Computers & Operations Research 145 (2022) 105861

Contents lists available at ScienceDirect

Computers and Operations Research

journal homepage: www.elsevier.com/locate/cor

An inventory system with time-dependent demand and partial backordering under return on inventory investment maximization

Luis A. San-José[a,*], Joaquín Sicilia[b], Valentín Pando[c], David Alcaide-López-de-Pablo[b]

[a] IMUVA, Dpto. de Matemática Aplicada, Universidad de Valladolid, Valladolid, Spain
[b] Dpto. de Matemáticas, Estadística e Investigación, Operativa, Universidad de La Laguna, Tenerife, Spain
[c] Departamento de Estadística e Investigación, Operativa, iuFOR, Universidad de Valladolid, Palencia, Spain

ARTICLE INFO

Keywords:
Inventory systems
Return on Inventory Investment
Time-dependent demand
Partial backordering
Shortage costs

ABSTRACT

In this article, we study an inventory system for items that have a power demand pattern and where shortages are allowed. We suppose that only a fixed proportion of demand during the stock-out period is backordered. The decision variables are the inventory cycle and the ratio between the initial stock and the total quantity demanded throughout the inventory cycle. The objective is to maximize the Return on Inventory Investment (ROII) defined as the ratio of the profit per unit time over the average inventory cost. After analysing the objective function, the optimal global solutions for all the possible cases of the inventory problem are determined. These optimal policies that maximize the ROII are, in general, different from those that minimize the total inventory cost per unit time. Finally, a numerical sensitivity analysis of the optimal inventory policy with respect to the system input parameters and some useful managerial insights derived from the results are presented.

1. Introduction

As it is well known, stock management models answer two important questions: when and how much to order so as to optimize a certain objective function related to the inventory control. Generally, the objective function represents the profit per unit of time or the average cost of inventory. However, in some companies, it may be more interesting to maximize the return on investment (ROI), instead of maximizing the profit per unit of time or minimizing the cost of inventory. This approach to maximizing return on investment in stock management has already been used in the literature on the topic. Thus, Otake and Min (2001) studied inventory and investment in quality improvement policies under return on investment maximization. Li et al. (2008) developed a return on inventory investment maximization model under an investment budget constraint for inventory and capital investment in setup and quality operations. They obtained various managerial insights into inventory reduction and the uniqueness of the global optimal solution. Wee et al. (2009) developed a joint replenishment inventory model with stock-dependent demand and shortage cost constraint under profit and ROI maximization. Yaghin and Ghomi (2012) studied a hybrid multi objective integrated pricing and lot sizing model in a fuzzy environment, considering three decision criteria: profit, return on inventory investment and a qualitative objective related to customer satisfaction. Yaghin et al. (2013) formulated a fuzzy inventory model integrating the marketing-inventory and price discrimination decisions, which maximized the total profit and the return on inventory investment (ROII) concurrently. Chen and Liao (2014) studied the return on inventory investment maximization problem for an intermediary firm of a deteriorating item. Yaghin et al. (2017) considered the return on inventory investment maximization in a joint pricing and lot-sizing problem in a fuzzy environment. Misook (2017) analysed the difference of return on inventory investment by the firm and industry characteristics and showed that the ROII of high-growth, low-leverage, large firms was greater than that of all other firms. Yaghin et al. (2018) developed an integrated model of ordering, shipping and differential pricing in a two-echelon supply chain under return on inventory investment maximization. More recently, Pando et al. (2019, 2020) developed inventory models with stock-dependent demand and non-linear holding cost under the return on investment maximization. Pando et al. (2021) studied an inventory system where the demand rate potentially depends on both selling price and stock level, in which the goal was the maximization of the profitability index. Baker et al. (2021) analysed how households can derive substantial financial returns from strategic shopping behaviour and optimal inventory management of consumer goods. Kouvelis and

* Correspondence to: Dpto. de Matemática Aplicada, Escuela de Ingeniería Informática, Paseo de Belén 15, 47011, Valladolid, Spain.
E-mail addresses: augusto@mat.uva.es (L.A. San-José), jsicilia@ull.edu.es (J. Sicilia), vpando@uva.es (V. Pando), dalcaide@ull.edu.es (D. Alcaide-López-de-Pablo).

https://doi.org/10.1016/j.cor.2022.105861
Received 23 September 2021; Received in revised form 11 March 2022; Accepted 28 April 2022
Available online 14 May 2022
0305-0548/

Qiu (2021) studied how capital efficiency metrics, such as return on investment, affect orders at a stand-alone single stocking stage under demand uncertainty or within bilateral supply chains of a supplier and a buyer interacting with the use of a trade-credit-contract.

When shortages occur during the inventory cycle, there are some customers who are willing to wait for the next replenishment, while others decide not to wait and they buy the products from other sellers. This realistic situation is modelled in inventory systems assuming partial backordering. Some recent papers on inventory models allowing partial backlogging are the articles of Mishra and Singh (2010), Roy et al. (2011), Pentico and Drake (2011), Sicilia et al. (2012) , Hasanov et al. (2012), San-José et al. (2017, 2018), Alfares and Ghaithan (2019) and Shaikh et al. (2019). Usually, in the stock-out situation, the customers make the decision to wait or not until the next replenishment, depending on the time they would have to wait and the possible compensation of the company if they wait. Thus, the future willingness to do business with the company also depends on the time remaining until the arrival of the next replenishment. This approach was partially assumed in Chern et al. (2005). Later, San-José et al. (2009) developed an inventory system where both backorder unit cost and lost unit sale cost have an affine structure: a variable cost depending on the period of time where shortages exist and a fixed cost. Among other works that use this last approach to model the shortage cost, we can mention the papers of Sicilia et al. (2012), and San-José et al. (2014, 2017).

In real-life inventory systems, the demand rate of an item usually depends on time so, based on this assumption, several approaches have been proposed in the literature to model this situation. One of these approaches considers that demand follows a power pattern during the scheduling period or inventory cycle. It leads to model different ways of drawing units from the inventory. These ways are characterized by the demand pattern indices, which describe the behaviour of demand. Thus, this pattern allows us to model situations where either a greater part of demand occurs at the beginning of the period, or scenarios where a larger portion of demand occurs toward the end of the inventory cycle. In this line, Dye (2004) studied a deteriorating inventory model with power demand pattern, time-varying deterioration and general time-proportional backlogging rate. Rajeswari and Vanjikkodi (2012) presented an inventory model where the demand rate potentially depends on time and a Weibull deterioration rate for three different scenarios: complete, partial and no backlogging. Keshavarzfard et al. (2019a) studied an economic production model for multiple items with full backlogging, production rate proportional to demand rate and demand rate depending linearly on price and time with a power demand pattern. Adaraniwon and Omar (2019) developed an EOQ model with a power demand pattern and partial backlogging for a delayed deteriorating item. Keshavarzfard et al. (2019b) analysed a production–inventory model with a power demand pattern, a production rate proportional to the demand rate and defective items, where either all the imperfect units are recovered or a certain fraction of these defective units are reworked and others are removed. Keshavarzfard et al. (2019c) developed a production system with power demand rate, dependent production rate and defective items. They considered three different situations for the inventory system regarding the date that imperfect products are withdrawn from the stock. San-José et al. (2019) developed an inventory system where customer demand has a power pattern, shortages are allowed and the inventory cycle must be an integer multiple of a fixed time period. San-José et al. (2021) studied a new lot-size inventory problem for products whose demand pattern is dependent on price, advertising frequency and time.

The main contribution of this paper is to provide the optimal inventory policy that maximizes the return on inventory investment (that is, the ratio of the profit per unit time to the average inventory cost), when customer demand depends on time and shortages are partially backlogged. To the best of our knowledge, this is the first paper that simultaneously assumes the following issues, which have not been considered together in the literature:

(a) the demand rate of the item follows a power demand pattern,

(b) shortages are allowed, but only a proportion of the demand during the shortage period is backlogged,

(c) both backorder unit cost and lost unit sale cost are composed of a variable cost, which depends on the length of the waiting time until the next replenishment and a fixed cost, and

(d) the objective is the maximization of the return on inventory investment.

Simultaneous consideration of the above assumptions allows us to model a wide variety of real-life situations and, therefore, makes the inventory model more realistic.

The rest of the paper is organized as follows. Section 2 presents the assumptions and notation used throughout the paper. Section 3 formulates mathematically the inventory problem. Optimal inventory policies considering partial backlogging, complete backlogging and full lost sales are developed in Section 4. In the same section, the particular case with constant demand rate is also analysed. Several numerical examples are solved in Section 5. A numerical sensitivity analysis of the optimal inventory policy with respect to the system input parameters and some useful managerial insights derived from the results are presented in Section 6. Finally, some conclusions and future research lines are set up in Section 7.

2. Hypothesis of the inventory system

The inventory system analysed in this paper has the following properties. A single item is considered in the inventory system. The replenishment is instantaneous. The inventory cycle T is a decision variable of the system. The fluctuations of the inventory level during the period T are continuously repeated in subsequent periods. The lead-time is zero or negligible. The average demand of the item is deterministic, with a rate of r units per inventory cycle. The way in which quantities are taken from the inventory depends on the time when they are withdrawn. Let $\lambda(t)$ denote the demand rate at time t $(0 < t < T)$. This demand rate is supposed to be the function

$$\lambda(t) = \frac{r}{n}\left(\frac{t}{T}\right)^{(1-n)/n}$$

where n is the index of demand pattern, with $n > 0$. Note that if $n > 1$, then a greater part of demand occurs at the beginning of the period. If $n = 1$, the demand rate is constant throughout the inventory cycle and, if $n < 1$ then a larger portion of demand occurs toward the end of the inventory cycle.

