Profitability ratio maximization in an inventory model with stock-dependent demand rate and non-linear holding cost

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Abstract

This paper studies a deterministic inventory model with a stock-dependent demand pattern where the cumulative holding cost is a non-linear function of both time and stock level. When the monetary resources are limited and the inventory manager can invest his/her money in buying different products, it seems reasonable to select the ones that provide a higher profitability. Thus, a new approach with the aim of maximizing the profitability ratio (defined as the profit/cost quotient) is considered in this paper. We prove that the profitability ratio maximization is equivalent to minimizing the inventory cost per unit of an item. The optimal policy is obtained in a closed form, whose general expression is a generalization of the classical EOQ formula for inventory models with a stock-dependent demand rate and a non-linear holding cost. This optimal solution is different from the other policies proposed for the problems of minimum cost or maximum profit per unit time. A complete sensitivity analysis of the optimal solution with respect to all the parameters of the model is developed. Finally, numerical examples are solved to illustrate the theoretical results and the solution methodology.

Keywords: inventory management; profitability ratio maximization; stock-dependent demand rate; non-linear holding cost.

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1 Introduction

The economic order quantity (EOQ) model is probably the most studied inventory model in management science and operations research literature. After more than a century, it remains the focus of attention for new lines of research in inventory theory. Cárdenas-Barrón et al. [1] managed to gather 41 new research papers referring to this model in a special issue celebrating the century since its appearance. To illustrate its great development, we can say that Bushuev et al. [2] provided a review of 39 review papers found in the literature of inventory lot sizing, with more than 14 research lines. Among them, there are the models with an inventory-level-dependent demand or with a time-dependent demand, which also gathers an extensive literature on management science. Some recent papers on this subject are Banerjee and Agrawal [3], Duan et al. [4], Onal et al. [5], Pervin et al. [6, 7], and San-José et al. [8].

Deterministic inventory models usually consider a single decision variable, which is the economic order quantity, and the basic models assume that a new order is not carried out until the inventory is depleted. With constant demand rate and without shortages, it is the right decision because, otherwise, the sum of the ordering cost and the holding cost per unit time would be increased and the minimum cost policy for the inventory system would not be obtained. However, such an action may not be suitable if the demand rate depends on the inventory level, as increased costs can be offset by increased sales due to a higher stock level. This type of demand can appear for many items in a supermarket, where large piles of goods displayed brings about increased sales. The paper by Balakrishnan et al. [9] is an excellent work where the stock-dependent demand rate is strongly justified.

As a starting point, Baker and Urban [10] proposed an inventory model with a stock-dependent demand rate focused on maximizing profits per unit time. They considered that the demand rate was a concave power function of the inventory level, which was subsequently used by many other authors. To get a convenient formulation of the problem, they assumed that the holding cost rate per unit and per unit time was constant. Two decision variables were necessary to solve the mathematical problem: the reorder level r (the inventory level when a new order should be carried out) and the order-up-to level S (the quantity to order up to); so it is known as the (r, S)-inventory model. Nevertheless, the objective function did not satisfy the usual quasi-concavity condition and a general solution to the problem in a closed form could not be obtained. The cited authors found an approximate solution using separable programming, because the profit per unit time was a quotient of two separable functions on the decision variables. Later on, Urban [11] introduced a model with a stock-dependent demand pattern at the beginning of the inventory cycle followed by a constant demand rate without a zero stock condition at the end of the inventory cycle.

Another extension of the basic EOQ models, widely used by researchers, was the relaxing of the assumption stating that the holding cost per unit and per unit time is constant. Thus, some researchers have considered that the holding cost of the inventory system could be a power function of the stock level, or of the length of the cycle time, or even both. In this way, many inventory models with non-linear holding cost have appeared in the literature on the subject, including the papers by Alfares [12], Ferguson et al. [13], Roy [14], and San-José et al. [15, 16].

Although the numerical example solved approximately by Baker and Urban [10] provided an optimal solution with reorder level $r > 0$, other authors have considered inventory models with a stock-dependent demand and $r = 0$, reducing the problem to a single decision variable. Thus, Goh [17] found the solution with minimum inventory cost per unit time for this problem, assuming that the holding cost rate per item is a convex power function of the time (Model A) or the holding cost rate per unit time is a convex power function of the stock level (Model B). Pando et al. [18, 19] performed a comprehensive analysis of these two models from the perspective of maximizing profit per unit time. From these results, Pando et al. [20] also assumed a zero ending order point and found the optimal solution to the inventory problem of the maximum profit per unit time, considering the most general situation with a non-linear holding cost in both time and stock level.

After a comprehensive review of the literature on the subject with a stock-dependent demand, Urban [21] analyzed inventory systems with periodic review in this environment. Moreover, Urban [22] studied the (r, S)-model with two decision variables and discretely variable holding costs, which were calculated as the product of the average inventory multiplied by a fixed value that estimates the holding cost per unit and per unit time. Other inventory models with a stock-dependent demand rate and a non-zero ending condition for the inventory cycle were proposed by Chang [23], Teng et al. [24] and Shah et al. [25].

However, an important issue that should be considered in (r, S)-inventory models with stock-dependent demand and profit maximization is that the increase in the profit occurs at the expense of an increase in the inventory costs due to the non-zero ending condition. If the maximum profit policy entails a greater investment in the inventory costs, it seems logical to analyze whether this is interesting or not. That is, if a small increase in the profit per unit time requires a very large increase in the inventory costs, the manager may not be interested in that increase, especially if he/she has other investment alternatives with a higher profitability. Then, perhaps the problem should be handled with the objective of the maximum profitability ratio (defined as the profit/cost quotient), instead of the maximum profit per unit time. If the increase in the profit does not lead to a greater profitability ratio of the inventory system, the manager could consider another choice for this investment. When the monetary resources are limited and the inventory manager can invest in different products, it would seem reasonable to select the one that provides a higher profitability ratio. The solution for the maximum profit per unit time does not necessarily match the maximum profitability ratio solution. As a rule, Van Horne [26] defined the profitability index (PI) of a project as the income/cost ratio. Later, Arcelus and Srinivasan [27] adapted the EOQ model to maximize the return on investment (ROI), defined as the profit/cost ratio, and they proved that, in a single period model, allocation on the basis of the ROI is equivalent to allocation on the basis of the profitability index. Trietsch [28] compared the solution of ROI maximization (ROQ solution) with the classical EOQ solution when the inventory items are ordered independently of each other. However, to the best of our knowledge, since then, this line of research in inventory models has not been highly developed.

From this background, the purpose of this paper is to solve the general (r, S)-inventory model, with a stock-dependent demand pattern and a cumulative holding cost that is non-linear over both time and stock level, from the approach of the profitability ratio maximization. To do so, the work is organized as follows. Notations and assumptions are given in Section 2 and the model is developed to get the mathematical problem formulation. Section 3 includes the theoretical results that lead to the solution of the model, distinguishing two cases with respect to the initial parameters of the inventory system. A sensitivity analysis of the optimal solution for all the parameters of the model is set out in Section 4. Section 5 illustrates the application of the model with numerical examples, which are solved using the proposed solution methodology, while also including a sensitivity analysis for the optimal lot size and the maximum profitability ratio. Finally, conclusions and future research lines are given in Section 6.

2 Model formulation

The model to be considered in this paper is formulated with the following basic assumptions: (i) the item is a single product and the planning horizon is infinite; (ii) the inventory is continuously reviewed and the replenishments are instantaneous; (iii) the unit purchasing cost and the selling price are known and constant; (iv) the order cost is fixed regardless of the lot size; and (v) shortages are not allowed. Assumptions (i)-(iv) are common in the economic order quantity literature for a single item with current consumption if the life cycle of the product has attained its maturity stage. Assumption (v) is necessary in the model because the functional form of the demand rate leads to no demand if no stock. This situation can happen for many items in a supermarket where there are no sales if the product runs out.

In addition, the cumulative holding cost for x items held in stock t units of time is defined by the function $H(t,x) = ht^{\gamma_1}x^{\gamma_2}$, with $h > 0$, $\gamma_1 \geq 1$ and $\gamma_2 \geq 1$, so it can be non-linear in both time and stock level when $\gamma_1 > 1$ and $\gamma_2 > 1$. This cost structure allows a wide range of practical situations to be analyzed for inventory models, also including other situations considered by some authors in the past, which are now obtained by setting $\gamma_1 = 1$ or $\gamma_2 = 1$. Obviously, it also generalizes the basic case with constant holding cost rate per unit and per unit time, which is obtained here when $\gamma_1 = \gamma_2 = 1$. Note that $h = H(1,1)$ and that this parameter depicts the holding cost for one unit of product held in stock during one unit of time. The other two parameters, γ_1 and γ_2, set the effects of time and stock level on the holding cost; so $\gamma_1 > 1$ if longer times lead to a higher cost per unit time and $\gamma_2 > 1$ if greater quantities of items lead to a higher cost per item. These situations can occur for perishable goods or space-constrained inventories where time or storage space are limiting factors for the retailer, and therefore the holding cost greatly increases over a long time and/or with a large quantity of product. Pando et al. [20] illustrated the use of this functional form in an inventory model with an example, assuming that the holding cost is a non-linear function with respect to both the time held in stock and the quantity of items stored.

Furthermore, it is supposed that the demand rate $D(t)$ is deterministic and depends on the inventory level $I(t)$ through the concave power function given by $\lambda\,(I(t))^{\beta}$ with $\lambda > 0$ and $0 \leq \beta < 1$. This assumption has been widely used in Inventory Theory, generalizing the basic case with a constant demand rate, which is obtained when $\beta = 0$. With this functional form, as the inventory level decreases, so does the demand rate. That is, at time $t = 0$, the inventory level and the demand rate are at their highest level. As more inventory is depleted, the rate of decrease in the stock level slows down. The parameter β is referred to as the elasticity of the demand rate with respect to the stock level and it represents the relative change in the demand rate with respect to the corresponding relative change in the stock level (that is: $\beta = \frac{\partial D(t)/\partial I(t)}{D(t)/I(t)}$), in such a way that it is a measure of the responsiveness of the demand rate to changes in the inventory level. Varying the values of λ and β, this function is expected to provide a good approximation for many situations in which the demand rate decreases as the inventory level decreases. As proven by Pando et al. [20], if ρ is the proportion of sales in the second half of an inventory cycle with $0 < \rho \leq 0.5$, then the parameter β can be estimated as $\beta = 1 + \ln 2/\ln\rho$, and β is usually less than 0.5. Moreover, as $\ln\rho < 0$, the condition $\beta < 1$ is necessary.

Unlike the models with a constant demand rate, where it is optimal to let the inventory level reach zero before reordering, relaxing the condition of zero ending can now lead to an improvement in the profit, because the increase in sales can offset the higher ordering and holding costs caused by maintaining a higher on-hand inventory. As a result, two decision variables need to be considered: the reorder level r (the inventory level at which to place an order), and the order-up-to level S (the quantity to order up to), so that the lot size is $q = S - r$. Note that $S = \lim_{t\to 0^+} I(t) = I(0+)$ and the length of a cycle time T is given by the equation $I(T-) = \lim_{t\to T^-} I(t) = r$.

