SOME EXAMPLES OF m-ISOMETRIES

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Abstract. We obtain the admissible sets on the unit circle to be the spectrum of a strict m-isometry on an n-finite dimensional Hilbert space. This property gives a better picture of the correct spectrum of an m-isometry. We determine that the only m-isometries on $\mathbb{R}^2$ are 3-isometries and isometries giving by $\pm I + Q$, where Q is a nilpotent operator. Moreover, on real Hilbert space, we obtain that m-isometries preserve volumes. Also we present a way to construct a strict $(m+1)$-isometry with an m-isometry given, using ideas of Aleman and Suciu [7, Proposition 5.2] on infinite dimensional Hilbert space.

1. Introduction

Let H be a Hilbert space. Denote by $L(H)$ the algebra of bounded linear operators on H. For $T \in L(H)$ we consider the adjoint operator $T^* \in L(H)$, which is the unique map that satisfies

$$\langle Tx, Ty \rangle = \langle T^*Tx, y \rangle \ ,$$

for every $x, y \in H$. Given $T \in L(H)$, denote by $Ker(T)$ and $R(T)$, the kernel and range of T, respectively. For a positive integer m, an *m-isometry* is an operator $T \in L(H)$ which satisfies the condition

$$(yx-1)^m(T) := \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} T^{*k}T^k = 0 \ ; \tag{1.1}$$

equivalently

$$\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \|T^k x\|^2 = 0 \ , \tag{1.2}$$

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for every $x \in H$. A *strict m-isometry* is an m-isometry which is not an $(m-1)$-isometry. This class of operators was introduced by Agler in [2] and was studied by Agler and Stankus in [4, 5, 6].

Let n be a positive integer. Recall that $Q \in L(H)$ is *n-nilpotent* if $Q^n = 0$ and $Q^{n-1} \neq 0$.

A notion related with m-isometries is the following. An operator $T \in L(H)$ is *isometric n-Jordan* if there exist an isometry $A \in L(H)$ and an n-nilpotent $Q \in L(H)$ such that $T = A + Q$ with $AQ = QA$.

Theorem 1.1. [13, Theorem 2.2] *Any isometric n-Jordan operator is a strict $(2n-1)$-isometry.*

Actually, a much stronger result is true. Indeed in [15, Theorem 3], it is obtained a generalization of Theorem 1.1 for m-isometries: if T is an m-isometry, Q is an n-nilpotent operator and they commute, then $T+Q$ is a $(2n+m-2)$-isometry. See also [25, 28]. Moreover, the study of isometric n-Jordan operators concerning with m-isometries on Banach spaces context has been studied in [15].

Another way of generalization was obtained in [13, Proposition 2.6] for sub-isometry n-Jordan operator. Recall that T is a *sub-isometry n-Jordan* operator if T is the restriction of an isometry n-Jordan operator J to an invariant subspace of J.

Notice that Theorem 1.1 gives an easy way to construct examples of m-isometries, for an odd m. It is sufficient to choose the identity operator as the isometry and any n-nilpotent operator with $n = \frac{m+1}{2}$.

At a first glance, we could think that all the m-isometries come from isometric n-Jordan. However, this is not true, since there are strict m-isometries for even m, see [8, Proposition 9]. What can we say about m-isometries with odd m? Recently, Yarmahmoodi and Hedayatian have proven that the only isometric n-Jordan weighted shift operators are isometries [30, Theorem 1]. So, there are m-isometries that are not isometric n-Jordan, since Athavale in [8] gave examples of strict m-isometries with the weighted shift operator for all integers m.

Whenever, if H is finite dimensional is possible to say more.

Some authors have given examples of m-isometries. For example with the unilateral or bilateral weighted shift [1, 12, 14, 18] and with the composition operator [14, 16, 23].

Another way to construct examples of m-isometries is developing different tools like tensor product [19], functional calculus [24], on Hilbert-Schmidt class [17] and with C_0-semigroups [10, 21, 29].

