-isometries.
2. The m-isometries on finite dimensional spaces for even m are never strict. See [2, Proposition 1.23].
3. If T is an m-isometry, then $\sigma(T) = \overline{\mathbb{D}}$ or $\sigma(T) \subseteq \partial\mathbb{D}$, [2, Lemma 1.2].

The remainder of the paper is organized as follows. In Section 2, we show that $T(t)$ is an m-isometry for every t if and only if the mapping $t \in \mathbb{R}^+ \to \|T(t)x\|^2$ is a polynomial of degree at most m for all $x \in H$. This property is used in Section 4 to study m-isometric right translation semigroup on weighted L^p-spaces. Furthermore, we present a characterization of the above property in terms of conditions on the generator operator and on the cogenerator operator of the C_0-semigroup. Moreover, if $\{T(t)\}_{t\geq 0}$ is a C_0-semigroup, then we show that $T(t)$ is an m-isometry for all $t > 0$ if and only if $T(t)$ is an m-isometry, for all $t \in [0, t_1]$ with $t_1 > 0$ or on $[t_1, t_2]$ with $0 < t_1 < t_2$.

Section 3 is devoted to embedding m-isometries into C_0-semigroups. Namely, given an m-isometry T finds a C_0-semigroup $\{T(t)\}_{t\geq 0}$ such that $T(1) = T$. By using the model for 2-isometries we conclude that a non-unitary 2-isometry on a Hilbert space which satisfies the *kernel condition*, that is

$$T^*T(Ker(T^*)) \subset Ker(T^*) \ ,$$

can be embedded into a C_0-semigroup if and only if $\dim(KerT^*) = \infty$.

Finally, in Section 4, we obtain a characterization so that the right translation C_0-semigroup on some weighted space is a semigroup of m-isometries for all $t > 0$.

2 C_0-semigroups of m-isometries

Recall that any C_0-semigroup $\{T(t)\}_{t\geq 0}$ have associated some real quantities such as:

1. The *spectral bound*, $s(A)$, that is, the supremum of real part of λ such that $\lambda \in \sigma(A)$.
2. The *growth bound*, w_0, that is, the infimum of all real numbers w such that there exists a constant $M_w \geq 1$ with $\|T(t)\| \leq M_w e^{wt}$ for all $t \geq 0$.

The above quantities are related in the following way, $s(A) \leq w_0$ and

$$w_0 = \frac{1}{t} \log r(T(t)) \;, \tag{1}$$

for all $t > 0$, where $r(T(t))$ denotes the spectral radius of the operator $T(t)$.

The following lemma shows that the cogenerator of a C_0-semigroup of m-isometries exists.

Lemma 2.1. *Let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup on a separable Hilbert space H consisting of m-isometries and A its generator. Then $1 \in \rho(A)$ and therefore the cogenerator V of $\{T(t)\}_{t\geq 0}$ is well-defined.*

Proof. If $T(t)$ is an m-isometry for any t, then the spectral radius of $T(t)$, $r(T(t))$ is 1 for all t. By considering equality (1), then $s(A) \leq w_0 = 0$. Hence $1 \in \rho(A)$. $\square$

The following combinatorial lemma will be useful for the proof of Theorem 2.1. Let us consider $\binom{m}{k} = 0$ if $m < k$ or $k < 0$.

Lemma 2.2. *Let m be a positive integer and p, q be integers such that $0 \leq p, q \leq m$.*

1. *If $p + q \neq m$, then*

$$\sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \left\{ \sum_{i=0}^{p} \binom{m-k}{i} \binom{k}{p-i} (-1)^i \right\} \left\{ \sum_{j=0}^{q} \binom{m-k}{j} \binom{k}{q-j} (-1)^j \right\} = 0$$

2. *If $p + q = m$, then*

$$\sum_{k=0}^{m} \binom{m}{k} \left\{ \sum_{i=0}^{q} \binom{m-k}{i} \binom{k}{q-i} (-1)^i \right\}^2 = 2^m \binom{m}{q} = 2^m \binom{m}{p} \;.$$

Proof. We define the polynomials r and s by $r(x,y) := 2^m (x+y)^m$ and

$$s(x,y) := ((x+1)(y+1) - (x-1)(y-1))^m \;.$$

Then

$$r(x,y) \;=\; 2^m \sum_{k=0}^{m} \binom{m}{k} x^k y^{m-k}$$

and

$$\begin{aligned}
s(x,y) &= \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} (x+1)^k (x-1)^{m-k} (y+1)^k (y-1)^{m-k} \\
&= \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \sum_{h,\,\ell=0}^{k} \binom{k}{\ell}\binom{k}{h} \sum_{i,\,j=0}^{m-k} \binom{m-k}{i}\binom{m-k}{j} (-1)^{i+j} x^{j+h} y^{i+\ell} \\
&= \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \left\{ \sum_{i=0}^{p} \binom{m-k}{m-k-i}\binom{k}{k+i-p} (-1)^i \right\} \\
&\quad \left\{ \sum_{j=0}^{q} \binom{m-k}{m-k-j}\binom{k}{k+j-q} (-1)^j \right\} x^{j+h} y^{i+\ell} \\
&= \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \left\{ \sum_{i=0}^{p} \binom{m-k}{i}\binom{k}{p-i} (-1)^i \right\} \\
&\quad \left\{ \sum_{j=0}^{q} \binom{m-k}{j}\binom{k}{q-j} (-1)^j \right\} x^{j+h} y^{i+\ell} \, .
\end{aligned}$$

