Trabajo de Fin de Grado

Interaction-Free Measurements, Hardy's Paradox and Bell's Theorem

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Dedico este trabajo a mi abuelo Domingo, que aunque ya no sea de forma física, siempre me acompaña. Gracias por enseñarme a ser un hombre tremendo.

Contents

Resumen

Desde principios del pasado siglo, la *mecánica de los cuántos* adoptó rápidamente un carácter enigmático, rodeándose tanto de defensores como de detractores que quedaban, a la vez, maravillados y horrorizados por sus implicaciones. Desde la "catástrofe ultravioleta" hasta la moderna teoría cuántica de campos, el desarrollo de este pilar de la física siempre ha ido acompañado de una palabra: *paradoja.* El motor dialéctico de la cuántica es la contradicción entre cómo nuestra mente macroscópica "piensa" que el mundo se comporta y cómo realmente lo hace. Los cuántos no solo han supuesto un reto para gatos desafortunados (o no), sino para la física teórica, que a día de hoy sigue debatiendo sus múltiples interpretaciones. En este Trabajo de Fin de Grado estudiamos una de estas *paradojas*: la conocida "paradoja de Hardy" pone de manifiesto dos de los grandes enigmas de la interpretación de la teoría cuántica. Por un lado, la dualidad onda-corpúsculo, esencial para el funcionamiento de los interferómetros Mach-Zehnder en los que está basada y por el otro, la no localidad. Para su análisis, partimos de la "medida libre de interacción", propuesta por A. C. Elitzur y L. Vaidman, contextualizando todos los "experimentos mentales" necesarios en el formalismo de dos vectores de estado desarrollado por Y. Aharonov y colaboradores. De forma complementaria, ayudándonos del formalismo de Feynman, resolveremos esta situación, aparentemente paradójica, causada por una interpretación contrafactual de la misma. La no localidad siempre fue una fuente de debate en este contexto, siendo célebremente discutida por Einstein, Podolsky y Rosen. La respuesta cuantitativa a esta discusión, las desigualdades de Bell, colocan este debate en el laboratorio, pasando de ser una cuestión filosófica a una cuestión experimental. Las famosas desigualdades se pueden resumir en el teorema de Bell, el cual mostraremos, sin necesidad de usar desigualdades, mediante el dispositivo propuesto por L. Hardy.

Palabras Clave: mecánica cuántica, medida libre de interacción, paradoja de Hardy, formalismo de dos vectores de estado, interpretación contrafactual, integral de camino, paradoja EPR, teorema de Bell.

Abstract

Since the beginning of the last century, the *mechanics of the quanta* rapidly took an enigmatic character, surrounded by both advocates and detractors who were, simultaneously, amazed and horrified by its implications. From the "ultraviolet catastrophe" to the recent "quantum field theory", the development of this cornerstone of physics has always been accompanied by one word: *paradox.* The dialectical engine of quantum mechanics is the contradiction between how our macroscopic mind "wants" the world to behave and how it actually does. Quanta have not only posed a challenge to unfortunate (or not) cats but also to theoretical physicists, who continue to debate its various interpretations. In this Bachelor Thesis, we study one of these *paradoxes*: "Hardy's paradox" highlights two of the greater enigmas in the interpretation of the quantum theory. On the one hand, the wave-particle duality, fundamental for the functioning of the Mach-Zehnder interferometer on which it is based, and, on the other, nonlocality. In order to analyse it, we depart from the "interaction free measurements", proposed by A. C. Elitzur and L. Vaidman, placing all the needed "thought experiments" in the context of the two state vector formalism developed by Y. Aharonov and collaborators. Additionally, supported by Feynman's formalism, we solve this seemingly paradoxical situation caused by a counterfactual interpretation. Nonlocality has always been a source of debate in this context and was famously discussed by Einstein, Podolsky and Rosen. The quantitative response to this discussion, Bell's inequalities, places the debate at the laboratory, shifting from a philosophical to an experimental matter. The renowned inequalities can be summarized in Bell's theorem, which we will prove without the need of inequalities, through Hardy's setup.

Keywords: quantum mechanics, interaction free measurement, Hardy's paradox, two state vector formalism, counterfactual interpretation, path integral, EPR paradox, Bell's theorem.

Chapter 1

Introduction

Resumen

La mecánica cuántica, a pesar de ser una de las teorías más exitosas e importantes de la física, presenta muchas dificultades desde el punto de vista de su interpretación. En esta sección contextualizamos el trabajo mediante una breve introducción histórica pasando por importantes etapas en el desarrollo de la cuántica. Se introduce el hilo conductor del trabajo, el interferómetro de Mach-Zehnder, y además, el formalismo de dos vectores de estado, que será clave durante el desarrollo del mismo.

Abstract

Quantum mechanics, despite being one of the most successful and important theories in physics, presents many difficulties from the perspective of its interpretation. In this section, we contextualize the work through a brief historical introduction covering significant stages in the development of quantum mechanics. We introduce the guiding thread of the work, the Mach-Zehnder interferometer. Additionally, the two state vector formalism is presented, which will be crucial throughout the development of the Thesis.