The purpose of this paper is to make a clear picture of m-isometries on finite dimensional Hilbert space. In Section 2, we begin with the study of m-isometries on $\mathbb{R}^2$ and on $\mathbb{R}^n$, with $n \geq 3$. We give all the 3-isometries on $\mathbb{R}^2$. Also, we obtain the expression of m-isometries and study how this class of operators change volumes on $\mathbb{R}^n$. Moreover, we study the case of complex Hilbert space, where we prove the admissible sets on the unit circle to be the spectrum of an m-isometry. In Section 3, we reproduce similar ideas of Aleman and Suciu [7, Proposition 5.2] to define a 3-isometry using a given 2-isometry. In fact, we obtain a way to construct a strict $(m+1)$-isometry using a weaker condition than a strict m-isometry.

In particular, we will answer the following problems.

Problem 1.2. *Let $T \in L(H)$ with H an n-finite dimensional Hilbert space and m an odd integer. Are all strict m-isometries of the form $\lambda I + Q$, where Q is a nilpotent operator and λ is a complex number with modulus 1?*

Problem 1.3. *Let $T \in L(\mathbb{R}^n)$. How does an m-isometry T change volumes?*

Problem 1.4. *Let H be any n-finite dimensional Hilbert space and let T be an m-isometry with odd m. What can we say about the spectrum?*

2. m-isometries on finite dimensional Hilbert space

Recall some important properties of the spectrum of an m-isometry.

Denote $\overline{\mathbb{D}}$ and $\partial\mathbb{D}$ the closed unit disk and the unit circle, respectively.

Lemma 2.1. *Let m be a positive integer, H be a Hilbert space and $T \in L(H)$ be an m-isometry. Then*

1. [4, Lemma 1.21] $\sigma(T) = \overline{\mathbb{D}}$ *or* $\sigma(T) \subseteq \partial\mathbb{D}$.
2. [3, Lemma 19] *The eigenvectors of T corresponding to distinct eigenvalues are orthogonal.*

Remark 2.2. (1) Notice that any m-isometry on a finite dimensional space is bijective.

(2) It is well known that if Q is k-nilpotent on an n-dimensional vector space, then $k \leq n$.

Denote

$$I_m(H) := \{T \in L(H) \quad : \quad T \text{ is an } m\text{-isometry}\} \, .$$

The following theorem gives a nice picture of m-isometries on finite dimensional spaces.

Theorem 2.3. *(*[13, Theorem 2.7], [3, page 134]*) Let H be an n-finite dimensional Hilbert space and $T \in L(H)$. Then*

(1) T is a strict m-isometry if and only if T is an isometric k-Jordan operator, where $m = 2k - 1$ with $k \leq n$.

(2) $I_1(H) = I_2(H) \subsetneq I_3(H) = I_4(H) \subsetneq \ldots \subsetneq I_{2n-1}(H) = I_j(H)$ for all $j \geq 2n - 1$.

Proof. We include the proofs for completeness.

(1) Assume that T is a strict m-isometry on H. Then the spectrum of T, $\sigma(T) = \{\lambda_1, \lambda_2, \ldots, \lambda_s\}$, where λ_i are eigenvalues of modulus 1, since the spectrum of T must be in the unit circle and m is odd [4, Lemma 1.21 & Proposition 1.23]. By part (2) of Lemma 2.1, the spectral subspaces of T, $H_i := Ker(T - \lambda_i)^{n_i}$ are mutually orthogonal and

$$T \cong T_{|H_1} \oplus \cdots \oplus T_{|H_s} \, ,$$

where $n_1, \ldots, n_s$ are positive integers such that $Ker(T-\lambda_i)^{n_i} = Ker(T-\lambda_i)^N$ for all $N \geq n_i$. Moreover, for all $j \in \{1, \ldots, s\}$, we have that $\sigma(T_{|H_j}) = \{\lambda_j\}$ and $T_{|H_j}$ is of the form $\lambda_j + Q_j$ for some nilpotent operator Q_j. So, $T = A + Q$ for some isometry, in fact unitary diagonal operator A and some nilpotent operator Q such that $AQ = QA$.

The converse is consequence of Theorem 1.1.

