Best pricing and optimal policy for an inventory system under time-and-price-dependent demand and backordering

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Abstract

In this paper, we study an inventory system for products where demand depends on time and price. Shortages are allowed and are fully backordered. We suppose that the demand rate is the product of a power time pattern and a three-parametric exponential price function. The objective is to determine the economic lot size, the optimal shortage level and the best selling price to maximize the total profit per unit time. We present an efficient procedure to determine the optimal solution of the inventory problem for all possible scenarios. This procedure is illustrated with several numerical examples. A sensitivity analysis of the optimal inventory policy with respect to the parameters of the demand rate function is also given. Finally, the main contributions of this paper are highlighted and future research directions are introduced.

Keywords: Inventory System; Optimal pricing; Profit maximization; Power demand pattern; Backordering

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1 Introduction

One of the main objectives in any company or organization that distributes products to other companies (or supplying directly to customers) is to establish the sufficient quantity of products in stock to meet customer demand at an acceptable price and in a reasonable period of time.

In general, this demand for articles is not constant, but fluctuates over time due to external causes that occur in the market. Therefore, product stock control is a dynamic activity requiring the continuous revision of the operational methods to reflect possible changes. As a consequence, inventory systems must adapt to new situations by applying the most efficient inventory policies at any particular time. The development and evolution of inventory models from the very beginning can be seen in Cárdenas-Barrón et al. (2014).

Inventory management seeks to control the level of available stock so that customer demand is covered while obtaining the greatest possible profit. To perform this control, it is necessary to work with mathematical models that allow the properties of inventory systems to be represented. In this way, the optimal policies that must be applied for an adequate inventory management can be obtained.

In the literature, there exist several papers where the demand rate depends on the selling price. Thus, for example, Ghoreishi et al. (2015) studied an EOQ model with permissible delay in payments and price- and inflation-induced demand. Jaggi et al. (2017) analyzed a two warehouse inventory model for non-instantaneous deteriorating items, supposing that the demand rate depends on the selling price and full backlogging. Feng et al. (2017) presented an inventory model for a perishable product that considers demand as a multivariate function of its unit price, freshness and stock level. Mishra et al. (2017) developed an EOQ inventory model where shortages are permitted, assuming the demand rate as a function of stock and selling price.

In this paper, we present and study an inventory model for determining the optimal policy for items in which demand depends on time and the selling price of the article. Thus, the demand rate is the product of a power time-function and a tri-exponential price-function. This price-function is a generalization of the commonly exponential price-dependent demand functions used in the inventory literature.

The power-time demand pattern was introduced by Naddor (1966). Since then, more papers have appeared with this type of demand. For example, Datta and Pal (1988) developed an inventory model with power demand pattern, assuming a variable rate of deterioration. Lee and Wu (2002) studied an inventory model with power demand pattern for items with Weibull distributed deterioration and full backlogging. Rajeswari and Vanjikkodi (2011) analyzed a deterministic inventory model for which items are subject to constant deterioration, assuming power demand pattern and partial backlogging. Mishra et al (2012) presented an EOQ model for perishable items with power demand pattern and two-parameter Weibull distribution for deterioration under the influence of inflation and time-value of money. Mishra and Singh (2013) developed an EOQ model for perishable items with power demand pattern and quadratic deterioration rate, assuming that shortages are partially backlogged. Rajeswari and Indrani (2015) studied a deterministic inventory model for linear time dependent deteriorating items with power demand pattern and partial backlogging under total cost minimization. A common characteristic of all the above papers is that the length of the inventory cycle is always known and fixed. However, Sicilia et al. (2012, 2013) developed some inventory systems where the length of the inventory cycle was a decision variable and, therefore, not constant. More recently, San-José et al. (2017) developed the optimal policy for an inventory system with power demand pattern and partial backlogging in which the selling price is a known constant and the length of the inventory cycle is a decision variable.

The demand rate, as a separable function of time and selling price, has also been considered in the inventory literature. For instance, Smith et al. (2007) studied the benefit of sequential versus simultaneous optimization within a newsboy problem in a pricing framework. Valliathal and Uthayakumar (2011) discussed the effects of an economic order quantity model for non-instantaneous deteriorating items under price and time dependent selling rate and time dependent partial backlogging. Avinadav et al. (2013) formulated an inventory model in which the demand rate function is a multiplication of two factors: selling price and time after replenishment. Soni (2013) analyzed an inventory model with demand influenced by both displayed stock level and selling price under delay in payment. Wu et al. (2014) revisited Soni's model and developed an optimization procedure to find the optimal replenishment policies. Sarkar et al. (2014) considered a production-inventory model for the selling price and the time dependent demand pattern in an imperfect production system. Avinadav et al. (2014) developed two inventory models for determining the optimal pricing, order quantity and replenishment period for items whose demand function can be separated into components of price and inventory age. Hossen et al. (2016) developed a fuzzy inventory model for deteriorating items with price and time dependent demand considering the effect of inflation on the system.

In some practical situations, it may be economically advantageous for the inventory manager to allow stock-out in the inventory cycle. In this case, the customer must wait to receive his/her order until the arrival of the next replenishment. For this reason, we suppose that shortages are allowed and fully backordered. That is, all customers that arrive to the system in the stock-out period are willing to wait for the next order. Thus, all backorders are met when a new order arrives to the system. An inventory model with full backlogging and all-units quantity discounts, where the cost of a backorder includes a fixed cost and a cost which is proportional to the length of time the backorder exists, was developed by San-José and García-Laguna (2009). Birbil et al. (2015) analyzed the impact of general ordering cost functions in EOQ-type models with shortages, which are completely backlogged. Jakšič and Fransoo (2015) studied the inventory control problem of a retailer working under stochastic demand and stochastic limited supply, assuming that the unfilled part of the retailer's order is fully backordered. Mishra et al. (2015) presented the state-of-art application of a nuanced technique of fuzzified EOQ control model allowing shortage with full backlogging. Prasad and Mukherjee (2015) analyzed a deterministic inventory model for deteriorating items with stock and time dependent demand where shortages are completely backordered. This situation of full backordering is also considered in the hypothesis of this paper.

In the inventory literature, no paper simultaneously considers a demand pattern which is potentially time-dependent and exponentially price-dependent, with full backordering and the length of the inventory cycle as an unfixed variable. These are precisely the assumptions on which the inventory system presented here is based. Thus, the objective is to determine the economic lot size, the optimal shortage and the best selling price to maximize the total profit per unit time. We suppose that the profit is represented by the difference between the revenues from product sales and the sum of ordering cost, purchasing cost, holding cost and backordering cost. Taking into account the above considerations, a procedure is developed for determining the optimal policy and the best selling price of the product for all possible scenarios.