We denote the coefficient of the power $x^p y^q$ in polynomial f by $\widehat{x^p y^q}^{f}$. Since $s(x,y) = r(x,y)$, then $\widehat{x^p y^q}^{s} = \widehat{x^p y^q}^{r}$ for every p and q. So, if $p+q \neq m$, then $\widehat{x^p y^q}^{s} = \widehat{x^p y^q}^{r} = 0$. If $p+q = m$, then

$$\widehat{x^p y^q}^{s} = \widehat{x^p y^q}^{r} = \widehat{x^p y^{m-p}}^{r} = 2^m \binom{m}{p} = 2^m \binom{m}{m-p} = 2^m \binom{m}{q} \, .$$

On the other hand, it is not difficult to satisfy that, if $p+q = m$, then

$$\begin{aligned}
\widehat{x^p y^q}^{s} &= \sum_{k=0}^{m} \binom{m}{k} \left\{ \sum_{i=0}^{q} \binom{m-k}{m-k-i}\binom{k}{k+i-q} (-1)^i \right\}^2 \\
&= \sum_{k=0}^{m} \binom{m}{k} \left\{ \sum_{i=0}^{q} \binom{m-k}{i}\binom{k}{q-i} (-1)^i \right\}^2 \, .
\end{aligned}$$

This completes the proof. □

Given a C_0-semigroup $\{T(t)\}_{t\geq 0}$, we have two different operators associated to $\{T(t))\}_{t\geq 0}$: The infinitesimal generator A and the cogenerator V, if $1 \in \rho(A)$. This two operators will be useful in the following result, where we obtain a generalization of [12, Proposition 2.2] to C_0-semigroup of m-isometries. See also [13, Proposition 2.6] and [21, Theorem 2].

Theorem 2.1. *Let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup on a separable Hilbert space H. Then the following assertions are equivalent:*

(i) $T(t)$ *is an m-isometry for every t.*

(ii) The mapping $t \in \mathbb{R}^+ \to \|T(t)x\|^2$ is a polynomial of degree at most m for each $x \in H$.

(iii) Equality

$$\sum_{k=0}^{m} \binom{m}{k} \langle A^{m-k}x, A^k x\rangle = 0 \ ,$$

holds for any $x \in D(A^m)$, where A is the generator of the semigroup.

(iv) The cogenerator V of $\{T(t)\}_{t\geq 0}$ exists and is an m-isometry.

Proof. *(i)* $\Leftrightarrow$ *(ii)* If $T(t)$ is an m-isometry for every t, then

$$\sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \|T(t+k\tau)x\|^2 = 0 \ , \tag{2}$$

for every $t,\ \tau > 0$ and $x \in H$. From the assumption on the semigroup, it is clear that function $t \in \mathbb{R}^+ \to f(t) := \|T(t)x\|^2$ is continuous. By [15, Theorem 13.7], the function $f(t)$ is a polynomial of degree at most m for each $x \in H$.

Conversely, if the mapping $t \in \mathbb{R}^+ \to \|T(t)x\|^2$ is a polynomial of degree at most m for each $x \in H$, then $T(t)$ is an m-isometry for every t, [15, page 271].

(ii) $\Leftrightarrow$ *(iii)* Let $y \in D(A^m)$. The function $t \in \mathbb{R}^+ \to \|T(t)y\|^2$ has *mth*-derivative and it is given by

$$\sum_{k=0}^{m} \binom{m}{k} \langle A^{m-k}T(t)y, A^k T(t)y\rangle \ . \tag{3}$$

By *(ii)* and (3) at $t = 0$ we have that

$$\sum_{k=0}^{m} \binom{m}{k} \langle A^{m-k}y, A^k y\rangle = 0 \ ,$$

for any $y \in D(A^m)$.

Conversely, if (3) holds on $D(A^m)$, then the *mth*-derivative of the function $t \in \mathbb{R}^+ \longrightarrow \|T(t)x\|^2$ is

$$\sum_{k=0}^{m} \binom{m}{k} \langle A^{m-k}T(t)x, A^k T(t)x\rangle \ ,$$

for every $t > 0$ and $x \in D(A^m)$. $D(A^m)$ is dense by [17, Theorem 2.7], so that we obtain the desired result.