(2) Let us prove that $I_{2\ell-1}(H) = I_{2\ell}(H)$ for all $\ell \in \mathbb{N}$. Recall that if T is (2ℓ)-isometry, then T is bijective and so T is $(2\ell - 1)$-isometry [4, Proposstion 1.23]. Moreover, the highest degree of nilpotent operator on n-dimensional Hilbert space is n. The result is a consequence of Theorem 1.1. $\square$

2.1. **m-isometries on real Hilbert spaces.** Next, we study the m-isometries on $\mathbb{R}^n$.

Based on the above results, we obtain all m-isometries on $\mathbb{R}^2$.

Theorem 2.4. *If $T \in L(\mathbb{R}^2)$ is a strict m-isometry, then $m = 1$ or $m = 3$ and $T = A + Q$, where A is an isometry and Q is a nilpotent operator of order 2 that commutes.*

Recall that isometries on $\mathbb{R}^2$ are given by

$$R_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \text{ and } S_\theta := \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} ,$$

where

(1) R_θ is a *rotation* (about 0) and its determinant, $\det(R_\theta)$ is 1 and
(2) S_θ is a symmetry respect to the straight line of equation $x_2 = \tan(\theta/2)x_1$ and $\det(S_\theta) = -1$.

And the non-zero nilpotent operators on $\mathbb{R}^2$ are λM, λN and λQ_k where

$$M := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \quad N := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \; Q_k := \begin{pmatrix} 1 & k \\ -\frac{1}{k} & -1 \end{pmatrix} , \tag{2.3}$$

with $k \neq 0$ and $\lambda \in \mathbb{C} \setminus \{0\}$.

We are interested in studying isometries that commute with nilpotent operators on $\mathbb{R}^2$.

Lemma 2.5. *The unique isometries on $L(\mathbb{R}^2)$ that commute with a non-zero nilpotent operator are the trivial cases, that is, $\pm I$.*

Proof. Simple calculations prove that

$$R_\theta M = MR_\theta \Longleftrightarrow R_\theta N = NR_\theta \Longleftrightarrow R_\theta Q_k = Q_k R_\theta \Longleftrightarrow \sin\theta = 0 \Longleftrightarrow \theta = 0 \text{ or } \theta = \pi .$$

That is, the unique isometries of type R_θ which commute with some non-zero nilpotent (hence with all the nilpotent) are $R_0 = I$ and $R_\pi = -I$.

Analogously, we have that

$$S_\theta M = MS_\theta \Longleftrightarrow S_\theta N = NS_\theta \Longleftrightarrow S_\theta Q_k = Q_k S_\theta \Longleftrightarrow \sin\theta = \cos\theta = 0 ,$$

which it is impossible. Hence there are not isometries S_θ which commute with some non-zero nilpotent operator. □

Taking into account Theorem 2.3 we give the unique strict 3-isometries on $\mathbb{R}^2$. Indeed, we answer Problem 1.2 for $n = 2$ in the following result.

Theorem 2.6. *The strict 3-isometries on $\mathbb{R}^2$ are of the form $\pm I + Q$, where Q is a non-zero nilpotent operator given in (2.3).*

Proof. It is immediate by Theorem 2.4 and Lemma 2.5.

□

Let $T \in L(\mathbb{R}^n)$ with $n \geq 3$ and let us consider the following n conditions:

$$(M_k) \qquad S_k(Tx_1, Tx_2, \ldots, Tx_k) = S_k(x_1, x_2, \ldots, x_k)$$

for all $x_1, x_2, \ldots, x_k \in \mathbb{R}^n$ and $k = 1, 2, \ldots, n$, where $S_k(x_1, x_2, \ldots, x_k)$ denotes the k-dimensional measure of the set

$$\{\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_k x_k \quad : \quad 0 \leq \lambda_i \leq 1, \quad \text{for } i = 1, 2, \ldots, k\} \, .$$

Lemma 2.7. *Let $T \in L(\mathbb{R}^n)$. Then*

(1) [26, Teorema II] *T satisfies the conditions (M_1), (M_2), $\cdots$, (M_{n-1}) if and only if T is an isometry.*
(2) [20] *The condition (M_n) is equivalent to $det(T) = \pm 1$.*

An easy application of Theorem 1.1 gives that, for example in $\mathbb{R}^3$, we have strict 3-isometries giving by $\pm I + Q$, where Q is a 2-nilpotent operator and strict 5-isometries giving by $\pm I + Q$, where Q is a 3-nilpotent operator.

