Maximization of the return on inventory management expense in a system with price- and stock-dependent demand rate

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Abstract

This paper considers an inventory model where the demand rate depends on the selling price and the stock level. A lower price or higher stock level lead to a higher demand rate. Three decision variables are considered: the selling price, the order-level and the reorder point. The goal is the maximization of the return on inventory management expense (ROIME), which is defined as the ratio between the profit and the total cost of the inventory system. The optimal values of the selling price, the order level, the reorder point, the lot size, the maximum ROIME and the cycle time are proposed, and the condition that ensures the profitability of the inventory system is established. The partial derivatives of these optimal values with respect to the initial parameters are calculated to analyse the sensitivity of the optimal policy concerning the parameters of the model. The profitability thresholds for each parameter, keeping all the others fixed, are also evaluated. A comparison between the solution with maximum ROIME and the solution with maximum profit per unit time is illustrated by using a numerical example. The solutions can be very different. Maximizing the return on inventory management expense leads to a zero-ending policy at the end of an inventory cycle, so the order-level is equal to the lot size. On the other hand, maximizing the profit per unit time requires a lower selling price, a higher lot size and a non-zero reorder point.

Keywords: Inventory; EOQ models; pricing decision; ROIME maximization; price- and stock-dependent demand rate

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1 Introduction

The literature on deterministic inventory models began by considering that the demand rate was constant along the inventory cycle. This assumption was a serious restriction because, in real life, the demand of an item can depend on such multiple factors as the selling price, stock level, quality, lead time, advertising or rebate.

Levin et al. (1972) and Silver & Peterson (1985) showed that the demand rate of some items may be influenced by the stock level. Indeed, large piles of goods displayed in a supermarket sometimes lead customers to buy more. This issue called into question two common rules for inventory managers: always keep low stock levels to minimize the costs of the inventory management, and match a new order just when the stock is depleted. The reason for this lies in the fact that the high stock-levels and the removal of stock-out increase the sales of the item and the profit of the inventory system per unit time, although the inventory costs are also increased. Then, it might be profitable to raise the order-level in each cycle and request a new order before the stock runs out. Thereby, Baker & Urban (1988) devised a deterministic inventory model with a stock-dependent demand rate, where the goal was the maximization of the profit per unit time and two decision variables were used: the order-level and the reorder point. Nevertheless, the objective function did not satisfy the usual quasi-concavity condition and a general solution to the problem in a closed-form could not be obtained. However, they solved a numerical example by using separable programming and found that the optimal solution had a non-zero reorder point. Since then, many papers on inventory models have considered stock-dependent demand rate. Urban (2005) published an overview of the inventory-level-dependent demand literature with more than 50 papers on the subject, and many others have appeared later. Choudhury et al. (2015), Chen et al. (2016), Pervin et al. (2017), Duan et al. (2017), and Pando et al. (2012, 2018) are some of the most recent papers.

Another factor that clearly affects the demand is the selling price of the item, because lower selling prices usually tend to increase sales. There are also many research papers considering that the demand rate is a deterministic function of the selling price. Eliashberg & Steinberg (1987), Petruzzi & Dada (1999), Chou & Parlar (2006), and Alfares & Ghaithan (2016) used decreasing linear functions. Instead, Ray et al. (2005), Chen et al. (2006), and Agrawal & Ferguson (2007) chose potential functions with a negative exponent. On the other hand, Jeuland & Shugan (1988), Hanssens & Parsons (1993), and Song et al. (2008) preferred negative exponential functions. Other inventory models with price-dependent demand have been derived by Modak & Kelle (2019) and Rapolu & Kandpal (2020).

However, it seems more realistic to assume that demand depends simultaneously on both factors: the stock-level and the selling price. A remarkable number of papers in the recent literature on inventory models have considered this topic. Teng & Chang (2005) studied an economic production quantity model with an additive effect of the selling price and the level of stock on the demand rate. Dye & Hsieh (2011) used this assumption in an EOQ model under fluctuating cost and limited capacity. Also, Soni (2013) and Mishra et al. (2017) did the same for inventories with deteriorating items. Instead, Pal et al. (2014), Onal et al. (2016) and Feng et al. (2017) considered a multiplicative effect of selling price and stock-level on the demand rate. San-José et al. (2020) also looked at the multiplicative effect of two factors on the demand rate, but they used the selling price and the time.

Nevertheless, all these papers are focused on the maximization of the profit per unit time, or the minimization of the inventory cost per unit time. Working with this type of models, we have observed that the policy of maximum profit per unit time leads to a great inventory cost per unit time, while on the other hand, the optimal policy for the minimum inventory cost per unit time results in a low profit per unit time. Thus, the inventory manager could perhaps prefer a balance between both policies with a good profit per unit time without a large inventory cost. That is, perhaps the goal should be the maximization of the ratio between the profit and the total cost of the inventory system. This ratio can be named as the return on inventory management expense (ROIME), so that a ROIME with value 0.40 means that the profit is 40% of the total expense in the inventory management. In this way, if diverse investment options are possible, the manager could allocate the available resources to the most profitable products to get the highest yield from the money. Then, the ROIME is a profitability ratio similar to the return on investment (ROI) defined as the ratio of the net profit over the cost of an investment, which was considered by other authors in inventory management. Profit and profitability do not always go together. Some business can yield a high profit per unit time but low profitability because they need a lot of money or resources to run. The option with the maximum profit per unit time does not always lead to the highest profitability. Therefore, in inventory theory, it can also be interesting to know which is the optimal policy with maximum profitability as an alternative to the maximum profit per unit time. Thus, the manager could allocate the available capital to the most profitable products of the supply chain, instead of concentrating resources on the product with the highest profit per unit time.

Although multiple optimization criteria have been used in inventory theory (see, for example, Arcelus & Srinivasan, 1987), the literature on inventory models with the aim of maximizing profitability ratios is not so extensive. Schroeder & Krishnan (1976) enumerated the conditions under which ROI is an appropriate criterion and contrasted it with the usual cost minimization and profit maximization criteria. Morse & Scheiner (1979) proposed as objective the maximization of the residual income (RI), which is the ratio of the excess of net income over the opportunity cost of invested capital. Later, Trietsch (1995) adapted the economic order quantity model to the objective of maximizing return on investment in inventory. Since then, Otake et al. (1999), Li et al. (2008), Choi & Chiu (2012), and Hidayat & Fauzi (2015) are some of the papers in this research line. Recently, Pando et al. (2019, 2020) obtained the optimal policy with the maximum return on inventory management expense in both inventory models with stock-dependent demand rate and non-linear holding cost.

Nonetheless, all these papers with profitability ratios maximization consider that the selling price of the item is a preset parameter of the model and the demand rate does not depend on this value. We have not found research papers on inventory models with price and stock-dependent demand rate that also assumes the maximization of the return on inventory management expense as objective. This is the gap in the published literature that this manuscript could fill. To compare the contribution of the cited references to the inventory theory, Table 1 collects the most relevant papers previously cited, classifying them according to the type of demand (constant, stock-dependent or price-dependent) and the objective considered (profit or profitability). As far we know, it can be seen that Table 1 shows the absence of papers with price- and stock-dependent demand rate, which assume a profitability ratio as objective.

Table 1. Summary of most relevant literature cited in this paper

	Demand rate:			Objective function:	
Paper	Constant	Stock-dependent	Price-dependent	Profit	Profitability
Schroeder & Krishnan (1976)	✓			✓	✓
Morse & Scheiner (1979)	✓			✓	✓
Arcelus & Srinivasan (1987)			✓	✓	✓
Eliashberg & Steinberg (1987)			✓	✓	
Baker & Urban (1988)		✓		✓	
Trietsch (1995)	✓				✓
Otake et al. (1999)	✓				✓
Chou & Parlar (2006)			✓	✓	
Li et al. (2008)	✓				✓
Pando et al. (2012)		✓		✓	
Soni (2013)		✓	✓	✓	
Pal et al. (2014)		✓	✓	✓	
Alfares & Ghaithan (2016)			✓	✓	
Chen et al. (2016)		✓		✓	
Onal et al. (2016)		✓	✓	✓	
Mishra et al. (2017)		✓	✓	✓	
Feng et al. (2017)		✓	✓	✓	
Pando et al. (2018)		✓		✓	
Pando et al. (2019)		✓			✓
San-José et al. (2020)			✓	✓	
Pando et al. (2020)		✓			✓
This paper		✓	✓		✓

The rest of the paper is organized as follows. Section 2 presents the assumptions and notation of the model. In Section 3, the model is formulated and the return on inventory management expense is evaluated to obtain the objective function to maximize, using the selling price as a decision variable along with the order-level and the reorder point. Section 4 provides the solution of the model for the three decision variables, and calculates the optimal return on inventory management expense. Also, some interesting properties and profitability thresholds for the initial parameters are proposed. Section 5 presents a sensitivity analysis by calculating the partial derivatives of the optimal values with respect to the parameters of the model, and some interesting rules about the relative role of the parameters on the solution are given. In Section 6, numerical examples are presented to illustrate all the obtained results, and to compare the optimal policy of the proposed model with the other one for the maximization of the profit per unit time. Finally, some conclusions and future research lines are given in Section 7.

2 Assumptions and notation

This paper considers an inventory system with the following basic assumptions: (i) there is a single item, (ii) the planning horizon is infinite, (iii) the inventory is continuously reviewed, (iv) the replenishment is instantaneous, and (v) shortages are not allowed. Also, the unit purchasing cost, $c > 0$, the ordering cost, $K > 0$, and the holding cost per unit and per unit time, $h > 0$, are all fixed, regardless of the order size.