(iii) $\Leftrightarrow$ *(iv)*. It is enough to prove that

$$2^m \sum_{k=0}^{m} \binom{m}{k} \langle A^{m-k}x, A^k x\rangle = \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \langle (A+I)^k (A-I)^{m-k}x, (A+I)^k (A-I)^{m-k}x\rangle \tag{4}$$

for all $x \in D(A^m)$, since (4) is equivalent to

$$2^m \sum_{k=0}^{m} \binom{m}{k} \langle A^{m-k}(A-I)^{-m}y, A^k(A-I)^{-m}y\rangle$$
$$= \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \langle (A+I)^k(A-I)^{-k}y, (A+I)^k(A-I)^{-k}y\rangle$$
$$= \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \|V^k y\|^2 ,$$

for all $y \in R(A-I)^m$.

Note that the second part of equality (4) is given by

$$\sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \sum_{\ell,\, h=0}^{k} \binom{k}{\ell}\binom{k}{h} \sum_{i,\, j=0}^{m-k} \binom{m-k}{i}\binom{m-k}{j} (-1)^{i+j} \langle A^{\ell+i}x, A^{h+j}x\rangle \ . \tag{5}$$

We denote the numerical coefficient of $\langle A^p x, A^q x\rangle$ by $\widehat{A}_{p,q}$. It is not difficult to prove that if $p+q=m$, then $\widehat{A}_{p,q} = \widehat{A}_{q,m-q} = \widehat{A}_{m-q,q}$ and

$$\begin{aligned}
\widehat{A}_{q,m-q} &= \sum_{k=0}^{m} \binom{m}{k} \left\{ \sum_{i=0}^{q} \binom{m-k}{m-k-i}\binom{k}{k+i-q} (-1)^i \right\} \\
&\quad \sum_{k=0}^{m} \binom{m}{k} \left\{ \sum_{i=0}^{q} \binom{m-k}{i}\binom{k}{q-i} (-1)^i \right\} \\
&= \sum_{k=0}^{m} \binom{m}{k} \left\{ \sum_{i=0}^{q} \binom{m-k}{m-k-i}\binom{k}{k+i-q} (-1)^i \right\}^2 \\
&= \sum_{k=0}^{m} \binom{m}{k} \left\{ \sum_{i=0}^{q} \binom{m-k}{i}\binom{k}{q-i} (-1)^i \right\}^2 = 2^m \binom{m}{q} ,
\end{aligned}$$

where the last equality is obtained by applying part (2) of Lemma 2.2.

If $p+q \neq m$, then

$$\begin{aligned}
\widehat{A}_{p,q} &= \widehat{A}_{m-p,m-q} \\
&= \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \left\{ \sum_{i=0}^{p} \binom{m-k}{m-k-i}\binom{k}{k+i-p} (-1)^i \right\} \\
&\quad \left\{ \sum_{j=0}^{q} \binom{m-k}{m-k-j}\binom{k}{k+j-q} (-1)^j \right\}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \left\{ \sum_{i=0}^{p} \binom{m-k}{i}\binom{k}{p-i} (-1)^i \right\} \left\{ \sum_{j=0}^{q} \binom{m-k}{j}\binom{k}{q-j} (-1)^j \right\} \\
&= 0 ,
\end{aligned}$$

where the last equality is obtained by applying part (1) of Lemma 2.2. So we get the desired result. □

Corollary 2.1. *Let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup on a separable Hilbert space H. Then $T(t)$ is a strict m-isometry for every $t > 0$ if and only if the cogenerator V of $\{T(t)\}_{t\geq 0}$ is a strict m-isometry.*

In the following corollary, we give an example of m-isometric semigroup.

Corollary 2.2. *Let $Q \in B(H)$ be a nilpotent operator of order n on a separable Hilbert space H. Then the C_0-semigroup generated by Q is a strict $(2n-1)$-isometric semigroup.*

Proof. Since Q is the generator of $\{T(t)\}_{t\geq 0}$ and $1 \in \rho(Q)$, then the cogenerator is well-defined and given by

$$V := (Q + I)(Q - I)^{-1} .$$

Thus $V = -(Q + I)(I + Q + \cdots + Q^{n-1}) = -I - 2Q(I + Q + \cdots + Q^{n-2})$, that is, the sum of an isometry and a nilpotent operator of order n. By [7, Theorem 2.2], the cogenerator is a strict $(2n-1)$-isometry. Then $\{T(t)\}_{t\geq 0}$ is a strict $(2n-1)$-isometric semigroup by Corollary 2.1. □

For a positive integer m, a closed linear operator A defined on a dense set $D(A) \subset H$ is called an *m-symmetry* if

$$\sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \langle A^{m-k}x, A^k x\rangle = 0 ,$$

for all $x \in D(A^m)$. We say that A is a *strict m-symmetry* if A is an m-symmetry but it is not an $(m-1)$-symmetry. There exists no strict m-symmetry bounded for even m. See [1, Page 7].

