(A,m)-ISOMETRIES ON HILBERT SPACES

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Abstract. A bounded linear operator T on a Hilbert space H is called an (A,m)-isometry, for some positive operator A on H and integer m if

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}T^{*k}AT^{k}=0.$$

We give some properties of (A,m)-isometries. In particular, we focus on spectral properties and the relation between (A,m')-isometries and m-isometries. Also, we obtain some dynamic properties of (A,m)-isometries as: a negative answer to [22, Question 1] with an example of an A-isometric which is N-supercyclic and sufficient conditions for an (A,m)-isometry to be not N-supercyclic. Moreover, we prove that the perturbation of (A,m)-isometry by a bigger class than nilpotent operators is not N-supercyclic.

1. Introduction

Throughout this paper, H denotes a Hilbert space, X a Banach space and $L(X)$ the algebra of all bounded and linear operators on X.

Given any positive integer m, it is said that the operator $T \in L(H)$ is an *m-isometry* if

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}T^{*k}T^{k}=0\ , \tag{1}$$

where T^* denotes the adjoint operator of T. The notion of m-isometric operator was introduced by Agler in [1] and was thoroughly studied by Agler and Stankus in [4, 5, 6]. This definition is one of the generalizations of isometry.

It is clear that (1) is equivalent to

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\|T^{k}x\|^2=0, \tag{2}$$

for all $x \in H$.

Let $A \in L(H)$ be a positive operator and let m be a positive integer. An operator $T \in L(H)$ is said an *(A,m)-isometry* if

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}T^{*k}AT^{k}=0\ . \tag{3}$$

An operator T is a *strict (A,m)-isometry* if T is an (A,m)-isometry and is not an $(A,m-1)$-isometry. If $m=1$, it is called A-isometry, that is, T is an A-isometry if $T^*AT=A$. The class of (A,m)-isometries has been introduced by Sid Ahmed and Saddi [23], and studied by other authors. See [14, 15, 18, 20, 22, 23, 24].

For any $T \in L(H)$, we denote by $R(T)$ and $\ker(T)$, the range and the null space of T, respectively. The following are trivial examples of this class.

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(1) If $A :\equiv I$, then

$$T \text{ is an } m\text{-isometry if and only if } T \text{ is } (A,m)\text{-isometry} .$$

(2) If $A :\equiv 0$, then any operator on $L(H)$ is an A-isometry.
(3) If T is A-isometry, then any operator of the form $T + L(H, \ker(A))$ is an A-isometry.

Notice that an (A,m)-isometry can not be injective.

Example 1.1. Let $A := \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $T := \begin{pmatrix} 2 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ be operators defined on $\mathbb{C}^3$. It is not difficult to prove that T is a strict $(A,3)$-isometry which is not injective.

Let us recall that, for $T \in L(H)$, the *orbit* of a subset $E \subseteq H$ under T is defined by

$$Orb(T,E) := \{T^n x \quad : \quad x \in E, \;\; n \in \mathbb{N}\}.$$

An operator T is said to be *N-supercyclic* if there exists an N-dimensional subspace E of H such that $Orb(T,E)$ is dense in H. If $N = 1$, we say that T is *supercyclic*.

In 1997, Ansari and Bourdon [7] proved that an isometry on a Banach space can not be supercyclic. Later, Faghih and Hedayatain [16] extended this result to m-isometric operators on a Hilbert space. In [8] Bayart proved that an m-isometry on a Banach space isn't N-supercyclic.

The paper is organized as follows. In Section 2, we study the relation between m-isometries and (A,m')-isometries. In general, these classes are different. However, with some additional hypotheses we get that (A,m)-isometry has similar properties as the class of m-isometries. Also, we prove that the spectrum of an (A,m)-isometry with a non zero operator A must intersect the unit circle. Indeed, for any compact subset K of $\mathbb{C}$ with intersection in the unit circle, we find a Hilbert space, a positive operator A and an A-isometry with spectrum K. In Section 3, we study the relationship between m-isometries and (A,m')-isometries, for particular cases (finite-dimensional case and infinite-dimensional case with the unilateral weighted shift operator). In the final section, we prove that an A-isometry with 0 not in the point spectrum of A, can not be N-supercyclic. Also, we give a negative answer of [22, Question 1], with an example of an A-isometry which is N-supercyclic. Moreover, we extend the result of [25, Theorem 2.2] and [18, Theorem 2.3] in the following way: the sum of an (A,m)-isometry S and an A-nilpotent operator Q on a Hilbert space H that commutes with S can not be N-supercyclic if $\ker(A)$ is invariant under S and Q and $\dim(H/\ker(A)) > N$.

2. Properties of (A,m)-isometries

Henceforth, A will denote a positive operator unless explicitly stated otherwise.

The purpose of this section is to present some properties of the class of (A,m)-isometries. We give some similar and different properties between (A,m)-isometries and m-isometries.

A first natural question is the following: Is there any relation between m-isometries and (A,m)-isometries?