The rest of the paper is organized as follows. The second section establishes the properties of the inventory system and presents the notation that will be followed throughout the paper. We then determine the profit as a function of the lot size, the total shortage and the selling price of the product, and we formulate the optimization problem to be solved related to inventory management. In the fourth section, some results that allow the optimal policy to be determined are presented. Moreover, the special case of the exponential price function most commonly used in the inventory literature is analyzed and we develop a simpler algorithm for this case. Then, in the fifth section, some numerical examples are introduced to illustrate the application of the optimization procedure previously described. Moreover, a numerical sensitivity analysis for the optimal policy and maximum profit with respect to the parameters of the demand rate function is also presented. Finally, the main contributions of this paper are synthesized and possible future research directions within the area of inventory management are presented.

2 Assumptions and notation

The inventory model for a single item here developed is based on the following hypothesis:

1. The replenishment is instantaneous, the lead time is negligible and the planning horizon is infinite.
2. The inventory is continuously revised.
3. Stock-outs are allowed and the demand during that period is fully backordered.
4. A lot of Q units (the total demand during the inventory cycle) is ordered when the number of units pending to serve reaches the amount B (total shortage amount).
5. The unit purchasing cost c and the holding cost h per unit and unit time are known and constant.
6. The ordering cost A is constant and independent of the lot size.
7. The shortage cost π per backordered unit and per unit time is constant.
8. The demand rate $\lambda(t, p)$ is a function dependent on time and on unit selling price p. It is assumed that the demand rate is the product of a power time-function and a tri-exponential price-function.

   The power demand pattern has been used by various authors in recent years (see, for example, Mishra and Singh 2013, and Rajeswani and Indrani 2015). The assumption of exponentially price-dependent demand has been used previously in the literature (see, for example, Jeuland and Shugan 1988, Hanssens and Parsons 1993, and Song et al. 2008). However, to the best of our knowledge, this is the first work that considers a tri-parameter exponential function more general than those of the previously cited authors.

The notation used throughout the paper is shown in Table 1.

Table 1. Notation

Q	Lot size per cycle (> 0, *decision variable*)
B	Maximum shortage quantity per cycle or the reorder point (≥ 0, *decision variable*)
S	Inventory level at the beginning of the inventory cycle (≥ 0)
ϕ	Length of the inventory cycle where the net stock is positive (≥ 0)
σ	Length of the inventory cycle where the net stock is negative (≥ 0)
T	Length of the inventory cycle, that is, $T = \phi + \sigma$ (> 0)
c	Unit purchasing cost (> 0)
p	Unit selling price ($p \geq c$, *decision variable*)
A	Fixed ordering cost per order (> 0)
h	Holding cost per unit and per unit time (> 0)
π	Shortage cost per backordered unit and per unit time (> 0)
$\lambda(t,p)$	Demand rate at time t when the selling price is p, with $0 < t < T$ and $p \geq c$
$I(t,p)$	Inventory level at time t when the selling price is p, with $0 \leq t < T$ and $p \geq c$
n	Demand pattern index (> 0)
$P(Q,B,p)$	Total profit per unit time

3 The mathematical model

Next, we develop the mathematical model for the inventory system according to the assumptions previously exposed.

We have assumed that the demand rate $\lambda(t,p)$ multiplies the effects of a power time demand pattern and an exponential function which depends on the selling price. Thus, $\lambda(t,p) = \lambda_1(t)\lambda_2(p)$, where $\lambda_1(t)$ represents the power time demand pattern given by

$$\lambda_1(t) = \tfrac{1}{n}\left(\frac{t}{T}\right)^{1/n-1}, \text{ with } n > 0, \tag{1}$$

(the parameter n is the index of the power time demand pattern) and $\lambda_2(p)$ is the tri-parameter exponential function defined by

$$\lambda_2(p) = \alpha e^{-\beta p^{\gamma}}, \text{ with } \alpha > 0,\ \beta > 0 \text{ and } \gamma > 0. \tag{2}$$

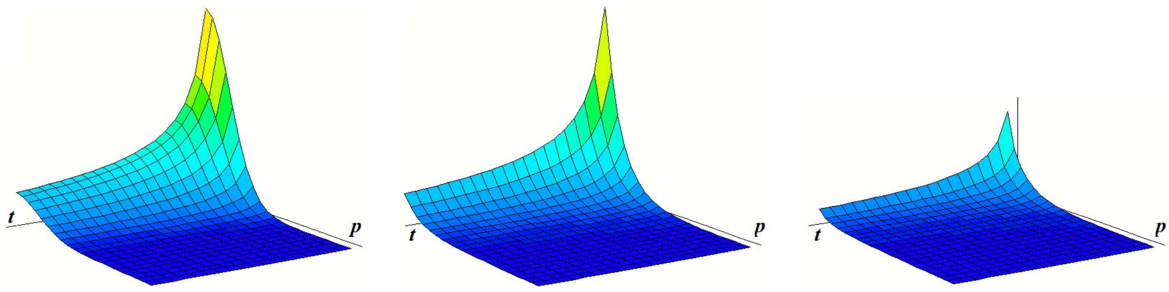
The consideration of the inventory cycle T in the function $\lambda_1(t)$ allows the influence of the replenishment policy on the behavior of the customers to be described. Thus, let us suppose that a customer wants to buy an item with expiry date, which is sold in two shops located in the same neighborhood and close to each other. The prices of the item are identical in both shops. The length of the inventory cycle at the first shop is longer than in the second one. This leads to a situation where, in the second shop, there is a greater stock rotation. In which of the two shops would a customer buy the item? It is more likely that the customer will buy in the second shop because the length of the inventory cycle is shorter and there is a greater stock rotation. This means that the article is more likely to have been produced more recently and, logically, the customers prefer these new items. Other explanation of the practical utility of the function $\lambda_1(t)$ to describe the demand for certain products can be seen in San-José et al. (2017). In the function $\lambda_2(p)$, the parameter α can be interpreted as the maximum possible level of demand, and β and γ are coefficients of the selling price sensitivity. In the exponential price-dependent demand models used in the inventory literature, demand is commonly described by means of the function $\alpha e^{-\beta p}$. Several bi-exponential price models have been used by researchers to characterize the sales in different markets (e.g., Cowling and Cubbin, 1971; Krishnamurthi and Raj, 1988; Bolton, 1989; Song et al., 2008). In this paper, we extend this exponential price-dependent function, incorporating the new parameter γ. Note that this parameter γ can be interpreted analogously to the exponent parameter of the power price-dependent model $\alpha - \beta p^\gamma$ (see, for example Chen et al. 2006, Huang et al. 2013, and Avinadav et al. 2014). So, the price-dependent function presented in this paper may be useful to represent the real demand for some products because it allows a better adjust in empirical applications.