The next result gives answer to Problems 1.2 and 1.3 for $n \geq 3$, where n is the dimension of the Hilbert space.

Theorem 2.8. *Let $n \geq 3$. Then the following properties follow:*

(1) There are non-trivial strict m-isometries on $L(\mathbb{R}^n)$ for any odd m less than $2n - 1$, that is, there exists an isometry different from $\pm I$ such that commutes with a non-zero k-nilpotent operator with $k \in \{1, 2, \cdots, n-1\}$.
(2) The m-isometries preserve volumes.

Proof. *(1)* Define

$$\begin{aligned} A(x_1, x_2, \dots, x_n) : &= (-x_1, x_2, \dots, x_n) \\ Q_j(x_1, x_2, \dots, x_n) : &= (0, x_3, x_4, \cdots, x_{j+1}, 0, \cdots, 0) . \end{aligned}$$

Then A is an isometry and Q_j is a j-nilpotent operator such that

$$AQ_j(x_1, x_2, \dots, x_n) = Q_j A(x_1, x_2, \dots, x_n) = (0, x_3, x_4, \dots, x_{j+1}, 0, \dots, 0) ,$$

for all $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. By Theorem 1.1, we get that $A + Q_j$ is a non trivial strict $(2j-1)$-isometry for $j = 1, \dots, n-1$.

(2) By Lemma 2.7, it will be enough to prove that $det(A + Q) = \pm 1$ for all isometries A that commute with a nilpotent operator Q. Since $AQ = QA$, then $\sigma(A + Q) = \sigma(A)$ by [31, Proposition 1.1]. According to the spectrum of an isometry on a finite dimensional space, we have that the spectrum of A is a closed subset of the unit circle. By [9, page 150], the determinant of T is the product of the eigenvalues of T, counting multiplicity. Hence $det(T) = \pm 1$.

□

The converse of part (2) of Theorem 2.8 is not true, as prove the following example.

Example 2.9. *Let* $T := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$. *Then* $det(T) = 1$ *and* T *is not a 3-isometry, since*

$$\|T^3 x\|^2 - 3\|T^2 x\|^2 + 3\|Tx\|^2 - \|x\|^2 \neq 0 ,$$

for $x := (1, 1, 0)$.

2.2. **On complex Hilbert space.** We recall the following results about the spectrum of m-isometries.

Lemma 2.10. [13, Theorem 4.4] *Let* H *be an infinite dimensional Hilbert space.*

(1) If K *is any compact subset of* $\partial\mathbb{D}$*, then there exists a strict* m*-isometry for any odd number* m *such that* $\sigma(T) = K$.

(2) If K is the closed unit disk, then there exits a strict m-isometry for any integer number m.

The main aim of this section is to solve Problem 1.4.

Let $T \in L(\mathbb{C}^n)$ be an m-isometry. It is clear that $\sigma(T) \subseteq \partial\mathbb{D}$ by part (1) of Lemma 2.1 and $\sigma(T)$ has at most n different eigenvalues. Indeed if $K := \{\lambda_1, \cdots, \lambda_n\}$ with λ_i different complex numbers on the unit circle, then it is possible to define an isometry T such that $\sigma(T) = K$. In particular, the following operator

$$T(x_1, \cdots, x_n) := (\lambda_1 x_1, \cdots, \lambda_n x_n)$$

is an isometry on $\mathbb{C}^n$ with $\sigma(T) = \{\lambda_1, \cdots, \lambda_n\}$.

In the following theorem we prove that any m-isometry with $m \geq 3$ on $\mathbb{C}^n$ can not have n different eigenvalues.