Example 1.1 shows that in general an (A,m)-isometry isn't an m'-isometry, for any m'. On the other hand, the following example proves that for a fixed positive operator A, we have no relation between the class of m-isometries and (A,m)-isometries.

Example 2.1. Let $A := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ and $T := \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \in L(\mathbb{C}^3)$. Then it is clear that T is an isometry and isn't A-isometry.

Let T be an (A,m)-isometry. We define the *A-covariance operator of T* by

$$\Delta_T^A := \frac{1}{(m-1)!} \sum_{k=0}^{m-1} (-1)^{m-1-k} \binom{m-1}{k} T^{*k} A T^k \,.$$

By [23, Theorem 2.1], the A-covariance operator, Δ_T^A, is a positive operator. An important property says that

$$T^*\Delta_T^A T - \Delta_T^A = 0\ . \tag{4}$$

Hence an (A,m)-isometry (m-isometry) is Δ_T^A-isometry (Δ_T-isometry). See [23] for more properties of the A-covariance operator.

Assume that T is an (A,m)-isometry and an A'-isometry, for some positive operators A and A'. Is $A' = a\Delta_T^A$ for some $a>0$?

This is not true in general.

Example 2.2. Consider the operators $T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $A := \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ and $A' := \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$ on $\mathbb{R}^2$. Then it is not difficult to check that T is a strict $(A,3)$-isometry and A'-isometry with $A' \neq a\Delta_T^A$, for all $a>0$.

In the next proposition, we obtain some immediate properties.

Proposition 2.1. *Let $T \in L(H)$ be an (A,m)-isometry. Then the following properties hold:*

1. *If $S \in L(H)$ is unitarily equivalent to T, then S is an (A,m)-isometry.*
2. *If H_1 is a closed invariant subspace of T, then the restriction of T to H_1, $T_{|H_1}$ is a $(P_{H_1}A_{|H_1}, m)$-isometry where P_{H_1} is the orthogonal projection with range H_1 and $P_{H_1}A_{|H_1}$ is a positive operator of $L(H_1)$.*

Proof. *(2)* Let us prove that $P_{H_1}A_{|H_1}$ is a positive operator of $L(H_1)$. Let $x \in H_1$. Then

$$\langle P_{H_1}A_{|H_1}x, x\rangle = \langle Ax, P_{H_1}x\rangle = \langle Ax, x\rangle \geq 0\ .$$

□

Denote the approximate point spectrum of A by $\sigma_{ap}(A)$.

With some extra spectral condition on A, the classes of (A,m)-isometries and m-isometries are "almost the same", as showed the following result.

Proposition 2.2. *Let $T \in L(H)$ be an (A,m)-isometry such that $0 \notin \sigma_{ap}(A)$. Then T is an m-isometry on $(H, ||.||_A)$, where $||x||_A := \|A^{1/2}x\|$ and $A^{1/2}$ is the square root of A.*

Proof. It is immediate since $0 \notin \sigma_{ap}(A)$ implies that $(H, ||.||_A)$ is a Hilbert space. □

In general, we are interested in obtaining sufficient conditions on T to be an (A,m)-isometry for some $A \geq 0$. Using the same ideas of the theory of m-isometries for perturbation by commuting nilpotent operators obtained in [9, 10, 11], we have the following for (A,m)-isometries.

Theorem 2.1. *Let $T \in L(H)$ be an (A,m)-isometry. Then*

1. *For every $x \in H$, we have that $\|A^{1/2}T^n x\|$ is a polynomial at n of degree at least $m-1$.*
2. *If Q is a nilpotent operator of order n that commutes with T, then $T+Q$ is an $(A, 2n+m-2)$-isometry.*

Proof. The proof of the first part is similar to [10, Theorem 2.1] and the second part to [9, Theorem 3]. □

It is well-known, in the theory of m-isometries, that if T is an isometry that commutes with a nilpotent operator of order exactly n, then $T+Q$ is a strict $(2n-1)$-isometry, that is, $T+Q$ is a $(2n-1)$-isometry and not a $(2n-2)$-isometry, [11, Theorem 2.2]. However, strictly part of this result is not valid for the class of (A,m)-isometries as proves the following example.

Example 2.3. Consider the operators $A := \begin{pmatrix} 0 & 0 \\ 0 & \alpha \end{pmatrix}$, with $\alpha > 0$, $T := \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}$ and $Q := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ on $\mathbb{C}^2$, when it is seen that T and $T+Q$ are A-isometries.

The following proposition is an immediate consequence of the definition of (A,m)-isometry.