Quantum mechanics has been one of the most successful theories in the physical sciences. However, its foundations are still debatable as the very nature of the quantum world clearly contradicts almost every aspect of our physical intuition coming from the classical world. In the year 1900, Max Planck had no other option but to accept the concept of *quantum* in order to explain the energy exchange between light and matter. Just five years later, Albert Einstein generalises this concept stating that light itself is composed of these *elemental quanta* that have very specific values of energy and momentum; the so called quantum light particles were later called *photons*. A few years passed and, in 1909, Einstein holds a lecture in Salzburg [1] during which he argues that light possesses some sort of double nature, corpuscular and undulatory, which supposed a real challenge to our understanding on the nature of light. Since then, quantum mechanics has never failed to, both, intrigue and surprise us. One might think that a theory that started its development in the beginning of the last century is a closed subject with no open research paths. Nevertheless, as the large amount of published literature shows, the foundations of the quantum theory are nowadays a vast field of research and its interpretation still presents itself with an enigmatic character. Even so, quantum mechanics supposed a real revolution in our understanding of the world, and the basic concept in which it is supported, the wave-particle duality, is one of the tougher ideas to understand from our macroscopic experience. In the first chapter dedicated to quantum mechanics, written in the early sixties, Richard P. Feynman comments on the well known Young's double slit experiment:

"In that chapter we shall tackle immediately the basic element of the mysterious behaviour in its most estrange form. We choose to examine a phenomenon which is impossible, absolutely impossible, to explain in any classical way, and which has in it the heart of Quantum Mechanics. In reality, it contains the only mistery" [2].

Nonetheless, another fundamental mystery surrounds quantum mechanics. In 1935, Einstein, along with his collaborators Boris Podolsky and Nathan Rosen [3] (from now on, "EPR") discovered that the mathematical formalism in which quantum mechanics is developed allowed for the existence of states associated to two or more particles, that possess surprising properties. These states were previously described by Schrödinger, who called them *entangled states.* EPR showed that, when we consider the correlations associated to the properties of the system composed by two entangled particles, it resulted quite difficult to consider the quantum theory to be complete. They proposed that a more profound description of reality, in which these particles possess additional properties, which are not taken into account by the established formalism, should exist. Niels Bohr quickly opposed this conclusion and the debate started by these giants of the physical sciences extended until his death, twenty years later.

Three decades had passed since what it is considered one of the most influential articles in physics was published when the EPR article was answered by one of the greatest scientists from the last century, John Bell. In what is probably the most renowned article on the foundations of quantum physics [4], Bell proved that if we accept Einstein's views on quantum mechanics, we would get a quantitative contradiction concerning some of the predictions of the quantum theory. From this moment forwards, the debate held between Einstein and Bohr ceased to be a matter of interpretation of the theory becoming an open question on whether nature behaves as predicted by the quantum formalism or, on the contrary, it did not, as supported by Einstein. It should be pointed that the circumstances in which these conflicts occur are rare, in fact, they can only manifest in situations known as *EPR situations*, in which entangled particles are involved. Even in these said situations, extremely precise measurements must be performed in order to manifest this conflict.

As the years went by, it became clearer that the "strangeness" resulting from entangled states was something different from the difficulties posed by the wave-particle duality. As an illustration of the train of thought coming from the EPR problem, we return to Feynman's *"Lectures on Physics"*, in which, the author reflects on the shared opinion by many physicists on the matter: *"This point was never accepted by Einstein... it became known as the EPR paradox. But when the situation is described as we have done it here, there does not seem to be any paradox at all"* [2]. However, twenty years later, Feynman seemed to change his opinion on the conflict between Einstein's ideas and Bell's inequalities:

"We always have had a great deal of difficulty in understanding the world view that Quantum Mechanics represents... I have entertained myself always by squeezing the difficulty of Quantum Mechanics into a smaller and smaller place, so as to get more and more worried about that particular item. It seems to be almost ridiculous that you can squeeze it to a numerical question that one thing is bigger than another. But there you are - it is bigger!" [5].

To highlight the radical difference between the conceptual problems associated to the wave-particle duality on one side, and entanglement on the other, Alain Aspect called the *"Second Quantum Revolution"* [6] the breakthroughs achieved from the seventies onwards, which were based on Bell's analysis of the EPR paradox. Still, as one of my professors once taught me during a laboratory course, *"Physics is not only made in the blackboard; where it is really made is in the lab"*. Hence, this new revolution stands on the development of innovative techniques allowing to control, trap and observe individual microscopic objects such as molecules, atoms, electrons, or even, ions and photons. These methods, that were once only reserved for the study of large collectives, are nowadays applied to the study of individual systems. This has been crucial in order to obtain a clearer understanding of questions that, until recently, were only considered in thought experiments or, *gedankenexperiments*, as they were first called in the German works. We can ask ourselves if this conceptual "Second Revolution" will have an influence on a new technological revolution that influences our society in the same way that the "First Revolution" did. It is still too early to answer this kind of questions, nevertheless, there have been some breakthroughs in this direction. Take as an example the emerging fields of *quantum information* and, in particular, *quantum cryptography.* Be that as it may, the so called "Second Quantum Revolution" has taken its first baby steps and it is our role as scientists to lead it by perfecting our description of the wonders of the Universe.