Moreover, the demand rate for the item depends on the selling price, p, and the quantity of items in stock, x, by the following function $D(p,x) = \lambda e^{-\alpha p} x^{\beta}$, with $\lambda > 0$, $\alpha > 0$ and $0 \leq \beta < 1$. As a consequence, the demand rate is greater with a low price and a high stock level. Note that $\alpha = -\frac{\partial D(p,x)/\partial p}{D(p,x)}$ and this parameter represents the relative decrease in the demand rate per unit of increase in the selling price. Similarly, $\beta = \frac{\partial D(p,x)/\partial x}{D(p,x)/x}$ and it represents the relative increase in the demand rate per unit of the relative increase in the level of stock. For example, if $\alpha = 0.01$ and $\beta = 0.1$, the demand rate decreases by 1% if the selling price increases one unit and it increases by 0.1% if the inventory level increases by 1%. As it was proposed by Feng et al. (2017), the λ coefficient is the scale parameter, and it can be seen as the maximum number of potential consumers, because the demand rate would be λ if there was only one item in stock and it was free. Thus, varying the three parameters of the demand rate, a lot of real practical situations can be modelled by the manager.

Denoting by t the elapsed time in the inventory, and by $I(t)$ the inventory level at time t, three decision variables are considered in the model: the order-level S (inventory level to order-up-to) with $S > 0$, the reorder point s (inventory level to set an order) with $0 \leq s < S$, and the selling price p. Then, the lot size is $q = S - s$, it is clear that $S = \lim_{t \to 0^+} I(t) = I(0+)$ and the length of the inventory cycle T is given by the equation $I(T-) = s$, where $I(T-) = \lim_{t \to T^-} I(t)$.

The notation used in this paper is resumed in Table 2.

Table 2. Notation for the inventory model

c	Unit purchasing cost ($c > 0$)
K	Ordering cost per order ($K > 0$)
h	Holding cost per unit and per unit time ($h > 0$)
T	Length of the inventory cycle ($T > 0$)
t	Elapsed time in the inventory ($0 \leq t \leq T$)
$I(t)$	Inventory level at time t
x	Quantity of items in stock at time t ($x = I(t)$)
p	Unit selling price, *decision variable* ($p > c$)
S	Order level, *decision variable* ($S > 0$)
s	Reorder point, *decision variable* ($0 \leq s < S$)
q	Lot size ($q = S - s$)
λ	Scale parameter of the demand rate ($\lambda > 0$)
α	Elasticity parameter of the demand rate regarding the selling price ($\alpha > 0$)
β	Elasticity parameter of the demand rate regarding the stock level ($0 \leq \beta < 1$)
$D(p,x)$	Demand rate with selling price p and x items in stock ($D(p,x) = \lambda e^{-\alpha p} x^{\beta}$)

3 Model formulation

Taking into account the assumptions described in previous section, it follows that, for $0 \leq t < T$, the inventory level curve $I(t)$ is obtained by solving the differential equation

$$\frac{d}{dt} I(t) = -\lambda e^{-\alpha p} \left(I(t)\right)^{\beta} \tag{1}$$

with initial condition $I(0+) = S$. The solution can be written as

$$I(t) = \left(S^{1-\beta} - (1-\beta)\lambda e^{-\alpha p} t\right)^{1/(1-\beta)} \tag{2}$$

Moreover, as $I(T-) = s$, the value for the cycle time is given by

$$T = \frac{S^{1-\beta} - s^{1-\beta}}{(1-\beta)\lambda e^{-\alpha p}} \tag{3}$$

and the holding cost $H(p, S, s)$ along an inventory cycle can be evaluated as

$$H(p, S, s) = \int_0^T hI(t)dt = \frac{-h}{\lambda e^{-\alpha p}} \int_0^T (I(t))^{1-\beta} \, dI(t) = \frac{h\left(S^{2-\beta} - s^{2-\beta}\right)}{(2-\beta)\lambda e^{-\alpha p}} \tag{4}$$

The income obtained in each inventory cycle is $pq = p(S-s)$. The total inventory management expense is the sum of the purchasing cost $c(S-s)$, the ordering cost K, and the holding cost $H(p, S, s)$. Then, the total inventory management expense per unit time $C(p, S, s)$ is

$$C(p, S, s) = \frac{c(S-s) + K + H(p, S, s)}{T} \tag{5}$$

The profit or gain per unit time $G(p, S, s)$ is given by

$$G(p, S, s) = \frac{(p-c)(S-s) - K - H(p, S, s)}{T} \tag{6}$$

and the return on inventory management expense, named as $R(p, S, s)$, is the ratio between these two quantities, that is,

$$R(p, S, s) = \frac{G(p, S, s)}{C(p, S, s)} = \frac{p}{c + r(p, S, s)} - 1, \tag{7}$$

where

$$r(p, S, s) = \frac{K + H(p, S, s)}{S - s} = \frac{K}{S - s} + \frac{h\left(S^{2-\beta} - s^{2-\beta}\right)}{\lambda e^{-\alpha p}(2-\beta)(S-s)} \tag{8}$$

depicts the average inventory cost for each item (excluding the purchasing cost), instead of the inventory cost per unit time mostly used in inventory theory. Moreover, as $r(p, S, s) > 0$, the next bounds for $R(p, S, s)$ can be established: $-1 < R(p, S, s) < p/c - 1$.

Remark 1 *From the perspective of financial investment, the management of an inventory system is acceptable if the return on inventory management expense is positive (that is, if the inventory system is profitable), otherwise, it is rejectable. Thus, from (7) it is clear that if $p \leq c$, then $R(p, S, s) < 0$ for any (p, S, s), and, from the perspective of financial investment, the management of the inventory system is rejectable. For this reason, the usual case in the literature, $p > c$, is considered.*

The goal of the inventory model is to maximize the return on inventory management expense $R(p, S, s)$. Therefore, the mathematical problem is

$$\max_{(p,S,s)\in\Omega} R(p, S, s) \tag{9}$$

where $\Omega = \left\{(p, S, s) \in \mathbb{R}^3 / p > c, S > 0, 0 \leq s < S\right\}$ is the feasible region.

4 Solution of the model

From (7) it is clear that, for each given value of the selling price p, maximizing the function $R_p(S, s) = R(p, S, s)$ is equivalent to minimizing the function $r_p(S, s) = r(p, S, s)$ given by (8). Moreover, if $0 < s < S$ then $K/(S-s) > K/S$ and $S^{2-\beta} - s^{2-\beta} > S^{2-\beta} - S^{1-\beta}s = S^{1-\beta}(S-s)$ and, therefore,

$$r_p(S, s) > r_p(S, 0) = \frac{K}{S} + \frac{hS^{1-\beta}}{\lambda e^{-\alpha p}(2-\beta)} \tag{10}$$

As a consequence, for each given value p, the optimal reorder point is $s_p^* = 0$ and the optimal order-level S_p^* can be obtained by minimizing the function $r_p(S, 0)$ given by (10). The following lemma gives the solution to this problem.

Lemma 1 *Let the function* $f(x) = \dfrac{K}{x} + \dfrac{hx^{1-\beta}}{\lambda e^{-\alpha p}(2-\beta)}$ *with* $x > 0$*. The minimum value of* $f(x)$ *is obtained for*

$$x^* = \left(\frac{(2-\beta)K\lambda e^{-\alpha p}}{(1-\beta)\,h}\right)^{1/(2-\beta)}$$

with

$$f(x^*) = \frac{(2-\beta)K}{(1-\beta)\,x^*} = Ae^{\alpha p/(2-\beta)}$$

where

$$A = \left(\frac{(2-\beta)K}{1-\beta}\right)^{(1-\beta)/(2-\beta)} \left(\frac{h}{\lambda}\right)^{1/(2-\beta)} \tag{11}$$

Proof. Please see the proof in Appendix A. □

From this lemma, it is clear that, for each given value p, the optimal order-level S_p^* is

$$S_p^* = \left(\frac{(2-\beta)K\lambda e^{-\alpha p}}{(1-\beta)\,h}\right)^{1/(2-\beta)} \tag{12}$$

and the optimal ROIME is

$$R^*(p) = R(p, S_p^*, 0) = \frac{p}{c + Ae^{\alpha p/(2-\beta)}} - 1 \tag{13}$$

Therefore, it is only necessary to find the best selling price p which maximizes the function given by (13). To do this, the following lemma introduces an auxiliary parameter in the model which will be useful for finding the stationary points of the function $R^*(p)$.

Lemma 2 *(Auxiliary parameter* B*) Let the function* $\varphi(x) = ce^{-x} + A(1-x)$ *with* $c > 0$ *and* $A > 0$*. Then, the equation* $\varphi(x) = 0$ *has a unique real root* B*. Moreover, it satisfies that* $1 < B < \sqrt{1 + c/A}$*.*

Proof. Please see the proof in Appendix A. □

In order to evaluate this root B, the Newton-Fourier method for solving equations can be used with the following iteration function

$$x - \frac{\varphi(x)}{\varphi'(x)} = x + \frac{ce^{-x} + A(1-x)}{ce^{-x} + A} = 1 + \frac{cx}{c + Ae^{x}}$$

Next, a numerical algorithm to evaluate the auxiliary parameter B with this method is provided. It can be easily implemented with any programming software.

Algorithm 1 *(Evaluation of the auxiliary parameter* B*)*

Step 1 *Calculate* $A = \left(\dfrac{(2-\beta)K}{1-\beta}\right)^{(1-\beta)/(2-\beta)} \left(\dfrac{h}{\lambda}\right)^{1/(2-\beta)}$.

Step 2 *Define the function* $\varphi(x) = ce^{-x} + A(1-x)$.

Step 3 *Select the tolerance* $TOL > 0$ *for the evaluation of* B.

Step 4 *Start with the initial value* $x_o = 1$.

Step 5 *Calculate* $x_i = 1 + \dfrac{cx_{i-1}}{c + Ae^{x_{i-1}}}$ *for* $i = 1, 2, 3, ...,$ *until* $\varphi(x_i + TOL) < 0$ *is satisfied.*

Step 6 *Take* $B = x_i$.

The next theorem uses the auxiliary parameters A and B to find the best selling price which maximizes the return on inventory management expense $R^*(p)$ given by (13).