Proposition 2.3. *Let $T,\ A \in L(H)$ such that $A \geq 0$ and T be an (A,m)-isometry. Then*

(1) $\ker(T) \subseteq \ker(A)$. *In particular, if* $\ker(A) = \{0\}$*, then* T *is injective.*
(2) *If* T *is invertible, then* T^{-1} *is an* (A, m)*-isometry.*

It is well-known that the approximate point spectrum of an isometry T is contained in the unit circle. Hence the spectrum of T is the closed unit disc if it is not invertible or a closed subset of the unit circle if it is invertible. Indeed, Agler and Stankus proved the same property for m-isometries [4]. So, a first glance, it could be interpreted that isometries and m-isometries have similar spectra. However, this is not totally true, see [12] for more details. Moreover, (A, m)-isometries have not the same spectrum as m-isometries. Indeed, an easy statement is that (A, m)-isometries can not be bounded below, even with points outside of the unit disc as proves the following example.

Example 2.4. If $A := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha \end{pmatrix}$ with $\alpha > 0$ and $T := \begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in L(\mathbb{C}^3)$, then T is A-isometry and $\sigma(T) = \{0, 1, 2\}$.

The next result gives a necessary spectral condition to be an (A, m)-isometry.

Theorem 2.2. *The spectrum of an* (A, m)*-isometry with* $A \neq 0$ *must intersect the unit circle.*

Proof. It is enough to prove the result for $m = 1$, since every (A, m)-isometry is an A'-isometry, where A' is the A-covariance operator of T.

Assume that T is an A-isometry such that $\sigma(T) \cap \partial\mathbb{D} = \varnothing$.

Let us prove that A is the null operator. By functional calculus we get that $H = H_1 \oplus H_2$, $T = T_1 \oplus T_2$, with $T_i := T_{|H_i}$, $i = 1, 2$, such that $\sigma_1 := \sigma(T_1) = \sigma(T) \cap \mathbb{D}$ and $\sigma_2 := \sigma(T_2) = \sigma(T) \cap \overline{\mathbb{D}}^c$.

Let us prove that $A_{|H_1} \equiv 0$. Using that H_1 is an invariant subspace of T and T is A-isometry, so by Proposition 2.1, we get that T_1 is a $P_{H_1} A_{|H_1}$-isometry, and with spectral radius less than 1, since $\sigma(T_1) \subset \mathbb{D}$. Hence $T_1^n h_1$ converges to 0 for any $h_1 \in H_1$. Then for any $h \in H_1$, we obtain that

$$\langle Ah, h\rangle = \langle AT^n h, T^n h\rangle = \langle AT_1^n h, T_1^n h\rangle \to 0 \quad \text{as} \quad n \to \infty.$$

This means that $\langle Ah, h\rangle = 0$ for all $h \in H_1$, then $A_{|H_1} \equiv 0$.

Let us prove that $A_{|H_2} \equiv 0$. Since $\sigma(T_2) = \sigma(T) \cap \overline{\mathbb{D}}^c$, then T_2 is an invertible and $P_{H_2} A_{|H_2}$-isometry. By Proposition 2.3 and the proof of the fist part, we derive the result. ☐

Theorem 2.3. *Let* K *be a compact subset of* $\mathbb{C}$ *such that* $K \cap \partial\mathbb{D} \neq \varnothing$*. Then there exists an infinite dimensional Hilbert space* H *and* $T,\ A \in L(H)$*, with* $A \geq 0$ *such that* T *is an* A*-isometry with* $\sigma(T) = K$*.*

Proof. Let $H := \ell_2(\mathbb{N}) \oplus \mathbb{C}$. Consider a positive operator of $L(H)$, $A := \begin{pmatrix} 0 & 0 \\ 0 & \alpha \end{pmatrix}$, with $\alpha > 0$.

Since $K \cap \partial\mathbb{D} \neq \varnothing$, we define a linear operator T on H as $T := \begin{pmatrix} D & 0 \\ 0 & \lambda \end{pmatrix}$, where $\lambda \in K \cap \partial\mathbb{D}$ and $D(x_1, x_2, ...) := (\beta_1 x_1, \beta_2 x_2, ...)$ with $\overline{\{\beta_n \ : \ n \in \mathbb{N}\}} = K$. Then we obtain that T is an A-isometry and $\sigma(T) = \sigma(D) \cup \{\lambda\} = K$. ☐

Assume that A is a non negative positive operator of $L(H)$. We will present some non trivial examples.

2.1. **On finite dimensional Hilbert space.** By Agler, Helton and Stankus, any m-isometry on a finite dimensional Hilbert space is of the form a unitary operator plus a commuting nilpotent operator [3]. Moreover, by [11, Theorem 2.7] on $\mathbb{R}^2$ there are only isometries and 3-isometries. Indeed, by [12] the strictly 3-isometries on $\mathbb{R}^2$ have a particular form.

Lemma 2.1. [12] *The strict* 3*-isometries on* $\mathbb{R}^2$ *are of the form* $\pm I + Q_i$*, where* Q_i *is a non-zero nilpotent operator given by:*

$$Q_1 := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ Q_2 := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \ \text{or} \ Q_3 := \begin{pmatrix} 1 & \lambda \\ -\lambda^{-1} & -1 \end{pmatrix}, \quad \text{with} \quad \lambda \neq 0.$$

The above result gives some ideas for $(A, 3)$-isometries on $\mathbb{R}^2$.