Thus, the total quantity demanded along the inventory cycle is $\int_0^T \lambda(t)dt = rT$. At the beginning of the inventory cycle, there are S units in stock. This amount, which is unknown and must be determined, is a fraction ρ of the demand during the inventory cycle. That is, $S = \rho rT$, with $0 \leq \rho \leq 1$. Shortages are allowed and let b denote the total number of shortages during the inventory cycle. Only a fraction β, with $0 \leq \beta \leq 1$, of that unsatisfied demand will be backordered. When the number of backorders is βb, the inventory must be replenished. The ordering cost A is constant and independent of the ordered amount. The price c of acquisition or purchasing and the selling price s of a unit of the item are known constants. The holding cost per unit and per unit time h is also a known constant. The unit backorder cost considers a constant cost ω_0 plus a variable cost $\omega\varphi$, where φ is the amount of time the customers wait before receiving the item. The goodwill cost of a lost sale is also described by a linear function of time for which lost sales exist with slope π and intercept π_0.

The notation used throughout the paper is summarized in Table 1.

3. Formulation of the problem

In this section, an inventory model for a single item over an infinite horizon under power demand pattern is developed. Let $I(t)$ denote the net inventory level at time t, with $0 \leq t \leq T$. At the beginning of the inventory cycle, the replenishment of products raises the inventory

Table 1
List of notation.

τ	Length of the inventory cycle where the net stock is positive (≥ 0).
Ψ	Length of the inventory cycle when net stock is less than or equal to zero (≥ 0).
T	Scheduling period or inventory cycle, that is, $T = \tau + \Psi$ (> 0, *decision variable*).
$I(t)$	Inventory level at time t, with $0 \leq t \leq T$.
$\lambda(t)$	Demand rate at time t.
r	Average demand per cycle (> 0).
n	Demand pattern index (> 0).
A	Replenishment cost (> 0).
c	Unit acquisition cost (> 0).
s	Selling price per unit ($s \geq c$).
h	Unit holding cost per unit time (> 0).
ω_0	Constant cost per backordered unit (≥ 0).
ω	Shortage cost per backordered unit and per unit time (≥ 0). We assume that $\omega_o + \omega\varphi$ is the backorder cost per unit, when the shortage time is φ and the demand is backordered.
π_0	Constant goodwill cost per lost unit (≥ 0).
π	Unit goodwill cost per unit time (≥ 0). We consider that $\pi_0 + \pi\varphi$ is the lost sale cost per unit, when the shortage time is φ and the demand is lost.
S	Maximum level of the stock (≥ 0).
b	Demanded quantity during the stock-out period (≥ 0).
β	Fraction of demand which is backordered ($0 \leq \beta \leq 1$).
Q	Lot size per cycle, that is, $Q = \beta b + S$ (≥ 0).
ρ	Ratio between the initial inventory and the total quantity demanded during the inventory cycle, that is, $\rho = S/(rT)$ (≥ 0, *decision variable*).
α_0	Fixed unit shortage cost, that is, $\alpha_0 = \omega_0\beta + \pi_0(1-\beta)$.
α_1	Time-dependent average shortage cost, that is, $\alpha_1 = \omega\beta + \pi(1-\beta)$.
$g_1(\rho)$	Auxiliary function, defined as $g_1(\rho) = (1-\beta)\rho + \beta$.
$g_2(\rho)$	Auxiliary function, defined as $g_2(\rho) = (h+\alpha_1)\rho^{n+1} - (n+1)\alpha_1\rho + n\alpha_1$.

level up to the maximum level S. Next, in the stock-in period τ, the inventory decreases due to demand. Thus, the inventory level at time t is given by

$$I(t) = S - rT\left(\frac{t}{T}\right)^{1/n} = rT\left(\rho - \left(\frac{t}{T}\right)^{1/n}\right) \quad \text{for } t \in [0,\tau].$$

Taking into account that at $t = \tau$ the inventory level is zero, it then leads to $\tau = \rho^n T$. Next, shortages occur during the time period $(\tau, T]$ and a fraction β of shortages are backordered. Thus, the net inventory level during the stock-out period is given by

$$I(t) = \beta rT\left(\rho - \left(\frac{t}{T}\right)^{1/n}\right) \quad \text{for } t \in [\tau, T).$$

Therefore, the minimum net stock level is $I(T) = -\beta(1-\rho)rT$. The total quantity demanded during the stock-out period is $b = \int_\tau^T \lambda(t)dt = (1-\rho)rT$. The lot size is

$$Q = S + \beta b = ((1-\beta)\rho + \beta)rT. \tag{1}$$

Taking into account the above assumptions, the total profit per cycle $P(\rho,T)$ is the difference between the revenue per cycle sQ and the sum of the ordering cost A, the purchasing cost cQ, the holding cost, the backordering cost and the lost sale cost per cycle. The holding cost per cycle is given by

$$HC(\rho,T) = h\int_0^\tau I(t)dt = \frac{hr}{n+1}\rho^{n+1}T^2$$

The backordering cost is

$$\begin{aligned} BC(\rho,T) &= \int_\tau^T \left(\omega_0\beta\lambda(t) + \omega\int_\tau^t \beta\lambda(u)du\right)dt \\ &= \omega_0\beta(1-\rho)rT + \omega\beta rT^2\left(\frac{n}{n+1} - \rho + \frac{\rho^{n+1}}{n+1}\right) \end{aligned}$$

and the goodwill lost sale cost is given by

$$\begin{aligned} LC(\rho,T) &= \int_\tau^T \left(\pi_0(1-\beta)\lambda(t) + \pi\int_\tau^t (1-\beta)\lambda(u)du\right)dt \\ &= \pi_0(1-\beta)(1-\rho)rT + \pi(1-\beta)rT^2\left(\frac{n}{n+1} - \rho + \frac{\rho^{n+1}}{n+1}\right) \end{aligned}$$

Thus, the total profit along an inventory cycle is

$$PC(\rho,T) = (s-c)Q - (A + HC(\rho,T) + BC(\rho,T) + LC(\rho,T))$$

and, consequently, the return on inventory investment (ROII) is defined by

$$ROII(\rho,T) = \frac{PC(\rho,T)}{CC(\rho,T)}, \tag{2}$$

where $CC(\rho,T)$ is the total cost per cycle, which is given by $cQ + A + HC(\rho,T) + BC(\rho,T) + LC(\rho,T)$.

4. Optimal solution

To find the optimal inventory policy, three scenarios can be considered: (i) partial backordering (i.e., $0 < \beta < 1$); (ii) full backordering (i.e., $\beta = 1$) and (iii) full lost sales (i.e., $\beta = 0$). Next, we study the case when $\beta \in (0,1)$.

4.1. *Partial backordering scenario* ($0 < \beta < 1$)

Since $\beta > 0$, after a few algebraic manipulations, and using (1), the return on inventory investment given in Eq. (2) can be expressed as

$$ROII(\rho,T) = \frac{s}{c + AC(\rho,T)} - 1, \tag{3}$$

where $AC(\rho,T)$ represents the average inventory cost (without including the purchasing cost) per ordered unit of item. That is,

$$\begin{aligned} AC(\rho,T) &= \frac{CC(\rho,T) - cQ}{Q} \\ &= \frac{1}{((1-\beta)\rho + \beta)rT}(A + HC(\rho,T) + BC(\rho,T) + LC(\rho,T)) \\ &= \frac{A}{rg_1(\rho)T} + \frac{g_2(\rho)}{(n+1)g_1(\rho)}T + \frac{\alpha_0(1-\rho)}{g_1(\rho)}, \end{aligned}$$

where $\alpha_0 = \beta\omega_0 + (1-\beta)\pi_0$, $g_1(\rho) = (1-\beta)\rho + \beta$ and $g_2(\rho) = (h+\alpha_1)\rho^{n+1} - (n+1)\alpha_1\rho + n\alpha_1$, with $\alpha_1 = \beta\omega + (1-\beta)\pi$. Note that $g_1(\rho)$ is a linear function on $[0,\infty)$ with positive slope. Also, $g_2(\rho)$ is a positive and convex function on $[0,\infty)$ and has a minimum at point

$$\rho_a = \left(\alpha_1/(\alpha_1 + h)\right)^{1/n} \in [0,1), \tag{4}$$

with $g_2(\rho_a) = \alpha_1 n\left(1 - \left(\alpha_1/(\alpha_1+h)\right)^{1/n}\right) > 0$.