In the following result, we present a connection between condition (iii) of Theorem 2.1 and m-symmetric operators.

Corollary 2.3. *Let $A \in B(H)$ be an m-*symmetry *on a separable Hilbert space H. Then the C_0-semigroup generated by iA is an m-isometric semigroup.*

Proof. If A is an m-symmetry, then

$$\sum_{k=0}^{m} \binom{m}{k} \langle (iA)^{m-k}x, (iA)^k x\rangle = (-i)^m \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \langle A^{m-k}x, A^k x\rangle = 0 ,$$

for all $x \in D(A^m)$. The proof is completed by Theorem 2.1. □

Let w be a weighted function on $\mathbb{T}$. It is defined

$$L^2_w(\mathbb{T}) := \left\{ f : \mathbb{T} \to \mathbb{C} \;\; : \;\; \int_{\mathbb{T}} |f(z)|^2 w(z) dz < \infty \right\} .$$

Let $\{T(t)\}_{t\geq 0}$ be a C_0-group defined on $L^2_w(\mathbb{T})$ with non-constant weighted function w by $T(t)f(z) := f(e^{it}z)$. Then $T(t)$ is an isometry if and only if t is a multiple of 2π. See [8, page 8].

Proposition 2.1. *Let $\{T(t)\}_{t\in\mathbb{R}}$ be a C_0-group on a Hilbert space H. Then the following statements are equivalent:*

(i) *$T(t)$ is an m-isometry for every t.*

(ii) *$T(t)$ is an m-isometry for t_1 and t_2 where $\frac{t_1}{t_2}$ is irrational.*

Proof. For every t, we have that

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\|T(t+kt_i)x\|^2 = 0 \text{ for } i = 1, 2 .$$

Montel's Theorem ([15, Theorem 13.5] & [19, Theorem 1.1]) implies that $\|T(t)x\|^2$ is a polynomial of degree at most m for each $x \in H$, since $\frac{t_1}{t_2}$ is irrational. Thus by Theorem 2.1 we have that $T(t)$ is an m-isometry for every t. □

If T is an m-isometry, then any power T^r is also an m-isometry, [14, Theorem 2.3]. In general, the converse is not true. However, if T^r and T^{r+1} are m-isometries for a positive integer r, then T is an m-isometry (see, [5, Corollary 3.7]). The stability of powers of m-isometries is fundamental to give necessary and sufficient conditions for a C_0-semigroup $\{T(t)\}_{t\geq 0}$ to be $T(t)$ an m-isometry for each $t \geq 0$.

Theorem 2.2. *Let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup on a Hilbert space H. Then the following assertions are equivalent:*

(i) *$T(t)$ is an m-isometry for every $t \geq 0$.*

(ii) *$T(t)$ is an m-isometry for every $t \in [0, t_1]$ for some $t_1 > 0$.*

(iii) *$T(t)$ is an m-isometry on an interval of the form $[t_1, t_2]$ with $t_1 < t_2$.*

Proof. The implications $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$ are clear.

$(ii) \Rightarrow (i)$. Fixed $t > 0$, there exist $n \in \mathbb{N}$ and $t' \in [0, t_1]$ such that $t = nt'$. Since $T(t')$ is an m-isometry, then any power of $T(t')$ is an m-isometry by [5, Theorem 3.1]. So, $T(t)$ is an m-isometry.

$(iii) \Rightarrow (i)$. Let us prove that $T(t)$ is an m-isometry for every $t \in (0, \frac{t_2-t_1}{4}]$.

Choose $k := [\frac{t_1}{t}] + 1$, where $[s]$ denotes the greatest integer less than or equal to s. Then kt and $(k+1)t$ belong to (t_1, t_2). Thus $T(t)^k$ and $T(t)^{k+1}$ are m-isometries. Hence $T(t)$ is an m-isometry by [5, Corollary 3.7]. □

3 Embedding m-isometries into C_0-semigroups

We are interested on the following question, when can an m-isometry be embedded into a continuous C_0-semigroup? In other words, for a given an m-isometry T, is there a C_0-semigroup $\{T(t)\}_{t\geq 0}$ such that $T(1) = T$?

Recall that an isometry T on a Hilbert space can be embedded into a C_0-semigroup if and only if T is unitary or $codim(R(T)) = \infty$, where $R(T)$ denotes the range of

T. In this case, it is also possible to embed T into an isometric C_0-semigroup [10, Theorem V.1.19].

Note that, if T can be embedded into C_0-semigroup, then $dim(KerT)$ and $dim(KerT^*)$ are zero or infinite [10, Theorem V.1.7].