Theorem 2.4. *Let $T_i := \pm I + Q_i$ be a strict 3-isometry on $\mathbb{R}^2$. Then T_i is an A-isometry, with $A \geq 0$ if and only if $A \equiv A_i$, $i = 1, 2, 3$, where*

(1) $A_1 := \begin{pmatrix} 0 & b \\ -b & a \end{pmatrix}$, $a \geq 0$ *and* $b \in \mathbb{R}$.

(2) $A_2 := \begin{pmatrix} a & b \\ -b & 0 \end{pmatrix}$, $a \geq 0$ *and* $b \in \mathbb{R}$.

(3) $A_3 := \begin{pmatrix} \frac{b+c}{2\lambda} & b \\ c & \frac{\lambda(b+c)}{2} \end{pmatrix}$, $\frac{\lambda(b+c)}{2} \geq 0$.

Proof. Let us prove part (3), the other cases are similar. Assume that $T_3 = \begin{pmatrix} 2 & \lambda \\ -\lambda^{-1} & 0 \end{pmatrix}$.
Consider a positive operator of $L(\mathbb{R}^2)$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.
Suppose that T_3 is A-isometry, that is, $T_3^* A T_3 - A \equiv 0$. Then we get the following system of equations

$$\begin{cases} 3a - 2\lambda^{-1}(b + c) + \lambda^{-2} d = 0 \\ a\lambda^2 - d = 0 \\ 2a\lambda - (c + b) = 0 \end{cases}$$

and the solution is given by $a = \frac{b+c}{2\lambda}$ and $d = \frac{\lambda(b+c)}{2}$.

Moreover, it is not difficult to prove that $A_3 := \begin{pmatrix} \frac{b+c}{2\lambda} & b \\ c & \frac{\lambda(b+c)}{2} \end{pmatrix}$ is a positive operator if and only if $\frac{\lambda(b+c)}{2} \geq 0$. Hence we obtain the result. □

The m-isometries on a finite dimensional Hilbert space has a concrete form: isometries plus commuting nilpotent operator. So, what can we say about (A, m)-isometries on finite dimensional Hilbert space?

Question 2.1. Let H be a finite dimensional Hilbert space and $T \in L(H)$ be an (A, m)-isometry. Is it possible to write T as the sum of an A-isometry and nilpotent operator which commutes?

In general, we do not know the answer of that question. Our examples satisfy that decomposition, that is, A-isometry plus a commuting nilpotent operator.

Example 2.5. If $T_1 := \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$ and $A_1 := \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \in L(\mathbb{C}^2)$, then T_1 is a strict 3-isometry and a strict $(A_1, 3)$-isometry with $T_1 := I + Q_1$ where $Q_1^2 = 0$. Moreover, if $A := \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}$ and $T := \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix}$, then we obtain that T isn't an m-isometry, for any integer m, and it is a strict $(A, 3)$-isometry with $T := S + Q$ where $S := \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ is an A-isometry and $Q := \begin{pmatrix} Q_1 & 0 \\ 0 & 0 \end{pmatrix}$ is a 2-nilpotent operator such that $SQ = QS$.

2.2. **Unilateral weighted shift.** We will assume that $(e_n)_{n \geq 1}$ is an orthonormal basis of $\ell^2(\mathbb{N})$. The unilateral weighted shift T on $\ell^2(\mathbb{N})$ with weight sequence $(w_n)_{n \geq 1}$ it is defined by

$$Te_n \; := \; w_n e_{n+1}, \quad \text{for all } n \geq 1.$$

Corollary 2.1. [2, Corollary 3] *A unilateral weighted shift T on $\ell^2(\mathbb{N})$ with weight sequence $(w_n)_{n\geq 1}$ is strictly m-isometric if and only if there exists a polynomial p of degree $m-1$ with real coefficients such that for all integers $n \geq 1$, we have $p(n) > 0$ and*

$$|w_n|^2 \quad = \quad \frac{p(n+1)}{p(n)}. \tag{5}$$

Corollary 2.2. *Let T be a unilateral weighted shift with weight sequence $(w_n)_{n\geq 1}$ and p be the monic polynomial satisfying (5). If T is a strict m-isometry with even m, then there exists a root α_{j_0} of p such that $\alpha_{j_0} \in (-\infty, 1)$.*

Proof. By Corollary 2.1, there exists a polynomial p of odd degree $m-1$ that satisfies (5). Suppose that there exists a root $\alpha_{j_1} \in (1, \infty) \setminus \mathbb{N}$, then there exist integer $n_1 \in \mathbb{N}$ and a root $\alpha_{j_1'}$ of p, such that:

$$\alpha_{j_1'},\ \alpha_{j_1} \in (n_1, n_1 + 1).$$

That is, for all integers $n \geq 1$, we have $(n-\alpha_{j_1})(n-\alpha_{j_1'}) > 0$, since $p(n) > 0$, for all integers $n \geq 1$ and the numbers of roots of p is odd. Hence there exists a root α_{j_0} such that $\alpha_{j_0} \in (-\infty, 1)$. □