The notation used throughout the paper is summarized in Table 1.

Table 1. List of notation

S	order-up-to level (quantity to order up to). It is a decision variable with $S > 0$
r	reorder level (inventory level to set an order). It is a a decision variable with $0 \leq r < S$
q	lot size, $q = S - r$ (> 0)
T	length of the inventory cycle (> 0)
t	time spent in inventory $(\leq T)$
$I(t)$	inventory level at time t $(r \leq I(t) \leq S)$
p	unit purchasing cost (> 0)
v	unit selling price $(> p)$
K	ordering cost per order (> 0)
h	scale parameter for the holding cost $(h > 0)$
γ_1	elasticity of the holding cost with respect to the time (≥ 1)
γ_2	elasticity of the holding cost with respect to the stock level (≥ 1)
$H(t,x)$	cumulative holding cost for x items during t units of time, $H(t,x) = ht^{\gamma_1}x^{\gamma_2}$
λ	scale parameter for the demand rate (> 0)
β	elasticity of the demand rate with respect to the stock level $(0 \leq \beta < 1)$
$D(t)$	demand rate at time t, $D(t) = \lambda\,(I(t))^\beta$
α	auxiliary parameter, $\alpha = 1 - \beta$ $(0 < \alpha \leq 1)$
ξ	auxiliary parameter, $\xi = \alpha\gamma_1 + \gamma_2$ $(\geq 1 + \alpha)$
τ	auxiliary decision variable, $\tau = 1 - (r/S)^\alpha$ $(0 < \tau \leq 1)$

With these assumptions and the decision variables (r, S), the inventory level $I(t)$ is the solution of the differential equation

$$\frac{dI}{dt} = -\lambda\,(I(t))^\beta \qquad 0 \leq t \leq T \tag{1}$$

with boundary conditions $I(0+) = S$ and $I(T-) = r$. Since $0 \leq \beta < 1$, the solution is

$$I(t) = (S^\alpha - \alpha\lambda t)^{1/\alpha} = S\left(1 - \frac{\alpha\lambda}{S^\alpha}t\right)^{1/\alpha} \tag{2}$$

where $\alpha = 1 - \beta > 0$. Moreover, as $I(T-) = r$, the length of the cycle time is given by

$$T = \frac{S^\alpha - r^\alpha}{\alpha\lambda} = \frac{S^\alpha}{\alpha\lambda}\left(1 - \left(\frac{r}{S}\right)^\alpha\right) \tag{3}$$

For a given S, the maximum length of the cycle time (obtained for r=0) is denoted by $T_{\max}(S)$, with

$$T_{\max}(S) = \frac{S^\alpha}{\alpha\lambda} \tag{4}$$

Then, another expression for the inventory level is

$$I(t) = S\left(1 - \frac{t}{T_{\max}(S)}\right)^{1/\alpha} \tag{5}$$

with $0 \le t \le T \le T_{\max}(S)$.

In Figure 1, the function $I(t)$ is plotted for $r = 0$ and $r > 0$ with the same order-up-to level S, showing the difference between the models with and without zero ending condition.

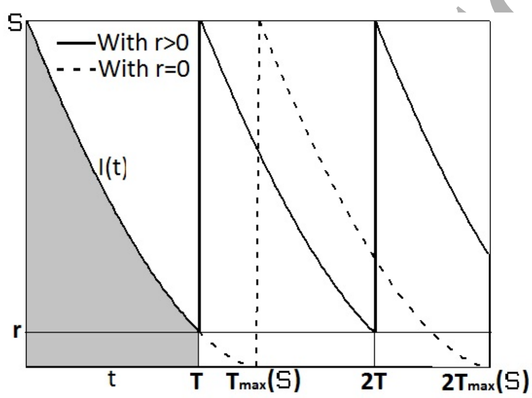

Figure 1. Inventory level curve with $r > 0$ and $r = 0$

The following lemma evaluates the holding cost in an inventory cycle.

Lemma 1 *Let us suppose that the cumulative holding cost for x items during t units of time is given by the function $H(t, x) = ht^{\gamma_1}x^{\gamma_2}$ with $h > 0$, $\gamma_1 \ge 1$ and $\gamma_2 \ge 1$. Then, in an inventory cycle, the holding cost is given by*

$$HC(r, S) = \frac{\gamma_1 h S^\xi}{(\alpha\lambda)^{\gamma_1}}\Upsilon_{beta}\left(1 - \left(\frac{r}{S}\right)^\alpha\right) \tag{6}$$

where $\alpha = 1 - \beta$, $\xi = \alpha\gamma_1 + \gamma_2$ and $\Upsilon_{beta}(z)$, with $z \in [0, 1]$, is the incomplete beta function with parameters γ_1 and $\gamma_2/\alpha + 1$, that is,

$$\Upsilon_{beta}(z) = \int_0^z u^{\gamma_1 - 1}(1 - u)^{\gamma_2/\alpha}du \tag{7}$$

Proof. Please see the proof in the Appendix. $\square$

For each cycle, the total cost is the sum of the ordering cost K, the purchasing cost $pq = p(S - r)$ and the holding cost $HC(r, S)$. Thus, the profit in a cycle is $(v - p)(S - r) - K - HC(r, S)$ and the profitability ratio is given by

$$W(r, S) = \frac{(v - p)(S - r) - K - HC(r, S)}{p(S - r) + K + HC(r, S)} = \frac{v}{p + w(r, S)} - 1 \tag{8}$$

where

$$w(r, S) = \frac{K + \frac{\gamma_1 h S^\xi}{(\alpha\lambda)^{\gamma_1}}\Upsilon_{beta}\left(1 - \left(\frac{r}{S}\right)^\alpha\right)}{S - r} \tag{9}$$

is the average inventory cost per unit of an item (excluding the purchasing cost).

An essential consequence of the expressions (8) and (9) is that the problem of the maximum profitability ratio given by $W(r,S)$ is equivalent to the problem of the minimum cost per unit of an item given by $w(r,S)$. That is, if the purpose of the inventory manager is to maximize the profitability ratio, he/she needs to minimize the average inventory cost per item, instead of minimizing the inventory cost per unit time. Thus, the purpose of this paper is to solve the problem of the maximum profitability ratio, that is,

$$\max_{\substack{S>0 \\ 0\leq r<S}} W(r,S) \tag{10}$$

or, equivalently, to solve the problem of the minimum average inventory cost per unit of an item, that is

$$\min_{\substack{S>0 \\ 0\leq r<S}} w(r,S) \tag{11}$$

Note that, as the cycle time T is given by (3), the inventory cost (excluding the purchasing cost) per unit time is

$$C(r,S) = \frac{K + \frac{\gamma_1 h S^{\xi}}{(\alpha\lambda)^{\gamma_1}} \Upsilon_{beta}\left(1-\left(\frac{r}{S}\right)^{\alpha}\right)}{\frac{S^{\alpha}}{\alpha\lambda}\left(1-\left(\frac{r}{S}\right)^{\alpha}\right)} \tag{12}$$

and the profit per unit time is

$$G(r,S) = \frac{(v-p)(S-r) - K - \frac{\gamma_1 h S^{\xi}}{(\alpha\lambda)^{\gamma_1}} \Upsilon_{beta}\left(1-\left(\frac{r}{S}\right)^{\alpha}\right)}{\frac{S^{\alpha}}{\alpha\lambda}\left(1-\left(\frac{r}{S}\right)^{\alpha}\right)} \tag{13}$$

Furthermore, the profitability ratio per inventory cycle is equal to the profitability ratio per unit time.

If the demand rate is constant (that is, $\beta = 0$ or, equivalently, $\alpha = 1$), then we have $C(r,S) = \lambda w(r,S)$ and $G(r,S) = \lambda(v - p - w(r,S))$. Therefore, the three problems (i) to minimize the inventory cost per unit time $C(r,S)$, (ii) to maximize the profit per unit time $G(r,S)$ and (iii) to maximize the profitability ratio per inventory cycle $W(r,S)$, are equivalent to the problem (11) with $w(r,S)$. But this is not true when the demand rate depends on the inventory level (that is, $0 < \alpha < 1$).

Note that the three approaches can be used, depending on the type of inventory. Non-profit inventory systems may prefer to minimize the inventory cost per unit time because the goal is not to make a profit, but to provide a service to customers. If the objective of the inventory is to obtain the greatest profit and the manager has no other investment alternative, he/she may prefer to maximize the profit per unit time. Finally, if the goal of the inventory is to make a profit while investing the lowest amount of money, and the manager has other investment alternatives (for example, inventories with several items), he/she may prefer the solution with the highest profitability, that is, the maximum profitability ratio solution. The approach used in this paper is just another choice for inventory models with a stock-dependent demand rate.

3 Solution methodology

In this section, the resolution of the problem (10) is addressed. For that, we take into account that:

(i) The function $W(r,S)$ is upper bounded by $v/p - 1$. Even more, $W(r,S) \in (-1, v/p - 1)$, which is derived from $w(r,S) > 0$.

(ii) If $B(a,b) = \int_0^1 u^{a-1}(1-u)^{b-1}du$ is the well-known beta function, for each $y \in [0, B(\gamma_1, \gamma_2/\alpha + 1)]$ there exists the inverse of the incomplete beta function given by Eq. (7), let us say $\Upsilon_{beta}^{-1}(y)$. This follows from the fact that $\Upsilon_{beta}(z)$ is an increasing monotone function on the interval $[0,1]$, and that $\Upsilon_{beta}(1) = B(\gamma_1, \gamma_2/\alpha + 1)$.

Two cases will be considered, depending on the values for the elasticity parameters β, γ_1 and γ_2.

3.1 Case $\beta > \gamma_2/\gamma_1$ (or, equivalently, $\gamma_1 > \xi$)

The following theorem proves that, if $\beta > \gamma_2/\gamma_1$, then the maximum profitability ratio is obtained with infinitely large values for the order-up-to level S, the reorder level r and the lot size $q = S - r$, and its value is equal to the upper bound $v/p - 1$.

Theorem 1 *Suppose that the parameters of the inventory system satisfy the condition $\beta > \gamma_2/\gamma_1$. Let $\alpha = 1 - \beta$, $\xi = \alpha\gamma_1 + \gamma_2$, $\Upsilon_{beta}(z)$ be given by (7) and $\Upsilon_{beta}^{-1}(y)$ be the inverse of the function $\Upsilon_{beta}(z)$, with $y \in [0, \Upsilon_{beta}(1)]$. For any order-up-to level S with $S > \left(\frac{K(\alpha\lambda)^{\gamma_1}}{(\xi-1)\gamma_1 h B(\gamma_1,\gamma_2/\alpha+1)}\right)^{1/\xi}$, let us consider the reorder level*

$$r_S = S\left(1 - z_S\right)^{1/\alpha}$$

where

$$z_S = \Upsilon_{beta}^{-1}\left(\frac{K(\alpha\lambda)^{\gamma_1}}{(\xi-1)\gamma_1 h S^{\xi}}\right) \tag{14}$$

Then, it is satisfied that

$$\lim_{S\to\infty} W(S, r_S) = v/p - 1$$

and, moreover,

$$W(r_S, S) = \frac{v}{p + \frac{\xi K}{(\xi-1)(S-r_S)}} - 1$$

where $S - r_S \approx \lambda\left(\frac{K}{(\xi-1)h}\right)^{1/\gamma_1} S^{1-\xi/\gamma_1}$ with large values for S.