The point-price elasticity of the demand rate function presented here (that is, $(p/\lambda(p,t))\,(\partial\lambda(t,p)/\partial p) = -\beta\gamma p^\gamma$) depends on the selling price p and is always decreasing; moreover, it is concave when $\gamma \geq 1$ and convex if $\gamma \leq 1$. Thus, we can say that the parameter γ of the tri-parametric exponential price-function is an index of convexity of the demand rate function. Moreover, the price curvature of $\lambda(p,t)$ (that is, $\lambda(p,t)(\partial^2\lambda(t,p)/\partial p^2)/(\partial\lambda(t,p)/\partial p)^2 = 1 + (1-\gamma)/(\beta\gamma p^\gamma)$) depends on the price decision variable p and the coefficients β and γ of the selling price sensitivity, unlike what happens in the bi-parametric exponential price models where the curvature is equal to one.

In order to illustrate the effect of the parameters n and γ on the demand, we have depicted the function $\lambda(t,p)$ for different potential indexes in Figures 1, 2 and 3.

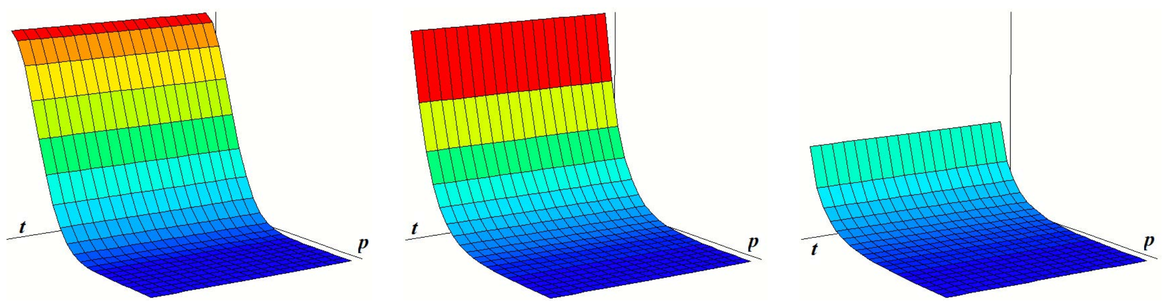
Taking into account that the demand during the stock-out period is fully backordered, the lot size Q must coincide with the demand during the inventory cycle. Thus,

$$Q = \int_0^T \lambda(x,p)dx = \alpha e^{-\beta p^\gamma} \int_0^T \lambda_1(x)dx = \alpha e^{-\beta p^\gamma} T. \tag{3}$$



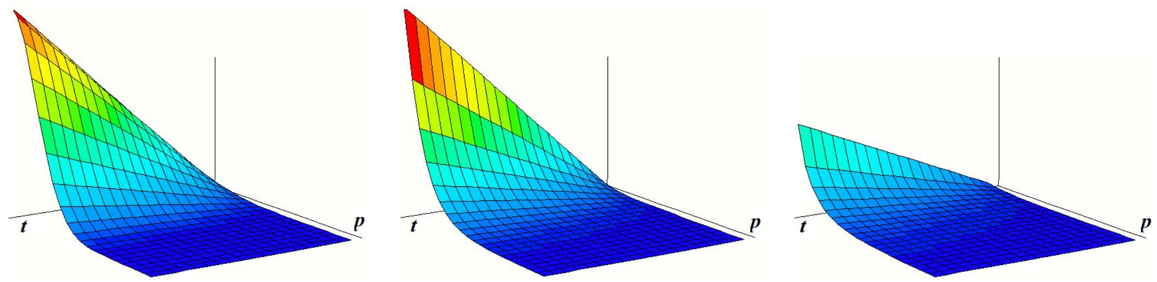
$\gamma > 1$ $\gamma = 1$ $\gamma < 1$

Figure 1 Demand rate functions $\lambda(t, p)$ when $n > 1$



$\gamma > 1$ $\gamma = 1$ $\gamma < 1$

Figure 2 Demand rate functions $\lambda(t, p)$ when $n = 1$



$\gamma > 1$ $\gamma = 1$ $\gamma < 1$

Figure 3 Demand rate functions $\lambda(t, p)$ when $n < 1$

That is, the inventory cycle T is given by

$$T = e^{\beta p^\gamma} Q/\alpha. \tag{4}$$

The total shortage amount during the inventory cycle is B. Thus, the initial inventory level is $S = Q - B$. This level has to be equal to demand during the positive stock cycle ϕ, that is,

$$S = Q - B = \int_0^\phi \lambda(x, p)dx = e^{-\beta p^\gamma/n} Q^{1-1/n} (\alpha\phi)^{1/n} \tag{5}$$

Therefore, the length of the inventory cycle where the net stock is positive is given by

$$\phi = \frac{e^{\beta p^{\gamma}}}{\alpha} Q^{1-n}(Q-B)^{n}. \tag{6}$$

The net stock level at time t, $I(t,p)$, is

$$I(t,p) = \int_t^{\phi} \lambda(x,p)dx = S - \int_0^t \lambda(x,p)dx = Q\left[1-\left(\frac{\alpha e^{-\beta p^{\gamma}}}{Q}t\right)^{1/n}\right] - B, \text{ for } 0 \le t < T$$

From (4) and (6), it follows that demand B during the stock-out period is given by

$$B = \int_{\phi}^{T} \lambda(x,p)dx = \alpha e^{-\beta p^{\gamma}} T\left[1-\left(\frac{\phi}{T}\right)^{1/n}\right] \tag{7}$$

The length of the stock-out period is

$$\sigma = T - \phi = \frac{e^{\beta p^{\gamma}}}{\alpha} Q\left[1-\left(1-\frac{B}{Q}\right)^{n}\right]. \tag{8}$$

3.1 The objective function

Our objective is to maximize the average profit per unit time. The total profit per cycle is determined as the difference between the revenue per cycle and the sum of the ordering, the purchasing, the holding and the backordering costs. The revenue per cycle is pQ, the ordering cost is A, the acquisition or purchasing cost is cQ. The holding cost is

$$\begin{aligned} HC &= h\int_0^{\phi} I(t,p)dt = h\int_0^{\phi}\left[\int_t^{\phi}\lambda(x,p)dx\right]dt = h\int_0^{\phi} x\lambda(x,p)dx \\ &= \frac{he^{\beta p^{\gamma}}}{(n+1)\,\alpha}(Q-B)^{n+1}Q^{1-n} \end{aligned}$$

and the backordering cost is given by

$$\begin{aligned} \pi\int_{\phi}^{T}\left[-I(t,p)\right]dt &= \pi\int_{\phi}^{T}\left[\int_{\phi}^{t}\lambda(x,p)dx\right]dt = \pi\int_{\phi}^{T}(T-x)\lambda(x,p)dx \\ &= \frac{\pi e^{\beta p^{\gamma}}}{\alpha}\left[BQ - \frac{Q^2}{n+1} + \frac{(Q-B)^{n+1}Q^{1-n}}{n+1}\right]. \end{aligned}$$