Remark 2.1. From Corollary 2.2, we conclude that p could be written as:

$$p(x) = (x-\alpha_1)(x-\alpha_2)...(x-\alpha_{m-1}),$$

such that for all integers $n \geq 1$, where $j_0 \in (-\infty, 1)$ is taking as 1, without lost of generality. Then we have that $(n-\alpha_1) > 0$ and

$$\begin{gathered}(n-\alpha_2)(n-\alpha_3) > 0,\\ (n-\alpha_4)(n-\alpha_5) > 0,\\ \dots,\\ (n-\alpha_{m-2})(n-\alpha_{m-1}) > 0.\end{gathered}$$

Recall the following combinatorial result.

Lemma 2.2. [26, Eq. 0.154 (3)] *If n is a positive integer, then*

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} k^j \quad = \quad 0$$

for all $j \in \{0, 1, ..., n-1\}$.

In Section 2, we have recalled that an m-isometry is a Δ_T-isometry. Now, we are interested in the study of m-isometry with the unilateral weighted shift related with the concept of (A, m')-isometry, for some positive operator A and some integer m'.

Theorem 2.5. *Let T be a unilateral weighted shift with weight sequence $(w_n)_{n\geq 1}$ on $\ell^2(\mathbb{N})$, which is a strict m-isometry, let p be the monic polynomial satisfying (5) and let α_j's be the roots of p.*

(1) *If m is even, then T is a strict A_ℓ-$(m-\ell)$-isometry, where*

$$A_\ell e_n \quad := \quad \begin{cases} \dfrac{1}{\Pi_{j=2}^{\ell+1}(n-\alpha_j)} e_n, & \text{if } \ell \text{ is even} \\ \dfrac{1}{\Pi_{j=1}^{\ell}(n-\alpha_j)} e_n, & \text{if } \ell \text{ is odd} \end{cases}$$

for all integers $n \geq 1$ and $1 \leq \ell \leq m-1$, where $\alpha_1 \in (-\infty, 1)$.

(2) *If m is odd, then T is a strict $A_{2\ell}$-$(m-2\ell)$-isometry, where*

$$A_{2\ell} e_n \quad := \quad \frac{1}{\Pi_{j=1}^{2\ell}(n-\alpha_j)} e_n,$$

for all integers $n \geq 1$ and $1 \leq \ell \leq \frac{m-1}{2}$.

Proof. Let us prove it for even m. Assume that $1 \leq \ell \leq m-1$. By Remark 2.1, we obtain that if ℓ is even, then $\Pi_{j=2}^{\ell+1}(n-\alpha_j) > 0$ and if ℓ is odd, then $\Pi_{j=1}^{\ell}(n-\alpha_j) > 0$, for all $n \in \mathbb{N}$. Then A_ℓ is a positive operator.
For even ℓ, we consider the diagonal operator A_ℓ with diagonal

$$\lambda_n \quad := \quad \frac{1}{\Pi_{j=2}^{\ell+1}(n-\alpha_j)}.$$

Denote

$$\beta_\ell(A,T,x) := \frac{1}{\ell!}\sum_{k=0}^{\ell}(-1)^{\ell-k}\binom{\ell}{k}\langle AT^k x, T^k x\rangle \ ,$$

for any positive integer ℓ. Let $x = \sum_{n\geq 1} x_n e_n \in \ell^2(\mathbb{N})$. We have

$$\begin{aligned}
\beta_{m-\ell}(A_\ell,T,x) &= \frac{1}{(m-\ell)!} \quad \sum_{k=0}^{m-\ell}(-1)^{m-\ell-k}\binom{m-\ell}{k}\langle A_\ell T^k x, T^k x\rangle \\
&= \frac{1}{(m-\ell)!} \quad \sum_{k=0}^{m-\ell}(-1)^{m-\ell-k}\binom{m-\ell}{k}\sum_{n\geq 1}|x_n|^2\langle A_\ell T^k e_n \ , \ T^k e_n\rangle \\
&= \frac{1}{(m-\ell)!} \quad \sum_{n\geq 1}|x_n|^2\sum_{k=0}^{m-\ell}(-1)^{m-\ell-k}\binom{m-\ell}{k}\prod_{j=n}^{n+k-1}|w_j|^2\langle A_\ell e_{n+k} \ , \ e_{n+k}\rangle \\
&= \frac{1}{(m-\ell)!} \quad \sum_{n\geq 1}|x_n|^2\sum_{k=0}^{m-\ell}(-1)^{m-\ell-k}\binom{m-\ell}{k}\prod_{j=n}^{n+k-1}\frac{p(j+1)}{p(j)}\langle A_\ell e_{n+k} \ , \ e_{n+k}\rangle \\
&= \frac{1}{(m-\ell)!} \quad \sum_{n\geq 1}|x_n|^2\sum_{k=0}^{m-\ell}(-1)^{m-\ell-k}\binom{m-\ell}{k}\frac{p(n+k)}{p(n)}\frac{1}{\prod_{j=2}^{\ell+1}(n+k-\alpha_j)}.
\end{aligned}$$