The first thought experiment introduced to physics students is the well known double-slit experiment. Young's setup is an elegant way of describing the profoundly counterintuitive behaviour of the quantum world, in particular, it highlights the wave-particle duality, entanglement phenomena and interference. Many authors seem to think that *"[...] the double slit experiment captures the essence of all quantum interference phenomena"* [7]. Following the same line of thought, many other *gedankenexperiments* appeared in the literature. The connecting thread of this work is one of these experiments. In this project, we aim to understand the root of some of the existing questions in relation to the interpretation of the quantum theory. This is done through a device that has been particularly useful in the research on a great number of situations of the same kind as the ones presented in this introduction. This setup is the well known Mach-Zehnder interferometer, which is introduced in the following section.

Yet, another characteristic that surrounds the enigmas of quantum mechanics is the measurement problem. The projective (Von Neumann) postulate plays a key role in the problematic of the interpretation of quantum mechanics. A direct consequence of said postulate is that the standard quantum formalism, contrary to classical mechanics, cannot be considered a time symmetrical theory. Many authors have worked on the concepts surrounding the measurement problem, which remains nowadays an open question and has led to "alternative interpretations" such as the two state vector formalism of quantum mechanics.

The primary objective of this Bachelor Thesis is the study of fundamental concepts of quantum mechanics following a very specific timeline of scientific publications. First, we will briefly introduce the so called *two state vector formalism* (TSVF) developed by Y. Aharonov and collaborators [8], as well as its notions of pre- and postselection, aiming to analyse particular setups where nonlocality and "paradoxical situations" are manifested. While our connecting thread is the Mach-Zehnder interferometer, the central piece of the Thesis is the study of Hardy's paradox [9]. This *gedankenexperiment* appeared in the literature following another thought experiment proposed by A. C. Elitzur and L. Vaidman's, in which they first introduced the method to achieve *interaction free measurements* on quantum systems [10]. Therefore, we perform an study of Elitzur and Vaidman's experiment using the Aharonov-Bergmann-Lebowitz (ABL) rule [11], an essential result from the TSVF. Next, the existence of *empty waves*, proposed by L. Hardy [12], is discussed, being a necessary notion in order to understand Hardy's paradox. Following that, we dive into Hardy's setup aiming to shed light on this seemingly paradoxical situation using Feynman's interpretation of quantum mechanics. Following the works of D. Sokolovski and collaborators [13], we aim to show that Hardy's paradox is a product of a *counterfactual* interpretation of the ABL rule. Finally, given that Hardy's setup highlights the effects of nonlocality, we turn the attention into our last purpose: we present a proof of Bell's theorem, showing its relevance for the development of the quantum description.

Although the objective of the work is not an exhaustive epistemological, or theoretical study of alternative formalisms of quantum mechanics, our endeavors require the use of some concepts taken from the aforementioned formalisms. Our method consists on a review of the literature, while using the notion of preselected and postselected systems, as well as Feynman's interpretation. Additionally, one of the main efforts of this work will consist in "unifying" these publications using a single notation, which allows a better understanding of the involved concepts.

1.1 Mach-Zehnder Interferometer

A half-silvered mirror is an optical device able to split a light beam into two parts, it is usually called *beam splitter*. Even if the term *beam* usually refers to light rays, we will also use this term for any stream of particles[1] travelling in the same direction. These particles are objects such as neutrons, electrons, or even atoms. Throughout this work we will use the term *detector* referring to a device able to count (*click*) the particles arriving at it. We will also assume that the particle sources used in the description of the following experiments are able to emit a single particle at a time.

[1]During this text we employ the term *particle* in reference to any kind of quantum object.

1.1.1 First experiment

Single particles are shot towards a beam splitter allowing to reflect, $\mathcal{R}$, or transmit, $\mathcal{T}$, the particles. At the end of each possible path produced by the beamsplitter a detector is placed. After a large number of particles are sent, we can observe the following (see figure 1.1): (a) the detectors are never activated at the same time, which means that the particles reaching the beam splitter do not split: it is either reflected or transmitted[2]. (b) If we count how many particles took one path or the other, we will obtain that half of them have been transmitted and that the other half, have been reflected. This can be understood as a "heads or tails" situation when flipping a coin. We say of this situation that the associated probability of each event is 50% or 1/2. In this sense, we can say that the beam splitter has a transmission coefficient of 1/2.

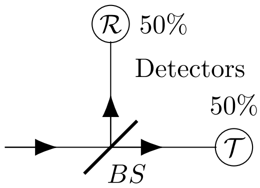


Figure 1.1: A single beam splitter (BS) reflects (R) or transmits (T) a particle.

In the light of this experience, can we consider the particles as physical objects behaving randomly? This innocent question has sparked rivers of ink throughout the last and present century. Indeed, randomness is a key component of quantum mechanics but, as we shall see, it is even more than that.

1.1.2 Second experiment

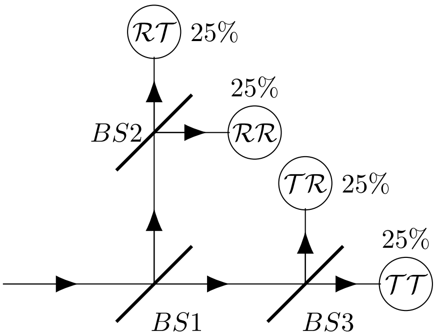


Figure 1.2: Three beam splitters define four equally probable paths.