Proposition 3.1. *(i) An m-isometry on a finite dimensional space is embeddable into a C_0-group.*

(ii) A normal m-isometry is embeddable.

(iii) A weighted forward shift m-isometry is not embeddable.

Proof. *(i)* On a finite-dimensional space an operator can be embeddable if and only if its spectrum does not contain 0 (see, [10, page 166]). Moreover, on finite-dimensional spaces the spectrum of m-isometries is contained in the unit circle. Hence any m-isometry on finite dimensional space is embeddable into a C_0-group.

(ii) If T is a normal m-isometry, then T^* is an m-isometry. By [2, Corollary 1.2.2] T is invertible and by [10, Theorem V.1.14] T is embeddable.

(iii) Assume that T is a weighted forward shift m-isometry. Then $dimKer(T^*) = 1$, ([3] & [6]). □

Let M_z be the multiplication operator on the Dirichlet space $D(\mu)$ for some finite non-negative Borel measure on $\mathbb{T}$ defined by

$$D(\mu) := \left\{ f : \mathbb{D} \to \mathbb{C} \ \text{ analytic } : \ \int_{\mathbb{D}} |f'(z)|^2 \varphi_\mu(z) dA(z) < \infty \right\} ,$$

where A denotes the normalized Lebesgue area measure in $\mathbb{D}$ and φ_μ is defined by

$$\varphi_\mu(z) := \frac{1}{2\pi} \int_{[0,2\pi)} \frac{1 - |z|^2}{|e^{it} - z|^2} d\mu(t) ,$$

for $z \in \mathbb{D}$.

Proposition 3.2. *M_z on $D(\mu)$ cannot be embedded into C_0-semigroup.*

Proof. By Richter's Theorem [20], M_z is an analytic 2-isometry with $dim(KerT^*) = 1$, then M_z can not be embedded into C_0-semigroup. □

Let Y be an infinite dimensional Hilbert space. The Hilbert space of all vector sequences $(h_n)_{n=1}^{\infty}$ such that $\sum_{n\geq 1} \|h_n\|^2 < \infty$ with the standard inner product is denoted ℓ^2_Y. If $(W_n)_{n=1}^{\infty} \subset B(Y)$ is an uniformly bounded sequence of operators, then the operator $S_W \in B(\ell^2_Y)$ defined by

$$S_W(h_1, h_2, \cdots) := (0, W_1 h_1, W_2 h_2, \cdots) ,$$

for any $(h_1, h_2, \cdots) \in \ell^2_Y$, is called the *operator valued unilateral forward weighted shifts* with weights $W := (W_n)_{n\geq 1}$.

Following some ideas of [10, Proposition V.1.18], we obtain the next result.

Lemma 3.1. *Let S_W be the operator valued unilateral forward weighted shift on ℓ^2_Y with weights $W = (W_n)_{n\geq 1}$ for an infinite dimensional Hilbert space Y. Then S_W can be embedded into a C_0-semigroup.*

Proof. The operator S_W is unitarily equivalent to the operator valued unilateral forward weighted shift on $\ell^2_{L^2([0,1),Y)}$. Moreover, $\ell^2_{L^2([0,1),Y)}$ can be identified with $L^2(\mathbb{R}^+, Y)$ by

$$(f_1, f_2 \cdots) \to (s \to f_n(s-n), \quad s \in [n, n+1)) \ .$$

The following family of operators is defined on $L^2(\mathbb{R}^+, Y)$ for $0 < t \leq 1$ by

$$(T(t)f)(s) := \begin{cases} 0 & s < t \\ f(s-t) & n-1+t \leq s < n \\ W_n f(s-t) & n \leq s < n+t \ , \end{cases}$$

namely,

$$(T(t)f)(s) = \begin{cases} 0 & s < t \\ \displaystyle\sum_{n\geq 1} \big(f(s-t)\chi_{[n-1+t,n)}(s) + W_n f(s-t)\chi_{[n,n+t)}(s)\big) & s \geq t \ . \end{cases}$$

In particular, for $t = 1$, we have that

$$(T(1)f)(s) = \begin{cases} 0 & s < 1 \\ \displaystyle\sum_{n\geq 1} W_n f(s-1)\chi_{[n,n+1)}(s) & s \geq 1 \ . \end{cases}$$

Hence $T(1)$ is unitarily equivalent to S_W.

We define $T(t) := T^{[t]}(1)T(t-[t])$, where $[t]$ denotes the greatest integer less than or equal to t, for $t > 1$.

Let us prove that $\{T(t)\}_{t\geq 0}$ is a C_0-semigroup. Given any $f \in L^2(\mathbb{R}^+, Y)$, we have that

$$\lim_{t\to 0^+} T(t)f(s) = \sum_{n\geq 1} \chi_{[n-1,n)} f(s) = f(s) \ ,$$

in the strong topology of $L^2(\mathbb{R}^+, Y)$.