Since,

$$\frac{p(n+k)}{\prod_{j=2}^{\ell+1}(n+k-\alpha_j)} \quad = \quad \begin{cases} (n+k-\alpha_1), & \text{if } \ \ell = m-2, \\ \\ \prod_{j=\ell+2}^{m-1}(n+k-\alpha_j)(n+k-\alpha_1), & \text{if } \ \ell \neq m-2. \end{cases}$$

By Lemma 2.2, we obtain that

$$\sum_{k=0}^{m-\ell}(-1)^{m-\ell-k}\binom{m-\ell}{k}\frac{p(n+k)}{\prod_{j=2}^{\ell+1}(n+k-\alpha_j)} \quad = \quad 0.$$

Hence $\beta_{m-\ell}(A_\ell,T,x) = 0$, which means that T is a strict A_ℓ-$(m-\ell)$-isometry.
The rest of the cases are similar. □

Corollary 2.3. *Let T be a unilateral weighted shift with weight sequence $(w_n)_{n\geq 1}$ on $\ell^2(\mathbb{N})$ and let p be the monic polynomial satisfying (5). If T is an m-isometry, then there exists a nonzero positive operator A_{m-1} given by*

$$A_{m-1}e_n \quad := \quad \frac{1}{p(n)}e_n, \ \textit{for all } n \geq 1,$$

such that T is an A_{m-1}-isometry, where $0 \in \sigma_{ap}(A_{m-1})\backslash\sigma_p(A_{m-1})$.

3. Dynamic properties

The purpose of this section is to give some dynamic properties of (A, m)-isometries.
Denote

$$L_A(H) := \{T \in L(H) \quad : \quad R(T^*A) \subset R(A)\} \, .$$

In the following result we summarize some positive results.

Theorem 3.1. [22] *Let $T \in L(H)$ be an (A, m)-isometry. Then*

- (a) *A power bounded A-isometry is never supercyclic.*
- (b) *If $(\|T^n x\|)_{n\in\mathbb{N}}$ is eventually increasing for any $x \in H$, then T is not supercyclic.*
- (c) *If $T \in L_A(H)$, $0 \notin \sigma(A)$ and Δ_T^A is injective, then T is not N-supercyclic.*

By a similar expression of (2), Bayart [8] and Hoffman, Mackey and Searcoid [19] have introduced the concept of m-isometries on Banach space context. That is, a bounded linear operator $T : X \longrightarrow X$ on a Banach space X is an (m, p)-isometry ($m \geq 1$ integer, $p > 0$ real) if

$$\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \|T^k x\|^p = 0, \tag{6}$$

for all $x \in H$. In [14] Duggal has introduced the following definition of (A, m, p)-isometry in a Banach space context using similar ideas. An operator $T \in L(X)$ is an (A, m, p)*-isometry* if

$$\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \|AT^k x\|^p = 0 \qquad (x \in X) \, .$$

If A is a positive operator defined in a Hilbert space, then $T \in L(H)$ is an (A, m)-isometry if and only if T is an $(A^{1/2}, m, 2)$-isometry with the definition given by Duggal, where $A^{1/2}$ is the square root of A.

We present a different and easy proof of Duggal's result on Hilbert space. A similar proof works on Banach spaces.

Corollary 3.1. [14, Corollary 2.6] *Let A, $T \in L(H)$ such that $0 \notin \sigma_{ap}(A)$. If T is an (A, m)-isometric, then T can not be N-supercyclic.*

Proof. By Proposition 2.2, T is an m-isometry on $(H, |||.|||_A)$. Moreover, Bayart has proved that an m-isometry can not be N-supercyclic, [8, Theorem 3.3]. So, the result is obtained. □

The above result can be improved. For that we need some lemmas.

Lemma 3.1. *Let A, $T \in L(H)$ such that T is A-isometry and $0 \notin \sigma_p(A)$. Then the following properties hold:*

- (a) *There exists $M > 0$ such that $\|AT^n\| \leq M$ for all positive integer n.*
- (b) *For any nonzero $x \in H$, $AT^n x \nrightarrow 0$ as $n \to \infty$.*

Proof. Suppose that T is an A-isometry. Then, $\|AT^n x\| = \|Ax\| \leq \|A\|\|x\|$. Hence we obtain the first part of the result.