We now place two other beam splitters in the reflected and transmitted paths (see figure 1.2). In this case, the system presents four different outcomes: the particle may be transmitted twice, $\mathcal{TT}$; transmitted by the first beam splitter and reflected by the second one, $\mathcal{TR}$; reflected by the first one and transmitted by the second one, $\mathcal{RT}$; and reflected twice, $\mathcal{RR}$.

We can now ask ourselves for the associated probability of each possible final situation. We can see that, just as if we flipped two coins, each outcome has a 25% probability assigned.

[2]If this experiment is performed with light, this would serve as a manifestation of its corpuscular nature.

1.1.3 Third experiment: interference

In the next experiment, we place two fully silvered mirrors after the first beam splitter in a way that both beams are redirected towards a second beam splitter as shown in figure 1.3. This way, one of the final detectors is reached by paths $\mathcal{RT}$ or $\mathcal{TR}$, while the other is reached by $\mathcal{TT}$ or $\mathcal{RR}$. Following the results from the second experiment, this new setup should not present anything different. As in path $\mathcal{RT}$ we found 25% of particles as well as in path $\mathcal{TR}$, then, in this experiment, we should now find 50% after each path. Hence, one can predict that half of the particles will reach one detector while the other receives the remaining half. However, this is not the observed result. Surprisingly, all particles are detected in just one of the two detectors! We shall call the non clicking detector the *dark output*.

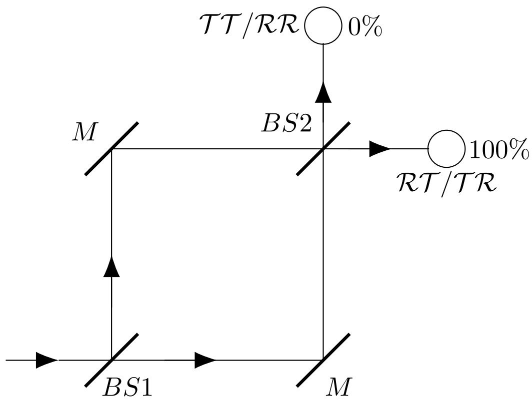


Figure 1.3: Mach-Zehnder interferometer composed of two beam-splitters with a transmission coefficient of 1/2 and two fully covered silver mirrors (M).

There is no doubt that we are missing something. The second experiment is irrefutable: 25% of the particles end up at detector $\mathcal{RR}$ and the remaining 25% at $\mathcal{TT}$. Therefore, one might expect to find 50% of particles in each detector, instead of a dark output. The beam splitters work in perfect condition and, even more, only a single particle is sent at a time so that it becomes impossible that, due to some unfortunate coincidence, two or more particles meet each other and collide, making the results unpredictable. Additionally, the first experiment proved that the particles are indivisible and that a *"half arrival"* is never observed. Everything seems to be in order, yet, nature behaves in a strange way. We are getting closer to the heart of quantum mechanics.

1.1.4 Even more surprising

In the setup described in the previous section, the paths travelled by the particles are constructed so that they are equally long. If the length of the paths varies in some way (see figure 1.4), some particles will arrive at the dark output. The greater the difference between the arms' lengths[3], the more particles reach the dark output. How is it possible that only by changing one of the path's length, the behaviour of all particles drastically changes? How is it possible that the particles travelling along the unchanged path "know" about the modification on the other path? Be that as it may, one is forced to accept this "bizarre" behaviour. There exists some "mysterious" mechanism that "informs" each particle on the paths that it can take. We can try to go further: suppose now that, indeed, we find some way to know which path the particle took after passing through the first beam splitter. Results from the first and second experiment will be confirmed, i.e.: the particle takes one path or the other with equal probability. Nevertheless, the third experiment will change completely. If we know the path "chosen" by the particle, then, half of them will be detected going through one path while the remaining half would be detected travelling through the other alternative.

[3]In particular, if the length difference between paths becomes greater than a particle's property called "coherence length", interference disappears.

In simpler words, if we know the path through which the particle travelled, the results from the third experiment change and the particles would behave as in the second experiment. This case is analogous to a double slit setup: when no measurement on the chosen slit is made, the experiment results in an interference pattern. On the other hand, when a measurement of the chosen slit is performed, the interference pattern is destroyed and the particles would behave as expected classically.

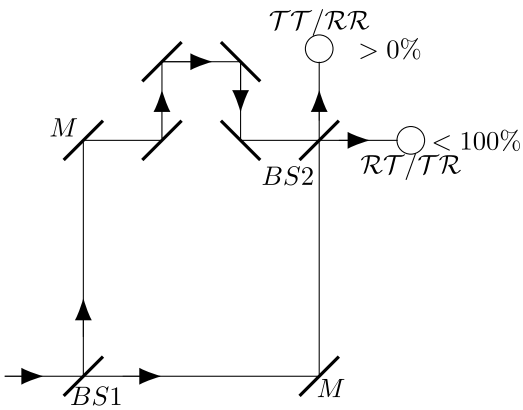


Figure 1.4: Mach-Zehnder interferometer with one of the arms' length modified.

When particles behave as in the third experiment, we say that the *particle interfered with itself.* Self interference is one of those strange physical phenomena coming out from the quantum world. It is a crucial concept in the working of the device that will be used throughout this work, the *Mach-Zehnder interferometer*.