Theorem 2.11. *Any strict $(2k-1)$-isometry on $\mathbb{C}^n$ with $2 \leq k \leq n$ has at most $n-1$ distinct eigenvalues.*

Proof. Assume that $T \in L(\mathbb{C}^n)$ is a strict $(2k-1)$-isometry with $\sigma(T) = \{\lambda_1, ..., \lambda_n\}$ where $\lambda_1, ..., \lambda_n$ are different eigenvalues of T. Then T could be written as $T = PSP^{-1}$, for some $P \in L(\mathbb{C}^n)$ where

$$S := \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \lambda_n \end{pmatrix}$$

and $|\lambda_i| = 1$ for $i \in \{1, \cdots, n\}$, by part (1) of Lemma 2.1. Since T is a strict $(2k-1)$-isometry, by part (2) of Lemma 2.1, the operator P is a unitary operator. This means that T is unitarily equivalent to S, therefore T is a unitary, which is a contradiction. □

Theorem 2.12. *The strict $(2k-1)$-isometries on $\mathbb{C}^n$, with $2 \leq k \leq n$ are of the form $(\lambda_1 I_{n_1} \oplus ... \oplus \lambda_\ell I_{n_\ell}) + Q$, with $\ell \in \{1, ..., n-k+1\}$, where Q is a k-nilpotent, $|\lambda_j| = 1$ for all $j \in \{1, ..., \ell\}$ and $n_1 + ... + n_\ell = n$.*

Proof. Suppose that T is a strict $(2k-1)$-isometry. By Theorem 2.3, we have that $T = U+Q$, where U is a unitary operator and Q is a k-nilpotent operator such that $UQ = QU$. Assume, by contradiction, that T has at least $n-k+2$ distinct eigenvalues. That means

$$\sigma(T) = \{\lambda_1, ..., \lambda_r\}, \quad \text{with} \ \ r \geq n-k+2.$$

Then $\mathbb{C}^n = H_{\lambda_1} \oplus ... \oplus H_{\lambda_r}$, where $H_{\lambda_i} := Ker(T-\lambda_i I)^{n_i}$ and n_i is the order of multiplicity of the eigenvalue λ_i. Denote $T_{|H_i}$ the restriction operator of T to H_i, for $1 \leq i \leq r$. Then $T_{|H_i} = \lambda_i I_{n_i} + Q_i$, where Q_i is a h_i-nilpotent with $1 \leq h_i \leq n_i$. By part (2) of Lemma 2.1, we conclude that T could be written as

$$T = (\lambda_1 I_{n_1} \oplus ... \oplus \lambda_r I_{n_r}) + (Q_1 \oplus ... \oplus Q_r) \ ,$$

where $Q_1 \oplus ... \oplus Q_r$ is a k_0-nilpotent, with $k_0 := \max_{i=1,...,r}\{h_i\}$ and $k_0 < k$. Then we get a contradiction. □

Corollary 2.13. *If $T \in L(\mathbb{C}^n)$ is a strict $(2k-1)$-isometry, with $2 \leq k \leq n$, then $\sigma(T) \subseteq \{\lambda_1, ..., \lambda_{n-k+1}\} \subseteq \partial\mathbb{D}$.*

Corollary 2.14. *Any $(2n-1)$-isometry on $\mathbb{C}^n$ is of the form $\lambda I + Q$, where Q is an n-nilpotent operator and $\lambda \in \partial\mathbb{D}$. In particular the spectrum is a single point on the unit circle.*

3. Construction of an $(m+1)$-isometry from an m-isometry

In this section we present a method to construct a Hilbert space H_k and an $(m+1)$-isometry on H_k from an m-isometry T^k on a Hilbert space for some integer k. Our result is based on the construction given by Aleman and Suciu in [7, Proposition 5.2] for $m = 2$ and $k = 1$.

Henceforth H will denote an infinite dimensional Hilbert space.

Given $S \in L(H)$, $x \in H$ and an integer $\ell \geq 1$, it is defined

$$\beta_\ell(S,x) := \frac{1}{\ell!}\sum_{j=0}^{\ell}(-1)^{\ell-j}\binom{\ell}{j}\|S^j x\|^2 \ .$$

Note that S is an m-isometry if and only if $\beta_m(S,x) = 0$ for all vector $x \in H$.