Let $t, t' \in [0, 1)$. Then $T(t)T(t')f(s) = T(t)\tilde{f}(s)$, where

$$\tilde{f}(s) := T(t')f(s) = \begin{cases} 0 & s < t' \\ \displaystyle\sum_{n\geq 1} (\chi_{[n-1+t',n)}(s) + W_n \chi_{[n,n+t')}(s)) f(s-t') & s \geq t' \ . \end{cases}$$

Then

$$T(t)T(t')f(s)$$

$$= \begin{cases} 0 & s < t + t' \\ \displaystyle\sum_{m\geq 1}\Bigg(\sum_{n\geq 1}\left(\chi_{[n-1+t',n)}(s-t) + W_n\chi_{[n,n+t')}(s-t)\right)\chi_{[m-1+t,m)}(s) + \\ W_m \displaystyle\sum_{n\geq 1}\left(\chi_{[n-1+t',n)}(s-t) + W_n\chi_{[n,n+t')}(s-t)\right) \\ \chi_{[m,m+t)}(s)\Bigg) f(s-t'-t) & s \geq t + t' \end{cases}$$

$$= \begin{cases} 0 & s < t + t' \\ \displaystyle\sum_{m\geq 1}\Bigg\{\sum_{n\geq 1}\left(\chi_{[n-1+t'+t,n+t)}(s) + W_n\chi_{[n+t,n+t'+t)}(s)\right)\chi_{[m-1+t,m)}(s) + \\ W_m \displaystyle\sum_{n\geq 1}\left(\chi_{[n-1+t'+t,n+t)}(s) + W_n\chi_{[n+t,n+t'+t)}(s)\right) \\ \chi_{[m,m+t)}(s)\Bigg\} f(s-t'-t) & s \geq t + t' \end{cases} \tag{6}$$

If $t' + t < 1$, then (6) is given by

$$\begin{aligned} T(t)T(t')f(s) &= \begin{cases} 0 & s < t' + t \\ \displaystyle\sum_{n\geq 1}\left(\chi_{[n-1+t'+t,n)}(s) + W_n\chi_{[n,n+t'+t)}(s)\right) f(s-t'-t) & s \geq t' + t \end{cases} \\ &= T(t+t')f(s)\ . \end{aligned}$$

Denote $t'' := t' + t - [t' + t]$. If $t' + t > 1$, then $t'' = t' + t - 1$ and

$$\begin{aligned} T(t+t')f(s) &= T^{[t+t']}(1)(T(t+t'-[t+t']))f(s) \\ &= T(1)T(t+t'-1)f(s) = T(1)T(t'')f(s) \\ &= \begin{cases} 0 & s < t'' \\ T(1)\displaystyle\sum_{n\geq 1}(\chi_{[n-1+t'',n)}(s) + W_n\chi_{[n,n+t'')}(s))f(s-t'') & s \geq t'' \end{cases} \\ &= \begin{cases} 0 & s - t'' < 1 \\ \displaystyle\sum_{m\geq 1} W_m \sum_{n\geq 1}\left(\chi_{[n-1+t'',n)}(s-1) + W_n\chi_{[n,n+t'')}(s-1)\right) \\ \chi_{[m,m+1)}(s)f(s-t''-1) & s - t'' \geq 1 \end{cases} \\ &= \begin{cases} 0 & s - t'' < 1 \\ \displaystyle\sum_{m\geq 1} W_m \sum_{n\geq 1}\left(\chi_{[n+t'',n+1)}(s) + W_n\chi_{[n+1,n+t''+1)}(s)\right) \\ \chi_{[m,m+1)}(s)f(s-t''-1) & s \geq 1 + t'' \end{cases} \end{aligned}$$

$$= \begin{cases} 0 & s < t+t' \\ \sum_{n\geq 1} \left(W_n \chi_{[n+t'',n+1)}(s) + W_{n+1}W_n \chi_{[n+1,n+t''+1)}(s) \right) f(s-t'-t) & s \geq t+t' \ . \end{cases}$$

On the other hand, by (6) we have that

$$T(t)T(t')f(s) =$$

$$\begin{cases} 0 & s < t+t' \\ \sum_{n\geq 1} \left(W_n \chi_{[n+t'+t-1,n+1)}(s) + W_{n+1}W_n \chi_{[n+1,n+t'+t)}(s) \right) f(s-t'-t) & s \geq t+t' \end{cases}$$

This completes the proof. □

We say that an operator $T \in B(H)$ satisfies the *kernel condition* if

$$T^*T(KerT^*) \subset KerT^* \ .$$

Corollary 3.1. *A non-unitary* 2*-isometry* T *on a Hilbert space satisfying the kernel condition can be embedded into* C_0*-semigroup if and only if* $dim(KerT^*) = \infty$*.*

Proof. If T is a non-unitary 2-isometry on a Hilbert space satisfying the kernel condition. Then, as a consequence of [4, Theorem 3.8], we obtain that $T \cong U \oplus W$ with U unitary and W a operator valued unilateral forward weighted shifts operator in ℓ^2_M with $dimM = dim(KerT^*)$. Thus by [10, Theorem V.1.19] and Lemma 3.1, T can be embedded into C_0-semigroup. □

The Wold-type Decomposition Theorem for 2-isometries, (see [16, 22]), states that any 2-isometry can be decomposed as a direct sum of an unitary operator and an analytic 2-isometry.