Evidently, in this case $\beta > 0$, the optimal solution that minimizes $AC(\rho,T)$ is the same as the optimal solution that maximizes the ROII function given by (2), because this Eq. (2) can be expressed as (3). Thus, our problem is to solve the following nonlinear program

$$\min_{\substack{T>0\\0\le\rho\le1}} AC(\rho,T). \tag{5}$$

To do so, we first consider fixed $\rho \in [0,1]$ and the variable $T > 0$. Thus, we obtain the function $AC_\rho(T) = AC(\rho,T)$, which is a strictly convex function that attains its minimum at

$$T^*_\rho = \sqrt{\frac{(n+1)A}{rg_2(\rho)}}, \tag{6}$$

with value

$$W(\rho) = AC_\rho(T^*_\rho) = \frac{1}{g_1(\rho)}\left(\sqrt{\frac{4Ag_2(\rho)}{(n+1)r}} + \alpha_0(1-\rho)\right) \tag{7}$$

$$= \frac{1}{(1-\beta)\rho+\beta}\left(\sqrt{\frac{4A\left((h+\alpha_1)\rho^{n+1}-(n+1)\alpha_1\rho+n\alpha_1\right)}{(n+1)r}} + \alpha_0(1-\rho)\right) \tag{8}$$

Next, taking into account the definition of the function $g_2(\rho)$, we can consider the following two cases: (i) $\alpha_1 > 0$ and (ii) $\alpha_1 = 0$. Remember that α_1 is the time-dependent average shortage cost.

4.1.1. Case $\alpha_1 > 0$

From (7), the first derivative of $W(\rho)$ can be expressed as

$$W'(\rho) = \frac{L(\rho)}{\sqrt{(n+1)rg_2(\rho)}g_1^2(\rho)}, \tag{9}$$

where

$$L(\rho) = \sqrt{A}\left(g_1(\rho)g_2'(\rho) - 2(1-\beta)g_2(\rho)\right) - \alpha_0\sqrt{(n+1)rg_2(\rho)} \tag{10}$$

Substituting the functions $g_1(\rho)$ and $g_2(\rho)$ into (10), we have

$$L(\rho) = (h+\alpha_1)\sqrt{A}\left(\rho(n-1)(1-\beta)+\beta(n+1)\right)\rho^n$$
$$-\alpha_0\sqrt{(n+1)r\left[(h+\alpha_1)\rho^{n+1}-\alpha_1\left(\rho(n+1)-n\right)\right]}$$
$$-\alpha_1\sqrt{A}\left(2n+\beta(1-n)-(n+1)(1-\beta)\rho\right). \tag{11}$$

From (9), it is clear that $sign(W'(\rho)) = sign(L(\rho))$. Moreover, since $g_2(\rho)$ is a positive and convex function, it follows that $g_2'(\rho) < 0$ for $\rho < \rho_a$, where ρ_a is the point at which $g_2(\rho)$ attains its minimum. Thus, from (10), we see that $L(\rho) < 0$ for $\rho \le \rho_a$. Therefore, we only need to analyse the function $L(\rho)$ for $\rho \in (\rho_a, 1]$.

Since the second derivative of $W(\rho)$ is

$$W''(\rho) = \frac{2g_2(\rho)g_1(\rho)L'(\rho) - L(\rho)\left(4g_2(\rho)g_1'(\rho)+g_1(\rho)g_2'(\rho)\right)}{2\sqrt{(n+1)rg_2^3(\rho)}g_1^3(\rho)},$$

it follows that if ρ^0 is a point with $L(\rho^0) = 0$ and $L'(\rho^0) > 0$, then $W''(\rho^0) > 0$ and ρ^0 is a local minimum of the function $W(\rho)$. Derivating Eq. (10) and taking into account that $g_1'(\rho) = 1-\beta$, we have

$$L'(\rho) = -\sqrt{A}(1-\beta)g_2'(\rho) + \sqrt{A}g_1(\rho)g_2''(\rho) - \alpha_0\sqrt{(n+1)r}\frac{g_2'(\rho)}{2\sqrt{g_2(\rho)}}$$

Therefore, if ρ^0 is a root of the function L, then, from (10), we obtain

$$\alpha_0\sqrt{(n+1)rg_2(\rho^0)} = \sqrt{A}\left(g_1(\rho^0)g_2'(\rho^0) - 2(1-\beta)g_2(\rho^0)\right)$$

and, consequently,

$$L'(\rho^0) = \frac{(n+1)\sqrt{A}g_1(\rho^0)}{2\rho^0 g_2(\rho^0)}f_1(\rho^0),$$

where

$$f_1(\rho) = \frac{\rho}{n+1}\left[2g_2(\rho)g_2''(\rho) - \left(g_2'(\rho)\right)^2\right]$$
$$= (h+\alpha_1)^2(n-1)\rho^{2n+1} + 2\alpha_1(h+\alpha_1)(1-n^2)\rho^{n+1}$$
$$+2\alpha_1(h+\alpha_1)n^2\rho^n - \alpha_1^2(n+1)\rho \tag{12}$$

This function $f_1(\rho)$ is, as can be seen in the following theorem, very useful for determining the optimal value ρ of the ratio between the initial inventory and the total quantity demanded during the inventory cycle.

Theorem 1. *Let $\alpha_0 = \beta\omega_0 + (1-\beta)\pi_0$, $\alpha_1 = \beta\omega + (1-\beta)\pi$, $\rho_a = \left(\alpha_1/(\alpha_1+h)\right)^{1/n}$ and $W(\rho)$, $L(\rho)$ and $f_1(\rho)$ be functions given, respectively, by* (8), (11) and (12). *Suppose $\alpha_1 > 0$. The function $W(\rho)$ attains its minimum value at the point ρ^*characterized as follows:*

1. *If $n < h/(2\alpha_1+h)$, then let $\rho_b = \arg_{\rho\in(\rho_a,1)}\left\{f_1(\rho)=0\right\}$ be the only root of the equation $f_1(\rho) = 0$ in the interval $(\rho_a, 1)$.*
   - (a) *If $L(\rho_b) < 0$, then $\rho^* = 1$.*
   - (b) *Otherwise, let $\rho_1 = \arg_{\rho\in(\rho_a,\rho_b]}\{L(\rho)=0\}$ be the unique root of the equation $L(\rho) = 0$ in the interval $(\rho_a, \rho_b]$. Thus, we have:*
     - i. *If $W(\rho_1) < \sqrt{\frac{4Ah}{(n+1)r}}$, then $\rho^* = \rho_1$.*
     - ii. *If $W(\rho_1) \ge \sqrt{\frac{4Ah}{(n+1)r}}$, then $\rho^* = 1$.*
2. *If $n \ge h/(2\alpha_1+h)$, then the following cases can occur:*
   - (a) *If $\alpha_0 < (2\beta+n-1)\sqrt{\frac{Ah}{(n+1)r}}$, then let ρ^* be the unique root of the equation $L(\rho) = 0$ in the interval $(\rho_a, 1]$.*
   - (b) *Otherwise, $\rho^* = 1$.*

Proof. See the Appendix. ■

Theorem 1 establishes the optimal ratio ρ^* between the initial stock and the total quantity demanded throughout the inventory cycle when the time-dependent average shortage cost (α_1) is not zero. Next, from Eq. (8), the minimum average inventory cost $W(\rho^*)$ per unit time (without including the purchasing cost) is calculated. In addition, from Eq. (6), the optimal inventory cycle T^* is determined.

Next, we study the case when the time-dependent average shortage cost is zero.

4.1.2. Case $\alpha_1 = 0$

From (7) it is clear that, in this scenario, the derivative of the function $W(\rho)$, for $\rho \in (0,1)$, is

$$W'(\rho) = \frac{M(\rho)}{\sqrt{(n+1)rg_1^2(\rho)}},$$

where $M(\rho)$ is defined on the interval $(0,1]$ by

$$M(\rho) = \sqrt{Ah\rho^{n-1}}\left((n-1)g_1(\rho)+2\beta\right) - \alpha_0\sqrt{(n+1)r} \tag{13}$$

$$= \sqrt{Ah\rho^{n-1}}\left((n-1)(1-\beta)\rho+(n+1)\beta\right) - \alpha_0\sqrt{(n+1)r} \tag{14}$$

The behaviour of the function $M(\rho)$ with respect to the value of the demand pattern index n can be seen in the Appendix.

The following theorem provides the optimal value of ρ in this situation.

Theorem 2. *Let $\alpha_0 = \beta\omega_0+(1-\beta)\pi_0$, $\alpha_1 = \beta\omega+(1-\beta)\pi = 0$, and $W(\rho)$ and $M(\rho)$ be functions given, respectively, by* (8) and (14). *The function $W(\rho)$ attains its minimum value at the point ρ^* characterized as follows:*

1. *Let $n < 1$:*
   - (a) *If $\alpha_0 \ge \beta\sqrt{\frac{4Ah}{(n+1)r}}$, then $\rho^* = 1$.*
   - (b) *Otherwise, $\rho^* = 0$.*
2. *Let $n = 1$:*
   - (a) *If $\alpha_0 > \beta\sqrt{\frac{2Ah}{r}}$, then $\rho^* = 1$.*
   - (b) *If $\alpha_0 = \beta\sqrt{\frac{2Ah}{r}}$, then the minimum is attained at any point of the interval $[0,1]$.*

(c) *Otherwise,* $\rho^* = 0$.

3. *Let* $n > 1$*:*

   (a) *If* $\alpha_0 \geq (2\beta + n - 1)\sqrt{\frac{Ah}{(n+1)r}}$, *then* $\rho^* = 1$.

   (b) *Otherwise, let* $\rho^* = \arg_{\rho \in (0,1)}\{M(\rho) = 0\}$ *be the unique root the equation* $M(\rho) = 0$ *in the interval* $(0, 1)$.

Proof. See the Appendix. ■

Theorem 2 determines the optimal ratio ρ^* between the initial stock and the total quantity demanded throughout the inventory cycle when the time-dependent average shortage cost (α_1) is zero. Next, from Eq. (8), the minimum average inventory cost $W(\rho^*)$ per unit time (without including the purchasing cost) is obtained. Then, from Eq. (6), the optimal inventory cycle T^* is established.