Proof. Please see the proof in the Appendix. □

This result shows that, if the demand elasticity exceeds a certain threshold (namely, $\beta > \gamma_2/\gamma_1$), then the sales in each cycle may be so high that the inventory cost per cycle is finite with an infinitely large lot size. This is the reason why the profitability ratio approaches its upper bound, which is $v/p - 1$.

Then, if there were a real practical situation with the assumption of Theorem 1 and $S_{\max}$ were the maximum capacity of the inventory, with $S_{\max} > \left(\frac{K(\alpha\lambda)^{\gamma_1}}{(\xi-1)\gamma_1 h B(\gamma_1,\gamma_2/\alpha+1)}\right)^{1/\xi}$, we could choose $S = S_{\max}$ and

$$r_{S_{\max}} = S_{\max}\left(1 - \Upsilon_{beta}^{-1}\left(\frac{K(\alpha\lambda)^{\gamma_1}}{(\xi-1)\gamma_1 h S_{\max}^{\xi}}\right)\right)^{1/\alpha} \tag{15}$$

to obtain

$$W(r_{S_{\max}}, S_{\max}) = \frac{v}{p + \frac{K\xi}{(\xi-1)(S_{\max}-r_{S_{\max}})}} - 1 \tag{16}$$

with $W(r_{S_{\max}}, S_{\max}) \approx v/p - 1$ if $S_{\max}$ is sufficiently large.

However, this case is unusual in practice and it cannot happen if we assume that $\lim_{q=S-r\to\infty} w(r, S) > 0$, which leads to $\gamma_1 \le \xi$ or, equivalently, $\beta \le \gamma_2/\gamma_1$.

3.2 Case $\beta \le \gamma_2/\gamma_1$ (or, equivalently, $\gamma_1 \le \xi$)

Note that, in real practical situations, this is usually true because it is common that $\beta \in [0, 0.5]$ and $\gamma_1, \gamma_2 \in [1, 2]$, which leads to $\gamma_2/\gamma_1 \ge 0.5 \ge \beta$.

To solve the model in this case, we consider the auxiliary decision variable

$$\tau = 1 - \left(\frac{r}{S}\right)^{\alpha}$$

so that

$$q = S - r = S\left(1 - (1-\tau)^{1/\alpha}\right)$$

Thus, the average inventory cost per unit of an item $w(r,S)$ is rewritten as

$$\eta(S,\tau)=\frac{K+\frac{\gamma_1 h \Upsilon_{beta}(\tau)}{(\alpha\lambda)^{\gamma_1}}S^{\xi}}{\left(1-(1-\tau)^{1/\alpha}\right)S} \tag{17}$$

with $S\in(0,\infty)$ and $\tau\in(0,1]$.

Then, the optimization problem (11) can be reformulated as

$$\min_{\substack{S>0\\0<\tau\leq 1}} \eta(S,\tau) \tag{18}$$

The function $\eta(S,\tau)$ satisfies the following result.

Lemma 2 *Let $\eta(S,\tau)$ be given by (17) with $S\in(0,\infty)$, $\tau\in(0,1]$, $\alpha=1-\beta$, $\xi=\alpha\gamma_1+\gamma_2$ and $\Upsilon_{beta}(\tau)$ be given by (7). Then, for a fixed $\tau\in(0,1]$, the minimum of the function $\eta(S,\tau)$ is obtained at the point*

$$S^*_{\tau}=\left(\frac{K(\alpha\lambda)^{\gamma_1}}{(\xi-1)\gamma_1 h\Upsilon_{beta}(\tau)}\right)^{1/\xi}$$

with the value

$$\eta(S^*_{\tau},\tau)=\left(\frac{\xi K}{\xi-1}\right)\left(\frac{(\xi-1)\gamma_1 h}{K(\alpha\lambda)^{\gamma_1}}\right)^{1/\xi}(\Phi(\tau))^{1/\xi}$$

where

$$\Phi(\tau)=\frac{\Upsilon_{beta}(\tau)}{\left(1-(1-\tau)^{1/\alpha}\right)^{\xi}} \tag{19}$$

Proof. Please see the proof in the Appendix. □

Taking into account the previous result, the resolution of the problem (18) is reduced to

$$\min_{\tau\in(0,1]}\Phi(\tau)$$

Hence, the following lemma provides the solution of this optimization problem.

Lemma 3 *Suppose that the parameters of the inventory system satisfy that $\beta\leq\gamma_2/\gamma_1$ and consider the function $\Phi(\tau)$ given by (19), where $\tau\in(0,1]$, $\alpha=1-\beta$, $\xi=\alpha\gamma_1+\gamma_2$ and $\Upsilon_{beta}(\tau)$ is given by (7). Then, the minimum of the function $\Phi(\tau)$ is obtained for $\tau^*=1$ with the value $\Phi^*=\Upsilon_{beta}(1)=B(\gamma_1,\gamma_2/\alpha+1)$, where $B(a,b)$ is the known beta function.*

Proof. Please see the proof in the Appendix. □

Using the previous lemma, we can already state our main result in the following theorem:

Theorem 2 *Let us suppose that the parameters of the inventory model satisfy that $\beta\leq\gamma_2/\gamma_1$. Let $\alpha=1-\beta$ and $\xi=\alpha\gamma_1+\gamma_2$. Then, the optimal policy of the inventory system is obtained for $r^*=0$ and*

$$S^*=q^*=\left(\frac{K(\alpha\lambda)^{\gamma_1}}{(\xi-1)\gamma_1 B(\gamma_1,\gamma_2/\alpha+1)h}\right)^{1/\xi} \tag{20}$$

with the maximum profitability ratio

$$W^*=W(0,S^*)=\frac{v}{p+\frac{\xi K}{(\xi-1)q^*}}-1 \tag{21}$$

Proof. Please see the proof in the Appendix. □

As a consequence of this theorem, for real practical situations with $\beta\leq\gamma_2/\gamma_1$, some interesting statements are highlighted below:

1. To obtain the maximum profitability ratio in the inventory system, the replacement should be done when the stock is depleted ($r^* = 0$). Therefore, the inventory manager does not need to process a new order while stock is available, if the goal is to maximize the profitability ratio. This is not true if the objective is to maximize the profit per unit time.

2. The optimal lot size q^* for the maximum profitability ratio is not dependent on the unit purchasing cost p or the unit selling price v. This is an interesting result for the inventory manager, because he/she does not need to change the lot size if the prices are modified. Furthermore, this is also true if the objective is to minimize the inventory cost per unit time, but it is not true if the objective is to maximize the profit per unit time.

3. From (21), the minimum average inventory cost per item is $w^* = \xi K / ((\xi - 1)\, q^*)$, where K/q^* is the optimal average ordering cost and $K/((\xi - 1)\, q^*)$ is the optimal average holding cost. Therefore, a rule for the maximum profitability ratio is
$$K = (\xi - 1)\, HC(0, q^*) \tag{22}$$
that is, the ordering cost is equal to the holding cost multiplied by $\xi - 1$. This rule is not true for the problem with minimum cost per unit time, or for the problem with maximum profit per unit time. Note that, if $\xi > 2$, then $HC(0, q^*) < K$ and the optimal holding cost is lower than the ordering cost (cheap storage) while, if $\xi \in (1, 2)$, then the opposite happens (expensive storage).

4. From (21), the inventory system is profitable (that is, $W^* > 0$) if and only if the condition $v - p > \xi K / ((\xi - 1)\, q^*)$ is satisfied. Then, if we introduce the expression (20) into this inequality, it is possible to know if the inventory system is profitable only from the parameters of the model. This is interesting for the inventory manager because he/she can know what the minimum selling price to reach profitability needs to be.

5. In the basic EOQ model, we have $\alpha = \gamma_1 = \gamma_2 = 1$, $\xi = 2$, and $B(1, 2) = 1/2$, then the expression (20) leads to $q^* = \sqrt{2K\lambda/h}$. Therefore, the expression (20) is a generalization of Wilson's formula. Similarly, the expression (22) is a generalization of Harris' well-known rule: the ordering cost must be equal to the holding cost.

6. As $r^* = 0$, from (3), it is clear that the optimal length of the inventory cycle is $T^* = (q^*)^\alpha / (\alpha\lambda)$.

7. From (12), the inventory cost per unit time for the maximum profitability ratio solution is
$$C(0, S^*) = \frac{w^* q^*}{T^*} = \frac{\alpha \xi \lambda K}{(\xi - 1)\,(q^*)^\alpha}$$

8. From (13), the profit per unit time for the maximum profitability ratio solution is
$$G(0, S^*) = \frac{(v - p)\, q^* - w^* q^*}{T^*} = \frac{\alpha\lambda \left((v - p)\, q^* - \dfrac{\xi K}{\xi - 1} \right)}{(q^*)^\alpha}$$

4 Sensitivity analysis

In this section, a sensitivity analysis of the optimal solution for the model is developed, supposing that $\beta \le \gamma_2/\gamma_1$, which is usually the case in real practical situations. As the solution has been given in a closed form, the analysis is developed by calculating the partial derivatives of the optimal lot size and the maximum profitability ratio with respect to the parameters of the inventory system. The following lemma gives the partial derivatives for the optimal lot size.

Lemma 4 *Let us suppose that the parameters of the inventory system satisfy the condition* $\beta \leq \gamma_2/\gamma_1$ *. Let* $\alpha = 1-\beta$, $\xi = \alpha\gamma_1 + \gamma_2$, $\psi(z)$ *be the digamma function defined by*

$$\psi(z) = \frac{d}{dz}(\ln\Gamma(z)) = \frac{\Gamma'(z)}{\Gamma(z)}, \tag{23}$$

where $\Gamma(z)$ *denotes the gamma function, and* q^* *the optimal lot size for the inventory model given by (20). Then:*

(a) $\dfrac{\partial q^*}{\partial K} = \dfrac{q^*}{\xi K}$

(b) $\dfrac{\partial q^*}{\partial h} = -\dfrac{q^*}{\xi h}$

(c) $\dfrac{\partial q^*}{\partial \lambda} = \dfrac{\gamma_1 q^*}{\xi \lambda}$

(d) $\dfrac{\partial q^*}{\partial \beta} = \left(\dfrac{q^*}{\xi}\right)(\gamma_1 \ln q^* - d_\beta)$, *with* $d_\beta = \dfrac{\gamma_1(\xi - 1 - \alpha)}{\alpha(\xi - 1)} - \dfrac{\gamma_2}{\alpha^2}\left(\psi(\gamma_1 + \gamma_2/\alpha + 1) - \psi(\gamma_2/\alpha + 1)\right)$

(e) $\dfrac{\partial q^*}{\partial \gamma_1} = \left(\dfrac{q^*}{\xi}\right)\left(d_{\gamma_1} - \ln\left(\dfrac{(q^*)^\alpha}{\alpha\lambda}\right)\right)$, *with* $d_{\gamma_1} = \psi(\gamma_1 + \gamma_2/\alpha + 1) - \psi(\gamma_1 + 1) - \dfrac{\alpha}{\xi - 1}$

(f) $\dfrac{\partial q^*}{\partial \gamma_2} = \left(\dfrac{q^*}{\xi}\right)(d_{\gamma_2} - \ln q^*)$, *with* $d_{\gamma_2} = \dfrac{\psi(\gamma_1 + \gamma_2/\alpha + 1) - \psi(\gamma_2/\alpha + 1)}{\alpha} - \dfrac{1}{\xi - 1}$

Proof. Please see the proof in the Appendix. □

As a consequence of the above lemma, the next theorem analyzes the effect of the parameters of the system on the optimal lot size.