Let $x \in H$ be a nonzero vector. Since $0 \notin \sigma_p(A)$, so we have $\|AT^n x\| = \|Ax\| \neq 0$. So

$$\|AT^n x\| \to \|Ax\| \neq 0 \text{ as } n \to \infty \, .$$

Thus $AT^n x \nrightarrow 0$ as $n \to \infty$, for any nonzero $x \in H$. □

Lemma 3.2. *Let T, $A \in L(H)$. If T and A satisfies properties (a) and (b) of Lemma 3.1, then T can be extended to an isometry on a Banach space.*

Proof. Define $F : \ \ell^\infty(\mathbb{C}) \ \longrightarrow \ \mathbb{C}$ a linear functional that satisfies:

1. If $x_n \leq y_n$, then $F((x_n)) \leq F((y_n))$.
2. For any $(x_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{C})$, $F((x_n)) = F((x_{n+1}))$.
3. $F((x_n))$ is the limit of a subsequence of $\dfrac{x_1 + ... + x_n}{n}$.

Define a new norm on X by $|||x||| := F((\|AT^n x\|))$. Let us prove that $|||x|||$ is a norm. Let $x \in H$ be a nonzero vector. Since $\|AT^n\| \leq M$, for all n, then $(\|AT^n x\|)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{C})$ for all $x \in H$. Further by (b), $AT^n x \nrightarrow 0$ as $n \to \infty$ for every $x \in H \setminus \{0\}$, that is, there exists $M_0 > 0$ such that:

$$\|AT^n x\| \geq M_0\ , \quad \text{for all } n\ .$$

Hence

$$\frac{\|ATx\| + \|AT^2x\| + ... + \|AT^n x\|}{n} \geq M_0 > 0, \quad \text{for all } n. \tag{7}$$

By definition of the linear functional and (7), we have that $|||x||| > 0$. Then $|||.|||$ is a norm on H. Let $x \in H$. Then

$$|||Tx||| = F((\|AT^{n+1}x\|)) = F((\|AT^n x\|)) = |||x|||\ .$$

Let $\widetilde{H}$ denote the completion of X with the norm $|||.|||$. Hence T extends to an isometry $\widetilde{T}$ on $(\widetilde{H}, |||.|||)$.

□

Theorem 3.2. *If $T \in L(H)$ is an A-isometry such that $0 \notin \sigma_p(A)$, then T can not be N-supercyclic.*

Proof. By Lemma 3.1 we have that T and A satisfies the hypothesis of Lemma 3.2. The result is consequence of [8, Theorem 3.4]. □

With some additional hypothesis, it is possible to prove that (A, m)-isometries are not N-supercyclic.

Corollary 3.2. *Let A, $T \in L(H)$ such that $0 \notin \sigma_p(\Delta_T^A)$. If T is an (A, m)-isometric, then T can not be N-supercyclic.*

Proof. Since T is an (A, m)-isometry, by (4) we have that T is a Δ_T^A-isometry. By Theorem 3.2 yields the result. □

In the following result Hedayatian proved a general result of Corollary 3.2.

Theorem 3.3. [18, Theorem 2.3] *If $T \in L(H)$ is an (A, m)-isometry such that* $\dim(H/\ker(\Delta_T^A)) >$ *N, for some $N \geq 1$, then T is not N-supercyclic.*

In [15, Theorem 4], Faghih-Ahmadi proved that any (A, m)-isometry is not supercyclic. However, in the next example, we prove that it is not correct, even for the N-supercyclic class. Also, this example gives a negative answer of [22, Question 1].

Example 3.1. Let $H := \mathbb{C}^N \oplus \ell_2(\mathbb{N})$. Consider the positive operator A of $L(H)$, $A := I_{\mathbb{C}^N} \oplus 0$, and $T \in L(H)$, $T := I_{\mathbb{C}^N} \oplus \lambda B$ where $|\lambda| > 1$ and $B(x_1, x_2, ...) := (x_2, x_3, ...)$. Then T is an A-isometry. Indeed,

$$\begin{aligned} \langle AT(z, (x_1, x_2, ...)), T(z, (x_1, x_2, ...))\rangle &= \langle (z, 0), (z, \lambda(x_2, x_3, ...))\rangle \\ &= |z|^2 \\ &= \langle A(z, x), (z, x)\rangle. \end{aligned}$$

However, T is N-supercyclic by [13].

In the literature there are some perturbation results for the class of m-isometries. See for example [9, 11, 17, 21]. We are interested in introducing a new concept that generalizes the class of the nilpotent operators, the K-nilpotent operator.

Definition 3.1. Let X and Y two Banach spaces, $Q \in L(X)$ and K a map from X to Y. We say that Q is K-*nilpotent* if there exists $n \geq 1$ such that

$$R(Q^n) \subseteq \ker(K). \tag{8}$$

Note that if $\ker(K) = \{0\}$, then Q is a classical *nilpotent* operator.

The next result generalizes [25, Theorem 2.2] and [18, Theorem 2.3]. See also [9, Theorem 3], [17, Theorem 4] and [21, Theorem 16].