Consider $\mathbb{C}[z]$ the space of all complex polynomials. Given $p \in \mathbb{C}[z]$, we write

$$p(z) = \sum_{n \geq 0} p_n z^n$$

and define $Lp \in \mathbb{C}[z]$ in the following way:

$$Lp(z) := \sum_{n \geq 1} p_n z^{n-1} = \frac{p(z) - p_0}{z} \, .$$

We have that $\mathbb{C}[z]$ is an inner product space with the norm $\|.\|_2$ given by

$$\|p\|_2^2 := \sum_{n \geq 0} |p_n|^2 \, .$$

Also if we consider a new norm on $\mathbb{C}[z]$ defined by

$$|||p|||_k^2 := \|p\|_2^2 + \sum_{n \geq 0} \|(L^{nk} p)(T) x_0\|^2 \, ,$$

it is obtained that $\mathbb{C}[z]$ is an inner product space with $|||.|||_k$. Denote H_k its completion with the new norm.

The following combinatorial result will be useful.

Lemma 3.1. [22, Eq. 0.151 (4)] *If m is any positive integer, then*

$$\sum_{k=0}^{m} (-1)^k \binom{n}{k} = (-1)^m \binom{n-1}{m} \, ,$$

for any integer $n \geq m + 1$.

Recall that the class of m-isometries is stable under powers. However, the converse is not true. See [11, 27].

Theorem 3.2. *Let $T \in L(H)$ such that T^k is a strict m-isometry on $R(T^k)$, for some k, and $x_0 \in H \setminus \{0\}$ such that $\beta_{m-1}(T^k, T^k x_0) \neq 0$.*

(1) For every $p \in \mathbb{C}[z]$ and $j \in \mathbb{N}$,

$$|||M_z^{kj} p|||_k^2 = |||p|||_k^2 + \sum_{i=1}^{j} \|T^{ki} p(T) x_0\|^2 \, ,$$

where M_z denotes the multiplication operator defined by $M_z p := zp$.

(2) For every $p \in \mathbb{C}[z]$ and $\ell \geq 1$,

$$\beta_{\ell+1}(M_z^k, p) = \frac{\ell!}{(\ell+1)!}\beta_\ell(T^k, T^k p(T)x_0) \ . \tag{3.4}$$

(3) The extension of M_z^k to H_k is an $(m+1)$-isometry.

Proof. (1) Let p be any polynomial and $j \in \mathbb{N}$. Then will prove that

$$|||M_z^{kj}p|||_k^2 = |||p|||_k^2 + \sum_{i=1}^{j} \|T^{ki}p(T)x_0\|^2 \ , \tag{3.5}$$

by induction. For $j = 1$ we need to prove that

$$|||M_z^k p|||_k^2 = |||p|||_k^2 + \|T^k p(T)x_0\|^2 \ , \tag{3.6}$$

for any polynomial p.
Let $p(z) := \sum_{n\geq 0} p_n z^n$. Then

$$\begin{aligned} |||M_z^k p|||_k^2 = |||z^k p|||_k^2 &= \|z^k p\|_2^2 + \sum_{n\geq 0} \|(L^{nk} z^k p)(T)x_0\|^2 \\ &= \|p\|_2^2 + \|(z^k p)(T)x_0\|^2 + \sum_{n\geq 1} \|(L^{nk} z^k p)(T)x_0\|^2 \\ &= \|p\|_2^2 + \|T^k p(T)x_0\|^2 + \sum_{n\geq 0} \|(L^{nk} p)(T)x_0\|^2 = |||p|||_k^2 + \|T^k p(T)x_0\|^2 \ . \end{aligned}$$

Then (3.6) holds.