Hence, we see that the total profit per cycle is

$$\begin{aligned} PT(Q,B,p) &= (p-c)Q - A - h\int_0^{\phi} I(t,p)dt + \pi\int_{\phi}^{T} I(t,p)dt \\ &= (p-c)Q - A - \frac{(h+\pi)\,e^{\beta p^{\gamma}}}{(n+1)\,\alpha}Q^{1-n}\,(Q-B)^{n+1} - \frac{\pi e^{\beta p^{\gamma}}}{\alpha}BQ + \frac{\pi e^{\beta p^{\gamma}}}{(n+1)\,\alpha}Q^2. \end{aligned}$$

Thus, the average profit per unit time is given by

$$P(Q,B,p) = (p-c)\alpha e^{-\beta p^\gamma} - \frac{\alpha A e^{-\beta p^\gamma}}{Q} - \frac{h+\pi}{n+1} Q^{-n} (Q-B)^{n+1} - \pi B + \frac{\pi}{n+1} Q. \tag{9}$$

Therefore, the aim is to determine the values of the decision variables Q, B and p, with $Q > 0, 0 \leq B \leq Q$ and $c \leq p$, which maximize the function $P(Q,B,p)$ given in (9).

4 Solution of the problem

Next, we present a first result that will allow us to reduce the above optimization problem with three decision variables to an optimization problem with a unique decision variable.

Lemma 1 *Let $P(Q,B,p)$ given by (9). Then:*

1. *The function $P(Q,B,p)$ is strictly concave on the region $\Lambda = \{(Q,B) : Q > 0, 0 \leq B \leq Q\}$ for any fixed value of p.*
2. *For any fixed value of p, the function $P(Q,B,p)$ attains its maximum value at the point (Q_p^*, B_p^*) given by*

$$Q_p^* = \sqrt{\frac{(n+1)\alpha A e^{-\beta p^\gamma}}{n\pi \left[1 - \left(\frac{\pi}{h+\pi}\right)^{1/n}\right]}} \tag{10}$$

$$B_p^* = Q_p^* \left[1 - \left(\frac{\pi}{h+\pi}\right)^{1/n}\right] = \sqrt{\frac{(n+1)\alpha A e^{-\beta p^\gamma} \left[1 - \left(\frac{\pi}{h+\pi}\right)^{1/n}\right]}{n\pi}} \tag{11}$$

Proof. Please, see Appendix. ■

Note that the point (Q_p^*, B_p^*) is in the interior of the feasible region Λ and is the solution of the system of nonlinear equations $(\partial P(Q,B,p)/\partial Q) = 0$ and $(\partial P(Q,B,p)/\partial B) = 0$.

By evaluating $P(Q,B,p)$ at the optimal point (Q_p^*, B_p^*), we obtain the function

$$G(p) = P(Q_p^*, B_p^*, p) = e^{-\beta p^\gamma/2} \left[\alpha (p-c) e^{-\beta p^\gamma/2} - 2\sqrt{\alpha}\theta\right], \tag{12}$$

where the parameter θ is defined as

$$\theta = \sqrt{\frac{n}{n+1} A\pi \left[1 - \left(\frac{\pi}{h+\pi}\right)^{1/n}\right]}. \tag{13}$$

Thus, our problem now consists of determining the selling price which maximizes the function $G(p)$ defined by (12).

Next, we give some properties of the function $G(p)$, which are easy to check.

1. The function $G(p)$ is continuous on the interval $[c, \infty)$. Moreover, $G(c) < 0$ and $\lim_{p\to\infty} G(p) = 0$.

2. The function $G(p)$ is of class C^1 on (c, ∞). Furthermore, the derivative is

$$G'(p) = \alpha e^{-\beta p^\gamma} p^\gamma f(p^\gamma), \tag{14}$$

where

$$f(x) = x^{-1/\gamma}\left(\frac{\beta\gamma\theta e^{\beta x/2}}{\sqrt{\alpha}} + \beta\gamma c\right) + \frac{1}{x} - \beta\gamma. \tag{15}$$

Note that since $p > 0$, sign$(G'(p))$ =sign$(f(p^\gamma))$. Therefore, we can determine the zeros of the function $G'(p)$ in the interval (c, ∞) through the study of the function $f(x)$ in the interval (c^γ, ∞). Next, we present some characteristics of this last function.

1. The function $f(x)$ is continuous on the interval $(0, \infty)$. Moreover, $f(c^\gamma) > 0$ and $\lim_{x\to\infty} f(x) = \infty$.

2. The first derivative of $f(x)$ is

$$f'(x) = x^{-(1+1/\gamma)}\left(\frac{\beta\theta e^{\beta x/2}(\beta\gamma x - 2)}{2\sqrt{\alpha}} - \beta c\right) - \frac{1}{x^2} \tag{16}$$

We can establish the following result.

Lemma 2 *The function $f(x)$ given by (15) is strictly convex on the interval $(0, \infty)$ and it attains its minimum value at the point*

$$x_1 = \arg_{x>0}\{f'(x) = 0\}. \tag{17}$$

Proof. Please, see Appendix. ■

Taking into account the previous results, we now present a new theorem which develops a simple approach to obtain the optimal price p^*.

Theorem 1 *Let $G(p)$, $f(x)$, $f'(x)$ and x_1 be given, respectively, by (12), (15), (16) and (17).*

1. *If $x_1 \leq c^\gamma$, then $p^* = \infty$ and $G(p^*) = 0$.*

2. *If $x_1 > c^\gamma$ and $f(x_1) \geq 0$, then $p^* = \infty$ and $G(p^*) = 0$.*

3. *Otherwise (if $x_1 > c^\gamma$ and $f(x_1) < 0$), let x_0 be given by $x_0 = \arg_{x\in(c^\gamma, x_1)}\{f(x) = 0\}$.*

   (a) If $G(x_0^{1/\gamma}) < 0$, then $p^* = \infty$ and $G(p^*) = 0$.

   (b) If $G(x_0^{1/\gamma}) \geq 0$, then $p^* = x_0^{1/\gamma}$ and the optimal profit is $G(p^*)$.