4.2. Full backordering scenario ($\beta = 1$)

In the same manner as in the previous section, we can see that the problem of the maximum $ROII(\rho, T)$ given by (2) is equivalent to the problem of the minimum cost per unit of item given by $AC(\rho, T)$. In this case, we have $g_1(\rho) = 1$, $\alpha_0 = \omega_0$ and $\alpha_1 = \omega$. Thus

$$AC(\rho, T) = \frac{1}{T}\left(\frac{A}{r} + \frac{g_2(\rho)}{n+1}T^2 + \omega_0(1-\rho)\right),$$

and, therefore, the problem is also equivalent to minimizing the average cost per unit time. Note that Theorems 1 and 2 also provide the optimal policies in this scenario.

4.3. Full lost sales scenario ($\beta = 0$)

In this case, we have $g_1(\rho) = \rho$, $\alpha_0 = \pi_0$ and $\alpha_1 = \pi$. Thus, the return on inventory investment (ROII) can be rewritten as

$$ROII(\rho, T) = \begin{cases} -1 & \text{if } \rho = 0 \\ \dfrac{s}{c + AC(\rho, T)} - 1 & \text{if } \rho > 0 \end{cases},$$

where now $AC(\rho, T) = \frac{A}{r\rho T} + \frac{(h+\pi)\rho^{n+1} - (n+1)\pi\rho + n\pi}{(n+1)\rho}T + \frac{\pi_0(1-\rho)}{\rho}$.

If $\alpha_1 = \pi > 0$, then it is easy to check that Theorem 1 remains valid (since Lemma 1 of the Appendix is also true).

Next, we analyse the case $\alpha_1 = \pi = 0$. Now, the function $M(\rho)$ is derived to $(n-1)\sqrt{Ah\rho^{n+1}} - \pi_0\sqrt{(n+1)r}$. Therefore, we can give the following result.

Theorem 3. *Let* $\beta = 0$ *and* $\pi = 0$. *The function* $W(\rho)$ *given by* (8) *attains its minimum value at the point* ρ^* *characterized as follows:*

1. *If* $n > 1$ *and* $\pi_0 < (n-1)\sqrt{\frac{Ah}{(n+1)r}}$, *then* $\rho^* = \rho_0 = \left(\frac{(n+1)r\pi_0^2}{(n-1)^2 Ah}\right)^{1/(n+1)}$.
2. *If* $n = 1$ *and* $\pi_0 = 0$, *then* ρ^* *is any point of the interval* $(0, 1]$.
3. *Otherwise,* $\rho^* = 1$.

Proof. See the Appendix. ■

Theorem 3 establishes the optimal ratio ρ^* between the initial stock and the total quantity demanded throughout the inventory cycle when the unit goodwill cost per unit time is zero, i.e. $\pi = 0$, and the fraction of backlogged demand (β) is zero, that is, all the shortages are lost sales. Next, from Eq. (8), the minimum average inventory cost $W(\rho^*)$ per unit time (without including the purchasing cost) is calculated. Also, from Eq. (6), the optimal inventory cycle T^* is determined.

4.4. Case $n = 1$ (constant demand rate)

Next, we discuss an inventory model which is a particular case of the model developed in this article. If we consider a demand pattern index n equal to 1, we obtain a uniform demand rate along the inventory cycle.

Table 2
Optimal value of ρ when $n = 1$.

		$\beta = 0$		$\beta > 0$
		$\pi_0 = 0$	$\pi_0 > 0$	
$\alpha_1 = 0$	$\alpha_0 < \beta\sqrt{2Ah/r}$	–	–	$\rho^* = 0$
	$\alpha_0 = \beta\sqrt{2Ah/r}$	$(0, 1]$	–	$(0, 1]$
	$\alpha_0 > \beta\sqrt{2Ah/r}$	–	$\rho^* = 1$	$\rho^* = 1$
$\alpha_1 > 0$	$\alpha_0 < \beta\sqrt{2Ah/r}$	–	–	$\rho^* = \rho_1$
	$\alpha_0 = \beta\sqrt{2Ah/r}$	$[0, 1]$	–	$\rho^* = 1$
	$\alpha_0 > \beta\sqrt{2Ah/r}$	–	$\rho^* = 1$	$\rho^* = 1$

Where $\rho_1 = \dfrac{\alpha_0\sqrt{\alpha_1 rh\left(2A\left(\alpha_1 + \beta^2 h\right) - r\alpha_0^2\right)} + \alpha_1\left(2A\left(\alpha_1 + \beta h\right) - r\alpha_0^2\right)}{2A\left(\alpha_1 + \beta h\right)^2 - \left(\alpha_1 + h\right) r\alpha_0^2}$.

According to the results obtained in the previous subsections, we can establish the following result for this particular situation.

Corollary 1. *Let* $n = 1$, $\alpha_0 = \beta\omega_0 + (1-\beta)\pi_0$ *and* $\alpha_1 = \beta\omega + (1-\beta)\pi$. *The value of* (ρ, T) *that maximizes the return on inventory investment (ROII) defined by* (2) *is* $(\rho^*, T^*) = \left(\rho^*, \sqrt{2A/\left[r\left(\alpha_1(1-\rho^*)^2 + h(\rho^*)^2\right)\right]}\right)$, *where* ρ^* *is given in Table 2.*

Proof. See the Appendix. ■

Corollary 1 presents the optimal inventory policy (ρ^*, T^*) in closed-form that maximizes the return on inventory investment when the demand rate is constant. Note that the optimal ratio ρ^* between the initial stock and the total quantity demanded throughout the inventory cycle can notably vary, depending on the scenario characterized by the input parameters of the inventory system.

5. Numerical examples

In this section, we include some numerical examples to illustrate the proposed model in different scenarios and their associated optimal policies.

Example 1. We consider the first numerical example proposed in San-José et al. (2017). That is, we suppose $n = 1$, $r = 1000$, $A = 500$, $c = 8$, $s = 10$, $h = 2$, $\omega_0 = 0.1$, $\omega = 3.2$, $\pi_0 = 2$ and $\pi = 0$. If $\beta = 0$, applying Theorem 3, we obtain $\rho^* = 1$ and, from (6), $T^* = 0.707107$. That is, the same policy that maximizes the total inventory profit per unit time given in San-José et al. (2017). If $\beta > 0$, applying now Theorem 1, we obtain $\rho^* = 1$ for $\beta < \left(380 - 200\sqrt{2}\right)/161$ and $\rho^*(\beta) = \arg_{\rho \in (\rho_a, 1)}\{L(\rho) = 0\}$. The optimal policies are shown in Table 3. From these results, we can make the following comments: The optimal inventory policy is constant when $\beta \leq 0.603461$. However, when $\beta > 0.603461$, if β increases then: (i) The ratio ρ^* between the initial inventory and the total demand throughout the inventory cycle, the stock-in period τ^* and the maximum stock level S^* are strictly decreasing; (ii) the stock-out period Ψ^*, the economic lot size Q^* and the maximum profit/cost ratio are strictly increasing and (iii) the inventory cycle T^* starts increasing but then decreases.

Example 2. This example assumes the parameters of Example 1, but modifying the value of n to $n = 2.5$. The optimal policies obtained are given in Table 4. These results present certain insights into the behaviour of the inventory system studied here. Thus, we can make the following observations: (i) the optimal inventory policy is constant when $\beta \leq 0.403570$; (ii) if $\beta > 0.403570$ and the value of β is increasing, then: (a) the ratio ρ^* between the initial inventory and the total demand throughout the inventory cycle and the stock-in period τ^* are strictly decreasing, (b) the inventory cycle T^* and the maximum stock level

Table 3
Numerical results associated with Example 1.

β	ρ^*	T^*	τ^*	Ψ^*	Q^*	S^*	b^*	$ROII^*$ (%)
0	1	0.707107	0.707107	0	707.107	707.107	0	6.22236
≤ 0.603461	1	0.707107	0.707107	0	707.107	707.107	0	6.22236
0.7	0.838052	0.826641	0.692768	0.133873	786.479	692.768	133.873	6.54693
0.8	0.745820	0.884613	0.659762	0.224851	839.643	659.762	224.851	7.30162
0.9	0.683918	0.904164	0.618374	0.285790	875.585	618.374	285.790	8.26320
1	0.636740	0.900521	0.573397	0.327123	900.521	573.397	327.123	9.32792

Table 4
Numerical results associated with Example 2.

β	ρ^*	T^*	τ^*	Ψ^*	Q^*	S^*	b^*	$ROII^*$ (%)
0	1	0.935414	0.935414	0	935.414	935.414	0	10.2652
≤ 0.403570	1	0.935414	0.935414	0	935.414	935.414	0	10.2652
0.5	0.947559	1.02211	0.893336	0.128777	995.312	968.512	53.6005	10.3600
0.6	0.910924	1.07777	0.853557	0.22421	1039.37	981.768	96.0030	10.6004
0.7	0.88445	1.10829	0.815338	0.292950	1069.87	980.227	128.061	10.9276
0.8	0.863954	1.12115	0.777837	0.343309	1090.64	968.619	152.527	11.3087
0.9	0.847140	1.12146	0.740750	0.380710	1104.32	950.033	171.426	11.7249
1	0.832665	1.11268	0.703955	0.408722	1112.68	926.487	186.190	12.1646

Table 5
Numerical results associated with Example 3.