Theorem 3 *With the assumptions and notation of Lemma 4, the optimal lot size* q^**:*

(a) increases as the ordering cost K *or the scale parameter of the demand rate* λ *increase.*

(b) decreases as the scale parameter of the holding cost h *increases.*

(c) increases as the elasticity parameter of the demand rate β *increases when* $q^* > \exp(d_\beta/\gamma_1)$*, and decreases otherwise.*

(d) decreases as the elasticity parameter of the holding cost with respect to time γ_1 *increases when the optimal length of the cycle time* T^* *satisfies that* $T^* > \exp\left(d_{\gamma_1}\right)$*, and increases otherwise.*

(e) decreases as the elasticity parameter of the holding cost with respect to the stock level γ_2 *increases when* $q^* > \exp\left(d_{\gamma_2}\right)$*, and increases otherwise.*

Proof. All the assertions follow from Lemma 4 and taking into account that $T^* = (q^*)^\alpha / (\alpha\lambda)$. □

It is interesting to remark that, from Lemma 4, the following equality is deduced

$$\gamma_1\left(\frac{K\dfrac{\partial q^*}{\partial K}}{q^*}\right) = -\gamma_1\left(\frac{h\dfrac{\partial q^*}{\partial h}}{q^*}\right) = \left(\frac{\lambda\dfrac{\partial q^*}{\partial \lambda}}{q^*}\right) \tag{24}$$

which means that the relative change in q^* with respect to the relative change in K or h are equal, but with the opposite sign, that is, from a relative point of view, a small relative increase in K causes the same effect on the optimal lot size q^* as the same relative decrease in h. On the other hand, for the scale parameter of the demand rate λ, if $\gamma_1 > 1$, from a relative point of view, the optimal lot size is more sensitive to changes in λ than in K or h. If there is no elasticity of the holding cost with respect to time ($\gamma_1 = 1$), the three parameters have the same relative effect on the optimal lot size, except for the sign.

In a similar way, the next lemma gives the partial derivatives for the maximum profitability ratio.

Lemma 5 *Let us suppose that the parameters of the inventory system satisfy the condition $\beta \leq \gamma_2/\gamma_1$. Let $\alpha = 1-\beta$, $\xi = \alpha\gamma_1 + \gamma_2$, the digamma function $\psi(z)$ be given by (23), q^* be the optimal lot size for the inventory model given by (20) and W^* the maximum profitability ratio for the inventory model given by (21). Then:*

(a) $\dfrac{\partial W^*}{\partial K} = -\dfrac{(1+W^*)^2}{vq^*}$

(b) $\dfrac{\partial W^*}{\partial h} = -\left(\dfrac{(1+W^*)^2 K}{v(\xi-1)q^*}\right)\left(\dfrac{1}{h}\right)$

(c) $\dfrac{\partial W^*}{\partial \lambda} = \left(\dfrac{(1+W^*)^2 K}{v(\xi-1)q^*}\right)\left(\dfrac{\gamma_1}{\lambda}\right)$

(d) $\dfrac{\partial W^*}{\partial \beta} = \left(\dfrac{(1+W^*)^2 K}{v(\xi-1)q^*}\right)(\gamma_1 \ln q^* - e_\beta)$, *with* $e_\beta = \dfrac{\gamma_1}{\alpha} - \dfrac{\gamma_2}{\alpha^2}(\psi(\gamma_1+\gamma_2/\alpha+1) - \psi(\gamma_2/\alpha+1))$

(e) $\dfrac{\partial W^*}{\partial \gamma_1} = \left(\dfrac{(1+W^*)^2 K}{v(\xi-1)q^*}\right)\left(e_{\gamma_1} - \ln\left(\dfrac{(q^*)^\alpha}{\alpha\lambda}\right)\right)$, *with* $e_{\gamma_1} = \psi(\gamma_1+\gamma_2/\alpha+1) - \psi(\gamma_1+1)$

(f) $\dfrac{\partial W^*}{\partial \gamma_2} = \left(\dfrac{(1+W^*)^2 K}{v(\xi-1)q^*}\right)(e_{\gamma_2} - \ln q^*)$, *with* $e_{\gamma_2} = \left(\dfrac{1}{\alpha}\right)(\psi(\gamma_1+\gamma_2/\alpha+1) - \psi(\gamma_2/\alpha+1))$

(g) $\dfrac{\partial W^*}{\partial v} = \dfrac{1+W^*}{v}$

(h) $\dfrac{\partial W^*}{\partial p} = -\dfrac{(1+W^*)^2}{v}$

Proof. Please see the proof in the Appendix. □

As a consequence of this lemma, the next theorem analyzes the effect of the parameters of the model on the maximum profitability ratio of the inventory system.

Theorem 4 *With the assumptions and notation of Lemma 5, the maximum profitability ratio of the inventory system W^*:*

(a) increases as the unit selling price v or the scale parameter of the demand rate λ increase.

(b) decreases as the unit purchasing cost p, the ordering cost K, or the scale parameter of the holding cost h, increase.

(c) increases as the elasticity parameter of the demand rate β increases when $q^* > \exp(e_\beta/\gamma_1)$, and decreases otherwise.

(d) decreases as the elasticity parameter of the holding cost with respect to time γ_1 increases when the optimal length of the cycle time T^* satisfies that $T^* > \exp(e_{\gamma_1})$, and increases otherwise.

(e) *decreases as the elasticity parameter of the holding cost with respect to the stock level* γ_2 *increases when* $q^* > \exp\left(e_{\gamma_2}\right)$, *and increases otherwise.*

Proof. All the assertions follow from Lemma 5 and taking into account that $T^* = (q^*)^\alpha / (\alpha\lambda)$. $\square$

Note that, from Lemma 5, the following equality is obtained:

$$-\left(\frac{1}{\xi-1}\right)\left(K\frac{\frac{\partial W^*}{\partial K}}{W^*}\right) = -\left(h\frac{\frac{\partial W^*}{\partial h}}{W^*}\right) = \left(\frac{1}{\gamma_1}\right)\left(\lambda\frac{\frac{\partial W^*}{\partial \lambda}}{W^*}\right) \tag{25}$$

Therefore, from a relative point of view, if $\gamma_1 > 1$, the maximum profitability ratio W^* is more sensitive to relative changes in λ than in h, but with the opposite sign. In a similar way, from a relative point of view, if $\xi < 1+\gamma_1$, the maximum profitability ratio W^* is more sensitive to relative changes in λ than in K, but with the opposite sign; while it is more sensitive to relative changes in K than in λ, but with the opposite sign, if $\xi > 1+\gamma_1$. Finally, from a relative point of view, if $\xi > 2$, the maximum profitability ratio W^* is more sensitive to relative changes in K than in h, and the opposite happens if $\xi < 2$. In the basic EOQ model, as $\alpha = \gamma_1 = \gamma_2 = 1$ and $\xi = 2$, the three parameters have the same relative effect, except for the sign.

For the unit purchasing cost and the unit selling price, from Lemma 5, we have

$$\left(v\frac{\frac{\partial W^*}{\partial v}}{W^*}\right) = -\left(\frac{v}{p\,(1+W^*)}\right)\left(p\frac{\frac{\partial W^*}{\partial p}}{W^*}\right) \tag{26}$$

Taking into account that $v/\left(p\,(1+W^*)\right) > 1$, because $W^* < v/p - 1$ then, from a relative point of view, the maximum profitability ratio W^* is more sensitive to relative changes in v than in p, but with the opposite sign.

5 Computational results

This section presents numerical examples to illustrate the proposed model and its solution procedure, including a sensitivity analysis of the optimal solution with respect to all the parameters of the inventory system. Assume that the ordering cost is $K = 1000$, the unit purchasing cost is $p = 40$ and the unit selling price is $v = 60$. The cumulative holding cost for x items held in stock t units of time is given by the function $H(t,x) = 25t^{1.5}x^{1.2}$, so the scale parameter is $h = 25$ and the elasticity parameters of the holding cost are $\gamma_1 = 1.5$ and $\gamma_2 = 1.2$. Finally, suppose that the scale parameter of the demand rate is $\lambda = 500$ and the imbalance in sales is 60% in the first half of a cycle and 40% in the second, that is, the proportion of sales in the second half of a cycle is $\rho = 0.4$. Then, the elasticity parameter of the demand rate with respect to the stock level can be estimated as $\beta = 1 + \ln 2/\ln\rho = 0.24$.

With these assumptions, the auxiliary parameters of the model are $\alpha = 0.76$ and $\xi = 2.34$, the condition $\beta \le \gamma_2/\gamma_1$ is satisfied and $B\,(\gamma_1, \gamma_2/\alpha + 1) = B\,(1.5, 2.58) = 0.1881$. By Theorem 2, the optimal reorder level is $r^* = 0$, the optimal order-up-to level is equal to the optimal lot size, and they can be evaluated by using the expression (20), which leads to $S^* = q^* = 330.28$. In a similar way, the maximum profitability ratio of the model can be obtained by the expression (21) in Theorem 2, which leads to $W^* = 0.325$, that is, the profitability in the inventory system is 32.5%.

For this optimal solution, the holding cost in an inventory cycle is $HC(0, q^*) = K/(\xi - 1) = 746.27$ and the minimum inventory cost per unit of an item is $w^* = w(0, q^*) = \xi K/((\xi - 1) q^*) = 5.29$, where $K/q^* = 3.03$ is the ordering cost and $K/((\xi - 1) q^*) = 2.26$ is the holding cost. The optimal length of the inventory cycle, given by (3), is $T^* = (q^*)^{\alpha} / (\alpha\lambda) = 0.216$, the inventory cost per unit time is $C(0, q^*) = w^* q^* / T^* = 8082.41$ and the profit per unit time is $G(0, q^*) = (v - p - w^*) q^* / T^* = 22491.17$.