Proof. Please, see Appendix. ■

Note that the inventory system studied here is profitable only in the case (3.b) of Theorem 1.

In the next section, we analyze the special case when $\gamma = 1$, because in that situation the conditions to obtain the optimal selling price can be formulated in an even simpler way.

4.1 Optimal inventory policy when $\gamma = 1$

Note that the function $f(x)$ given by (15) can be rewritten as

$$f(x) = x^{-1/\gamma} \left(\frac{\beta\gamma\theta e^{\beta x/2}}{\sqrt{\alpha}} + r(x) \right), \tag{18}$$

where $r(x) = \beta\gamma c + x^{1/\gamma - 1} - \beta\gamma x^{1/\gamma}$. When $\gamma = 1$, the rational function $r(x)$ has a unique term dependent on x, while if $\gamma \neq 1$, it always has two terms in the variable x. For this reason, if $\gamma = 1$, it is easier to work with the function $f_1(x) = xf(x)$ instead of $f(x)$. Moreover, since $\gamma = 1$, we see from (12) and (14) that

$$G(p) = e^{-\beta p/2} \left[\alpha \left(p - c \right) e^{-\beta p/2} - 2\sqrt{\alpha}\theta \right] \tag{19}$$

and

$$G'(p) = \sqrt{\alpha} e^{-\beta p/2} \left[\sqrt{\alpha} e^{-\beta p/2} \left(\beta(c - p) + 1 \right) + \beta\theta \right]. \tag{20}$$

Thus, sign($G'(p)$) =sign($f_1(p)$), where the function $f_1(x)$ has the following expression:

$$f_1(x) = \frac{\beta\theta e^{\beta x/2}}{\sqrt{\alpha}} + \beta(c - x) + 1. \tag{21}$$

For convenience, we consider the function $f_1(x)$ defined on the set of all real numbers. Next, we analyze some properties of this function.

Lemma 3 *The function $f_1(x)$ given by (21) is strictly convex on $\mathbb{R}$ and it attains its minimum value at the point*

$$p_1 = \frac{2}{\beta} \ln \left(\frac{2\sqrt{\alpha}}{\beta\theta} \right). \tag{22}$$

Proof. Please, see Appendix. ■

Reasoning as we did above with the function $f(x)$, but now taking the function $f_1(x)$, we have the following result to obtain the optimal solution.

Theorem 2 *Suppose that $\gamma = 1$. Let $G(p)$, $f_1(x)$ and p_1 be given, respectively, by (19), (20) and (22).*

1. *If* $p_1 \leq c$, *then* $p^* = \infty$ *and* $G(p^*) = 0$.

2. *If* $p_1 > c$ *and* $f_1(p_1) \geq 0$, *then* $p^* = \infty$ *and* $G(p^*) = 0$.

3. *If* $p_1 > c$ *and* $f_1(p_1) < 0$, *then calculate* $p_0 = \arg_{x \in (c.p_1)}\{f_1(x) = 0\}$.

   (a) *If* $G(p_0) < 0$, *then* $p^* = \infty$ *and* $G(p^*) = 0$.

   (b) *Otherwise,* $p^* = p_0$ *and the maximum profit is* $G(p_0)$.

Proof. Please, see Appendix. ■

In addition, we can obtain some of the above conditions as functions of only the input parameters.

Lemma 4 *We assume that* $\gamma = 1$. *Let* $G(p)$, $f_1(x)$ *and* p_1 *be given, respectively, by (19), (20) and (22). Let* $p_0 = \arg_{p \in (c.p_1)}\{f_1(p) = 0\}$. *Then:*

1. *The condition* $f_1(p_1) \geq 0$ *is equivalent to* $p_1 \leq c + 3/\beta$.

2. *The condition* $G(p_0) < 0$ *is equivalent to* $p_0 > c + 2/\beta$.

Proof. Please, see Appendix. ■

Taking into account this last result, we are now able to establish the following efficient algorithm to determine the optimal inventory policy when $\gamma = 1$.

Algorithm 1

Step 1 *Calculate* p_1 *by using (22).*

Step 2 *If* $p_1 \leq c + 3/\beta$, *then go to* Step 6.

Step 3 *Calculate* $p_0 = \arg_{p \in (c.p_1)}\{f_1(p) = 0\}$.

Step 4 *If* $p_0 > c + 2/\beta$, *then go to* Step 6.

Step 5 *Take* $p^* = p_0$.

Calculate $Q^* = Q^*_{p^*}$ *by using (10).*

Calculate $B^* = B^*_{p^*}$ *by using (11).*

Calculate $G^* = G(p^*)$ *by using (12). Stop.*

Step 6 *Consider* $p^* = \infty$. *Put* $Q^* = 0$, $B^* = 0$ *and* $G^* = 0$. *Stop.*

5 Numerical examples

Next, we illustrate with several examples the theoretical results previously presented in the last section.

Example 1 Consider an inventory system with the hypotheses assumed in this paper. The parameter values of the system are: $c = 8$, $A = 500$, $h = 2$, $\pi = 3.2$, $\alpha = 1250$, $\beta = 0.2$, $\gamma = 1$ and $n = 2.5$. From (13), we have $\theta = 14.2030$. Then $p_1 = 32.1458$. Since $c + 3/\beta = 23 < p_1$, we calculate $p_0 = 14.7572$. Taking into account that $c + 2/\beta = 18$, we see that $p^* = p_0$. From (10), we have $Q^* = 284.543$ and, from (11), $B^* = 50.2245$. Therefore, the optimal profit is $G^* = 211.853$ and the optimal cycle is $T^* = 4.35544$.

Example 2 We assume the same input data and parameters as in the previous example, but we change the value of β to $\beta = 0.4$. Now, we have $p_1 = 12.6072$ and $c+3/\beta = 15.5 \geq p_1$. Thus, we fall into the case described by step 2 of Algorithm 1. Consequently, the inventory system is non-profitable for any selling price p.

Example 3 We consider the same parameters as in Example 1, but modify the value of β to $\beta = 0.3$. Now, we have $p_1 = 18.7274$ and $c + 3/\beta = 18 < p_1$. Therefore, we calculate $p_0 = 15.3505$. Since $c + 2/\beta = 14.6667 < p_0$, we fall into the case described by step 4 of Algorithm 1. As in Example 2, the inventory system is always non-profitable, but in this case the function $G(p)$ has a local maximum at the point p_0.

Example 4 Now, we suppose the same parameters as in Example 1, but change the value of γ to $\gamma = 1.2$. From (15) and (17), we obtain $x_1 = 25.6839$, $c^\gamma = 12.1257$ and $f(x_1) = 0.01145210 \geq 0$. Applying Theorem 1, we deduce, as in the previous example, that the inventory system is always non-profitable.