β	ρ^*	T^*	τ^*	Ψ^*	Q^*	S^*	b^*	$ROII^*$ (%)
0	1	0.661438	0.661438	0	661.438	661.438	0	5.13193
≤ 0.360373	1	0.661438	0.661438	0	661.438	661.438	0	5.13193
0.4	0.867527	0.745461	0.670097	0.075365	686.209	646.708	98.7531	5.19438
0.5	0.720175	0.850255	0.664703	0.185552	731.293	612.33	237.923	5.65191
0.6	0.652434	0.89008	0.646150	0.243934	766.339	580.721	309.363	6.27145
0.7	0.612015	0.901582	0.623845	0.277736	796.642	551.782	349.800	6.93049
0.8	0.584545	0.898776	0.600848	0.297928	824.096	525.375	373.401	7.58461
0.9	0.564306	0.888313	0.57837	0.309948	849.610	501.280	387.033	8.21590
1	0.548545	0.873694	0.556888	0.316805	873.694	479.260	394.434	8.81734

S^* start increasing but then decrease and (c) the stock-out period Ψ^*, the economic lot size Q^* and the maximum profit/cost ratio are strictly increasing.

Example 3. This example considers the same parameters as in the first example, but modifying the values of n and π_0 to $n = 0.75$ and $\pi_0 = 0.5$, respectively. The optimal policies are given in Table 5. We can discuss the following issues: if $\beta > 0.360373$ and β increases, then: (a) the ratio ρ^* between the initial inventory and the total demand throughout the inventory cycle and the maximum stock level S^* are decreasing, (b) the inventory cycle T^* and the stock-in period τ^* start increasing but then decrease and (c) the stock-out period Ψ^*, the economic lot size Q^* and the maximum profit/cost ratio are strictly increasing.

Example 4. This example considers the same parameters as in Example 3, but modifying the values of h, ω_0 and π_0 to $h = 6.5$, $\omega_0 = 0$ and $\pi_0 = 0$, respectively. We have the optimal policies given in Table 6. We can make the following comments: if $\beta \geq 0.1$ and the value of β is increasing, then: (a) the ratio ρ^* between the initial inventory and the total demand throughout the inventory cycle starts decreasing but then increases, (b) the stock-in period τ^* and the maximum stock level S^* are strictly decreasing, (c) the inventory cycle T^* and the stock-out period Ψ^* start increasing but then decrease and (d) the economic lot size Q^* and the maximum profit/cost ratio are strictly increasing.

6. Sensitivity analysis and managerial insights

In this section, managerial implications based on the sensitivity analysis of the parameters are displayed. Some suggestions are provided to inventory managers that could help them to improve the efficiency of the inventory control.

It follows from the theoretical results presented in the previous sections that if the unit purchasing cost increases, then the total cost per cycle goes up and the total profit throughout the inventory cycle goes down. Consequently, the return on inventory investment decreases. Retailers can offset against this additional purchasing cost by buying in bulk or by applying discounting strategies that provide a lower unit cost of the product. Also, if the unit selling price increases, the total profit over the inventory cycle goes up and the return on investment in inventory increases.

The first numerical example presented in Section 5 reveals that if $\beta = 0$, that is, shortages are lost sales, then the optimal policy that maximizes the return on inventory investment coincides with the one that maximizes the profit per unit time. However, when there is a strict fraction of demand which is backordered, that is $0 < \beta < 1$, the optimal policy that maximizes the ROII is, in general, different from the one that maximizes the profit per unit time.

In addition, from the numerical examples we can deduce that there is a value β_0 such that the optimal inventory policy is constant when $\beta \leq \beta_0$. However, when $\beta > \beta_0$ the inventory policy varies. In this last case, if the fraction of backordered demand β increases, then the economic lot size Q^* and the return on inventory investment increase in the four examples.

Next, in order to study the effect of some parameters on the optimal inventory policy and the maximum ROII, two tables are included showing the variations, in percentage terms, of the optimal values ρ^*, T^*, τ^*, Ψ^*, Q^*, S^*, b^* and the maximum ROII for different changes in the parameters. First, we consider the parameters of Example 2, but modifying the value of π to $\pi = 0.5$ and the fraction of backordered demand β to $\beta = 0.8$. In this case, the optimal ratio between the initial inventory and the total quantity demanded during the inventory cycle is $\rho^* = 0.867611$, the optimal inventory cycle is $T^* = 1.11511$, the optimal stock-in cycle is $\tau^* = 0.781863$, the optimal stock-out period is $\Psi^* = 0.333248$, the optimal lot size is $Q^* = 1085.59$, the inventory level

Table 6
Numerical results associated with Example 4.

β	ρ^*	T^*	τ^*	Ψ^*	Q^*	S^*	b^*	$ROII^*$ (%)
0	1	0.366900	0.366900	0	366.900	366.900	0	−6.76461
0.05	1	0.366900	0.366900	0	366.900	366.900	0	−6.76461
0.1	1	0.366900	0.366900	0	366.900	366.900	0	−6.76461
0.2	0.254030	1.01790	0.364226	0.653679	410.443	258.578	759.326	−4.18143
0.3	0.238749	0.971428	0.331793	0.639635	453.778	231.928	739.500	−1.99654
0.4	0.232945	0.914667	0.306693	0.607974	493.707	213.067	701.599	−0.25428
0.5	0.230169	0.862925	0.286754	0.576172	530.772	198.619	664.306	1.17313
0.6	0.228747	0.817751	0.270481	0.547270	565.474	187.058	630.693	2.37062
0.7	0.228044	0.778487	0.256901	0.521586	598.200	177.530	600.958	3.39464
0.8	0.227769	0.744184	0.245359	0.498826	629.248	169.502	574.682	4.28400
0.9	0.227767	0.713986	0.235401	0.478585	658.849	162.622	551.363	5.06632
1	0.227949	0.687188	0.226701	0.460487	687.188	156.644	530.544	5.76184

Table 7
Effects of the parameters r, n, A and h on the optimal inventory policy and the maximum $ROII$.

	Δ	$\Delta\rho^*$(%)	ΔT^*(%)	$\Delta\tau^*$(%)	$\Delta\Psi^*$(%)	ΔQ^*(%)	ΔS^*(%)	Δb^*(%)	$\Delta ROII^*$ (%)
r	+25%	0.672237	−10.9663	−9.46247	−14.4946	11.4255	12.0403	6.38912	10.8113
	+10%	0.277824	−4.83007	−4.16768	−6.38417	4.73876	4.97776	2.78086	4.73223
	+5%	0.140543	−2.50059	−2.15766	−3.30518	2.40003	2.51826	1.43146	2.44418
	−5%	−0.144060	2.69388	2.32443	3.56069	−2.46586	−2.58136	−1.51976	−2.61942
	−10%	−0.291925	5.60750	4.83845	7.41184	−5.00271	−5.23072	−3.13488	−5.43667
	−25%	−0.761912	16.0210	13.8237	21.1764	−13.1024	−13.6472	−8.63936	−15.3737
n	+25%	2.28866	7.97862	6.04784	12.5086	8.41910	10.4499	−8.21679	9.03503
	+10%	1.01392	3.25717	2.45895	5.12996	3.44378	4.30412	−3.60400	3.87028
	+5%	0.525989	1.63993	1.23617	2.58725	1.73522	2.17455	−1.86367	1.98246
	−5%	−0.569066	−1.66314	−1.24950	−2.63362	−1.76288	−2.22274	2.00422	−2.08527
	−10%	−1.18722	−3.34988	−2.51212	−5.31543	−3.55441	−4.49733	4.16994	−4.28250
	−25%	−3.41768	−8.55280	−6.37317	−13.6666	−9.10987	−11.6782	11.9293	−11.6562
A	+25%	−0.600441	12.2280	10.5509	16.1627	12.1079	11.5541	16.6441	−11.7779
	+10%	−0.264743	5.06005	4.36608	6.68823	5.01048	4.78191	6.88284	−4.90858
	+5%	−0.137116	2.56085	2.20964	3.38485	2.53578	2.42022	3.48245	−2.49040
	−5%	0.147848	−2.62728	−2.26698	−3.47263	−2.60162	−2.48332	−3.57075	2.56835
	−10%	0.307912	−5.32645	−4.59599	−7.04025	−5.27449	−5.03494	−7.23686	5.22117
	−25%	0.881369	−13.9226	−12.0134	−18.4019	−13.7873	−13.1639	−18.8945	13.7673
h	+25%	−3.24630	−7.11800	−14.4735	10.1395	−7.65544	−10.1332	12.6423	−9.85981
	+10%	−1.34332	−3.16460	−6.38393	4.38857	−3.39646	−4.46541	5.36025	−4.18752
	+5%	−0.679466	−1.64389	−3.30613	2.25603	−1.76301	−2.31219	2.73579	−2.13855
	−5%	0.695605	1.78375	3.56303	−2.3908	1.90995	2.49176	−2.85622	2.23563
	−10%	1.40788	3.72729	7.41681	−4.92904	3.98759	5.18764	−5.84314	4.57680
	−25%	3.64988	10.7854	21.1726	−13.5850	11.5061	14.8289	−15.7139	12.3376

is $S^* = 967.483$, the quantity demanded during the stock-out period is $b^* = 147.628$ and the maximum return on inventory investment is $ROII^* = 11.2788\%$. Next, we calculate the variations, in percentage terms, of the optimal inventory policy varying each of the parameters, while keeping all the others fixed. Thus, Table 7 shows the effects of the parameters r, n, A and h when each of these parameters varies its value by ±25%, ±10% and ±5%. Next, we present some findings obtained from the sensitivity analysis.