Theorem 1 *(Optimal selling price) Consider the univariate real valued functions* $g(p) = \dfrac{p}{c + Ae^{\alpha p/(2-\beta)}}$ *and* $\varphi(x) = ce^{-x} + A(1-x)$, *with* $p > 0$, $A > 0$, $c > 0$, $\alpha > 0$ *and* $0 \leq \beta < 1$. *Let* B *be the only value with* $\varphi(B) = 0$. *Then, the global maximum of the function* $g(p)$ *on* $(0, \infty)$ *is obtained at the point*

$$p^* = \frac{(2-\beta)\,B}{\alpha} \tag{14}$$

and it satisfies that $\dfrac{2-\beta}{\alpha} < p^* < \dfrac{(2-\beta)\sqrt{1+c/A}}{\alpha}$.

Proof. Please see the proof in Appendix A. □

Once the optimal selling price p^* has been obtained, the optimal order-level S^* can be evaluated with the expression (12):

$$\begin{aligned} S^* &= S^*_{p^*} = \left(\frac{(2-\beta)K\lambda}{(1-\beta)\,h}\right)^{1/(2-\beta)} e^{-B} = \left(\frac{(2-\beta)K\lambda}{(1-\beta)\,h}\right)^{1/(2-\beta)} \frac{A\,(B-1)}{c} \\ &= \left(\frac{(2-\beta)K\lambda}{(1-\beta)\,h}\right)^{1/(2-\beta)} \left(\frac{(2-\beta)K}{1-\beta}\right)^{(1-\beta)/(2-\beta)} \left(\frac{h}{\lambda}\right)^{1/(2-\beta)} \left(\frac{B-1}{c}\right) = \frac{(2-\beta)\,(B-1)\,K}{(1-\beta)\,c} \end{aligned} \tag{15}$$

Since the optimal reorder point is $s^* = 0$, the optimal lot size q^* coincides with the optimal order-level S^* given by (15).

Remark 2 *Since a feasible selling price must be in the region* Ω, *from now on we assume that the parameters of the model satisfy that* $\alpha c < (2-\beta)B$. *This condition ensures that* $p > c$ *and, therefore, the point* $(p^*, S^*_{p^*}, 0)$ *belongs to* Ω *and it is the solution of the problem (9).*

Also, taking into account that $Ae^B = c/(B-1)$, the optimal ROIME can be evaluated with the expression (13), obtaining the following value:

$$R^* = R^*(p^*) = \frac{p^*}{c + Ae^B} - 1 = \frac{(2-\beta)\,B/\alpha}{c + c/(B-1)} - 1 = \frac{(2-\beta)\,(B-1)}{\alpha c} - 1 \tag{16}$$

Therefore, the condition that ensures the profitability of the inventory system is

$$\alpha < \frac{(2-\beta)\,(B-1)}{c} \tag{17}$$

Thus, from Remark 2, if $(2-\beta)\,(B-1)\,/c \leq \alpha < (2-\beta)\,B/c$, then the inventory system is unprofitable regardless of the selling price $p > c$. As a consequence, the manager should also give up the investment in the inventory in this case. Note that, for the limit case with $\alpha = (2-\beta)\,(B-1)\,/c$, the equality $(2-\beta)B = \alpha c + 2 - \beta$ is satisfied, and the optimal selling price is $p^* = c + (2-\beta)\,/\alpha$, with $R^* = 0$. Then, if the inventory system is profitable, the optimal selling price satisfies that

$$c + \frac{2-\beta}{\alpha} < p^* < \frac{(2-\beta)\sqrt{1+c/A}}{\alpha} \tag{18}$$

The optimal average inventory cost per each item (excluding the purchasing cost), given by (8), is $r^* = r(p^*, S^*, 0) = Ae^B = c/(B-1)$. That is, c is the unit purchasing cost and $c/(B-1)$ is the optimal average inventory cost per item. Then, $r^*/(c+r^*) = B^{-1}$ and, therefore, B^{-1} is the ratio between the inventory cost and the total expense per item.

Furthermore, as $B = \alpha p^*/(2-\beta)$ and $e^{-B} = A(B-1)/c$, using the expressions (3), (11) and (12), the optimal length of the cycle time can be evaluated as:

$$\begin{aligned} T^* &= \frac{(S^*)^{1-\beta}}{(1-\beta)\lambda e^{-\alpha p^*}} = \frac{\left(\frac{(2-\beta)K}{1-\beta}\right)^{(1-\beta)/(2-\beta)}\left(\frac{h}{\lambda}\right)^{1/(2-\beta)}\left(\frac{\lambda}{h}\right)}{(1-\beta)\lambda e^{-\alpha p^*/(2-\beta)}} \\ &= \frac{A}{(1-\beta)he^{-B}} = \frac{c}{(1-\beta)(B-1)h} \end{aligned} \tag{19}$$

Note that the optimal lot size q^* and the optimal cycle time T^* do not depend on the parameter α, because the auxiliary parameters A and B do not depend on α. Thus, the optimal inventory policy does not change if the dependence degree of the demand rate concerning the selling price changes. On the other hand, however, the optimal selling price and the ROIME naturally change.

Table 3 summarizes all the expressions for the optimal policy of the inventory system, based solely on the initial parameters and the auxiliary parameter B. The five columns contain, respectively, the optimal values for the selling price, the reorder point, the order level (which coincides with the lot size), the maximum ROIME, and the cycle time.

Table 3. Optimal values for the maximum ROIME policy

p^*	s^*	$S^* = q^*$	R^*	T^*
$\frac{(2-\beta)B}{\alpha}$	0	$\frac{(2-\beta)(B-1)K}{(1-\beta)c}$	$\frac{(2-\beta)(B-1)}{\alpha c} - 1$	$\frac{c}{(1-\beta)(B-1)h}$

From (5) and (8), the total cost per unit time for the optimal solution is

$$C(p^*, S^*, 0) = \frac{(c+r^*)S^*}{T^*} = \frac{\left(c + \frac{c}{B-1}\right)\left(\frac{(2-\beta)(B-1)K}{(1-\beta)c}\right)}{\frac{c}{(1-\beta)(B-1)h}} = \frac{(2-\beta)(B-1)BKh}{c}$$

and the profit per unit time is

$$\begin{aligned} G(p^*, S^*, 0) &= \frac{(p^* - c - r^*)S^*}{T^*} = \frac{\left(p^* - c - \frac{c}{B-1}\right)\left(\frac{(2-\beta)(B-1)K}{(1-\beta)c}\right)}{\frac{c}{(1-\beta)(B-1)h}} \\ &= \frac{\left(p^* - \frac{cB}{B-1}\right)(2-\beta)(B-1)^2 Kh}{c^2} = \left(\frac{p^*\left(1-B^{-1}\right)}{c} - 1\right) C(p^*, S^*, 0) \end{aligned}$$

From (14), $p^* B^{-1} = (2-\beta)/\alpha$ and the optimal ROIME can also be calculated as

$$R^* = \frac{G(p^*, S^*, 0)}{C(p^*, S^*, 0)} = \frac{p^* - \frac{2-\beta}{\alpha}}{c} - 1 \tag{20}$$

Moreover, from (20) and (18), an acceptable inventory system satisfies that

$$0 < R^* < \frac{(2-\beta)\left(\sqrt{1+c/A} - 1\right)}{\alpha c} - 1$$

Note that, the value $p^*/c - 1$ would be the return on inventory management expense, if there was no inventory cost. Therefore, if there was no inventory cost, to obtain the same ROIME, a decrease in the selling price of magnitude $(2-\beta)/\alpha$ could be made. Thus, $(2-\beta)/(\alpha c)$ is the loss of profitability caused by the inventory cost.

Furthermore, taking into account that $r^* S^* = (2-\beta)K/(1-\beta)$, the ratio between the average ordering cost per item and the average inventory cost per item for the optimal solution can be evaluated as

$$\frac{K/S^*}{r^*} = \frac{1-\beta}{2-\beta} \leq 0.5$$

and the ordering cost is less than or equal to 50% of the total inventory cost. Therefore, the ordering cost is less than or equal to the holding cost for the solution of the maximum return on inventory management expense. If the demand rate does not depend on the inventory level, that is $\beta = 0$, then the ordering cost is equal to the holding cost, just as in the basic EOQ model with a preset selling price.

In order to obtain the acceptability condition for the other parameters of the model, the following result is provided.

Lemma 3 *The inventory system is acceptable if, and only if, the initial parameters of the model satisfy the following inequality:*

$$\frac{K^{1-\beta}h}{\lambda} < \frac{\Gamma}{e^{\alpha c}} \tag{21}$$

with

$$\Gamma = \frac{(2-\beta)(1-\beta)^{1-\beta}}{\alpha^{2-\beta}e^{2-\beta}} \tag{22}$$

Proof. Please see the proof in Appendix A. □

Note that Γ is an auxiliary parameter which only depends on the elasticity parameters of the model.

Then, the expressions (21) and (22) can be used to obtain the acceptability conditions for the parameters K, h, λ, and c, and to evaluate the profitability threshold for each parameter, keeping all the others fixed. The results are given in Table 4.

Table 4. Profitability thresholds for the initial parameters of the inventory system

Parameter	K	h	c	λ	α
Threshold	$< \left(\frac{\lambda\Gamma}{he^{\alpha c}}\right)^{1/(1-\beta)}$	$< \frac{\lambda\Gamma}{K^{1-\beta}e^{\alpha c}}$	$< \frac{\ln\left(\frac{\lambda\Gamma}{K^{1-\beta}h}\right)}{\alpha}$	$> \frac{K^{1-\beta}he^{\alpha c}}{\Gamma}$	$< \frac{(2-\beta)(B-1)}{c}$

5 Sensitivity analysis

In this section, the expressions given in Table 3 are used to develop a sensitivity analysis by calculating the partial derivatives of p^*, q^*, R^* and T^* with respect to all the parameters of the model. Table 5 shows the expressions for all these partial derivatives, and all the proofs can be found in Appendix B.