Theorem 3.4. *Let $S \in L(H)$ be an (A,m)-isometry and $Q \in L(H)$ be an A-nilpotent such that $SQ = QS$. If $\ker(A)$ is invariant under S and Q and $\dim(H/\ker(A)) > N$, then $S+Q$ is not N-supercyclic.*

Proof. Suppose that $T := S + Q$ is N-supercyclic on $(H, ||.||)$. Since

$$||x||_A^2 = \langle Ax, x\rangle \leq ||A|| ||x||^2, \quad \forall x \in H,$$

then T is N-supercyclic on $(H, ||.||_A)$.
If $Q(\ker(A)) \subseteq \ker(A)$ and $S(\ker(A)) \subseteq \ker(A)$, then we can define $\widetilde{S_0}$ and $\widetilde{Q_0}$ on $H/\ker(A)$ by

$$\widetilde{S_0}[x] := [Sx], \;\; \widetilde{Q_0}[x] := [Qx]$$

and

$$\widetilde{T_0} := \widetilde{S_0} + \widetilde{Q_0}.$$

Moreover, for each $x \in H$, $||[x]||_A = ||x||_A$. Then $\widetilde{S_0}$ is an m-isometry. Indeed,

$$\begin{aligned}\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}||\widetilde{S}_0^k[x]||_A^2 &= \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}||[S^kx]||_A^2 \\ &= \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}||S^kx||_A^2 = 0.\end{aligned}$$

Let $n \in \mathbb{N}$ be such that $R(Q^n) \subseteq \ker(A)$.
For $x \in H$, we have $\widetilde{Q}_0^n[x] = [Q^nx] = [0]$. Thus $\widetilde{Q_0}$ is a nilpotent operator on $H/\ker(A)$ of order $n_0 \leq n$. We can consider the following commutative diagram

$$\begin{array}{ccc} H & \xrightarrow{T} & H \\ \varphi\downarrow & & \downarrow\varphi \\ H/\ker(A) & \xrightarrow{\widetilde{T_0}} & H/\ker(A) \end{array}$$

where φ is the canonical projection map. Then, $\widetilde{T_0}$ is supercyclic. Let $\mathcal{K}$ be the completion of $H/\ker(A)$ and $\widetilde{T}$, $\widetilde{Q}$ and $\widetilde{S}$ the extensions of $\widetilde{T_0}$, $\widetilde{Q_0}$ and $\widetilde{S_0}$ on the Hilbert space $\mathcal{K}$. Then $\widetilde{T} = \widetilde{S} + \widetilde{Q}$, where $\widetilde{S}$ is an m-isometry, $\widetilde{Q}$ is an n_0-nilpotent and $\widetilde{S}\widetilde{Q} = \widetilde{Q}\widetilde{S}$. However, $\widetilde{T}$ is N_0-supercyclic, with $1 \leq N_0 \leq N$. On the other hand, [9, Theorem 3.1] implies that $\widetilde{T}$ is a $(2n_0 + m - 2)$-isometry on the Hilbert space $\mathcal{K}$.
We suppose that $N < \dim(H/\ker(A)) < \infty$, so we get a contradiction by [13, Theorem 3.4]. If $\dim(H/\ker(A)) = \infty$, then we obtain that $\widetilde{T}$ is a $(2n_0 + m - 2)$-isometry which is N_0-supercyclic on an infinite dimensional Hilbert space, which is a contradiction [8, Theorem 3.3]. □

Remark 3.1. The condition of invariance of $\ker(A)$ under the operator S, in the above theorem is necessary. In fact, if $S := \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$ and $A := \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$ of $L(\mathbb{R}^2)$, then it is clear that A is a positive operator and S is a strict $(A,3)$-isometry which satisfies that $S(\ker(A)) \subsetneq \ker(A)$. So in this case is not well defined the operator $\widetilde{S}$.

Theorem 3.4 could be obtained in Banach space context using the definition of Duggal of (A, m, p)-isometry.

Define the map $N_p := (\beta_{m-1}^{(p)}(A,T,.))^{\frac{1}{p}} : X \to \mathbb{R}$. Then N_p is a semi-norm satisfying

$$(\beta_{m-1}^{(p)}(A,T,x))^{\frac{1}{p}} \leq ||A||(1 + ||T||^p)^{\frac{m-1}{p}}||x|| \,,$$

and

$$T(\ker(N_p)) \subseteq \ker(N_p).$$

where

$$\beta_\ell^{(p)}(A,T,x) := \frac{1}{\ell!}\sum_{k=0}^{\ell}(-1)^{\ell-k}\binom{\ell}{k}\|AT^k x\|^p .$$

The proof of the following result is similar to Theorem 3.4.

Theorem 3.5. *Let $S, Q \in L(X)$, S is an (A, m, p)-isometry and Q an N_p-nilpotent satisfying $Q(\ker(N_p)) \subseteq \ker(N_p)$ and $SQ = QS$. If $\dim(X/\ker(N_p)) > N$, then $T = S + Q$ is not N-supercyclic.*

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