Suppose that (3.5) is true for j. Let us prove it for $j+1$. Then

$$\begin{aligned}
|||M_z^{k(j+1)}p|||_k^2 &= |||M_z^{kj}(M_z^k p)|||_k^2 = |||M_z^k p|||_k^2 + \sum_{i=1}^{j} \|T^{ki}(M_z^k p)(T)x_0\|^2 \\
&= |||z^k p|||_k^2 + \sum_{i=1}^{j} \|T^{ki}T^k p(T)x_0\|^2 \\
&= \|p\|_2^2 + \sum_{n\geq 0} \|(L^{nk}z^k p)(T)x_0\|^2 + \sum_{i=1}^{j} \|T^{k(i+1)}p(T)x_0\|^2 \\
&= \|p\|_2^2 + \|T^k p(T)x_0\|^2 + \sum_{n\geq 0} \|(L^{nk}p)(T)x_0\|^2 + \sum_{i=2}^{j+1} \|T^{ki}p(T)x_0\|^2 \\
&= |||p|||_k^2 + \sum_{i=1}^{j+1} \|T^{ki}p(T)x_0\|^2 \,.
\end{aligned}$$

So we prove (3.5).

(2) For $\ell \in \mathbb{N}$, we have

$$\begin{aligned}
\beta_{\ell+1}(M_z^k, p) &= \frac{1}{(\ell+1)!}\sum_{j=0}^{\ell+1}(-1)^{\ell+1-j}\binom{\ell+1}{j}|||M_z^{kj}p|||_k^2 \\
&= \frac{1}{(\ell+1)!}\left((-1)^{\ell+1}|||p|||_k^2 + \sum_{j=1}^{\ell+1}(-1)^{\ell+1-j}\binom{\ell+1}{j}|||M_z^{kj}p|||_k^2\right) \\
&= \frac{1}{(\ell+1)!}\left((-1)^{\ell+1}|||p|||_k^2 + \sum_{j=1}^{\ell+1}(-1)^{\ell+1-j}\binom{\ell+1}{j}\left(|||p|||_k^2 + \sum_{i=1}^{j}\|T^{ki}p(T)x_0\|^2\right)\right) \\
&= \frac{1}{(\ell+1)!}\sum_{j=1}^{\ell+1}(-1)^{\ell+1-j}\binom{\ell+1}{j}\sum_{i=1}^{j}\|T^{ki}p(T)x_0\|^2 \\
&= \frac{1}{(\ell+1)!}\sum_{j=1}^{\ell+1}\|T^{kj}p(T)x_0\|^2\sum_{i=j}^{\ell+1}(-1)^{\ell+1-i}\binom{\ell+1}{i} \,,
\end{aligned}$$

where p is any polynomial.

Using Lemma 3.1, in the last sum, we have that

$$\sum_{i=j}^{\ell+1}(-1)^{\ell+1-i}\binom{\ell+1}{i} = -\sum_{i=0}^{j-1}(-1)^{\ell+1-j}\binom{l+1}{j} = (-1)^{\ell+j-1}\binom{\ell}{j-1} \,.$$

So,

$$\begin{aligned}\beta_{\ell+1}(M_z^k,p) &= \frac{1}{(\ell+1)!}\sum_{j=1}^{\ell+1}\|T^{kj}p(T)x_0\|^2(-1)^{\ell+j-1}\binom{\ell}{j-1}\\ &= \frac{1}{(\ell+1)!}\sum_{j=0}^{\ell}(-1)^{\ell-j}\binom{\ell}{j}\|T^{kj}p(T)T^kx_0\|^2\\ &= \frac{\ell!}{(\ell+1)!}\beta_\ell(T^k,T^kp(T)x_0)\ .\end{aligned}$$

So, (3.4) is proved.

(3) It is enough to prove that $\beta_{m+1}(M_z^k,p)=0$ for any $p\in\mathbb{C}[z]$. This is a consequence of (3.4), since T^k is an m-isometry on $R(T^k)$. □

Corollary 3.3. [7, Proposition 5.2] *Let T be a 2-isometry on a Hilbert space H. Fix $x_0\in H\setminus\{0\}$ and let H_1 be the completion of the space of analytic polynomials with respect to the norm*

$$\|p\|_1^2 := \|p\|_2^2 + \sum_{n\geq 0}\|(L^np)(T)x_0\|^2\ .$$

Then the multiplication operator by the independent variable $M_zp:=zp$ extends to a 3-isometry on H_1.

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