For this inventory model, we have also solved the problem of minimum inventory cost per unit time, by minimizing the function $C(r, S)$ given by (12). The solution is $C^* = C(0, 273.76) = 7905.66$, which is less than $C(0, q^*) = 8082.41$, as is to be expected. In a similar way, we have solved the problem of the maximum profit per unit time, maximizing the function $G(r, S)$ given by (13), and the solution is $G^* = G(70.89, 482.35) = 24875.52$, which is higher than $G(0, q^*) = 22491.17$, as is to be expected. All relevant quantities for the three solutions are given in Table 2, showing that they are all different. Moreover, the zero ending policy for the inventory cycle is optimal for the maximum profitability ratio, but this is not true for the maximum profit per unit time. It does not happen for inventory models with constant demand rate, where the three optimal solutions are equivalent. Moreover, in this example, the maximum profitability ratio solution is closer to the minimum cost per unit time solution than to the maximum profit per unit time solution.

Table 2. Relevant quantities for the three problems

	Minimum cost per unit time	Maximum profit per unit time	Maximum profitability ratio per unit time
Order-up-to level S	273.76	482.35	330.28
Reorder level r	0	70.89	0
Lot size q	273.76	411.46	330.28
Cycle time T	0.187	0.221	0.216
$C(r, S)$	*7905.66*	12355.90	8082.41
$G(r, S)$	21321.29	*24875.52*	22491.17
$w(r, S)$	5.41	6.64	**5.29**
$W(r, S)$	32.1%	28.7%	**32.5**%

In order to evaluate the sensitivity of the optimal solution with respect to the parameters of the model, we have computed the auxiliary constants d_β, d_{γ_1}, d_{γ_2}, e_β, e_{γ_1} and e_{γ_2} given in Lemmas 4 and 5, and the obtained values are included in Table 3.

Table 3. Auxiliary constants for the sensitivity analysis

d_β	d_{γ_1}	d_{γ_2}	e_β	e_{γ_1}	e_{γ_2}
−0.2616	0.0080	−0.0395	0.8578	0.5751	0.7068

Using these values and the partial derivatives given in Lemmas 4 and 5, we can evaluate the instantaneous rate of change of the optimal lot size and the maximum profitability ratio with respect to the parameters of the model, both in absolute and relative values. The results are included in Table 4.

Table 4. Sensitivity analysis for the optimal lot size and the maximum profitability ratio

	v	p	K	h	λ	β	γ_1	γ_2
Absolute change in q^*	0	0	0.14	-5.65	0.42	1264.9	217.4	-824.2
Relative change in q^*	0	0	0.43	-0.43	0.64	0.92	0.99	-2.99
Absolute change in W^*	0.022	-0.029	-0.00009	-0.0026	0.0002	0.52	0.14	-0.34
Relative change in W^*	4.08	-3.60	-0.27	-0.20	0.31	0.38	0.64	-1.24

All the statements given in Theorems 3 and 4 are supported: the optimal lot size increases with K and λ, and it decreases with h; the maximum profitability ratio decreases with K and h, and it increases with λ. For the elasticity parameters of the inventory system, in this example, both the lot size and the maximum profitability ratio increase with β and γ_1, and they decrease with γ_2 (note that, in this case, $q^* > \exp(e_\beta/\gamma_1) > \exp(d_\beta/\gamma_1)$, $T^* < \exp(d_{\gamma_1}) < \exp(e_{\gamma_1})$ and $q^* > \exp(e_{\gamma_2}) > \exp(d_{\gamma_2})$). The expressions (24) and (25) for the scale parameters of the inventory model (K, h and λ) are also satisfied. Moreover, in this example, the optimal lot size and the maximum profitability ratio are more sensitive to changes in the elasticity parameters (β, γ_1 and γ_2) than in the scale parameters (K, h and λ), especially highlighting the effect of the γ_2 parameter on the optimal solution.

Finally, a small decrease in the purchasing cost p leads to a greater increase in the maximum profitability ratio than a rise in the selling price v of the same size. Nevertheless, from a relative point of view, the effect of the selling price v is greater than the effect of the purchasing cost p, as is derived from Eq. (26). It is also worth noting that the maximum profitability ratio is much more sensitive to changes in the selling price or the purchasing cost than to the other parameters of the inventory system.

In order to confirm all the theoretical results given in Section 4, we have solved the inventory model assuming changes of between -50% and $+50\%$ in each of the parameters, while keeping all the others fixed. For each system, the optimal lot size q^* and the maximum profitability ratio W^* were obtained by using Theorem 2, and the percentual changes with respect to the initial solution were evaluated. The results are plotted in Figures 2 and 3, which support all the statements discussed above.

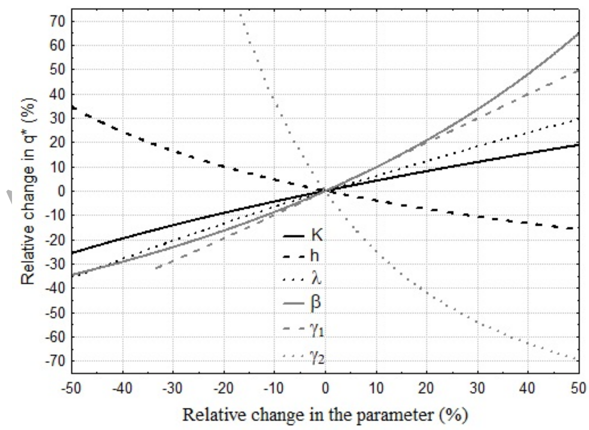

Figure 2. Relative changes in q^*

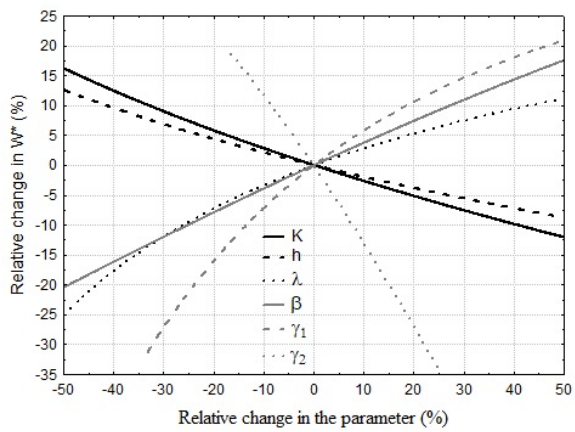

Figure 3. Relative changes in W^*

To finish the computational results, the elasticity parameters of the model were changed to $\beta = 0.5$, $\gamma_1 = 3.3$ and $\gamma_2 = 1.1$, keeping all the others fixed. Now, it is clear that $\beta > \gamma_2/\gamma_1$ and Theorem

1 is relevant. In this case, the expressions (20) and (21) in Theorem 2 lead to $S = q = 6049.4$ with $W(0, 6049.4) = 0.490$, which are not the optimal solutions for the inventory problem. Indeed, supposing that the maximum possible lot size is $S_{\max} = 40000$, we can use Theorem 1 to obtain $r_S = 32439.1$, $q = S_{\max} - r_S = 7560.9$ and $W(r_S, S_{\max}) = 0.492$, which is greater than the value given by Theorem 2. Moreover, if $S_{\max}$ converges to the infinite, we obtain solutions with $W \approx 0.5$, which is the upper bound for the profitability ratio in this inventory model. However, real practical situations like this are scarce.

6 Conclusions and future research

This paper analyzes a deterministic (r, S)-type inventory system with two decision variables, the reorder level r (inventory level when a new order is carried out) and the order-up-to level S (quantity to order up to). The system considers a stock-dependent demand rate and a cumulative holding cost which is non-linear in both time and stock level. The goal is the profitability ratio maximization, instead of the minimization of the inventory cost per unit time or the maximization of the profit per unit time. Usually, the minimum cost solution provides a low profit, and the maximum profit solution has a very large cost. This is an expected issue because, in those cases, the model only takes into account one of the two targets. Then, the main contribution of this paper is to obtain the maximum profitability ratio solution, which provides an optimal balance between the profit and the cost. In economic theory, when monetary resources are limited and different investment alternatives are possible, the best option is usually the solution with the maximum profitability. If the final objective is the sale of the item, the inventory manager will always be interested in obtaining the maximum return on the money.

It is proved that the profitability ratio maximization is equivalent to the minimization of the inventory cost per unit of an item. Two cases can occur depending on the parameters of the system. The first one leads to the maximum possible profitability ratio for the values of the unit purchasing cost and the selling price. It is obtained with infinitely large values of the order-up-to level, the reorder level, and the lot size. The second one, which is usually the case in real practical situations, leads to a zero stock condition at the end of the cycle time (that is, $r = 0$) and, therefore, the order-up-to level S is equal to the lot size q.

In this case, a closed expression for the optimal lot size is obtained, which is a generalization of the Harris-Wilson's formula for inventory systems with a stock-dependent demand rate and a holding cost non-linear with respect to both time and stock level. This solution is different from the others with minimum inventory cost per unit time or maximum profit per unit time and, moreover, it is not dependent on the unit purchasing cost or the unit selling price. The optimal policy can be identified by a rule that relates the ordering cost and the holding cost, which is also a generalization of Harris' well-known rule for inventory models with a stock-dependent demand rate and a non-linear holding cost in both time and stock level.

The sensitivity analysis for the maximum profitability ratio with respect to the parameters of the model shows that it increases when the unit selling price v or the scale parameter of the demand rate λ increase; whereas it decreases when the cost parameters p, K or h increase. Moreover, from a relative point of view, the maximum profitability ratio is more sensitive to changes in the selling price v than in the purchasing cost p, and more sensitive to changes in the scale parameter of the demand rate λ than in the scale parameter of the holding cost h. The relative sensitivity with respect to the ordering cost K can be greater or lower than the relative sensitivity with respect to the parameters λ or h, depending on the elasticity parameters of the model. Finally, the maximum profitability ratio can increase or decrease when the elasticity parameter of the demand rate or the elasticity parameters of the holding cost increase, depending on the value of the optimal lot size or the optimal length of the cycle time.

For the optimal lot size, the sensitivity analysis shows that it increases as the ordering cost K or the scale parameter of the demand rate λ increase; whereas it decreases when the scale parameter of the holding cost h increases. The relative sensitivity with respect to the parameters K or h are equal but with the opposite sign. With respect to the scale parameter of the demand rate λ, the relative sensitivity can be greater or lower than the one obtained by varying the parameters K or h, depending on the elasticity parameter γ_1 of the holding cost with respect to time. Also now, the optimal lot size can increase or decrease when the elasticity parameter of the demand rate or the elasticity parameters of the holding cost increase, depending on the value of the optimal lot size or the optimal length of the cycle time.

The proposed model can be extended in several ways. For example, it would seem interesting to consider deteriorating items, or to incorporate discounts in the purchasing cost, or to study the model with a demand rate which depends on the selling price and the stock level. Also, other mathematical functions for the cumulative holding cost could be considered.

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Appendix

In this appendix, the proofs for all the results obtained in this paper are included.

Proof of Lemma 1.