Table 5. Partial derivatives of p^*, q^*, R^* and T^* with respect to the parameters of the model

	$\partial p^*/\partial x$	$\partial q^*/\partial x$	$\partial R^*/\partial x$	$\partial T^*/\partial x$
$x = K$	$\frac{(1-\beta)(1-B)}{\alpha BK} < 0$	$\frac{(2-\beta)(B-1)\left(B-\left(\frac{1-\beta}{2-\beta}\right)\right)}{c(1-\beta)B} > 0$	$\frac{(1-\beta)(1-B)}{c\alpha KB} < 0$	$\frac{c}{(2-\beta)(B-1)BKh} > 0$
$x = h$	$\frac{1-B}{\alpha Bh} < 0$	$\frac{-K(B-1)}{ch(1-\beta)B} < 0$	$\frac{1-B}{c\alpha hB} < 0$	$\frac{-c\left(1-\frac{1}{(2-\beta)B}\right)}{(1-\beta)(B-1)h^2} < 0$
$x = \lambda$	$\frac{B-1}{\alpha B\lambda} > 0$	$\frac{K(B-1)}{c\lambda(1-\beta)B} > 0$	$\frac{B-1}{c\alpha\lambda B} > 0$	$\frac{-c}{(1-\beta)(2-\beta)(B-1)Bh\lambda} < 0$
$x = c$	$\frac{(2-\beta)(B-1)}{\alpha cB} > 0$	$\frac{-K(2-\beta)(B-1)^2}{(1-\beta)c^2B} < 0$	$\frac{-(2-\beta)(B-1)^2}{\alpha c^2B} < 0$	$\frac{1}{h(1-\beta)B} > 0$
$x = \alpha$	$\frac{-(2-\beta)B}{\alpha^2} < 0$	0	$\frac{-(2-\beta)(B-1)}{c\alpha^2} < 0$	0
$x = \beta$	$\frac{(1-B)\Delta}{(2-\beta)\alpha B} - \frac{B}{\alpha}$	$\frac{K(B-1)\left(1-\frac{(1-\beta)\Delta}{(2-\beta)B}\right)}{c(1-\beta)^2}$	$\frac{(1-B)\left(1+\frac{\Delta}{(2-\beta)B}\right)}{c\alpha}$	$\frac{c\left(1+\frac{(1-\beta)\Delta}{(2-\beta)^2B}\right)}{h(1-\beta)^2(B-1)}$

where

$$\Delta = 1 + \ln\left(\frac{(1-\beta)\,h}{(2-\beta)\,K\lambda}\right) \tag{23}$$

From these expressions, the next lemma provides the sensitivity analysis for the best selling price p^* with respect to all the parameters of the inventory system.

Lemma 4 *Let p^* be the optimal selling price given by (14), and the auxiliary parameters Δ and B given by (23) and Lemma 2, respectively. Then, it is satisfied that:*

(i) p^* decreases as one of the parameters K, h, or α increases.

(ii) p^* increases as one of the parameters c or λ increases.

(iii) If $\Delta < -\frac{(2-\beta)B^2}{B-1}$ then p^* increases as the parameter β increases. In any other case, p^* decreases as β increases.

Proof. (i) and (ii) follow directly from the second column of Table 5. Also, taking into account that $\frac{(1-B)\Delta}{(2-\beta)\alpha B} - \frac{B}{\alpha} > 0$ if, and only if, $\Delta < -\frac{(2-\beta)B^2}{B-1}$, the assertion (iii) is proven. □

Also, from Table 5 and (14), it is easily seen that $\dfrac{\partial p^*/\partial\alpha}{p^*/\alpha} = -1$, $\dfrac{\partial p^*/\partial c}{p^*/c} = \dfrac{B-1}{B^2}$ and

$$-\left(\frac{1}{1-\beta}\right)\left(\frac{\partial p^*/\partial K}{p^*/K}\right) = -\left(\frac{\partial p^*/\partial h}{p^*/h}\right) = \left(\frac{\partial p^*/\partial\lambda}{p^*/\lambda}\right) = \left(\frac{1}{2-\beta}\right)\left(\frac{\partial p^*/\partial c}{p^*/c}\right) = \frac{B-1}{(2-\beta)\,B^2}$$

Then, as $\frac{B-1}{(2-\beta)B^2} < \frac{B-1}{B^2} < 1$ and $1-\beta \le 1$, the comparison leads to

$$\left|\frac{\partial p^*/\partial K}{p^*/K}\right| \le \left|\frac{\partial p^*/\partial h}{p^*/h}\right| = \frac{\partial p^*/\partial\lambda}{p^*/\lambda} < \frac{\partial p^*/\partial c}{p^*/c} < \left|\frac{\partial p^*/\partial\alpha}{p^*/\alpha}\right|$$

Note that, each of these fractions is the ratio between the relative change in the best selling price and the relative change in each of the parameters. Therefore, from a comparative point of view, and in a decreasing order of degree of sensitiviy, the optimal selling price is more sensitive to changes in the value of the parameter α, then to changes in the value of c, and then to changes in λ or h. The optimal selling price is less sensitive with respect to K. If the demand rate does not depend on the stock level, that is $\beta = 0$, the relative effects of the parameters K, h and λ are equal except for the sign. Also, if $\beta = 0$, the comparative effect of the purchasing cost c on the optimal selling price is around two times the relative effects of the parameters K, h or λ.

In a similar way, the next lemma provides the sensitivity analysis for the optimal lot size q^* with respect to all the parameters of the inventory system.

Lemma 5 *Let q^* be the optimal lot size given by (15), and the auxiliary parameters Δ and B given by (23) and Lemma 2, respectively. Then, it is satisfied that:*

(i) q^* *increases as one of the parameters* K *or* λ *increases.*

(ii) q^* *decreases as one of the parameters* h *or* c *increases.*

(iii) q^* *does not depend on the parameter* α.

(iv) If $\Delta < \frac{(2-\beta)B}{1-\beta}$ *then* q^* *increases as the parameter* β *increases. In any other case,* q^* *decreases as* β *increases.*

Proof. (i), (ii) and (iii) follow directly from the third column of Table 5. Also, taking into account that $\frac{(1-\beta)\Delta}{(2-\beta)B} < 1$ if, and only if, $\Delta < \frac{(2-\beta)B}{1-\beta}$, the assertion (iv) is proven. □

For the relative changes, using the partial derivatives and the expression (15), you can see that

$$\frac{\left(\frac{\partial q^*/\partial K}{q^*/K}\right)}{1+(2-\beta)(B-1)} = -\left(\frac{\partial q^*/\partial h}{q^*/h}\right) = \left(\frac{\partial q^*/\partial \lambda}{q^*/\lambda}\right) = -\left(\frac{\frac{\partial q^*/\partial c}{q^*/c}}{(2-\beta)(B-1)}\right) = \frac{1}{(2-\beta)B}$$

Then, the comparison leads to

$$\left|\frac{\partial q^*/\partial h}{q^*/h}\right| = \frac{\partial q^*/\partial \lambda}{q^*/\lambda} < \frac{\partial q^*/\partial K}{q^*/K}$$

and

$$\left|\frac{\partial q^*/\partial c}{q^*/c}\right| < \frac{\partial q^*/\partial K}{q^*/K}$$

Therefore, from a comparative point of view, the optimal lot size is more sensitive to a relative change in K than in the parameters c, λ or h. Also, it is equally sensitive with respect to h and λ, but with the opposite sign. The relative changes with respect to the parameter c can be higher or lower than with respect to h or λ.

The sensitivity analysis for the maximum return on inventory management expense R^* is given by the next lemma.

Lemma 6 *Let* R^* *be the optimal return on inventory management expense given by (16), and the auxiliary parameters* Δ *and* B *given by (23) and Lemma 2, respectively. Then, it is satisfied that:*

(i) R^* *decreases as one of the parameters* K, h, c, *or* α *increases.*

(ii) R^* *increases as the parameter* λ *increases.*

(iii) If $\Delta < -(2-\beta)B$ *then* R^* *increases as the parameter* β *increases. In any other case,* R^* *decreases as* β *increases.*

Proof. (i) and (ii) follow directly from the fourth column of Table 5. Also, taking into account that $\frac{\Delta}{(2-\beta)B} + 1 < 0$ if, and only if, $\Delta < -(2-\beta)B$, the assertion (iii) is proven. □

For the relative changes, the relationship is

$$-\left(\frac{\frac{\partial R^*/\partial K}{R^*/K}}{1-\beta}\right) = -\left(\frac{\partial R^*/\partial h}{R^*/h}\right) = \left(\frac{\partial R^*/\partial \lambda}{R^*/\lambda}\right) = -\left(\frac{\frac{\partial R^*/\partial \alpha}{R^*/\alpha}}{(2-\beta)B}\right) = -\left(\frac{\frac{\partial R^*/\partial c}{R^*/c}}{(2-\beta)(B-1)}\right)$$

and, taking into account that $1-\beta \le 1 < (2-\beta)B$, the comparison leads to

$$\left|\frac{\partial R^*/\partial K}{R^*/K}\right| \le \left|\frac{\partial R^*/\partial h}{R^*/h}\right| = \frac{\partial R^*/\partial \lambda}{R^*/\lambda} < \left|\frac{\partial R^*/\partial \alpha}{R^*/\alpha}\right|$$

and

$$\left|\frac{\partial R^*/\partial c}{R^*/c}\right| < \left|\frac{\partial R^*/\partial \alpha}{R^*/\alpha}\right|$$

Then, from a comparative point of view, the maximum return on inventory management expense R^* is more sensitive to a relative change in the parameter α than in the parameters c, K, λ or h. Also, the sensitivity with respect to h and λ are equal again, but with the opposite sign. The relative changes with respect to the parameter c can be higher or lower than with respect to K, h or λ. Furthermore, now, if the demand rate does not depend on the stock level, that is, $\beta = 0$, the relative effects of the parameters K, h and λ are equal except for the sign.

Finally, the sensitivity analysis for the optimal length of the inventory cycle T^* is provided in the next lemma.

Lemma 7 *Let T^* be the optimal length of the inventory cycle given by (19), and the auxiliary parameters Δ and B given by (23) and Lemma 2, respectively. Then, it is satisfied that:*

(i) T^* increases as one of the parameters K or c increases.