1.2 Two State Vector Formalism: Preselected and Postselected Systems

The two state vector formalism provides some fundamental concepts that can be used in order to analyse Elitzur's setup and Hardy's paradox, as well as the measurement problem, as it was already introduced. This formalism contains two fundamental properties that will be used throughout this Bachelor Thesis. The projective measurement postulate (Von Neumann protocol) provides the mathematical description of a quantum measurement. The reader can find additional information on the standard projective postulate in [14]. A direct consequence of the well known wave-function *collapse* is time asymmetry in standard quantum mechanics. In addition, this measurement scheme also offers a possibility that is routinely used in quantum mechanics, state selection or preparation. This postulate not only gives the resulting state vector from a measurement, but also allows to *preselect* the system in a given state.

1.2.1 Postselection and the ABL rule

In contrast to the standard formalism, the two state vector formalism, developed by Aharonov and collaborators, consists on a time symmetrical formulation of quantum mechanics able to describe any physical system during the time interval between two consecutive projective (strong) measurements. An initial measurement at time t_i prepares the system in a *preselected state*, $|\psi_i(t_i)\rangle$. A final measurement results in the *postselected state*, $|\psi_f(t_f)\rangle$. Even if this formalism is perfectly coherent with the results that can be derived from the standard formalism of quantum mechanics, we will see that using a counterfactual interpretation of Aharonov's theory (we will later discuss what *counterfactual* actually means) is problematic. In this formalism, the statistics associated to non-performed intermediate measurements in an intermediate time t_m, such that $t_i < t_m < t_f$, is determined by two state vectors: the usual or *"forwards evolving"* state vector, $|\Psi_i(t_m)\rangle$, from now onwards, the preselected state and a *"backwards evolving"* state vector, $|\Psi_f(t_m)\rangle$, from now onwards, the postselected state.

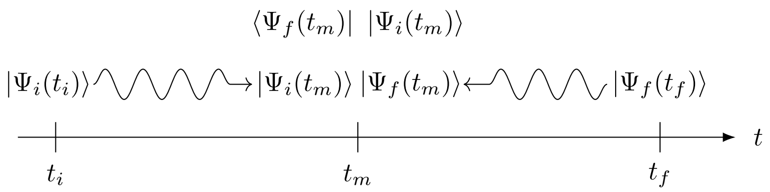


Figure 1.5: A preselected system at t_i and postselected at t_f. In TSVF the description of this system at the intermediate time t_m is given by a two state vector $\langle \Psi_f(t_m)| \;\; |\Psi_i(t_m)\rangle$ (not to be confused with the scalar product).

Postselection consists in choosing a final vector by performing a strong measurement of an operator, $\mathbf{F}$, at time t_f and propagating it from t_f to t_m. From which, it follows:

$$|\Psi_f(t_m)\rangle = \mathbf{U}(t_f, t_m)\, |\Psi_f(t_f)\rangle\,. \tag{1.1}$$

This proposed quantum formalism allowing to predict physical properties on preselected and postselected systems has been a quite fruitful research path, both theoretically and experimentally, as it is shown by the large amounts of literature on the two state vector formalism that has been published since the nineties' decade of the last century.

The origin of this formalism is found in another study developed by Y. Aharonov, in collaboration with P.G. Bergman and J. L. Lebowitz [11]. The Aharonov-Bergmann-Lebowitz rule (ABL) provides the associated probability of intermediate measurements between two succesive measurements. For a quantum system that has been preselected in a given state $|i\rangle$, at time $t = t_i$, and postselected, at time $t = t_f$, in the state $|f\rangle$, the probability of obtaining the eigenvalue s_n of an observable, $\mathbf{S}$, when it is measured at an intermediate time, $t = t_m$, is given by:

$$Prob(s_n|\, |i\rangle\,, |f\rangle) = \frac{|\,\langle f|\,\mathbf{P}_{s_n}\,|i\rangle\,|^2}{\sum_j |\,\langle f|\,\mathbf{P}_{s_j}\,|i\rangle\,|^2}, \tag{1.2}$$

where $\mathbf{P}_{s_n}$ and $\mathbf{P}_{s_j}$ are the corresponding projectors for the eigenvalues s_n and s_j (of the operator $\mathbf{S}$), respectively. A generalization of this expression and its relation with the usual expressions obtained for non postselected systems can be found in [13]. In order to understand this expression, let us return to the essential thought experiment of quantum mechanics, the multiple slit Young's setup shown in figure 1.6. We shoot a particle from an initial position towards a wall with N slits. A screen placed behind the wall records the arrival position. In this illustration, we can understand $\mathbf{I}$ to be an "initial position operator[4]", $\mathbf{S}$ a "slit number operator" and $\mathbf{F}$ a "final position operator" on the screen. We aim to assign a probability for the outcomes of a measurement, allowing us to "*determine which slit was chosen by the electron*" so that it reaches the postselected state.

Let $\{|s_n\rangle\,, n = 1, ..N\}$ be the (non degenerate) eigenstates of the operator $\mathbf{S}$. The probability associated to an intermediate measurement of operator $\mathbf{S}$ resulting in the eigenvalue s_n is given by:

$$Prob(s_n|\, |i\rangle) = \langle i|\,\mathbf{P}_{s_n}\,|i\rangle = |\,\langle s_n|i\rangle\,|^2, \tag{1.3}$$

where $\mathbf{P}_{s_n} = |s_n\rangle\,\langle s_n|$ is the projector operator for the eigenvalue s_n.