The optimal inventory cycle, the economic lot size, the initial inventory level and the optimal return on inventory investment are moderately sensitive to the parameters r, n, A and h. Thus, a 10% increase in the value of the average demand r per cycle leads to an increment of 4.74% in the economic lot size, a 4.98% increase in the initial stock and a 4.73% increase in the ROII. However, that increment in the average demand results in a reduction of the optimal inventory cycle of 4.83%. Therefore, to increase the ROII the decision maker should boost the demand by implementing marketing policies or quantity discount.

Also, a 25% increase in the demand pattern index n leads to an increment of 8.42% in the economic lot size, a 10.45% increase in the initial stock, a 7.98% increase in the length of the optimal cycle, and a 9.04% increase in the ROII.

An increment in the replenishment cost A results in an increase in the optimal inventory cycle, the economic lot size and the initial stock, but the return on inventory investment decreases. The impact of the replenishment cost A is positive on the stock-in period and is negative on the ratio between the initial inventory and the total quantity demanded during the inventory cycle. Thus, a 25% increase in the value of A leads to a 10.55% increase in the length of the stock-in period, and a 0.6% decrease of the ratio between the initial inventory and the total quantity demanded.

With respect to the unit holding cost h, if this cost increases then the optimal inventory cycle, the economic lot size, the initial inventory level, and the return on inventory investment decrease. A 10% increment in the unit holding cost results in a reduction of 3.40% in the economic lot size, a 4.47% decrease in the initial stock, a 3.16% decrease in the length of the optimal cycle, and a 4.19% decrease in the ROII.

Therefore, from the above comments, it is recommended that the decision-maker should be alert to the fluctuations in the parameters A and h.

Table 8 displays the variations, in percentage terms, of the optimal inventory policies when each of the four parameters that determine the shortage cost has changed by ±25%, ±10% and ±5%. From these results, we can establish the following insights.

The variation of the parameters that appear in the backlogging costs do not have great influence on the behaviour of the return on inventory investment. The effect of ω_0, or ω, on the ROII is almost negligible. Thus, a 10% decrease in the value of the parameter ω_0 or ω, leads to an increase in the ROII less than 0.2%, and 0.8%, respectively.

Also, the maximum ROII is not very sensitive to movements of the goodwill parameters π_0 and π. Thus, a 10% decrease in the value of the

Table 8
Effects of the parameters ω_0, ω, η_0 and η on the optimal inventory policy and the maximum $ROII$.

	Δ	$\Delta\rho^*(\%)$	$\Delta T^*(\%)$	$\Delta\tau^*(\%)$	$\Delta\Psi^*(\%)$	$\Delta Q^*(\%)$	$\Delta S^*(\%)$	$\Delta b^*(\%)$	$\Delta ROII^*$ (%)
ω_0	+25%	0.237158	−0.157510	0.435506	−1.54884	−0.115305	0.079275	−1.70928	−0.296134
	+10%	0.094847	−0.062488	0.174651	−0.618861	−0.045593	0.032300	−0.683682	−0.119047
	+5%	0.047421	−0.031158	0.087400	−0.309318	−0.022708	0.016249	−0.341836	−0.059622
	−5%	−0.047416	0.030986	−0.087550	0.309093	0.022531	−0.01645	0.341826	0.059820
	−10%	−0.094828	0.061799	−0.175250	0.617961	0.044886	−0.033088	0.683641	0.119838
	−25%	−0.237040	0.153202	−0.439251	1.54321	0.110887	−0.084201	1.70902	0.301080
ω	+25%	2.23234	−2.76562	2.75208	−15.7112	−2.37873	−0.595015	−16.9907	−1.38968
	+10%	0.980536	−1.23524	1.20365	−6.95735	−1.06263	−0.266817	−7.58181	−0.613623
	+5%	0.506885	−0.642649	0.621211	−3.60791	−0.552881	−0.139021	−3.94317	−0.317841
	−5%	−0.543836	0.699441	−0.664079	3.89852	0.601829	0.151802	4.28840	0.342514
	−10%	−1.12893	1.46370	−1.37574	8.12556	1.25953	0.318241	8.97043	0.712752
	−25%	−3.18662	4.25246	−3.85543	23.2751	3.66032	0.930328	26.0241	2.02917
π_0	+25%	1.18760	−0.830253	2.14038	−7.79993	−0.620330	0.347490	−8.54859	−1.43139
	+10%	0.474467	−0.319308	0.867282	−3.10328	−0.235008	0.153644	−3.41880	−0.587331
	+5%	0.237158	−0.157510	0.435506	−1.54884	−0.115305	0.079275	−1.70928	−0.296134
	−5%	−0.237040	0.153202	−0.439251	1.54321	0.110887	−0.084201	1.70902	0.301080
	−10%	−0.473993	0.302078	−0.882262	3.08077	0.217337	−0.173347	3.41777	0.607118
	−25%	−1.18465	0.722542	−2.23403	7.65922	0.509863	−0.470666	8.54223	1.55510
π	+25%	0.101781	−0.129753	0.124563	−0.726426	−0.111635	−0.028104	−0.795907	−0.063930
	+10%	0.040880	−0.052158	0.050019	−0.291884	−0.044875	−0.011299	−0.319922	−0.025683
	+5%	0.020468	−0.026122	0.025042	−0.146162	−0.022475	−0.005659	−0.160223	−0.012860
	−5%	−0.020524	0.026208	−0.025107	0.146605	0.022549	0.005679	0.160748	0.012898
	−10%	−0.041105	0.052504	−0.050281	0.293655	0.045173	0.011377	0.322025	0.025834
	−25%	−0.103188	0.131915	−0.126196	0.737495	0.113499	0.028591	0.809053	0.064870

parameter π_0, or π, leads to an increase in the ROII less than 0.7% and 0.03%, respectively. In particular, the effect of a modification of π on the ROII is also almost negligible. Note that a 25% decrease in the value of the parameter π leads to an increase in the ROII less than 0.07%.

An increment in any shortage cost ω_0, ω, π_0, or π, has a negative effect on the economic lot size, the length of the inventory cycle, the length of the stock-out period and the demanded quantity during the stock-out period. Thus, a 25% increase in the value of ω or π_0 leads to a 2.38% or 0.62% decrease of the economic lot size, and a 2.77% or 0.83% decrease of the length of the inventory cycle. However, the impact of any shortage cost ω_0, ω, π_0, or π, is positive on the length of the stock-in period and on the ratio between the initial inventory and the total quantity demanded during the inventory cycle. Thus, a 25% increase in the value of ω or π_0 leads to a 2.75% or 2.14% increase of the length of the stock-in period, and a 2.23% or 1.19% increase of the ratio between the initial inventory and the total quantity demanded.

Also, an increase in the fixed shortage costs ω_0, or π_0, results in an increase in the initial inventory level. However, an increment in some of the variable shortage costs ω or π, leads to a decrease in the initial inventory level.

Therefore, the parameters ω_0 and π have an insignificant influence on the return on inventory investment. Thus, the inventory manager should not worry about those parameters. However, the parameters ω and π_0 have a major effect on the ROII. For this reason, the decision-maker should try to reduce the shortage cost per backordered unit and the constant goodwill cost per lost unit as much as possible.

7. Conclusions

In the models of the inventory control, it is very common to consider as the objective the inventory policy that maximizes the profit per unit of time. However, from the point of view of investors or shareholders, it may be more interesting for the company to maximize the return on inventory investment than obtaining a larger profit. In this paper, we consider this approach for an inventory system with power demand pattern in which shortages are allowed. It is assumed that only a fixed fraction of the demand during the stock-out period is satisfied with the arrival of the next replenishment. We also consider that both the unit backorder cost and the unit lost sale cost are composed of a fixed cost plus a variable cost which depends on the length of the waiting time until the next replenishment.

We thoroughly analyse the inventory problem and obtain the optimal global solutions for all the possible scenarios of the inventory system (including the full lost sale case). The optimal policies obtained here turn out to be different, in general, from those that maximize the profit per unit time. As a particular case, we derive the optimal inventory policy in closed-form for the uniform demand case. To illustrate the results obtained in the paper, several numerical examples are provided. Furthermore, we analyse the sensitivity of the decision variables and the maximum ROII with respect to the system input parameters. From those numerical results, we derive some useful managerial insights. In particular, it is recommended that the decision-maker should be alert to the fluctuations in the replenishment cost and the holding cost. Also, we advise to the decision-maker should boost the demand by implementing marketing policies or quantity discount.

Some future research lines in this subject are the following: (i) to assume in the inventory system that the replenishment is non-instantaneous; (ii) to suppose that the demand rate also depends on the selling price; (iii) to develop the inventory system considering stochastic demand; (iv) to incorporate in the model the possibility that items may suffer some deterioration over time and (v) to consider a non-linear holding cost.

CRediT authorship contribution statement

Luis A. San-José: Conceptualization, Methodology, Investigation, Validation, Formal Analysis, Writing – original draft, Writing – review & editing. **Joaquín Sicilia:** Conceptualization, Methodology, Investigation, Validation, Formal Analysis, Writing – original draft, Writing – review & editing. **Valentín Pando:** Conceptualization, Methodology, Investigation, Validation, Formal Analysis, Writing – original draft, Writing – review & editing. **David Alcaide-López-de-Pablo:** Conceptualization, Methodology, Investigation, Validation, Formal Analysis, Writing – original draft, Writing – review & editing.