(ii) T^* decreases as one of the parameters h or λ increases.

(iii) T^* does not depend on the parameter α.

(iv) If $\Delta < -\frac{(2-\beta)^2 B}{1-\beta}$ then T^* decreases as the parameter β increases. In any other case, R^* increases as β increases.

Proof. (i), (ii) and (iii) follow directly from the last column of Table 5. Also, taking into account that $1 + \frac{(1-\beta)\Delta}{(2-\beta)^2 B} < 0$ if, and only if, $\Delta < -\frac{(2-\beta)^2 B}{1-\beta}$, the assertion (iv) is proven. □

Now, the relationship for the relative changes is:

$$\frac{\frac{\partial T^*/\partial K}{T^*/K}}{1-\beta} = -\left(\frac{\frac{\partial T^*/\partial h}{T^*/h}}{(2-\beta)\,B-1}\right) = -\left(\frac{\partial T^*/\partial \lambda}{T^*/\lambda}\right) = \left(\frac{\frac{\partial T^*/\partial c}{T^*/c}}{(2-\beta)(B-1)}\right) = \frac{1}{(2-\beta)\,B}$$

with $1-\beta \leq 1 < (2-\beta)\,B-1$ and $(2-\beta)(B-1) < (2-\beta)\,B-1$. Then, the comparison between them leads to $\left|\frac{\partial T^*/\partial h}{T^*/h}\right| > \frac{\partial T^*/\partial c}{T^*/c}$, $\left|\frac{\partial T^*/\partial h}{T^*/h}\right| > \frac{\partial T^*/\partial K}{T^*/K}$ and $\left|\frac{\partial T^*/\partial \lambda}{T^*/\lambda}\right| > \frac{\partial T^*/\partial K}{T^*/K}$. Therefore, the optimal length of the inventory cycle is more sensitive with respect to a change in the parameter h, than in the parameters c or K, and more sensitive with respect to λ than K.

6 Computational results

In this section, the proposed model and the solution methodology are illustrated with a numerical example. Furthermore, a numerical comparison between the optimal solution of the problem with maximum return on inventory management expense, given by the function $R\,(p, S, s)$, and the optimal solution of the problem with maximum profit per unit time, given by the function $G\,(p, S, s)$, is also shown.

Let us suppose, as input data for the numerical example, that the purchasing cost for the item is $c = 20$ currency units, the ordering cost for a new order is $K = 1000$ currency units, and the holding cost per unit and per unit time (a month) for the inventory system is $h = 15$ currency units. Consider that the parameters for the demand rate are $\lambda = 6000$, $\alpha = 0.1$ and $\beta = 0.3$. That is, the quantity of potential consumers is 6000, the demand rate decreases 10% if the selling price increases a currency unit, and the demand rate increases 0.3% if the stock level increases 1%.

First of all, the auxiliary parameters A, Γ and Δ are evaluated with the expressions (11), (22) and (23). Also, the parameter B is calculated with the Algorithm 1. The results are: $A = 0.7300$, $\Gamma = 12.1260$,

$\Delta = -12.7865$ and $B = 2.7505$. With this value B, the condition (17) is satisfied and the inventory system is acceptable. Then, the optimal selling price given by the expression (14) leads to $p^* = 46.8$ currency units; the reorder point is, as always, $s^* = 0$; the order-level, which coincides with the lot size given by (15), is $S^* = q^* = 212.6$ items; and the cycle time, given by (19), is $T^* = 1.09$ months.

For this optimal solution, the holding cost in a cycle, given by (4), is $H(46.8, 212.6, 0) = 1428.57$. The total expense of the system per unit time, given by (5), is $C(46.8, 212.6, 0) = 6138.8$ currency units, and the profit per unit time, given by (6), is $G(46.8, 212.6, 0) = 2995.2$ currency units. The optimal return on inventory management expense is the rate between these two last quantities, that is, $R(46.8, 212.6, 0) = 2995.2/6138.8 = 0.4879$, which coincides with the value given by the expression (16). That is, the profitability of the inventory system is 48.79%. Note that the holding cost (1428.57) is greater than the ordering cost (1000). Also, the average holding cost per item is $1428.57/212.6 = 6.7$, the average ordering cost per item is $1000/212.6 = 4.7$, and the average inventory cost per item is $r(46.8, 212.6, 0) = 11.4$. Then, the ratio between the inventory cost per item (11.4) and the total cost per item ($20 + 11.4 = 31.4$) is 0.3631, which coincides with $B^{-1} = 0.3631$. That is, the inventory cost is 36.31% of the total expense of the system.

It is also interesting to know the optimal solution for the problem of maximizing the profit per unit time given by the function $G(p, S, s)$. This problem is not solved in a closed form within the literature on inventory models. Supposing that the selling price is a preset parameter, instead of a decision variable, Baker & Urban (1988) studied the model. They only could give a numerical algorithm to obtain an approximate solution with fixed values for the parameters. This method showed that the optimal policy with maximum profit per unit time is reached with a non-zero reorder point. Hence, in order to compare the solutions for the problems of the maximum return on inventory management expense and the maximum profit per unit time, a numerical study has been developed for this particular example, using the following methodology. For each fixed $p \in (20, 75)$ with a step of one-tenth, the problem of maximum profit per unit time is solved by using a numerical algorithm (as in Baker & Urban, 1988) to obtain the optimal order-level S_p^G, the optimal reorder point s_p^G, the maximum profit per unit time $G_p^* = G\left(p, S_p^G, s_p^G\right)$, and the return on inventory management expense for this optimal solution $R_p^G = R\left(p, S_p^G, s_p^G\right)$. The point $\left(p, S_p^G, s_p^G\right)$ with a greater value of G_p^* provides the optimal selling price p^G, the optimal order-level S^G, the optimal reorder point s^G and the maximum profit per unit time G^*. Also, for this optimal solution, the return on inventory management expense $R^G = R\left(p^G, S^G, s^G\right)$, the lot size $q^G = S^G - s^G$, the cycle time $T^G = T\left(p^G, S^G, s^G\right)$, the average inventory cost per item $r^G = r\left(p^G, S^G, s^G\right)$, and the total expense of the system per unit time $C^G = C\left(p^G, S^G, s^G\right)$ are evaluated. All these quantities are included in Table 6, together with the corresponding quantities for the optimal solution with the maximum return on inventory management expense.

Table 6. Optimal solutions for the two problems, $R(p, S, s)$ and $G(p, S, s)$

Problem	p	S	s	q	T	r	C	G	R
$R(p, S, s)$	46.8	212.6	0	212.6	1.09	11.4	6138.8	2995.2	48.79%
$G(p, S, s)$	31.2	916.2	59.5	856.7	0.54	5.3	39890.3	9216.6	23.10%

Both optimal solutions are so different. The selling price with the maximum profit per unit time is $p^G = 31.2$, which is 33% lower than the optimal selling price with the maximum ROIME, which is $p^* = 46.8$. The order-level and the reorder point with the maximum profit per unit time are, respectively, $S^G = 916.2$ and $s^G = 59.5$. Note that now the reorder point is not zero, and the lot size is $q^G = 856.7$, which is 303% higher than the optimal lot size with the maximum ROIME, which is $q^* = 212.6$. Also, the cycle time with the maximum profit per unit time is $T^G = 0.54$ months, which is 50% lower than the optimal cycle time with the maximum ROIME, which is $T^* = 1.09$ months. The average inventory cost per item is much smaller in the solution with maximum profit per unit time ($r^G = 5.3$) than in the solution with maximum ROIME, where it is $r^* = 11.4$. Nevertheless, the total inventory management expense per unit time is much larger in the solution with maximum profit per unit time, $C^G = 39890.3$, than in the solution with maximum ROIME, where it is $C(p^*, S^*, 0) = 6138.8$. The maximum profit per unit time is $G^* = 9216.6$, which is 208% higher than the profit per unit time for the solution with maximum ROIME, which is $G^R = G(p^*, S^*, 0) = 2995.2$. On the other hand, the ROIME for the solution with the maximum profit per unit time is $R^G = 23.10\%$, which is 53% lower than the maximum ROIME $R^* = 48.79\%$.

Note that, if the inventory manager had other investment alternatives with the same ROIME (48.79%), using the same resources needed for the solution of maximum profit per unit time (39890.3 currency units), he/she could obtain a profit per unit time of 19462.5 currency units, which is 111% greater than the 9216.6 currency units that provides the solution with maximum profit per unit time. This issue should be taken into account by the inventory manager when choosing between the maximum profit per unit time and the maximum ROIME policies.

Also, the results obtained in this numerical example suggest that, concerning the solution of maximum profit per unit time, the solution with maximum ROIME requires a higher selling price, smaller lot size, longer cycle time, higher inventory cost per item, and a lower total expense of the system per unit time.

To illustrate these results graphically, for each $p \in (20, 75)$ with a step of one-tenth, Figure 1 plots the values $R_p^* = R^*(p)$ and R_p^G in the right vertical axis, and the values G_p^* and $G_p^R = G\left(p, S_p^*, 0\right)$ in the left vertical axis.