Some natural questions arise.

Question 3.1. Let T be an analytic 2-isometry on a Hilbert space. Can be embedded T into C_0-semigroup if and only if $dim(KerT^*) = \infty$?

Question 3.2. Is it possible to characterize all m-isometries on Hilbert spaces having the embedding property?

4 Translation semigroups of m-isometries

In this section, we discuss examples of semigroups of m-isometries.

Definition 4.1. By a *right admissible weighted function* in $(0, \infty)$, we mean a measurable function $\rho : (0, \infty) \to \mathbb{R}$ satisfying the following conditions:

1. $\rho(\tau) > 0$ for all $\tau \in (0, \infty)$,

2. there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\rho(t+\tau) \leq Me^{\omega t}\rho(\tau)$ holds for all $\tau \in (0,\infty)$ and $t > 0$.

For a right admissible weighted function, we define the *weighted space*, $L^2(\mathbb{R}^+, \rho)$, of measurable functions $f : \mathbb{R}^+ \to \mathbb{C}$ such that

$$\|f\|_{L^2(\mathbb{R}^+,\, \rho)} = \int_0^\infty |f(s)|^2 \rho(s) ds < \infty \; .$$

Then the *right translation semigroup* $\{S(t)\}_{t\geq 0}$ given for $t \geq 0$ and $f \in L^2(\mathbb{R}^+, \rho)$ by

$$(S(t)f)(s) := \begin{cases} 0 & \text{if } s \leq t \\ f(s-t) & \text{if } s > t \; , \end{cases}$$

is a strongly continuous semigroup and straightforward computation shows that for $s, t \geq 0$ and $f \in L^2(\mathbb{R}^+, \rho)$

$$(S^*(t)f)(s) = \frac{\rho(s+t)}{\rho(s)} f(s+t) \; .$$

Theorem 4.1. *Let $\{S(t)\}_{t\geq 0}$ be the right translation C_0-semigroup on $L^2(\mathbb{R}^+, \rho)$ with ρ a continuous function. The operator $S(t)$ is an m-isometry for every $t > 0$ if and only if $\rho(s)$ is a polynomial of degree at most m.*

Proof. By definition, $S(t)$ is an m-isometry for every $t > 0$ if

$$\sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \|S^k(t)f\|^2 = 0 \; ,$$

for all $t \geq 0$ and $f \in L^2(\mathbb{R}^+, \rho)$. That is,

$$\int_0^\infty \left(\sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \frac{\rho(s+kt)}{\rho(s)} \right) |f(s)|^2 \rho(s) ds = 0 \; , \tag{7}$$

for all $t \geq 0$ and $f \in L^2(\mathbb{R}^+, \rho)$. Fixed $t \geq 0$, we define

$$g(s) := \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \frac{\rho(s+kt)}{\rho(s)} \; .$$

If $g(s) \neq 0$, we can suppose without lost of generality that, $g(s) > 0$, for some $s \geq 0$. Then by continuity of ρ and $\rho(\tau) > 0$ for all $\tau \geq 0$, we obtain that there exists an interval $I \subset \mathbb{R}^+$, with finite measure, such that there exists $M > 0$ with $g(s) > M$ for all $s \in I$.

Let $f_1(s) := \frac{1}{\sqrt{\rho(s)}} \chi_I(s) \in L^2(\mathbb{R}^+, \rho)$. Then by (7) we have that

$$0 = \int_0^\infty g(s)|f_1(s)|^2 \rho(s) ds = \int_I g(s) ds > M\mu(I) \; ,$$

which it is an absurd. So,

$$\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}\frac{\rho(s+kt)}{\rho(s)}=0\ ,$$

for all $s\geq 0$ and $t\geq 0$. Then by [15, Theorem 13.5], the function ρ is a polynomial of degree at most m. □

Corollary 4.1. *Let $\{S(t)\}_{t\geq 0}$ be the right translation C_0-semigroup on $L^2(\mathbb{R}^+,\rho)$ with ρ a continuous function. Then $S(t)$ is a strict m-isometry for every $t>0$ if and only if $\rho(s)$ is a polynomial of degree $m-1$.*

Consider the *right weighted translation C_0-semigroup*, S_ρ, defined on $L^2(\mathbb{R}^+)$ as

$$(S_\rho(t)f)(s):=\begin{cases}0 & \text{if } s\leq t\\ \dfrac{\rho(s)}{\rho(s-t)}f(s-t) & \text{if } s>t\ .\end{cases}$$

Then $\{S_\rho(t)\}_{t\geq 0}$ is a strongly continuous semigroup if and only if ρ is a right admissible weighted function. For $s,t\geq 0$ and $f\in L^2(\mathbb{R}^+)$

$$(S_\rho^*(t)f)(s)=\frac{\rho(s+t)}{\rho(s)}f(s+t)\ .$$

See [9] for further details.