Acknowledgements

This work is partially supported by the Spanish Ministry of Science, Innovation and Universities through the Research Project MTM2017-84150-P, which is co-financed by the European Community under European Regional Development Fund (ERDF).

Appendix

In this appendix, we give the proofs of the main results and also provide some properties of the functions $f_1(\rho)$ and $M(\rho)$, which are given, respectively, by (12) and (13).

We begin by analysing the behaviour of the function $f_1(\rho)$, function required to prove Theorem 1.

Lemma 1. *Let $\rho_a = \left(\alpha_1/(\alpha_1+h)\right)^{1/n}$ and $f_1(\rho)$ be given by (12). Then:*

1. *$f_1(\rho_a) > 0$ and $f_1'(\rho_a) > 0$.*
2. *f_1 is a strictly increasing linear function when $n = 1$.*
3. *f_1 is a strictly convex function on the interval $(\rho_a, 1)$ when $n > 1$.*
4. *f_1 is a strictly concave function on the interval $(\rho_a, 1)$ when $n < 1$.*

Proof.

1. It is immediate because $\rho_a < 1$, $f_1(\rho_a) = 2\alpha_1^2 n^2(1-\rho_a)$, $f_1'(\rho) = (h+\alpha_1)^2(2n^2-n-1)\rho^{2n} + 2\alpha_1(h+\alpha_1)(1-n^2)(1+n)\rho^n + 2\alpha_1(h+\alpha_1)n^3\rho^{n-1} - \alpha_1^2(n+1)$ and $f_1'(\rho_a) = 2\alpha_1^2 n^3(1/\rho_a - 1)$.
2. It is obvious, since in this case $f_1(\rho)$ is reduced to $2\alpha_1 h\rho$.
3. The second derivative of the function f_1 can be written as $f_1''(\rho) = 2(h+\alpha_1)n(n-1)\rho^{n-2}f_2(\rho)$, where $f_2(\rho) = (h+\alpha_1)(2n+1)\rho^{n+1} + \alpha_1\left(n^2-(n+1)^2\rho\right)$. Thus, as $n > 1$, we only need to show that f_2 is a positive function in the interval $(\rho_a, 1)$ to complete the proof. Since $f_2'(\rho) = (h+\alpha_1)(2n^2+3n+1)\rho^n - \alpha_1(n+1)^2$ and $f_2''(\rho) > 0$ for all $\rho > 0$, then $f_2'(\rho)$ is a strictly increasing function with $f_2'(\rho_a) = \alpha_1 n(n+1) > 0$. Therefore, $f_2'(\rho) > 0$ for all $\rho \in (\rho_a, 1)$ and we deduce that f_2 is also an increasing function on $\left(\rho_a, 1\right)$. The rest of the proof follows from $f_2(\rho_a) = \alpha_1 n^2(1-\rho_a) > 0$. Consequently, $f_2(\rho) > 0$ on $\left(\rho_a, 1\right)$.
4. As $n > 1$ and $f_2(\rho)$ is a positive function on $\left(\rho_a, 1\right)$, this leads to $f_1''(\rho) < 0$ and, therefore, $f_1(\rho)$ is a strictly concave function on $\left(\rho_a, 1\right)$. ∎

Now, we prove Theorem 1.

Proof of Theorem 1.

1. If $n < h/(2\alpha_1+h)$, then $n < 1$ and, from (12), $f_1(1) = h\left(2\alpha_1 n + (n-1)h\right) < 0$. Applying Lemma 1 and Bolzano's Theorem, f_1 has a unique root ρ_b on the interval $(\rho_a, 1)$, because $f_1(\rho)$ is concave and $f_1'(\rho_a) > 0$. Thus, as $f_1(\rho_a) > 0$, then $f_1(\rho) > 0$ for $\rho \in (\rho_a, \rho_b)$ and $f_1(\rho) < 0$ for $\rho \in (\rho_b, 1)$. Note that, as $L(\rho_a) = -2n\alpha_1(1-\beta)(1-\rho_a)\sqrt{A} - \alpha_0\sqrt{n(n+1)\alpha_1 r(1-\rho_a)} < 0$, we can consider the following cases:
   (a) If $L(\rho_b) < 0$, then necessarily $L(\rho) \le 0$ for $\rho \in (\rho_a, 1)$. Thus, $W(\rho)$ is a strictly decreasing function which attains its minimum at $\rho^* = 1$.
   (b) If $L(\rho_b) \ge 0$, then the function $L(\rho)$ has a unique root ρ_1 on the interval $(\rho_a, \rho_b]$. Comparing the values of $W(\rho_1)$ and $W(1) = \sqrt{4Ah/((n+1)r)}$, we obtain the optimal solution.
2. If $n \ge h/(2\alpha_1+h)$, then three scenarios can occur:
   A. $h/\left(2\alpha_1+h\right) \le n < 1$. Now $f_1(1) \ge 0$ and, by Lemma 1, the function $f_1(\rho)$ is always positive on $(\rho_a, 1)$. Thus, the function $W(\rho)$ has at most a local minimum on the interval $(\rho_a, 1)$.
   B. $n = 1$. Since f_1 is a strictly increasing function with $f_1(\rho_a) > 0$, it follows that $W(\rho)$ has at most a local minimum on the interval $(\rho_a, 1)$.
   C. $n > 1$. Taking into account that f_1 is a strictly convex function with $f_1'(\rho_a) > 0$, we deduce, as in the above two cases, that the function $W(\rho)$ has at most a local minimum on the interval $(\rho_a, 1)$.

   Therefore, in the three scenarios, we can ensure that the function $L(\rho)$ has at most a root ρ on the interval $(\rho_a, 1)$. Since $L(\rho_a) < 0$, $L(\rho)$ has a unique root in such interval if and only if $L(1) > 0$. Taking into account that $L(1) = (2\beta+n-1)h\sqrt{A} - \alpha_0\sqrt{(n+1)rh}$, we obtain $L(1) > 0$ if and only if the condition proposed in 2(a) is satisfied. ∎

The following lemma provides some characteristics of the function $M(\rho)$ given by (13).

Lemma 2. *Let $M(\rho)$ be given by (13). Then:*

1. *If $n > 1$, then $M(\rho)$ is a strictly increasing function with $M(0) = -\alpha_0\sqrt{(n+1)r} < 0$.*
2. *If $n = 1$, then $M(\rho)$ is a constant function with $M(\rho) = 2\beta\sqrt{Ah} - \alpha_0\sqrt{2r}$.*
3. *If $n < 1$, then $M(\rho)$ is a strictly decreasing function with $\lim_{\rho\to 0^+} M(\rho) = \infty$ when $\beta > 0$ and $\lim_{\rho\to 0^+} M(\rho) = -\pi_0\sqrt{(n+1)r}$, if $\beta = 0$.*

Proof. From (13), we obtain

$$M'(\rho) = \frac{(n^2-1)g_1(\rho)\sqrt{Ah\rho^{n-3}}}{2}.$$

The rest of the proof is already immediate. ∎

Now, we prove Theorem 2.

Proof of Theorem 2. It is immediate taking into account Lemma 2, $M(1) = (2\beta+n-1)\sqrt{Ah} - \alpha_0\sqrt{(n+1)r}$, $W(0) = \alpha_0/\beta$ and $W(1) = \sqrt{4Ah/((n+1)r)}$. ∎

The proof of Theorem 3 is given below.

Proof of Theorem 3. Note that, in this case, $M(0) = -\pi_0\sqrt{(n+1)r} < 0$. We can consider the following scenarios:

(a) If $n < 1$, then $M(\rho) < 0$ for $\rho \in (0,1)$ and, therefore, $W(\rho)$ attains its minimum at $\rho^* = 1$.
(b) If $n = 1$, then: (i) If $\pi_0 > 0$, then $M(\rho) < 0$ for $\rho \in (0,1)$ and, as in the previous case, $\rho^* = 1$ and (ii) if $\pi_0 = 0$, then $M(\rho) = 0$ and $W(\rho) = \sqrt{2Ah/r}$ for $\rho \in (0,1]$. Thus, the minimum is attained at all points of the interval $(0,1]$.
(c) If $n > 1$, then the function $M(\rho)$ is strictly increasing with $M(1) = (n-1)\sqrt{Ah} - \pi_0\sqrt{(n+1)r}$. Thus, if $M(1) \le 0$, then $W(\rho)$ is strictly decreasing and it attains its minimum at $\rho^* = 1$ and, if $M(1) > 0$ (or equivalently, $\pi_0 < (n-1)\sqrt{Ah/((n+1)r)}$), then $W(\rho)$ attains its minimum at $\arg_{\rho\in(0,1)}\{M(\rho) = 0\} = \rho_0$. ∎

Proof of Corollary 1. The cases with $\alpha_1 = 0$ are immediate from Theorems 2 and 3. Also, if $\alpha_1 > 0$ and $\alpha_0 \ge \sqrt{2Ah/r}\beta$, the solution $\rho^* = 1$ is obtained from Theorems 1 and 3.