Example 5 Now, we assume the same parameters as in the previous example, but modify the value of β to $\beta = 0.1$. We obtain $x_1 = 53.7625$, $f(x_1) = -0.0410956$, $x_0 = 24.2391$ and $G(x_0^{1/\gamma}) = 392.908$. Therefore, the optimal selling price is $x_0^{1/\gamma} = 14.2483$, the economic lot size is $Q^* = 370.424$, the maximum shortage quantity is $B^* = 65.3833$ and the optimal cycle $T^* = 3.34565$.

Example 6 We consider the same parameters as in Example 1, but change the value of γ to $\gamma = 0.8$. We obtain $x_1 = 33.7827$, $c^\gamma = 5.27803$, $f(x_1) = -0.0915442$, $x_0 = 11.2920$ and $G(x_0^{1/\gamma}) = 1334.49$. Thus, the optimal selling price is $p^* = x_0^{1/\gamma} = 20.6996$ and the optimal inventory policy is $(Q^*, B^*) = (402.384, 71.0245)$.

5.1 Sensitivity analysis

Next, we include an analysis of the behavior of the best selling price and the optimal inventory policy when the price-dependent demand parameters or the demand pattern index are moved.

We consider the following parameters of the inventory system: $c = 8$, $A = 500$, $h = 2$ and $\pi = 3.2$. We present three tables to show the behavior of p^*, Q^*, B^* and G^*as functions of α, β, γ and n. Tables 2, 3 and 4 display computational results when $\alpha \in \{1000, 1250, 1500\}$, $\beta \in \{0.16, 0.18, 0.20, 0.22\}$, $\gamma \in \{0.8, 0.9, 1, 1.1, 1.2\}$ and $n \in \{0.5, 1, 2\}$. According to the obtained results, we can establish the following issues:

1. Having fixed the price-dependent parameters α, β and γ, if the value of n is increasing, then there is a point $\widehat{n}$ such that $p^*(n) = \infty$ for all $n \leq \widehat{n}$ and $p^*(n)$ is finite when $n > \widehat{n}$. Furthermore, when $n > \widehat{n}$, the optimal selling price p^* and the maximum shortage quantity B^* are strictly decreasing as n increases, while the economic lot size and the maximum profit are strictly increasing.
2. The optimal inventory policy and the best selling price are not very sensitive to changes in the demand pattern index n. The same occurs with the optimal profit. However, the optimal reorder point is more sensitive to changes in the value of n.
3. Having fixed α, γ and n, if the value of β is increasing, then there is a point $\widehat{\beta}$ such that $G(p^*(\beta)) > 0$ for all $\beta < \widehat{\beta}$ and $G(p^*(\beta)) = 0$ if $\beta \geq \widehat{\beta}$. Moreover, when $\beta < \widehat{\beta}$, the economic lot size, the maximum shortage and the optimal profit are strictly decreasing as the parameter β increases . The sensitivity of the optimal policy to the parameter β is slightly greater when the value n is small.
   The same conclusion can be drawn for the parameter γ, if the parameters α, β and n are fixed.
4. Having fixed β, γ and n, if the value of α is increasing, then there exists a point $\widehat{\alpha}$ such that $G(p^*(\alpha)) > 0$ for all $\alpha > \widehat{\alpha}$ and $G(p^*(\alpha)) = 0$ if $\alpha \leq \widehat{\alpha}$. When $\alpha > \widehat{\alpha}$, the optimal selling price is strictly decreasing as the parameter α increases, while the economic lot size, the maximum shortage and the optimal profit are strictly increasing. The sensitivity of the optimal policy to this parameter α is slightly greater when the value n is small.
5. In general, note that the demand sensitivity on price is high for high γ values, which makes the system non-profitable.

Table 2. Effects of α, β and γ on the optimal policy when $n = 0.5$

β	γ	$\alpha = 1000$				$\alpha = 1250$				$\alpha = 1500$			
		p^*	Q^*	B^*	G^*	p^*	Q^*	B^*	G^*	p^*	Q^*	B^*	G^*
0.16	0.8	24.4161	309.807	192.483	1677.76	24.2107	348.787	216.702	2151.58	24.0609	384.021	238.593	2630.35
	0.9	19.1265	278.044	172.749	771.583	18.9071	314.542	195.426	1013.40	18.7485	347.511	215.909	1259.86
	1.0	16.3852	234.175	145.493	299.086	16.1196	267.437	166.159	415.260	15.9312	297.413	184.783	535.656
	1.1	15.1555	176.921	109.921	62.3690	14.7546	207.166	128.713	109.648	14.4850	234.090	145.440	160.743
	1.2	∞	0	0	0	∞	0	0	0	∞	0	0	0
0.18	0.8	22.6832	291.026	180.815	1262.60	22.4637	328.146	203.878	1629.37	22.3039	361.694	224.721	2000.86
	0.9	18.2134	254.827	158.324	541.314	17.9694	289.153	179.651	721.548	17.7939	320.143	198.906	906.144
	1.0	15.9810	206.147	128.080	176.310	15.6639	237.152	147.343	256.970	15.4422	265.023	164.659	341.561
	1.1	15.4029	140.421	87.2440	7.35553	14.8038	169.698	105.434	34.7729	14.4299	195.115	121.225	65.8355
	1.2	∞	0	0	0	∞	0	0	0	∞	0	0	0
0.20	0.8	21.3621	272.835	169.513	956.732	21.1263	308.176	191.470	1243.88	20.9553	340.111	211.311	1535.53
	0.9	17.5486	232.567	144.495	376.281	17.2747	264.880	164.570	511.412	17.0791	294.029	182.681	650.647
	1.0	15.7927	179.041	111.239	93.7861	15.4022	208.145	129.321	149.178	15.1351	234.184	145.499	208.252
	1.1	∞	0	0	0	∞	0	0	0	14.6834	155.820	96.8112	8.55024
	1.2	∞	0	0	0	∞	0	0	0	∞	0	0	0
0.22	0.8	20.3379	255.229	158.574	726.982	20.0836	288.876	179.479	953.647	19.8998	319.272	198.364	1184.59
	0.9	17.0755	211.161	131.195	256.483	16.7647	241.624	150.122	357.974	16.5448	269.072	167.175	463.332
	1.0	15.8290	152.276	94.6094	38.7852	15.3235	179.985	111.825	75.8887	14.9901	204.527	127.073	116.479
	1.1	∞	0	0	0	∞	0	0	0	∞	0	0	0
	1.2	∞	0	0	0	∞	0	0	0	∞	0	0	0