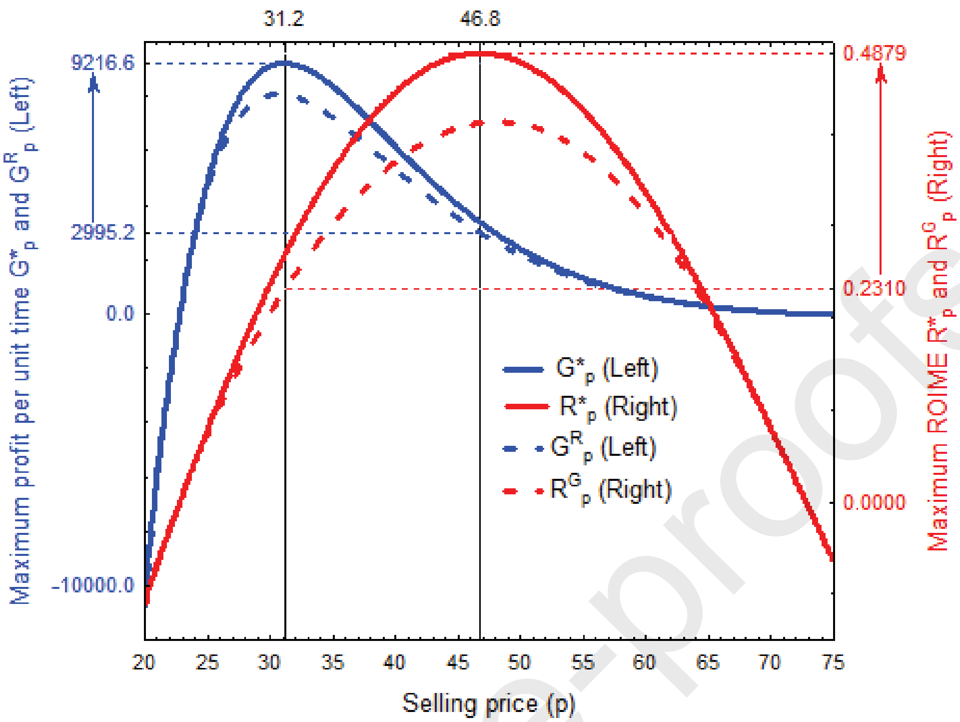

Figure 1. Optimal profit per unit time and optimal ROIME for each p

Next, a sensitivity analysis of the optimal solution with respect to all the initial parameters of the inventory system is included. For this sensitivity analysis, the partial derivatives of the optimal solution, concerning the initial parameteres of the model, both in absolute and relative values, are evaluated with the expressions in Table 5. The results are included in Table 7.

Table 7. Absolute and relative rates of change in p^*, q^*, R^* and T^* with respect to each parameter

	$x=K=1000$	$x=h=15$	$x=c=20$	$x=\lambda=6000$	$x=\alpha=0.1$	$x=\beta=0.3$
$\partial p^*/\partial x$	$-0.4E{-}2$	-0.42	0.54	$0.1E{-}2$	-467.6	20.4
$(\partial p^*/\partial x)\,/\,(p^*/x)$	-0.10%	-0.14%	0.23%	0.14%	-1%	0.13%
$\partial q^*/\partial x$	0.18	-3.03	-6.76	$0.8E{-}2$	0	520.5
$(\partial q^*/\partial x)\,/\,(q^*/x)$	0.85%	-0.21%	-0.64%	0.21%	0%	0.73%
$\partial R^*/\partial x$	$-0.2E{-}3$	$-0.2E{-}1$	$-0.5E{-}2$	$0.5E{-}6$	-14.9	1.52
$(\partial R^*/\partial x)\,/\,(R^*/x)$	-0.46%	-0.65%	-1.94%	0.65%	-3.05%	0.93%
$\partial T^*/\partial x$	$0.2E{-}3$	$-0.6E{-}1$	$0.3E{-}1$	$-0.4E{-}4$	0	-0.20
$(\partial T^*/\partial x)\,/\,(T^*/x)$	0.15%	-0.79%	0.64%	-0.21%	0%	-0.05%

The parameter K has an increasing effect on q^* and T^*, a decreasing effect on p^* and R^*, and it is the most influential on the optimal lot size q^*. The parameter h always has a decreasing effect on all the optimal values, and the relative rate has a major influence on the optimal cycle time T^*. The parameter c has an increasing effect on p^* and T^*, a decreasing effect on q^* and R^*, and the relative rates on q^* and T^* are equal but with opposite sign. The parameter λ has a decreasing effect on T^*, an increasing effect on the other optimal values, and its relative effect on p^*, q^* and R^* is the same as that of h, except for the sign. The parameter α has no effect on q^* and T^*, and it is the most influential on p^* and R^*, with a decreasing effect. Finally, in this numerical example, the parameter β has a decreasing effect on T^* and an increasing effect on the other optimal values. Everything agrees with the results given in Section 5.

Besides, Figure 2 plots the percentual changes in the selling price, the lot size, the return on inventory management expense, and the cycle time for percentual changes between -50% and 50% in each of the parameters, keeping all the others fixed. Notice that, for all the curves, the slope at the origin of the coordinates matches the relative values given in Table 7.

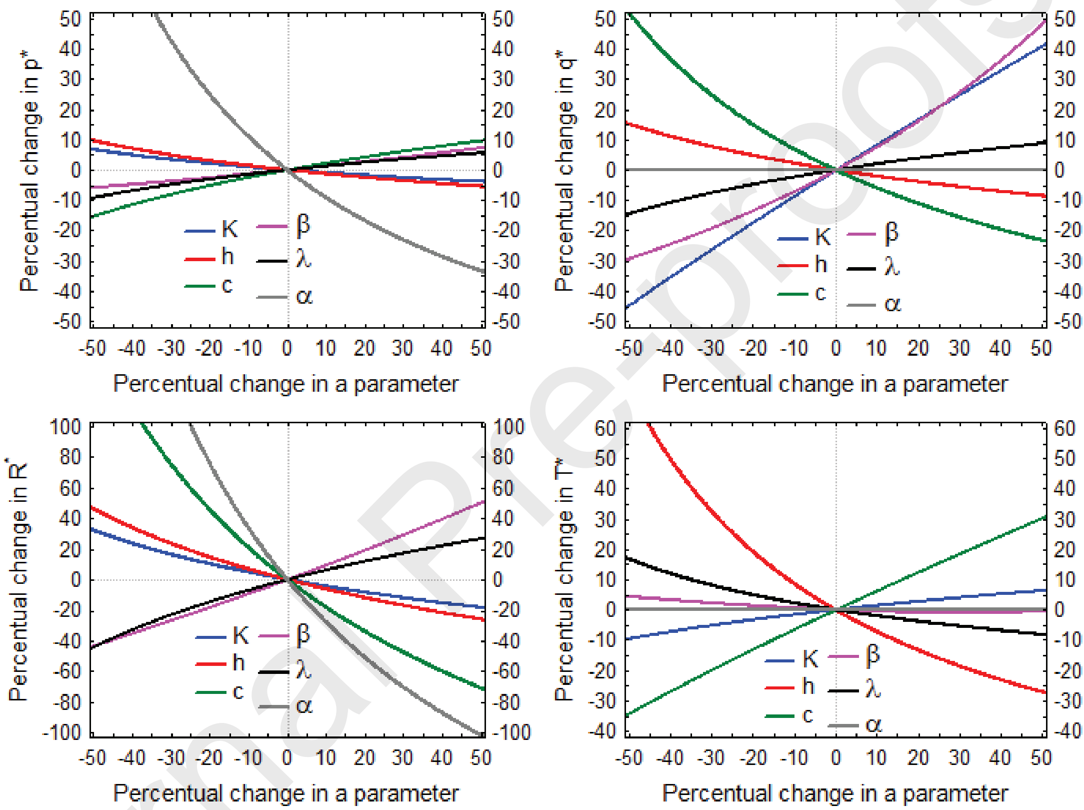

Figure 2. Percentual changes in p^*, q^*, R^* and T^* versus percentual changes in each parameter

To finish the numerical results of this example, the profitability threshold for each parameter, keeping all the others fixed, is calculated by using the expressions given in Table 4. The obtained values are shown in Table 8.

Table 8. Profitability thresholds for the initial parameters of the analysed inventory system

	K	h	c	λ	α	β
Actual value	1000	15	20	6000	0.1	0.3
Profitability threshold	$K < 10581.8$	$h < 78.3$	$c < 36.6$	$\lambda > 1150.6$	$\alpha < 0.149$	$0 \leq \beta < 1$

Then, the maximum purchasing cost is 36.6 and the potential consumers must be, at least, 1151. The maximum values for the holding cost per unit and per unit time, and the ordering cost are, respectively, 78.3 and 10581.8. For the parameter α, the maximum value that allows a profitable system is 0.149. Also, it is not difficult to check that, for all $\beta \in [0, 1)$, the inequality given in (21) is satisfied and, therefore, in this numerical example, the inventory system is always profitable for any $\beta \in [0, 1)$.

7 Conclusions

This paper studies a deterministic inventory model where the demand rate depends on both the selling price and the stock level. A lower selling price or a higher stock level lead to a higher demand rate. Three decision variables are considered: the selling price p, the order-level S and the reorder point s. The goal is the maximization of the return on inventory management expense (ROIME), that is, the ratio between the profit and the total cost of the system.

Many times, the maximum ROIME policy is a better alternative to the policy of the maximum profit per unit time because, although this criterion reduces the profit, it requires a lower investment cost in the management of the inventory. If the manager has several business options, he/she can make a better decision diversifying the available resources on the most profitable products instead of concentrating all resources on the products with the highest profit per unit time. This is always a good investment strategy on business, also in inventory management. The proposed model in this paper allows the manager to know the optimal inventory policy, and also the optimal selling price, which lead to the highest profitability of the system.

The zero-ending policy at the end of an inventory cycle is optimal for the maximum ROIME policy and, therefore, the replacement must be done when the stock is depleted ($s^* = 0$). As a consequence, the optimal order-level S^* coincides with the optimal lot size q^*. Also, the optimal selling price p^*, the optimal lot size q^*, the maximum ROIME R^*, and the optimal cycle time T^* are obtained. Curiously, the optimal lot size q^* and the optimal cycle time T^* do not depend on the elasticity parameter of the demand rate regarding the selling price. Thus, the manager knows that the optimal order policy does not change if the demand is more o less sensitive to the selling price, which is an interesting managerial insight.

The optimal selling price p^* increases if the purchasing cost c, or the scale parameter λ of the demand rate, increase. On the other hand, it decreases if the ordering cost K, the holding cost h per unit and per unit time, or the elasticity parameter α of the demand rate with respect to the selling price, increase. The optimal lot size q^* increases if the parameters K or λ increase, and it decreases if the parameters h or c increase. The maximum ROIME R^* increases if the parameter λ increases, and it decreases if any of the parameters K, h, c, or α increases. The optimal cycle time T^* increases if the parameters K or c increase, and it decreases if the parameters h or λ increase. Regarding the elasticity of the demand rate with respect to the stock level β, all the optimal quantities can increase or decrease if this parameter increases, depending on the values of the other parameters of the model.