To prove the result for $\alpha_1 > 0$ and $\alpha_0 < \sqrt{2Ah/r}\beta$, we observe that, from the part 2(a) of Theorem 1, the optimal solution ρ^* can be obtained by solving the equation $L(\rho) = 0$ with $\rho \in \left(\rho_a, 1\right)$ and $\rho_a = \alpha_1/\left(\alpha_1+h\right)$. As $n = 1$, we have $g_2(\rho) = \alpha_1(1-\rho)^2 + h\rho^2$ and the function $L(\rho)$ is simplified to

$$L(\rho) = 2\left(\beta h\rho - (1-\rho)\alpha_1\right)\sqrt{A} - \alpha_0\sqrt{2r\left(h\rho^2 + \alpha_1(1-\rho)^2\right)}$$

To explicitly obtain the value of ρ^*, we consider the function $\Lambda(\rho) = L(\rho)L_1(\rho)$, where

$$L_1(\rho) = 2\left(\beta h\rho - (1-\rho)\alpha_1\right)\sqrt{A} + \alpha_0\sqrt{2r\left(h\rho^2 + \alpha_1(1-\rho)^2\right)}.$$

It is easy to check that $\Lambda(\rho)$ can be written as $2\left(q_2\rho^2 + 2q_1\rho + q_0\right)$, where $q_2 = 2A(\alpha_1+\beta h)^2 - \alpha_0^2(\alpha_1+h)r$, $q_1 = \alpha_1\left(\alpha_0^2 r - 2A(\alpha_1+\beta h)\right)$ and $q_0 = \alpha_1(2\alpha_1 A - \alpha_0^2 r)$. That is, $\Lambda(\rho)$ is a quadratic and strictly convex function with two real roots (or a double root), one of which is ρ^*. Since $\Lambda(1) > 0$, ρ^* is necessarily the largest of the roots of $\Lambda(\rho)$. Thus,

$\rho^* = (-q_1 + \sqrt{q_1^2 - q_0 q_2})/q_2$. Now, substituting the values of q_0, q_1 and q_2, we obtain $\rho^* = \rho_1$.

Finally, using Eq. (6), we obtain the optimal cycle time T^* as

$$T^* = \sqrt{\frac{2A}{r\left(h\left(\rho^*\right)^2 + \alpha_1\left(1-\rho^*\right)^2\right)}}. \quad \blacksquare$$

References

Adaraniwon, A.O., Omar, M.B., 2019. An inventory model for delayed deteriorating items with power demand considering shortages and lost sales. J. Intell. Fuzzy Systems 36, 5397–5408.

Alfares, H.K., Ghaithan, A.M., 2019. EOQ and EPQ production–inventory models with variable holding cost: State-of-the-art review. Arab. J. Sci. Eng. 44, 1737–1755.

Baker, S.R., Johnson, S., Kueng, L., 2021. Financial returns to household inventory management. http://dx.doi.org/10.2139/ssrn.3677084, Swiss Finance Institute Research, Paper No. 20–114.

Chen, C.K., Liao, Y.X., 2014. A deteriorating inventory model for an intermediary firm under return on inventory investment maximization. J. Ind. Manag. Optim. 10, 989–1000.

Chern, M.S., Chan, Y.L., Teng, J.T., 2005. A comparison among various inventory shortage models for deteriorating items on the basic of maximising profit. Asia-Pac. J. Oper. Res. 22, 121–134.

Dye, C.Y., 2004. A note on an EOQ model for items with Weibull distributed deterioration, shortages and power demand pattern. Inf. Manag. Sci. 15, 81–84.

Hasanov, P., Jaber, M.Y., Zolfaghari, S., 2012. Production, remanufacturing and waste disposal models for the cases of pure and partial backordering. Appl. Math. Model. 36, 5249–5261.

Keshavarzfard, R., Makui, A., Tavakkoli-Moghaddam, R., 2019a. A multi-product pricing and inventory model with production rate proportional to power demand rate. Adv. Prod. Eng. Manag. 14, 112–124.

Keshavarzfard, R., Makui, A., Tavakkoli-Moghaddam, R., Taleizadeh, A.A., 2019b. EPQ models with a production rate proportional to the power demand rate, rework process and scrapped items. Int. J. Ind. Eng. 26, 173–198.

Keshavarzfard, R., Makui, A., Tavakkoli-Moghaddam, R., Taleizadeh, A.A., 2019c. Optimization of imperfect economic manufacturing models with a power demand rate dependent production rate. Sādhanā 44, 206.

Kouvelis, P., Qiu, Y.Z., 2021. Financing inventories with an investment efficiency objective: ROI-maximising newsvendor, bank loans and trade credit contracts. Int. J. Prod. Res. http://dx.doi.org/10.1080/00207543.2021.1987555.

Li, J.L., Min, K.J., Otake, T., Van Voorhis, T., 2008. Inventory and investment in setup and quality operations under return on investment maximization. European J. Oper. Res. 185, 593–605.

Mishra, V.K., Singh, L.S., 2010. Deteriorating inventory model with time dependent demand and partial backlogging. Appl. Math. Sci. 4, 3611–3619.

Misook, L., 2017. The difference analysis of return on inventory investment by the firm's characteristics and industry. J. Korean Data Anal. Soc. 19, 2623–2635.

Otake, T., Min, K.J., 2001. Inventory and investment in quality improvement under return on investment maximization. Comput. Oper. Res. 28, 997–1012.

Pando, V., San-José, L.A., Sicilia, J., 2019. Profitability ratio maximization in an inventory model with stock-dependent demand rate and non-linear holding cost. Appl. Math. Model. 66, 643–661.

Pando, V., San-José, L.A., Sicilia, J., 2020. A new approach to maximize the profit/cost ratio in a stock-dependent demand inventory model. Comput. Oper. Res. 120, 104940.

Pando, V., San-José, L.A., Sicilia, J., Alcaide-López-de Pablo, D., 2021. Profitability index maximization in an inventory model with a price- and stock-dependent demand rate in a power-form. Mathematics 9, 1157.

Pentico, D.W., Drake, M.J., 2011. A survey of for the EOQ and EPQ with partial backordering. European J. Oper. Res. 214, 179–198.

Rajeswari, N., Vanjikkodi, T., 2012. An inventory model for items with two parameter Weibull distribution deterioration and backlogging. Am. J. Oper. Res. 2, 247–252.

Roy, M.D., Sana, S.S., Chaudhuri, K., 2011. An economic order quantity model of imperfect quality items with partial backlogging. Internat. J. Systems Sci. 42, 1409–1419.

San-José, L.A., Sicilia, J., Abdul-Jalbar, B., 2021. Optimal policy for an inventory system with demand dependent on price, time and frequency of advertisement. Comput. Oper. Res. 128, 105169.

San-José, L.A., Sicilia, J., Febles-Acosta, J., Gonzalez-De la Rosa, M., 2017. Optimal inventory policy under power demand pattern and partial backlogging. Appl. Math. Model. 46, 618–630.

San-José, L.A., Sicilia, J., García-Laguna, J., 2009. A general model for EOQ inventory systems with partial backlogging and linear shortage costs. Internat. J. Systems Sci. 40, 59–71.

San-José, L.A., Sicilia, J., García-Laguna, J., 2014. Optimal lot size for a production–inventory system with partial backlogging and mixture of dispatching policies. Int. J. Prod. Econ. 155, 194–203.

San-José, L.A., Sicilia, J., González-de-la Rosa, M., Febles-Acosta, J., 2018. An economic order quantity model with nonlinear holding cost, partial backlogging and ramp-type demand. Eng. Optim. 50, 1164–1177.

San-José, L.A., Sicilia, J., González-de-la Rosa, M., Febles-Acosta, J., 2019. Analysis of an inventory system with discrete scheduling period, time-dependent demand and backlogged shortages. Comput. Oper. Res. 109, 200–208.

Shaikh, A.A., Khan, M.A.A., Panda, G.C., Konstantaras, I., 2019. Price discount facility in an EOQ model for deteriorating items with stock-dependent demand and partial backlogging. Int. Trans. Oper. Res. 26, 1365–1395.

Sicilia, J., San-José, L.A., García-Laguna, J., 2012. An inventory model where backordered demand ratio is exponentially decreasing with the waiting time. Ann. Oper. Res. 199, 137–155.

Wee, H.W., Lo, C.C., Hsu, P.H., 2009. A multi-objective joint replenishment inventory model of deteriorated items in a fuzzy environment. European J. Oper. Res. 197, 620–631.

Yaghin, R.G., Ghomi, S.F., 2012. A hybrid fuzzy multiple objective approach to lotsizing, pricing, and marketing planning model. In: Meier, A., Donzé, L. (Eds.), Fuzzy Methods for Customer Relationship Management and Marketing: Applications and Classifications. IGI Global, pp. 271–289.

Yaghin, R.G., Ghomia, S.F., Torabi, S.A., 2013. A possibilistic multiple objective pricing and lot-sizing model with multiple demand classes. Fuzzy Sets and Systems 231, 26–44.

Yaghin, R.G., Ghomia, S.F., Torabi, S.A., 2017. Incorporating return on inventory investment into joint lot-sizing and price discriminating decisions: A fuzzy chance constraint programming model. Iran. J. Manag. Stud. 10, 929–959.

Yaghin, R.G., Mosadegh, H., Ghomi, S.M., 2018. Differential return on investment optimization: pricing, lotsizing, and shipment considerations in a two-echelon supply chain. In: Yılmaz, Ö. Faruk, Tüfekçí, S. (Eds.), Handbook of Research on Applied Optimization Methodologies in Manufacturing Systems. IGI Global, pp. 189–211.