Additionally, the associated probability of obtaining the eigenvalue f given that the particle went through s_n is:

[4]Not to be confused with the unity matrix, $\mathbb{1}$

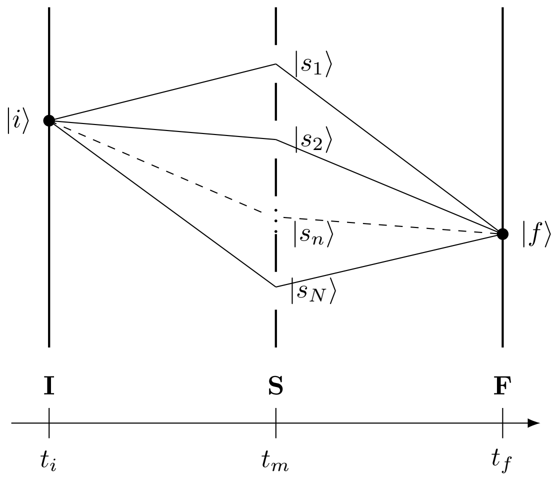


Figure 1.6: Preselection and postselection on an N-slit setup. Preselection implies knowing the initial state $|i\rangle$, while postselection consists on choosing the final state $|f\rangle$ after an intermediate measurement of $\mathbf{S}$.

$$Prob(f|\,|s_n\rangle) = \langle s_n|\,\mathbf{P}_f\,|s_n\rangle = |\,\langle f|s_n\rangle\,|^2, \tag{1.4}$$

where $\mathbf{P}_f = |f\rangle\langle f|$ is the projector operator for the eigenvalue f.

The joint probability of obtaining the outcomes $|s_n\rangle$, in the first measurement, and $|f\rangle$, in the second measurement, is the product of the associated probabilities:

$$Prob(s_n, f|\,|i\rangle) = |\,\langle f|s_n\rangle\,|^2|\,\langle s_n|i\rangle\,|^2. \tag{1.5}$$

Taking into account that the system can be in any of the intermediate states $\{|s_n\rangle\}$, the total probability of obtaining $|f\rangle$ starting from $|i\rangle$ is:

$$Prob(f|\,|i\rangle) = \sum_n Prob(s_n, f|\,|i\rangle) = \sum_n |\,\langle f|s_n\rangle\,|^2|\,\langle s_n|i\rangle\,|^2. \tag{1.6}$$

Introducing Bayes' theorem of conditioned probabilities, the probability associated to the outcome s_n of an intermediate measurement is given by the quotient of probabilities computed on equations 1.5 and 1.6:

$$Prob(s_n|\,|i\rangle, |f\rangle) = \frac{Prob(s_n, f|\,|i\rangle)}{Prob(f|\,|i\rangle)} = \frac{|\,\langle f|s_n\rangle\,|^2|\,\langle s_n|i\rangle\,|^2}{\sum_j |\,\langle f|s_j\rangle\,|^2|\,\langle s_j|i\rangle\,|^2}. \tag{1.7}$$

We can generalize this expression to the degenerate case by introducing the projection operators onto the respective subspaces corresponding to the eigenvalues s_n and s_j: $\mathbf{P}_{s_n} = \sum_n |s_n\rangle\langle s_n|$ and $\mathbf{P}_{s_j} = \sum_j |s_j\rangle\langle s_j|$. We finally get the ABL rule:

$$Prob(s_n|\,|i\rangle, |f\rangle) = \frac{|\,\langle f|\,\mathbf{P}_{s_n}\,|i\rangle\,|^2}{\sum_j |\,\langle f|\,\mathbf{P}_{s_j}\,|i\rangle\,|^2}. \tag{1.8}$$

Although a more general approach would take time evolution into consideration, during this work, we will always assume that the discussed systems have zero Hamiltonian between measurements, $\mathbf{H} = 0$, in all situations where the TSVF and the ABL rule is applied. This implies that no time evolution takes place between $[t_i, t_m - 0^+]$ and $[t_m + 0^+, t_f]$, where 0^+ represent an infinitesimal time.

In order to illustrate this question we can use a Young's double slit setup. If the experiment is run without any kind of intermediate measurement, an interference pattern is produced in the screen. However, if we perform an intermediate measurement of the slit chosen by the particle, the interference pattern is destroyed. The measurement produces a different system! The ABL rule provides the probabilities of going through each slit if this is measured in an intermediate time, which may lead to *counterfactual* reasoning. For example, if we want to determine through which slit the particle went through, we would obtain the probabilities $Prob(s_1| |i\rangle, |f\rangle)$ and $Prob(s_2| |i\rangle, |f\rangle)$ for slits labelled as s_1 and s_2, respectively. Assigning these probabilities to the unmeasured system, in order to answer the *which slit?* question, would constitute a counterfactual statement. This is clear because the probabilities of arriving at a given final state in the screen depend on whether the system is observed, or not observed. Even further, the choice of the slit that is being measured also produces a different system. The interpretation of this expression has been involved with much controversy in the scientific community, as it can be interpreted in a *counterfactual* way, as we just saw. This kind of reasoning must be avoided, as it results from taking as true facts statements that have not actually happened. The renowned theoretical physicist and mathematician H.P. Stapp warns us:

"The word "counterfactual" engenders in the minds of most physicists a feeling of deep suspicion. This wariness is appropriate because counterfactuals, misused, can lead to all sorts of nonsense."[15].