From a comparative point of view, the parameter K is the most influential on the optimal lot size q^*, while the parameter h is the most influential on the optimal cycle time T^*. The relative effects of the parameter c on q^* and T^* are equal, but with opposite sign. The relative effects of the parameter λ on p^*, q^* and R^* are the same as that of h, except for the sign. Finally, the parameter α is the most influential on p^* and R^*.

Profitability thresholds for each parameter, keeping all the others fixed, are also obtained. They allow us to know the values that ensure obtaining an acceptable inventory system.

If the demand rate is price-sensitive and the selling price is a decision variable, the solution with the maximum return on inventory management expense, and the maximum profit per unit time solution could be very different. The computational results suggest that, concerning the solution of maximum profit per unit time, the maximum ROIME policy requires a higher selling price, smaller lot size, longer cycle time, higher inventory cost per item, and a lower total expense of the system per unit time. Then, the inventory manager could invest the saved resources in other businesses more profitable to obtain a greater profit per unit time. Only if there are no other investment alternatives, the inventory manager could prefer the maximum profit per unit time solution instead of the maximum ROIME policy.

Some possible extensions of the model that can be future research topics are: (i) to consider other functions for the demand rate; (ii) to suppose a non-linear holding cost; (iii) to incorporate discounts in the unit purchasing cost; and (iv) to study the case of perishable or deteriorating items over time.

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Appendix A

In this appendix, the proofs of Lemma 1, Lemma 2, Lemma 3 and Theorem 1 are included.

Proof of Lemma 1. The function $f(x)$ satisfies that $f(x) > 0$ for all $x > 0$ and $\lim_{x\to 0^+} f(x) = \lim_{x\to\infty} f(x) = \infty$. Moreover, it is a differentiable function with

$$f'(x) = \frac{-(2-\beta)\,K\lambda e^{-\alpha p} + (1-\beta)\,hx^{2-\beta}}{(c + Ae^{x})^2}$$

and therefore

$$x^* = \left(\frac{(2-\beta)K\lambda e^{-\alpha p}}{(1-\beta)\,h}\right)^{1/(2-\beta)}$$

is the only stationary point. Thus, the global minimum of the function $f(x)$ is obtained at point x^*, with

$$\begin{aligned} f(x^*) &= \frac{K}{x^*} + \frac{h\,(x^*)^{1-\beta}}{(2-\beta)\lambda e^{-\alpha p}} = \frac{(2-\beta)K\lambda e^{-\alpha p} + h\,(x^*)^{2-\beta}}{(2-\beta)\lambda e^{-\alpha p} x^*} = \frac{h\,(x^*)^{2-\beta}}{\lambda e^{-\alpha p} x^*} \\ &= \frac{(2-\beta)K}{(1-\beta)\,x^*} = \left(\frac{(2-\beta)K}{1-\beta}\right)\left(\frac{(1-\beta)\,h}{(2-\beta)K\lambda e^{-\alpha p}}\right)^{1/(2-\beta)} = Ae^{\alpha p/(2-\beta)} \end{aligned}$$

where

$$A = \left(\frac{(2-\beta)K}{1-\beta}\right)^{(1-\beta)/(2-\beta)} \left(\frac{h}{\lambda}\right)^{1/(2-\beta)}$$

and the proof is finished. □

Proof of Lemma 2. If $\varphi(x) = ce^{-x} + A(1-x)$, with $c > 0$ and $A > 0$, it is clear that $\varphi(x)$ is a continuous and twice differentiable function, with $\varphi'(x) = -ce^{-x} - A < 0$ and $\varphi''(x) = ce^{-x} > 0$ for all $x \in \mathbb{R}$. Moreover, $\lim_{x\to-\infty} \varphi(x) = \infty$ and $\lim_{x\to\infty} \varphi(x) = -\infty$. Therefore, the equation $\varphi(x) = 0$ necessarily has a unique real root B with $\varphi(x) > 0$ if $x < B$, and $\varphi(x) < 0$ if $x > B$. Also, taking into account that $e^x > 1 + x$ and $e^{-x} < 1/(1+x)$ if $x > 0$, then

$$\varphi\left(\sqrt{1 + c/A}\right) < \frac{c}{1 + \sqrt{1 + c/A}} + A\left(1 - \sqrt{1 + c/A}\right) = \frac{c + A\,(-c/A)}{1 + \sqrt{1 + c/A}} = 0$$

Finally, as $\varphi(1) = ce^{-1} > 0$, we can ensure that $1 < B < \sqrt{1 + c/A}$. □

Proof of Theorem 1. The function $g(p)$ satisfies that $g(p) > 0$ for all $p > 0$ and $\lim_{p\to 0^+} g(p) = \lim_{p\to\infty} g(p) = 0$. Moreover, it is a differentiable function with

$$g'(p) = \frac{c + Ae^{\alpha p/(2-\beta)} - \frac{\alpha p}{2-\beta} Ae^{\alpha p/(2-\beta)}}{\left(c + Ae^{\alpha p/(2-\beta)}\right)^2} = \frac{ce^{-\alpha p/(2-\beta)} + A\left(1 - \frac{\alpha p}{2-\beta}\right)}{e^{-\alpha p/(2-\beta)}\left(c + Ae^{\alpha p/(2-\beta)}\right)^2} = \frac{e^{\alpha p/(2-\beta)}\varphi(\frac{\alpha p}{2-\beta})}{\left(c + Ae^{\alpha p/(2-\beta)}\right)^2}$$

Therefore, from Lemma 2, the unique stationary point p^* of the function $g(p)$ is obtained when $\frac{\alpha p^*}{2-\beta} = B$, and therefore $p^* = \frac{(2-\beta)B}{\alpha}$. Moreover, it is neccesarily the global maximum of the function $g(p)$ because $g'(p) > 0$ (the function $g(p)$ increases) on $(0, p^*)$, and $g'(p) < 0$ (the function $g(p)$ decreases) on (p^*, ∞). Furthermore, $g(p^*) = \frac{p^*}{c+Ae^B}$ and, taking into account that $1 < B < \sqrt{1 + c/A}$, it satisfies that

$$\frac{2-\beta}{\alpha} < p^* < \frac{(2-\beta)\sqrt{1 + c/A}}{\alpha}$$

and the proof is finished. □

Proof of Lemma 3. The inequality given by (17) can be rewritten in the following equivalent ways (take into account that $x < B$ if, and only if, $\varphi(x) > 0$):

$$\begin{aligned}
\alpha &< \frac{(2-\beta)(B-1)}{c} \Leftrightarrow 1 + \frac{\alpha c}{2-\beta} < B \Leftrightarrow \varphi\left(1 + \frac{\alpha c}{2-\beta}\right) > 0 \Leftrightarrow ce^{-\left(1+\frac{\alpha c}{2-\beta}\right)} > \frac{A\alpha c}{2-\beta} \\
\Leftrightarrow \quad A &< \frac{(2-\beta)e^{-\left(1+\frac{\alpha c}{2-\beta}\right)}}{\alpha} \Leftrightarrow \left(\frac{(2-\beta)K}{1-\beta}\right)^{(1-\beta)/(2-\beta)}\left(\frac{h}{\lambda}\right)^{1/(2-\beta)} < \frac{(2-\beta)e^{-\left(1+\frac{\alpha c}{2-\beta}\right)}}{\alpha} \\
\Leftrightarrow \quad & \left(\frac{(2-\beta)K}{1-\beta}\right)^{1-\beta}\left(\frac{h}{\lambda}\right) < \left(\frac{2-\beta}{\alpha}\right)^{2-\beta} e^{-(2-\beta+\alpha c)} \Leftrightarrow \frac{K^{1-\beta}h}{\lambda} < \frac{(2-\beta)(1-\beta)^{1-\beta}}{\alpha^{2-\beta}e^{2-\beta+\alpha c}}
\end{aligned}$$

Therefore, the acceptability condition for the inventory system is

$$\frac{K^{1-\beta}h}{\lambda} < \frac{\Gamma}{e^{\alpha c}}$$

where

$$\Gamma = \frac{(2-\beta)(1-\beta)^{1-\beta}}{\alpha^{2-\beta}e^{2-\beta}}$$

This completes the proof. □

Appendix B

In this appendix, all the partial derivatives given in Table 5 are calculated.

As a starting point, the partial derivatives of the auxiliary parameters A and B with respect to the initial parameters are calculated. For the parameter A, from (11), we have

$$\ln A = \left(\frac{1-\beta}{2-\beta}\right)\left(-\ln\left(\frac{1-\beta}{2-\beta}\right) + \ln K\right) + \left(\frac{\ln h - \ln\lambda}{2-\beta}\right)$$

and the logarithmic differentiation can be used to obtain:

$$\frac{\partial A}{\partial K} = A\frac{\partial \ln(A)}{\partial K} = \frac{(1-\beta)A}{(2-\beta)K}$$

$$\frac{\partial A}{\partial h} = A\frac{\partial \ln(A)}{\partial h} = \frac{A}{(2-\beta)h}$$

$$\frac{\partial A}{\partial \lambda} = A\frac{\partial \ln(A)}{\partial \lambda} = \frac{-A}{(2-\beta)\,\lambda}$$

$$\begin{aligned}\frac{\partial A}{\partial \beta} &= A\frac{\partial \ln(A)}{\partial \beta} = A\left(\frac{\ln\left(\frac{1-\beta}{2-\beta}\right)-\ln K}{(2-\beta)^2}+\left(\frac{1}{2-\beta}\right)^2+\frac{\ln h-\ln\lambda}{(2-\beta)^2}\right)\\ &= A\left(\frac{\ln(1-\beta)-\ln(2-\beta)-\ln K+1+\ln h-\ln\lambda}{(2-\beta)^2}\right) = \frac{\Delta A}{(2-\beta)^2}\end{aligned}$$