If $H(t,x) = ht^{\gamma_1}x^{\gamma_2}$ is the cumulative holding cost, then the holding cost rate per unit and per unit time is given by

$$\frac{\partial^2}{\partial t \partial x} H(t,x) = h\gamma_1\gamma_2 t^{\gamma_1-1}x^{\gamma_2-1}$$

As a consequence, if T is the length of an inventory cycle and $I(t)$ is given by (5), the holding cost for the region of the (t,x)-plane under the inventory level curve $x = I(t)$ with $0 \leq t \leq T$ (which is shaded in Figure 1) can be evaluated as

$$\begin{aligned} HC(r,S) &= \int_0^T \int_0^{I(t)} h\gamma_1\gamma_2 t^{\gamma_1-1}x^{\gamma_2-1} dx\, dt = \int_0^T h\gamma_1 t^{\gamma_1-1} (I(t))^{\gamma_2} dt \\ &= h\gamma_1 S^{\gamma_2} (T_{\max}(S))^{\gamma_1} \int_0^T \left(\frac{t}{T_{\max}(S)}\right)^{\gamma_1-1} \left(1 - \frac{t}{T_{\max}(S)}\right)^{\gamma_2/\alpha} \frac{dt}{T_{\max}(S)} \\ &= h\gamma_1 S^{\gamma_2} (T_{\max}(S))^{\gamma_1} \int_0^{\frac{T}{T_{\max}(S)}} u^{\gamma_1-1} (1-u)^{\gamma_2/\alpha} du = h\gamma_1 S^{\gamma_2} (T_{\max}(S))^{\gamma_1} \Upsilon_{beta}\left(\frac{T}{T_{\max}(S)}\right) \end{aligned}$$

Finally, taking into account the expressions (3) and (4) for T and $T_{\max}(S)$, we obtain

$$HC(r,S) = \frac{h\gamma_1}{(\alpha\lambda)^{\gamma_1}} S^{\alpha\gamma_1+\gamma_2}\Upsilon_{beta}\left(1-\left(\frac{r}{S}\right)^{\alpha}\right) = \frac{\gamma_1 h S^{\xi}}{(\alpha\lambda)^{\gamma_1}}\Upsilon_{beta}\left(1-\left(\frac{r}{S}\right)^{\alpha}\right)$$

with $\xi = \alpha\gamma_1 + \gamma_2$. $\square$

Proof of Theorem 1.

From the condition $S > \left(\frac{K(\alpha\lambda)^{\gamma_1}}{(\xi-1)\gamma_1 h B(\gamma_1,\gamma_2/\alpha+1)}\right)^{1/\xi}$, it is clear that $z_S \in (0,1)$ and $r_S \in (0,S)$. Using Lemma 1, we obtain

$$HC(r_S,S) = \frac{\gamma_1 h S^{\xi}}{(\alpha\lambda)^{\gamma_1}}\Upsilon_{beta}\left(\Upsilon_{beta}^{-1}\left(\frac{K(\alpha\lambda)^{\gamma_1}}{(\xi-1)\gamma_1 h S^{\xi}}\right)\right) = \frac{K}{\xi-1}$$

Then, taking into account that $W(r_S,S) = \dfrac{v}{p+w(r_S,S)} - 1$ and

$$w(r_S,S) = \frac{K + HC(r_S,S)}{S-r_S} = \frac{\xi K}{(\xi-1)(S-r_S)}$$

it is clear that

$$W(r_S,S) = \frac{v}{p+\frac{\xi K}{(\xi-1)(S-r_S)}} - 1$$

Moreover, it is satisfied that

$$\begin{aligned}\frac{S-r_S}{S^{1-\xi/\gamma_1}} &= \frac{1-(1-z_S)^{1/\alpha}}{S^{-\xi/\gamma_1}} = \left(\frac{1-(1-z_S)^{1/\alpha}}{z_S}\right)\left(\frac{z_S^{\gamma_1}}{\Upsilon_{beta}(z_S)}\right)^{1/\gamma_1}\left(S^{\xi}\Upsilon_{beta}(z_S)\right)^{1/\gamma_1}\\ &= \alpha\lambda\left(\frac{K}{(\xi-1)\gamma_1 h}\right)^{1/\gamma_1}\left(\frac{1-(1-z_S)^{1/\alpha}}{z_S}\right)\left(\frac{z_S^{\gamma_1}}{\Upsilon_{beta}(z_S)}\right)^{1/\gamma_1}\end{aligned}$$

Now, taking into account that $z_S \downarrow 0$ if $S \to \infty$ and $\Upsilon'_{beta}(z_S) = z_S^{\gamma_1-1}(1-z_S)^{\gamma_2/\alpha}$, it follows that $\lim_{z_s\downarrow 0}\left(\frac{z_S^{\gamma_1}}{\Upsilon_{beta}(z_S)}\right) = \lim_{z_s\downarrow 0}\left(\frac{\gamma_1 z_S^{\gamma_1-1}}{z_S^{\gamma_1-1}(1-z_S)^{\gamma_2/\alpha}}\right) = \gamma_1$ and $\lim_{z_s\downarrow 0}\frac{1-(1-z_S)^{1/\alpha}}{z_S} = 1/\alpha$. As a consequence,

$$\lim_{S\to\infty}\frac{S-r_S}{S^{1-\xi/\gamma_1}} = \alpha\lambda\left(\frac{K}{(\xi-1)\gamma_1 h}\right)^{1/\gamma_1}\lim_{z_s\downarrow 0}\left(\left(\frac{1-(1-z_S)^{1/\alpha}}{z_S}\right)\left(\frac{z_S^{\gamma_1}}{\Upsilon_{beta}(z_S)}\right)^{1/\gamma_1}\right) = \lambda\left(\frac{K}{(\xi-1)h}\right)^{1/\gamma_1}$$

and $S - r_S \approx \lambda\left(\frac{K}{(\xi-1)h}\right)^{1/\gamma_1} S^{1-\xi/\gamma_1}$ with large values for S.

Finally, as $\gamma_1 > \xi$ by the condition $\beta > \gamma_2/\gamma_1$, we have $\lim_{S\to\infty}(S-r_S) = \infty$ and

$$\lim_{S\to\infty} W(r_S,S) = \lim_{S\to\infty}\left(\frac{v}{p+\frac{\xi K}{(\xi-1)(S-r_S)}} - 1\right) = v/p - 1$$

This result ends the proof. $\square$

Proof of Lemma 2.

The partial derivatives of the function $\eta(S,\tau)$ with respect to S are:

$$\frac{\partial\eta(S,\tau)}{\partial S} = \frac{-(\alpha\lambda)^{\gamma_1}K + (\xi-1)\gamma_1 h\Upsilon_{beta}(\tau)S^{\xi}}{(\alpha\lambda)^{\gamma_1}\left(1-(1-\tau)^{1/\alpha}\right)S^2}$$

$$\begin{aligned}\frac{\partial^2\eta(S,\tau)}{\partial S^2} &= \frac{\xi(\xi-1)\gamma_1 h\Upsilon_{beta}(\tau)S^{\xi} + 2(\alpha\lambda)^{\gamma_1}K - 2(\xi-1)\gamma_1 h\Upsilon_{beta}(\tau)S^{\xi}}{(\alpha\lambda)^{\gamma_1}\left(1-(1-\tau)^{1/\alpha}\right)S^3}\\ &= \frac{\xi(\xi-1)\gamma_1 h\Upsilon_{beta}(\tau)S^{\xi-3}}{(\alpha\lambda)^{\gamma_1}\left(1-(1-\tau)^{1/\alpha}\right)} - \frac{2}{S}\frac{\partial\eta(S,\tau)}{\partial S}\end{aligned}$$

First of all, as $\xi > 1$, it is clear that, for any fixed $\tau \in (0,1]$, it is satisfied that

$$\lim_{S\downarrow 0} \eta(S,\tau) = \lim_{S\to\infty} \eta(S,\tau) = +\infty$$

Secondly, it is clear that the unique root of the equation $\dfrac{\partial \eta(S,\tau)}{\partial S} = 0$ is

$$S_\tau^* = \left(\frac{K(\alpha\lambda)^{\gamma_1}}{(\xi-1)\gamma_1 h \Upsilon_{beta}(\tau)}\right)^{1/\xi}$$

and, thirdly, it is clear that $\dfrac{\partial^2 \eta(S_\tau^*,\tau)}{\partial S^2} > 0$.

Therefore, from these three statements, we conclude that the minimum of the function $\eta(S,\tau)$ for a fixed τ is obtained at the point S_τ^* with value

$$\begin{aligned}
\eta(S_\tau^*,\tau) &= \frac{K + \frac{K}{\xi-1}}{\left(1-(1-\tau)^{1/\alpha}\right)\left(\frac{K(\alpha\lambda)^{\gamma_1}}{(\xi-1)\gamma_1 h \Upsilon_{beta}(\tau)}\right)^{1/\xi}} = \frac{\left(\dfrac{\xi K}{\xi-1}\right)(\Upsilon_{beta}(\tau))^{1/\xi}}{\left(\frac{K(\alpha\lambda)^{\gamma_1}}{(\xi-1)\gamma_1 h}\right)^{1/\xi}\left(1-(1-\tau)^{1/\alpha}\right)} \\
&= \left(\frac{\xi K}{\xi-1}\right)\left(\frac{(\xi-1)\gamma_1 h}{K(\alpha\lambda)^{\gamma_1}}\right)^{1/\xi}(\Phi(\tau))^{1/\xi} \qquad \square
\end{aligned}$$

Proof of Lemma 3.