Chapter 2

Interaction-Free Measurements

Resumen

En este capítulo se plantea el experimento mental propuesto por A. C. Elitzur y L. Vaidman a través del cual se explica el concepto de *medida libre de interacción.* Se introduce también las nociones de *colapso libre de interacción* y de *onda vacía*, las cuales constituyen la base teórica necesaria para entender la conocida paradoja de Hardy, que será discutida más adelante. Estos experimentos ponen de manifiesto la no localidad cuántica y, por tanto, carecen de análogo clásico.

Abstract

In this chapter, we consider the *gedankenexperiment* proposed by A. C. Elitzur and L. Vaidman through which the concept of *interaction free measurement* is introduced. We will also introduce the notions of *interaction free collapse* and of *empty wave*, which suppose the theoretical basis for the understanding of the well known Hardy's paradox. These experiments highlight the nonlocal characteristics of quantum mechanics and, therefore, do not have a classical analogue.

2.1 Interaction-Free Measurements

One of these "intriguing" characteristics of the quantum behaviour, already mentioned during the introduction, is nonlocality. The idea that a measurement on one object can influence another object's properties, even if they are placed at distant locations, surely is challenging. It probably represents one of the most difficult ideas to understand from our classical consciousness, as the EPR debate shows. If a nonlocal theory is acceptable, or not, was a question answered in favour of nonlocality by Bell's theorem, theoretically, and experimentally by Aspect's experiments. Bell's theorem will be discussed later in this work. For the moment, we will analyse a *gendankenexperiment* where nonlocality can be used in order to infer information about a quantum system.

Interaction free measurements are a kind of measurement able to yield information about a system "without interacting with it". In particular, Elitzur and Vaidman proposed a kind of *"[...] measurement which, when successful, is capable of ascertaining the existence of an object in a given region of space, though no particle and no light "touched this object""*[10]. Even further, Elitzur proved that it is possible to obtain information about the existence of an object in a given region of space without any prior information about the object. The experimental setup to be used in the following discussion is the already introduced Mach-Zehnder interferometer. During the introduction we obtained some understanding on the workings of the device, however, we need to translate the behaviour of the particle using the quantum formalism. As shown in figure 2.1, the interferometer's arms are labelled as u and v, thus, states $|u\rangle$ and $|v\rangle$ describe the particle travelling along the upper or lower arms, respectively. The injected particle is prepared in the initial state $|s\rangle$. States $|c\rangle$ and $|d\rangle$ describe the paths finally detected at C and D, respectively. Let us now see how we can model the effects of the optical devices in our mathematical formalism.

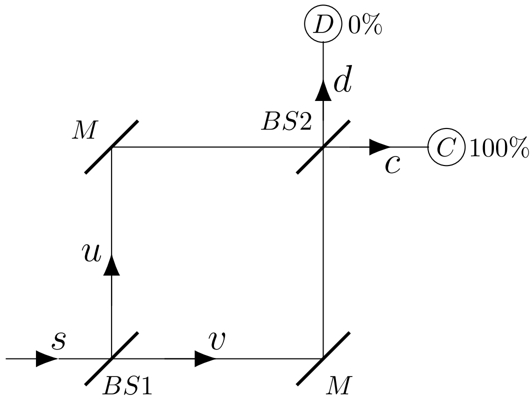


Figure 2.1: Mach-Zehnder interferometer composed of two beam-splitters with a transmission coefficient of 1/2 and two fully covered silver mirrors. The beams are directed towards two detectors, C and D. A single particle is injected into the first beam splitter $BS1$ in the state $|s\rangle$.

The operation of the beam splitters with transmission coefficient 1/2 is described as a superposition of the two possible paths with 1/2 probability associated:

$$\begin{aligned} &\bullet |s\rangle \xrightarrow{BS1} \frac{1}{\sqrt{2}}(i|u\rangle + |v\rangle). \\ &\bullet |u\rangle \xrightarrow{BS2} \frac{1}{\sqrt{2}}(|c\rangle + i|d\rangle). \\ &\bullet |v\rangle \xrightarrow{BS2} \frac{1}{\sqrt{2}}(i|c\rangle + |d\rangle). \end{aligned} \tag{2.1}$$

Applying these transformations to the state of the particle along the paths, when there is no object placed in the interferometer, leads to:

$$|s\rangle \xrightarrow{BS1} \frac{1}{\sqrt{2}}(i|u\rangle + |v\rangle) \xrightarrow{BS2} \frac{1}{2}(i|c\rangle - |d\rangle + i|c\rangle + |d\rangle) = i|c\rangle. \tag{2.2}$$

From equation 2.2 we can see that the quantum description matches the experimental setup (see subsection 1.1.3): the only possible outcome of a measurement (detection) is $|c\rangle$, which corresponds to detector C clicking with certainty in all possible runs of the experiment (interference effect).