Table 3. Effects of α, β and γ on the optimal policy when $n = 1$

β	γ	$\alpha = 1000$				$\alpha = 1250$				$\alpha = 1500$			
		p^*	Q^*	B^*	G^*	p^*	Q^*	B^*	G^*	p^*	Q^*	B^*	G^*
0.16	0.8	24.3450	322.274	123.952	1692.71	24.1480	362.723	139.509	2168.41	24.0041	399.284	153.571	2648.89
	0.9	19.0503	289.721	111.431	785.016	18.8406	327.584	125.994	1028.59	18.6887	361.789	139.150	1276.64
	1.0	16.2923	244.828	94.1646	310.419	16.0402	279.302	107.424	428.195	15.8609	310.379	119.376	550.036
	1.1	15.0122	186.664	71.7939	70.9706	14.6396	217.852	83.7891	119.703	14.3873	245.670	94.4886	172.093
	1.2	∞	0	0	0	∞	0	0	0	∞	0	0	0
0.18	0.8	22.6072	302.899	116.500	1276.65	22.3968	341.416	131.314	1645.21	22.2435	376.226	144.702	2018.32
	0.9	18.1285	265.811	102.235	553.632	17.8957	301.414	115.928	735.521	17.7281	333.561	128.293	921.610
	1.0	15.8694	216.088	83.1109	186.299	15.5701	248.190	95.4577	268.453	15.3602	277.065	106.563	354.387
	1.1	15.1816	149.977	57.6833	14.2256	14.6420	179.835	69.1675	43.0420	14.2987	205.943	79.2088	75.3236
	1.2	∞	0	0	0	∞	0	0	0	∞	0	0	0
0.20	0.8	21.2803	284.140	109.285	969.910	21.0546	320.808	123.388	1258.76	20.8908	353.943	136.132	1551.95
	0.9	17.4530	242.911	93.4273	387.530	17.1924	276.415	106.313	524.219	17.0060	306.645	117.940	664.859
	1.0	15.6540	188.394	72.4591	102.479	15.2887	218.468	84.0263	159.271	15.0374	245.409	94.3881	219.598
	1.1	∞	0	0	0	∞	0	0	0	14.4840	166.405	64.0019	16.1732
	1.2	∞	0	0	0	∞	0	0	0	∞	0	0	0
0.22	0.8	20.2495	265.993	102.305	739.314	20.0065	300.900	115.731	967.601	19.8306	332.436	127.860	1200.01
	0.9	16.9667	220.918	84.9685	266.706	16.6719	252.489	97.1111	369.665	16.4629	280.944	108.055	476.345
	1.0	15.6467	161.224	62.0093	46.2018	15.1807	189.735	72.9751	84.6355	14.8704	215.062	82.7163	126.406
	1.1	∞	0	0	0	∞	0	0	0	∞	0	0	0
	1.2	∞	0	0	0	∞	0	0	0	∞	0	0	0

Table 4. Effects of α, β and γ on the optimal policy when $n = 2$

β	γ	$\alpha = 1000$				$\alpha = 1250$				$\alpha = 1500$			
		p^*	Q^*	B^*	G^*	p^*	Q^*	B^*	G^*	p^*	Q^*	B^*	G^*
0.16	0.8	24.0924	376.030	81.0478	1746.73	23.9247	422.811	91.1308	2229.18	23.8020	465.098	100.245	2715.75
	0.9	18.7818	340.035	73.2896	833.719	18.6053	383.791	82.7206	1083.61	18.4772	423.328	91.2422	1337.36
	1.0	15.9705	290.620	62.6389	351.814	15.7636	330.346	71.2013	475.332	15.6154	366.190	78.9270	602.351
	1.1	14.5403	228.009	49.1439	103.005	14.2540	263.448	56.7823	156.893	14.0558	295.228	63.6322	213.895
	1.2	∞	0	0	0	13.8898	177.771	38.3159	7.69032	13.5079	207.170	44.6526	26.8773
0.18	0.8	22.3375	354.085	76.3180	1327.47	22.1590	398.625	85.9178	1702.46	22.0287	438.882	94.5947	2081.37
	0.9	17.8307	313.117	67.4878	598.399	17.6363	354.234	76.3500	786.224	17.4958	391.374	84.3551	977.675
	1.0	15.4882	258.712	55.7619	222.996	15.2471	295.595	63.7112	310.488	15.0761	328.829	70.8742	401.228
	1.1	14.5051	189.408	40.8242	40.4312	14.1230	222.479	47.9521	74.1141	13.8679	251.861	54.2849	110.689
	1.2	∞	0	0	0	∞	0	0	0	∞	0	0	0
0.20	0.8	20.9912	332.869	71.7450	1017.63	20.8006	375.262	80.8822	1312.60	20.6618	413.575	89.1400	1611.31
	0.9	17.1200	287.422	61.9496	428.534	16.9044	326.078	70.2814	570.806	16.7494	360.979	77.8038	716.485
	1.0	15.1901	228.295	49.2057	134.674	14.9036	262.662	56.6129	196.453	14.7033	293.553	63.2712	261.255
	1.1	14.8013	150.168	32.3665	2.94751	14.2280	182.316	39.2955	22.7111	13.8740	210.106	45.2853	45.2454
	1.2	∞	0	0	0	∞	0	0	0	∞	0	0	0
0.22	0.8	19.9383	312.379	67.3288	784.043	19.7340	352.723	76.0244	1018.15	19.5856	389.179	83.8819	1255.82
	0.9	16.5906	262.853	56.6540	304.103	16.3496	299.225	64.4935	412.319	16.1774	332.040	71.5664	523.740
	1.0	15.0581	198.991	42.8896	74.0225	14.7079	231.216	49.8353	117.156	14.4681	260.055	56.0510	163.117
	1.1	∞	0	0	0	∞	0	0	0	14.1454	168.069	36.2248	5.07824
	1.2	∞	0	0	0	∞	0	0	0	∞	0	0	0

6 Conclusions

In this paper, we study an inventory system for products whose demand combines multiplicatively a potential function of time and a tri-exponential function of the selling-price (more general than those previously used in the inventory literature). The possibility of stock-out is admitted and, in such a situation, unsatisfied demand is served when a new replenishment arrives in the system. The objective is to maximize the profit per unit time, assuming that the inventory cost is the sum of the costs of ordering, purchasing, holding and backordering.

We develop an approach to determine the optimal inventory policy, the best selling price and the maximum profit in all possible scenarios. This approach is based on the reduction of the inventory problem with three decision variables to an optimization problem with a single decision variable. Furthermore, we present a simpler algorithm if we consider the exponential price function more commonly used in the inventory literature.