The derivative of the function $\Phi(\tau)$ is

$$\Phi'(\tau) = \frac{\left(1-(1-\tau)^{1/\alpha}\right)\Upsilon'_{beta}(\tau) - \frac{\xi}{\alpha}(1-\tau)^{1/\alpha-1}\Upsilon_{beta}(\tau)}{\left(1-(1-\tau)^{1/\alpha}\right)^{\xi+1}} = \left(\frac{\Upsilon'_{beta}(\tau)}{\Upsilon_{beta}(\tau)} - \frac{\frac{\xi}{\alpha}(1-\tau)^{1/\alpha-1}}{1-(1-\tau)^{1/\alpha}}\right)\Phi(\tau)$$

Thus, we have

$$\frac{\Phi'(\tau)}{\Phi(\tau)} = \frac{\Upsilon'_{beta}(\tau)}{\Upsilon_{beta}(\tau)} - \xi\frac{\frac{1}{\alpha}(1-\tau)^{1/\alpha-1}}{1-(1-\tau)^{1/\alpha}} \tag{27}$$

Taking into account that, if $\tau \in (0,1)$,

$$\Upsilon''_{beta}(\tau) = \left(\frac{\gamma_1-1}{\tau} - \frac{\gamma_2/\alpha}{1-\tau}\right)\Upsilon'_{beta}(\tau)$$

and, by derivating again the two members of Eq. (27), we obtain

$$\begin{aligned}
\frac{\Phi''(\tau)}{\Phi(\tau)} - \left(\frac{\Phi'(\tau)}{\Phi(\tau)}\right)^2 &= \left(\frac{\gamma_1-1}{\tau} - \frac{\gamma_2/\alpha}{1-\tau}\right)\left(\frac{\Upsilon'_{beta}(\tau)}{\Upsilon_{beta}(\tau)}\right) - \left(\frac{\Upsilon'_{beta}(\tau)}{\Upsilon_{beta}(\tau)}\right)^2 \\
&\quad + \xi\left(\frac{\frac{1}{\alpha}-1}{1-\tau}\right)\left(\frac{\frac{1}{\alpha}(1-\tau)^{1/\alpha-1}}{1-(1-\tau)^{1/\alpha}}\right) + \xi\left(\frac{\frac{1}{\alpha}(1-\tau)^{1/\alpha-1}}{1-(1-\tau)^{1/\alpha}}\right)^2
\end{aligned}$$

If $\Phi'(\tau) = 0$, from (27), we have $\dfrac{\Upsilon'_{beta}(\tau)}{\Upsilon_{beta}(\tau)} = \xi\dfrac{\frac{1}{\alpha}(1-\tau)^{1/\alpha-1}}{1-(1-\tau)^{1/\alpha}}$, and therefore

$$\begin{aligned}
\frac{\Phi''(\tau)}{\Phi(\tau)} &= \left(\frac{\gamma_1-1}{\tau} - \frac{\frac{\gamma_2-1}{\alpha}+1}{1-\tau}\right)\xi\left(\frac{\frac{1}{\alpha}(1-\tau)^{1/\alpha-1}}{1-(1-\tau)^{1/\alpha}}\right) - \xi(\xi-1)\left(\frac{\frac{1}{\alpha}(1-\tau)^{1/\alpha-1}}{1-(1-\tau)^{1/\alpha}}\right)^2 \\
&= \left(\frac{\gamma_1 - 1 - \left(\frac{\xi-1}{\alpha}\right)\tau}{\tau(1-\tau)} - \frac{\frac{\xi-1}{\alpha}(1-\tau)^{1/\alpha-1}}{1-(1-\tau)^{1/\alpha}}\right)\xi\left(\frac{\frac{1}{\alpha}(1-\tau)^{1/\alpha-1}}{1-(1-\tau)^{1/\alpha}}\right) \\
&= \left(\frac{\gamma_1-1}{\tau(1-\tau)} - \frac{\frac{\xi-1}{\alpha}}{(1-\tau)\left(1-(1-\tau)^{1/\alpha}\right)}\right)\xi\left(\frac{\frac{1}{\alpha}(1-\tau)^{1/\alpha-1}}{1-(1-\tau)^{1/\alpha}}\right) \\
&= \left(\frac{\gamma_1-1}{\tau} - \frac{\frac{\xi-1}{\alpha}}{1-(1-\tau)^{1/\alpha}}\right)\left(\frac{\frac{\xi}{\alpha}(1-\tau)^{1/\alpha-2}}{1-(1-\tau)^{1/\alpha}}\right) = \varphi(\tau)\left(\frac{\xi(1-\tau)^{1/\alpha-2}}{\alpha\tau\left(1-(1-\tau)^{1/\alpha}\right)^2}\right)
\end{aligned}$$

where

$$\varphi(\tau) = (\gamma_1 - 1)\left(1 - (1-\tau)^{1/\alpha}\right) - \frac{\xi - 1}{\alpha}\tau$$

In addition, as $\beta \le \gamma_2/\gamma_1$ is equivalent to $\gamma_1 \le \xi$, for $\tau \in (0,1)$ we have

$$\varphi'(\tau) = \left(\frac{\gamma_1 - 1}{\alpha}\right)(1-\tau)^{1/\alpha - 1} - \frac{\xi - 1}{\alpha} < \frac{\gamma_1 - 1}{\alpha} - \frac{\xi - 1}{\alpha} = \frac{\gamma_1 - \xi}{\alpha} \le 0$$

This ensures that the function $\varphi(\tau)$ is a strictly decreasing function on the interval $(0,1)$, which leads to $\varphi(\tau) < \lim_{\tau \downarrow 0} \varphi(\tau) = 0$ for $\tau \in (0,1)$. Therefore, for any $\tau \in (0,1)$ with $\Phi'(\tau) = 0$, we have

$$\Phi''(\tau) = \left(\frac{\xi(1-\tau)^{1/\alpha - 2}}{\alpha\tau\left(1 - (1-\tau)^{1/\alpha}\right)^2}\right)\Phi(\tau)\varphi(\tau) < 0$$

and the function $\Phi(\tau)$ has no local minimums on the interval $(0,1)$.

As $\Phi(1) = B(\gamma_1, \gamma_2/\alpha + 1)$, we only need to prove that $\lim_{\tau\downarrow 0} \Phi(\tau) > B(\gamma_1, \gamma_2/\alpha + 1)$ as follows:

$$\lim_{\tau\downarrow 0}\Phi(\tau) = \lim_{\tau\downarrow 0}\left(\left(\frac{\Upsilon_{beta}(\tau)}{\tau^{\xi}}\right)\left(\frac{\tau}{1-(1-\tau)^{1/\alpha}}\right)^{\xi}\right) = \left(\lim_{\tau\downarrow 0}\frac{\Upsilon'_{beta}(\tau)}{\xi\tau^{\xi-1}}\right)\left(\lim_{\tau\downarrow 0}\frac{\alpha}{(1-\tau)^{1/\alpha - 1}}\right)^{\xi}$$

$$= \left(\frac{\alpha^{\xi}}{\xi}\right)\lim_{\tau\downarrow 0}\left(\tau^{\gamma_1 - \xi}(1-\tau)^{\gamma_2/\alpha}\right) = \left(\frac{\alpha^{\xi}}{\xi}\right)\lim_{\tau\downarrow 0}\tau^{\gamma_1 - \xi} = \begin{cases} \frac{\alpha^{\xi}}{\xi} & \text{if} \quad \gamma_1 = \xi \text{ (that is, } \beta = \gamma_2/\gamma_1) \\ \infty & \text{if} \quad \gamma_1 < \xi \text{ (that is, } \beta < \gamma_2/\gamma_1) \end{cases}$$

Then, it is sufficient to check that, if $\gamma_1 = \xi$, then

$$\frac{\alpha^{\xi}}{\xi} > B(\gamma_1, \gamma_2/\alpha + 1)$$

However, as $\gamma_1 = \xi$ and $\xi = \alpha\gamma_1 + \gamma_2$, we have $\gamma_1 > \gamma_2$, $\gamma_1 > 1$, $\alpha = 1 - \gamma_2/\gamma_1 < 1$, $\gamma_2/\alpha = \gamma_1/\alpha - \gamma_1$, and

$$\frac{\alpha^{\xi}}{\xi B(\gamma_1, \gamma_2/\alpha + 1)} = \frac{\alpha^{\gamma_1}}{\gamma_1 B(\gamma_1, \gamma_1/\alpha - \gamma_1 + 1)}$$

Then, it remains to prove that

$$\frac{\alpha^{\gamma_1}}{\gamma_1 B(\gamma_1, \gamma_1/\alpha - \gamma_1 + 1)} > 1$$

or, equivalently,

$$\ln\left(\frac{\alpha^{\gamma_1}}{\gamma_1 B(\gamma_1, \gamma_1/\alpha - \gamma_1 + 1)}\right) > 0$$

To do this, we consider the function defined on the interval $(0,1]$ by

$$g(\alpha) = \ln\left(\frac{\alpha^{\gamma_1}}{\gamma_1 B(\gamma_1, \gamma_1/\alpha - \gamma_1 + 1)}\right)$$

Taking into account that $\gamma_1 > 1$, $B(a,b) = \dfrac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ and $a\Gamma(a) = \Gamma(a+1)$, where $\Gamma(z)$ denotes the gamma function, we have

$$\begin{aligned} g(\alpha) &= \ln\left(\frac{\alpha^{\gamma_1}}{\gamma_1 B(\gamma_1, \gamma_1/\alpha - \gamma_1 + 1)}\right) = \ln\left(\frac{\alpha^{\gamma_1}\Gamma(\gamma_1/\alpha + 1)}{\gamma_1\Gamma(\gamma_1)\Gamma(\gamma_1/\alpha - \gamma_1 + 1)}\right) \\ &= \gamma_1\ln(\alpha) - \ln(\gamma_1\Gamma(\gamma_1)) - \ln(\Gamma(\gamma_1/\alpha - \gamma_1 + 1)) + \ln(\Gamma(\gamma_1/\alpha + 1)) \\ &= \gamma_1\ln(\alpha) - \ln(\Gamma(\gamma_1 + 1)) - \ln(\Gamma(\gamma_1/\alpha - \gamma_1 + 1)) + \ln(\Gamma(\gamma_1/\alpha + 1)) \end{aligned}$$

The derivative of the function $g(\alpha)$ is

$$g'(\alpha) = \frac{\gamma_1}{\alpha} - \frac{\gamma_1}{\alpha^2}\left(\psi\left(\gamma_1/\alpha + 1\right) - \psi\left(\gamma_1/\alpha + 1 - \gamma_1\right)\right)$$

where $\psi(z)$ is the digamma function, that is, $\psi(z) = \frac{d}{dz}\left(\ln \Gamma(z)\right) = \Gamma'(z)/\Gamma(z)$.

This function is strictly increasing on $(0, \infty)$ and satisfies that (see Abramowitz and Stegun [29], page 258)

$$\psi(n+z) - \psi(1+z) = \sum_{k=1}^{n-1} \frac{1}{k+z} > \frac{n-1}{n-1+z}$$

with $n > 2$ and $z > 0$. Then, if $x > 2$ and $z > 0$,

$$\psi(x+z) - \psi(1+z) > \frac{x-1}{x-1+z}$$

and, using this property with $x = \gamma_1 + 1 > 2$ and $z = \gamma_1/\alpha - \gamma_1 > 0$, we deduce that

$$\psi\left(\gamma_1/\alpha + 1\right) - \psi\left(\gamma_1/\alpha + 1 - \gamma_1\right) > \alpha$$

and, as a consequence,

$$g'(\alpha) < \frac{\gamma_1}{\alpha} - \frac{\gamma_1}{\alpha^2}\alpha = 0$$

That is, the function $g(\alpha)$ is strictly decreasing on the interval $(0, 1)$. Therefore, we can ensure that $g(\alpha) > g(1) = -\ln(1) = 0$ for any $\alpha \in (0, 1)$, which ends the proof. □

Proof of Theorem 2.