We now improve part (2) of [18, Corollary 3.3].

Theorem 4.2. *Let $\{S_\rho(t)\}_{t\geq 0}$ be the right weighted translation C_0-semigroup on $L^2(\mathbb{R}^+)$ with ρ a continuous function. Then $S_\rho(t)$ is an m-isometry for every $t>0$ if and only if $\rho(s)^2$ is a polynomial of degree at most m.*

Proof. Consider $M_\rho : L^2(\mathbb{R}^+,\rho^2)\to L^2(\mathbb{R}^+)$ defined by $M_\rho f=\rho f$. Then $S_\rho(t)=M_\rho S(t)M_\rho^{-1}$. Thus $S_\rho(t)$ is an m-isometry for every $t>0$ on $L^2(\mathbb{R}^+)$ if and only if $S(t)$ is an m-isometry for every $t>0$ on $L^2(\mathbb{R}^+,\rho^2)$ if and only if $\rho(s)^2$ is a polynomial of degree at most m. □

Corollary 4.2. *Let $\{S_\rho(t)\}_{t\geq 0}$ be the right weighted translation C_0-semigroup on $L^2(\mathbb{R}^+)$ with ρ a continuous function. Then $S_\rho(t)$ is a 2-isometry for every $t>0$ if and only if $\rho(s)^2=as+b$ for some constants a and b.*

We characterize below the weighted spaces where the adjoint of translation operator is an m-isometry.

Definition 4.2. By a *left admissible weighted function* in $(0,\infty)$, we mean a measurable function $w:(0,\infty)\to\mathbb{R}$ satisfying the following conditions:

1. $w(\tau)>0$ for all $\tau\in(0,\infty)$,

2. there exist constants $M \geq 1$ and $\alpha \in \mathbb{R}$ such that $w(\tau) \leq Me^{\alpha t} w(t+\tau)$ holds for all $\tau \in (0, \infty)$ and all $t > 0$.

Let $\{T(t)\}_{t\geq 0}$ be the *left shift semigroup* given for $t \geq 0$ and $f \in L^2(\mathbb{R}^+, w)$ by

$$(T(t)f)(s) := f(s+t) .$$

Then $\{T(t)\}_{t\geq 0}$ is a strongly continuous semigroup [11]. For $f \in L^2(\mathbb{R}^+, w)$,

$$(T^*(t)f)(s) = \begin{cases} 0 & \text{if } s \leq t \\ \dfrac{w(s-t)}{w(s)} f(s-t) & \text{if } s > t . \end{cases}$$

Theorem 4.3. *Let $\{T^*(t)\}_{t\geq 0}$ be the adjoint of left weighted translation C_0-semigroup on $L^2(\mathbb{R}^+, w)$, such that w is a continuous function. Then $T^*(t)$ is an m-isometry for every $t > 0$ if and only if $w(s) = \frac{1}{p(s)}$ for some polynomial p of degree at most m.*

Proof. $T^*(t)$ is an m-isometry for every $t > 0$ if and only if

$$\sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \|T^{*k}(t)f\|^2 = 0 ,$$

for all $t > 0$ and $f \in L^2(\mathbb{R}^+, w)$. Then

$$\begin{aligned} 0 &= \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} (-1)^{m-k} \int_{kt}^{\infty} \left| \frac{w(s-kt)}{w(s)} f(s-kt) \right|^2 w(s) ds \\ &= \int_0^{\infty} \left(\sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \frac{w(u)}{w(u+kt)} \right) |f(u)|^2 w(u) du , \end{aligned}$$

for all $f \in L^2(\mathbb{R}^+, w)$. As in the proof of Theorem 4.1 we get that $\frac{1}{w(s)}$ is a polynomial of degree at most m. □

Corollary 4.3. *Let $\{T^*(t)\}_{t\geq 0}$ be the adjoint of left weighted translation C_0-semigroup on $L^2(\mathbb{R}^+, w)$ such that w is a continuous function. Then $T^*(t)$ is a strict m-isometry for every $t > 0$ if and only if $w(s) = \frac{1}{p(s)}$ for some polynomial p of degree $m-1$.*

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