where

$$\Delta = 1+\ln\left(\frac{(1-\beta)\,h}{(2-\beta)\,K\lambda}\right)$$

Parameter B is defined by the implicit equation $F(K,h,\lambda,\beta,c,B)=0$ with

$$F(K,h,\lambda,\beta,c,B) = ce^{-B}+A(1-B)$$

Then, by implicit differentiation, the partial derivatives of B are:

$$\frac{\partial B}{\partial K} = -\frac{\partial F/\partial K}{\partial F/\partial B} = \frac{(1-B)\,(\partial A/\partial K)}{ce^{-B}+A} = \frac{(1-B)\,(\partial A/\partial K)}{AB} = \frac{(1-\beta)\,(1-B)}{(2-\beta)\,BK} < 0$$

$$\frac{\partial B}{\partial h} = -\frac{\partial F/\partial h}{\partial F/\partial B} = \frac{(1-B)\,(\partial A/\partial h)}{ce^{-B}+A} = \frac{(1-B)\,(\partial A/\partial h)}{AB} = \frac{1-B}{(2-\beta)\,Bh} < 0$$

$$\frac{\partial B}{\partial \lambda} = -\frac{\partial F/\partial \lambda}{\partial F/\partial B} = \frac{(1-B)\,(\partial A/\partial \lambda)}{ce^{-B}+A} = \frac{(1-B)\,(\partial A/\partial \lambda)}{AB} = \frac{B-1}{(2-\beta)\,B\lambda} > 0$$

$$\frac{\partial B}{\partial c} = -\frac{\partial F/\partial c}{\partial F/\partial B} = \frac{e^{-B}}{ce^{-B}+A} = \frac{A\,(B-1)\,/c}{AB} = \frac{B-1}{Bc} > 0$$

$$\frac{\partial B}{\partial \beta} = -\frac{\partial F/\partial \beta}{\partial F/\partial B} = \frac{(1-B)\,(\partial A/\partial \beta)}{ce^{-B}+A} = \frac{(1-B)\,(\partial A/\partial \beta)}{AB} = \frac{\Delta\,(1-B)}{(2-\beta)^2\,B}$$

Now, using the expression (14), the partial derivatives of p^* with respect to the initial parameters are:

$$\frac{\partial p^*}{\partial K} = \frac{(2-\beta)\,(\partial B/\partial K)}{\alpha} = \frac{(1-\beta)\,(1-B)}{\alpha BK} < 0$$

$$\frac{\partial p^*}{\partial h} = \frac{(2-\beta)\,(\partial B/\partial h)}{\alpha} = \frac{(1-B)}{\alpha Bh} < 0$$

$$\frac{\partial p^*}{\partial \lambda} = \frac{(2-\beta)\,(\partial B/\partial \lambda)}{\alpha} = \frac{B-1}{\alpha B\lambda} > 0$$

$$\frac{\partial p^*}{\partial c} = \frac{(2-\beta)\,(\partial B/\partial c)}{\alpha} = \frac{(2-\beta)\,(B-1)}{\alpha cB} > 0$$

$$\frac{\partial p^*}{\partial \alpha} = -\frac{(2-\beta)\,B}{\alpha^2} < 0$$

$$\frac{\partial p^*}{\partial \beta} = \frac{(2-\beta)\,(\partial B/\partial \beta)-B}{\alpha} = \frac{(1-B)\,\Delta}{(2-\beta)\,\alpha B} - \frac{B}{\alpha}$$

In a similar way, using the expression (15), the partial derivatives are:

$$\begin{aligned}\frac{\partial q^*}{\partial K} &= \left(\frac{2-\beta}{c(1-\beta)}\right)\left(B-1+K\left(\frac{\partial B}{\partial K}\right)\right) = \left(\frac{2-\beta}{c(1-\beta)}\right)\left(B-1-\frac{(1-\beta)\,(B-1)}{(2-\beta)\,B}\right)\\ &= \left(\frac{(2-\beta)\,(B-1)}{(1-\beta)cB}\right)\left(B-\frac{1-\beta}{2-\beta}\right) > 0\end{aligned}$$

$$\frac{\partial q^*}{\partial h} = \left(\frac{(2-\beta)\,K}{(1-\beta)c}\right)\left(\frac{\partial B}{\partial h}\right) = \frac{K\,(1-B)}{ch(1-\beta)B} < 0$$

$$\frac{\partial q^*}{\partial \lambda} = \left(\frac{K\,(2-\beta)}{c(1-\beta)}\right)\left(\frac{\partial B}{\partial \lambda}\right) = \frac{K\,(B-1)}{c\lambda(1-\beta)B} > 0$$

$$\frac{\partial q^*}{\partial c} = \left(\frac{K(2-\beta)}{1-\beta}\right)\left(\frac{c\left(\frac{\partial B}{\partial c}\right)-(B-1)}{c^2}\right) = \frac{-K(2-\beta)\,(B-1)^2}{(1-\beta)\,c^2B} < 0$$

$$\begin{aligned}\frac{\partial q^*}{\partial \beta} &= \left(\frac{K}{c}\right)\left(\frac{B-1}{(1-\beta)^2}+\left(\frac{2-\beta}{1-\beta}\right)\left(\frac{\partial B}{\partial \beta}\right)\right) = \left(\frac{K}{(1-\beta)\,c}\right)\left(\frac{B-1}{1-\beta}-\frac{\Delta\,(B-1)}{(2-\beta)\,B}\right)\\ &= \left(\frac{K\,(B-1)}{c\,(1-\beta)^2}\right)\left(1-\frac{(1-\beta)\,\Delta}{(2-\beta)\,B}\right)\end{aligned}$$

Regarding the optimal return on inventory management expense R^*, using the expression (20), it is possible to obtain:

$$\frac{\partial R^*}{\partial K} = \left(\frac{1}{c}\right)\left(\frac{\partial p^*}{\partial K}\right) = \frac{(1-\beta)\,(1-B)}{c\alpha KB} < 0$$

$$\frac{\partial R^*}{\partial h} = \left(\frac{1}{c}\right)\left(\frac{\partial p^*}{\partial h}\right) = \frac{(1-B)}{c\alpha hB} < 0$$

$$\frac{\partial R^*}{\partial \lambda} = \left(\frac{1}{c}\right)\left(\frac{\partial p^*}{\partial \lambda}\right) = \frac{B-1}{c\alpha\lambda B} > 0$$

$$\begin{aligned}\frac{\partial R^*}{\partial c} &= \left(\frac{1}{c^2}\right)\left(c\left(\frac{\partial p^*}{\partial c}\right)-\left(p^*-\frac{2-\beta}{\alpha}\right)\right) = \left(\frac{1}{c^2}\right)\left(\frac{(2-\beta)\,(B-1)}{\alpha B}-\frac{(2-\beta)\,(B-1)}{\alpha}\right)\\ &= \frac{-\,(2-\beta)\,(B-1)^2}{\alpha c^2 B} < 0\end{aligned}$$

$$\frac{\partial R^*}{\partial \alpha} = \left(\frac{1}{c}\right)\left(\left(\frac{\partial p^*}{\partial \alpha}\right)+\frac{2-\beta}{\alpha^2}\right) = \frac{-(2-\beta)(B-1)}{c\alpha^2} < 0$$

$$\frac{\partial R^*}{\partial \beta} = \left(\frac{1}{c}\right)\left(\left(\frac{\partial p^*}{\partial \beta}\right)+\frac{1}{\alpha}\right) = \frac{(1-B)\,\Delta}{(2-\beta)\,c\alpha B} - \frac{B-1}{c\alpha} = \frac{(1-B)\left(1+\frac{\Delta}{(2-\beta)B}\right)}{c\alpha}$$

Finally, from the expression (19), the partial derivatives of the optimal cycle time T^* are:

$$\frac{\partial T^*}{\partial K} = \frac{-c\left(\frac{\partial B}{\partial K}\right)}{(1-\beta)\,(B-1)^2\,h} = \frac{c\left(\frac{(1-\beta)(B-1)}{(2-\beta)BK}\right)}{(1-\beta)\,(B-1)^2\,h} = \frac{c}{(2-\beta)\,(B-1)\,BKh} > 0$$

$$\frac{\partial T^*}{\partial h} = \frac{-c\left(B-1+h\left(\frac{\partial B}{\partial h}\right)\right)}{(1-\beta)\,(B-1)^2\,h^2} = \frac{-c\left(B-1-\frac{B-1}{(2-\beta)B}\right)}{(1-\beta)\,(B-1)^2\,h^2} = \frac{-c\left(1-\frac{1}{(2-\beta)B}\right)}{(1-\beta)\,(B-1)\,h^2} < 0$$

$$\frac{\partial T^*}{\partial \lambda} = \frac{-c\left(\frac{\partial B}{\partial \lambda}\right)}{(1-\beta)\,(B-1)^2\,h} = \frac{-c\left(\frac{(B-1)}{(2-\beta)B\lambda}\right)}{(1-\beta)\,(B-1)^2\,h} = \frac{-c}{(1-\beta)(2-\beta)B\,(B-1)\,h\lambda} < 0$$

$$\frac{\partial T^*}{\partial c} = \frac{B-1-c\left(\frac{\partial B}{\partial c}\right)}{(1-\beta)\,(B-1)^2\,h} = \frac{B-1-\frac{B-1}{B}}{(1-\beta)\,(B-1)^2\,h} = \frac{1}{(1-\beta)Bh} > 0$$

$$\frac{\partial T^*}{\partial \beta} = \frac{-c\left(-(B-1)+(1-\beta)\left(\frac{\partial B}{\partial \beta}\right)\right)}{(1-\beta)^2\,(B-1)^2\,h} = \frac{-c\left(-(B-1)+(1-\beta)\left(\frac{\Delta(1-B)}{(2-\beta)^2B}\right)\right)}{(1-\beta)^2\,(B-1)^2\,h} = \frac{c\left(1+\frac{(1-\beta)\Delta}{(2-\beta)^2B}\right)}{(1-\beta)^2\,(B-1)\,h}$$