By Lemma 3, the minimum of the function $\Phi(\tau)$ is obtained for $\tau^*=1$ with value $\Phi^*=B\left(\gamma_1, \gamma_2/\alpha + 1\right)$. Then, from Lemma 2, the minimum of the function $\eta(S, \tau)$ is obtained for $\tau^* = 1$ and

$$S^* = S_1^* = \left(\frac{K(\alpha\lambda)^{\gamma_1}}{(\xi - 1)\gamma_1 B\left(\gamma_1, \gamma_2/\alpha + 1\right) h}\right)^{1/\xi}$$

with value

$$\eta^* = \eta\left(S_1^*, 1\right) = \left(\frac{\xi K}{\xi - 1}\right)\left(\frac{(\xi - 1)\gamma_1 B\left(\gamma_1, \gamma_2/\alpha + 1\right) h}{K(\alpha\lambda)^{\gamma_1}}\right)^{1/\xi} = \frac{\xi K}{(\xi - 1) S^*}$$

Thus, the minimum of the function $w(r, S) = \eta(S, \tau)$ is obtained for $S^* = S_1^*$ and $r^* = S^*\left(1 - \tau^*\right)^{1/\alpha} = 0$, with value $w^* = \eta^*$. Finally, the maximum of the function $W(r, S)$ is obtained for $r^* = 0$ and $S^* = q^*$, with value

$$W^* = W(0, S^*) = \frac{v}{p + w^*} - 1 = \frac{v}{p + \dfrac{\xi K}{(\xi - 1) q^*}} - 1$$

which ends the proof. □

Proof of Lemma 4.

As $B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ and $a\Gamma(a) = \Gamma(a+1)$, where $\Gamma(z)$ denotes the gamma function, we have

$$\ln q^* = \frac{\ln K - \ln h + \gamma_1 \ln \lambda + \gamma_1 \ln \alpha - \ln(\xi - 1) - \ln\left(\Gamma\left(\gamma_1 + 1\right)\right) - \ln\left(\Gamma\left(\gamma_2/\alpha + 1\right)\right) + \ln\left(\Gamma\left(\gamma_1 + \gamma_2/\alpha + 1\right)\right)}{\xi}$$

If we derive the two members of this equality with respect to the parameters K, h and λ, the first three assertions are easily obtained. For the other three, we take into account that

$$\frac{d}{dz}\left(\ln \Gamma(z)\right) = \psi(z)$$

where $\psi(z)$ is the digamma function. Then, for the β parameter,

$$\frac{\frac{\partial q^*}{\partial \beta}}{q^*} = -\frac{\frac{\partial q^*}{\partial \alpha}}{q^*} = -\left(\frac{\frac{\gamma_1}{\alpha} - \frac{\gamma_1}{\xi-1} + \frac{\gamma_2}{\alpha^2}\left(\psi\left(\gamma_2/\alpha+1\right) - \psi\left(\gamma_1+\gamma_2/\alpha+1\right)\right)}{\xi} - \frac{\gamma_1 \xi \ln q^*}{\xi^2}\right)$$

$$= \left(\frac{1}{\xi}\right)\left(\gamma_1 \ln q^* - \frac{\gamma_1(\xi-1-\alpha)}{\alpha(\xi-1)} + \frac{\gamma_2}{\alpha^2}\left(\psi\left(\gamma_1+\gamma_2/\alpha+1\right) - \psi\left(\gamma_2/\alpha+1\right)\right)\right) = \left(\frac{1}{\xi}\right)\left(\gamma_1 \ln q^* - d_\beta\right)$$

In a similar way, for the γ_1 parameter

$$\frac{\frac{\partial q^*}{\partial \gamma_1}}{q^*} = \frac{\ln(\alpha\lambda) - \frac{\alpha}{\xi-1} - \psi(\gamma_1+1) + \psi(\gamma_1+\gamma_2/\alpha+1)}{\xi} - \frac{\alpha\xi \ln q^*}{\xi^2}$$

$$= \left(\frac{1}{\xi}\right)\left(\psi(\gamma_1+\gamma_2/\alpha+1) - \psi(\gamma_1+1) - \frac{\alpha}{\xi-1} - \ln\left(\frac{(q^*)^\alpha}{\alpha\lambda}\right)\right) = \left(\frac{1}{\xi}\right)\left(d_{\gamma_1} - \ln\left(\frac{(q^*)^\alpha}{\alpha\lambda}\right)\right)$$

Finally, for the γ_2 parameter

$$\frac{\frac{\partial q^*}{\partial \gamma_2}}{q^*} = \frac{-\frac{1}{\xi-1} + \left(\frac{1}{\alpha}\right)\left(-\psi\left(\gamma_2/\alpha+1\right) + \psi\left(\gamma_1+\gamma_2/\alpha+1\right)\right)}{\xi} - \frac{\xi \ln q^*}{\xi^2}$$

$$= \left(\frac{1}{\xi}\right)\left(\frac{\psi(\gamma_1+\gamma_2/\alpha+1) - \psi(\gamma_2/\alpha+1)}{\alpha} - \frac{1}{\xi-1} - \ln q^*\right) = \left(\frac{1}{\xi}\right)\left(d_{\gamma_2} - \ln q^*\right)$$

and the proof is finished. □

Proof of Lemma 5.

The maximum profitability ratio W^* satisfies that

$$\frac{v}{1+W^*} - p = \frac{\xi K}{(\xi-1)\, q^*}$$

(a) By derivating the two members of this equality with respect to K, and using Lemma 4, we obtain:

$$\frac{-v\frac{\partial W^*}{\partial K}}{(1+W^*)^2} = \left(\frac{\xi}{\xi-1}\right)\left(\frac{1}{q^*} - \frac{K\frac{\partial q^*}{\partial K}}{(q^*)^2}\right) = \left(\frac{\xi}{\xi-1}\right)\left(\frac{\xi-1}{\xi q^*}\right) = \frac{1}{q^*}$$

Thus, $\dfrac{\partial W^*}{\partial K} = -\dfrac{(1+W^*)^2}{v q^*}$.

(b) In a similar way, by derivation with respect to h:

$$\frac{-v\frac{\partial W^*}{\partial h}}{(1+W^*)^2} = \left(\frac{\xi}{\xi-1}\right)\left(\frac{-K\frac{\partial q^*}{\partial h}}{(q^*)^2}\right) = \left(\frac{\xi}{\xi-1}\right)\left(\frac{K}{\xi h q^*}\right)$$

Thus, $\dfrac{\partial W^*}{\partial h} = -\left(\dfrac{(1+W^*)^2 K}{v(\xi-1)\, q^*}\right)\left(\dfrac{1}{h}\right)$.

(c) By derivation with respect to λ, the following equality is deduced:

$$\frac{-v\frac{\partial W^*}{\partial \lambda}}{(1+W^*)^2} = \left(\frac{\xi}{\xi-1}\right)\left(\frac{-K\frac{\partial q^*}{\partial \lambda}}{(q^*)^2}\right) = \left(\frac{\xi}{\xi-1}\right)\left(\frac{-\gamma_1 K}{\xi \lambda q^*}\right)$$

Thus, $\dfrac{\partial W^*}{\partial \lambda} = \left(\dfrac{(1+W^*)^2 K}{v(\xi-1)\, q^*}\right)\left(\dfrac{\gamma_1}{\lambda}\right)$.

(d) By derivation with respect to β, we obtain:

$$\begin{aligned}
\frac{-v\dfrac{\partial W^*}{\partial \beta}}{(1+W^*)^2} &= -\left(\left(\frac{-\gamma_1}{(\xi-1)^2}\right)\left(\frac{K}{q^*}\right)+\left(\frac{\xi}{\xi-1}\right)\left(\frac{K\dfrac{\partial q^*}{\partial \beta}}{(q^*)^2}\right)\right)\\
&= \frac{\gamma_1 K}{(\xi-1)^2 q^*}-\left(\frac{K}{(\xi-1)\,q^*}\right)(\gamma_1 \ln q^* - d_\beta)
\end{aligned}$$

Thus, taking into account that $e_\beta = \dfrac{\gamma_1}{\xi-1}+d_\beta$, we have

$$\frac{\partial W^*}{\partial \beta} = \left(\frac{(1+W^*)^2 K}{v\,(\xi-1)\,q^*}\right)\left(\gamma_1 \ln q^* - \frac{\gamma_1}{\xi-1} - d_\beta\right) = \left(\frac{(1+W^*)^2 K}{v\,(\xi-1)\,q^*}\right)(\gamma_1 \ln q^* - e_\beta)$$

(e) By derivation with respect to γ_1:

$$\begin{aligned}
\frac{-v\dfrac{\partial W^*}{\partial \gamma_1}}{(1+W^*)^2} &= \left(\frac{-\alpha}{(\xi-1)^2}\right)\left(\frac{K}{q^*}\right)+\left(\frac{\xi}{\xi-1}\right)\left(\frac{K\dfrac{\partial q^*}{\partial \gamma_1}}{(q^*)^2}\right)\\
&= \frac{-\alpha K}{(\xi-1)^2 q^*}-\left(\frac{K}{(\xi-1)\,q^*}\right)\left(d_{\gamma_1}-\ln\left(\frac{(q^*)^\alpha}{\alpha\lambda}\right)\right)
\end{aligned}$$

and, taking into account that $e_{\gamma_1} = \dfrac{\alpha}{\xi-1}+d_{\gamma_1}$, we obtain

$$\frac{\partial W^*}{\partial \gamma_1} = \left(\frac{(1+W^*)^2 K}{v\,(\xi-1)\,q^*}\right)\left(\frac{\alpha}{\xi-1}+d_{\gamma_1}-\ln\left(\frac{(q^*)^\alpha}{\alpha\lambda}\right)\right) = \left(\frac{(1+W^*)^2 K}{v\,(\xi-1)\,q^*}\right)\left(e_{\gamma_1}-\ln\left(\frac{(q^*)^\alpha}{\alpha\lambda}\right)\right)$$

(f) By derivation with respect to γ_2:

$$\frac{-v\dfrac{\partial W^*}{\partial \gamma_2}}{(1+W^*)^2} = \left(\frac{-1}{(\xi-1)^2}\right)\left(\frac{K}{q^*}\right)+\left(\frac{\xi}{\xi-1}\right)\left(\frac{K\dfrac{\partial q^*}{\partial \gamma_2}}{(q^*)^2}\right) = \frac{-K}{(\xi-1)^2 q^*}-\left(\frac{K}{(\xi-1)\,q^*}\right)(d_{\gamma_2}-\ln q^*)$$

Thus, taking into account that $e_{\gamma_2} = \dfrac{1}{\xi-1}+d_{\gamma_2}$, we see that

$$\frac{\partial W^*}{\partial \gamma_2} = \left(\frac{(1+W^*)^2 K}{v\,(\xi-1)\,q^*}\right)\left(\frac{1}{\xi-1}+d_{\gamma_2}-\ln(q^*)\right) = \left(\frac{(1+W^*)^2 K}{v\,(\xi-1)\,q^*}\right)(e_{\gamma_2}-\ln(q^*))$$

(g) By derivation with respect to v:

$$\frac{1}{1+W^*}-\frac{v\dfrac{\partial W^*}{\partial v}}{(1+W^*)^2}=0$$

Thus, $\dfrac{\partial W^*}{\partial v} = \dfrac{1+W^*}{v}$.

(h) Finally, by derivation with respect to p, we obtain

$$\frac{-v\dfrac{\partial W^*}{\partial p}}{(1+W^*)^2}-1=0$$

Thus, $\dfrac{\partial W^*}{\partial p} = -\dfrac{(1+W^*)^2}{v}$. $\square$

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