In order to study the effect of the parameters associated with demand rate on the optimal policy and on the maximum profit, we give computational results which permit a sensitivity analysis of the inventory policy to be established.

As future research related to this paper we can cite the following research lines: (i) to study the inventory system with the same hypotheses for perishable items; (ii) to analyze the inventory system under the same assumptions allowing partial backordering; (iii) to develop the inventory system considering stochastic demand; (iv) to assume an infinite rate of replenishment and, hence, to determine the economic production quantity and (v) to consider discounts in purchasing costs.

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Appendix

Proof of Lemma 1.

For a fixed value p, the function $P(Q,B,p)$ is twice-differentiable on the region

$$\Lambda = \{(Q,B) : Q > 0, 0 \leq B \leq Q\}$$

The first partial derivatives are

$$\begin{aligned}
\frac{\partial P(Q,B,p)}{\partial Q} &= \frac{A\alpha e^{-\beta p^{\gamma}}}{Q^2} - \frac{h+\pi}{n+1}\left(1-\frac{B}{Q}\right)^n\left(1+\frac{nB}{Q}\right)+\frac{\pi}{n+1} \qquad (23)\\
\frac{\partial P(Q,B,p)}{\partial B} &= (h+\pi)\left(1-\frac{B}{Q}\right)^n - \pi \qquad (24)
\end{aligned}$$

Thus, the second partial derivatives are given by

$$\begin{aligned}
\frac{\partial^2 P(Q,B,p)}{\partial Q^2} &= -\frac{2A\alpha e^{-\beta p^{\gamma}}}{Q^3} - \frac{n\,(h+\pi)}{Q}\left(1-\frac{B}{Q}\right)^{n-1}\left(\frac{B}{Q}\right)^2\\
\frac{\partial^2 P(Q,B,p)}{\partial B^2} &= -\frac{(h+\pi)n}{Q}\left(1-\frac{B}{Q}\right)^{n-1}\\
\frac{\partial^2 P(Q,B,p)}{\partial Q \partial B} &= \frac{(h+\pi)n}{Q}\left(1-\frac{B}{Q}\right)^{n-1}\frac{B}{Q}
\end{aligned}$$

If we prove that the determinant of the Hessian matrix is positive for all $(Q,B) \in \Lambda$, the first assertion follows, because $\partial^2 P(Q,B,p)/\partial Q^2 < 0$ for all $(Q,B) \in \Lambda$.

Indeed, the determinant of the Hessian matrix is

$$\frac{2A\alpha\,(h+\pi)\,n e^{-\beta p^{\gamma}}}{Q^4}\left(1-\frac{B}{Q}\right)^{n-1}$$

To prove the second assertion, it is sufficient to show that the point (Q_p^*, B_p^*) given by (10) and (11) belongs to Λ and that $\left.\frac{\partial P(Q,B,p)}{\partial Q}\right|_{(Q_p^*,B_p^*)} = \left.\frac{\partial P(Q,B,p)}{\partial B}\right|_{(Q_p^*,B_p^*)} = 0$, which is immediate. ■

Proof of Lemma 2.

The second derivative of $f(x)$ is

$$f''(x) = x^{-(2+1/\gamma)}\left(\frac{\beta\theta e^{\beta x/2}\left[(\beta\gamma x-2)^2+4\gamma\right]}{4\sqrt{\alpha}\gamma}+\frac{(\gamma+1)\,\beta c}{\gamma}\right)+\frac{2}{x^3} \qquad (25)$$

Note that $f''(x) > 0$ for all $x > 0$. Moreover, $\lim_{x\to 0^+} f(x) = \infty$ and $\lim_{x\to\infty} f(x) = \infty$. Therefore, the function $f(x)$ is strictly convex and it attains its minimum at point x_1 given by (17). ■

Proof of Theorem 1.

1. If $x_1 \leq c^{\gamma}$, then $f'(p^{\gamma}) > 0$ for $p > c$ and, therefore, $f(p^{\gamma}) > f(c^{\gamma}) > 0$ for all $p \geq c$. Since sign($G'(p)$) =sign($f(p^{\gamma})$), we see that the function $G(p)$ is strictly increasing on (c,∞).
2. If $x_1 > c^{\gamma}$ and $f(x_1) \geq 0$, then it is obvious that $f(p^{\gamma}) > f(x_1) \geq 0$ for all $p \neq x_1^{1/\gamma}$ and we conclude as in the previous case.

3. If $x_1 > c^\gamma$ and $f(x_1) < 0$, then there exist two roots x_0 and $\widetilde{x}$ of the equation $f(x) = 0$, with $c^\gamma < x_0 < x_1 < \widetilde{x}$, such that the function $f(x)$ is positive on (c^γ, x_0), negative on $(x_0, \widetilde{x})$ and positive on $(\widetilde{x}, \infty)$. Thus, the function $G(p)$ is strictly increasing on (c^γ, x_0), strictly decreasing on $(x_0, \widetilde{x})$ and strictly increasing on $(\widetilde{x}, \infty)$. Therefore, $G(p)$ attains its minimum at $p^* = x_0^{1/\gamma}$ or $p^* = \infty$. Comparing the values $G(x_0^{1/\gamma})$ and $\lim_{p\to\infty} G(p) = 0$, we obtain the optimal selling price. ■

Proof of Lemma 3.

From (21), we have

$$f_1'(x) = \beta\left(\frac{\beta\theta e^{\beta x/2}}{2\sqrt{\alpha}} - 1\right) \tag{26}$$

and

$$f''(x) = \frac{\beta^3\theta e^{\beta x/2}}{4\sqrt{\alpha}} > 0 \tag{27}$$

The rest of the proof follows from $\lim_{x\to-\infty} f_1(x) = \infty$ and $\lim_{x\to\infty} f_1(x) = \infty$. ■

Proof of Theorem 2.

This follows by the same method as in the proof of Theorem 1. ■

Proof of Lemma 4.

1. Substituting p_1 given by (22) in the function $f_1(x)$ given by (21), we have $f_1(p_1) = \beta(c - p_1) + 3$ and, therefore $f_1(p_1) \geq 0$ is equivalent to $p_1 \leq c + 3/\beta$.
2. Taking into account that $f_1(p_0) = 0$, it is verified that

$$\theta = -\sqrt{\alpha e^{-\beta p_0}}\left(c - p_o + \frac{1}{\beta}\right).$$

   Substituting this value into the expression (19) of the function, G becomes

$$G(p_0) = \alpha e^{-\beta p_0}\left(c - p_o + \frac{2}{\beta}\right).$$

   Thus, $G(p_0) < 0$ if and only if $p_o > c + 2/\beta$. ■

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