Let us now analyse the experiment using the two state vector formalism. First, the system is preselected in the state $|i\rangle = \frac{1}{\sqrt{2}}(i|u\rangle + |v\rangle)$ in all cases. Then, we intend to measure which arm of the interferometer is travelled by the particle in order to arrive at a final state, i.e.: we perform a measurement of the projectors in 2.3.

$$\bullet\, \mathbf{P}_u = |u\rangle\langle u|, \qquad \bullet \mathbf{P}_v = |v\rangle\langle v|. \tag{2.3}$$

If we postselect the system at the state $|f\rangle = |c\rangle$, then, the ABL rule (equation 1.8) gives the following probabilities[1]:

[1]Where we have used: $|c\rangle = \frac{1}{\sqrt{2}}(|u\rangle - i|v\rangle)$ and $|d\rangle = \frac{1}{\sqrt{2}}(-i|u\rangle + |v\rangle)$.

$$\begin{aligned} &\bullet\ Prob(u|\,|i\rangle\,,|f\rangle) = \frac{|\,\langle f|\,\mathbf{P}_u\,|i\rangle\,|^2}{|\,\langle f|\,\mathbf{P}_u\,|i\rangle\,|^2 + |\,\langle f|\,\mathbf{P}_v\,|i\rangle\,|^2} = \frac{1}{2}, \\ &\bullet\ Prob(v|\,|i\rangle\,,|f\rangle) = \frac{|\,\langle f|\,\mathbf{P}_v\,|i\rangle\,|^2}{|\,\langle f|\,\mathbf{P}_u\,|i\rangle\,|^2 + |\,\langle f|\,\mathbf{P}_v\,|i\rangle\,|^2} = \frac{1}{2}, \end{aligned} \tag{2.4}$$

meaning that it is equally probable to find the particle in either arm of the interferometer if a detection at C follows an intermediate measurement of the path followed by the particle.

Analogously, if we postselect the system at the dark output, D, i.e.: $|f\rangle = |d\rangle$, and perform an intermediate measurement on the arm followed by the particle, we get $Prob(u|\,|i\rangle\,,|f\rangle) = \frac{1}{2}$ and $Prob(v|\,|i\rangle\,,|f\rangle) = \frac{1}{2}$. One may argue that it would not be possible to postselect the system in the state $|d\rangle$ due to not being present in equation 2.2. However, this is not the case, because an intermediate measurement of any of the projectors in equations 2.3, would collapse the state of the system in either one of the states $|u\rangle$ or $|v\rangle$, these will recombine after passing $BS2$, leading to a possible detection at D. We can see that it is very easy to fall into *counterfactual* reasoning while using the ABL rule.

Returning to our main argument, we aim to determine the presence of an object placed in one of the interferometer's arms "without touching it". The object is constructed in a way that makes it completely opaque to the particle, meaning that if the particle travels through u, it is scattered and the experiment ends (see figure 2.2). The state describing the scattering effect is $|\gamma\rangle$.

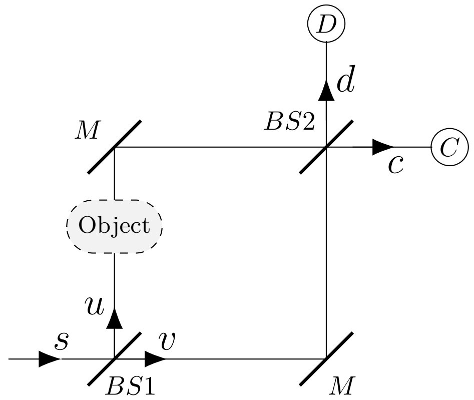


Figure 2.2: An object is placed in the upper arm of the Mach-Zehnder interferometer.

The resulting state vector, just after $BS2$, is a superposition of three possible final outcomes:

$$\begin{aligned} &|s\rangle \xrightarrow{BS1} \frac{1}{\sqrt{2}}(i\,|u\rangle + |v\rangle) \xrightarrow{Object} \frac{1}{\sqrt{2}}(i\,|\gamma\rangle + |v\rangle) \xrightarrow{BS2} \\ &\xrightarrow{BS2} \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(i\,|c\rangle + |d\rangle) + i\,|\gamma\rangle\right) = \frac{1}{2}(i\,|c\rangle + |d\rangle) + \frac{i}{\sqrt{2}}\,|\gamma\rangle\,. \end{aligned} \tag{2.5}$$

The result of this experiment can only lead to one out of three possible outcomes:

$$\begin{cases} i) & |\gamma\rangle \rightarrow \text{no detector clicks},\ Prob = 1/2. \\ ii) & |c\rangle \rightarrow C \text{ clicks},\ Prob = 1/4. \\ iii) & |d\rangle \rightarrow D \text{ clicks},\ Prob = 1/4. \end{cases} \tag{2.6}$$

We can clearly see that the mere presence of an object produces a change in the particle's state just after $BS2$ (as seen in equation 2.5). The first one is the most probable outcome and corresponds to an interaction between the particle and the object which collapses the state of the system into $|\gamma\rangle$. Detector C still clicks, with probability 1/4, when no interaction between the particle and the object takes place. Our main interest is in outcome *iii*). If detector D clicks, we have achieved an interaction free measurement: if the object is not present in the interferometer, the only possible outcome would be a detection at C as it can be seen from equation 2.2; therefore, the only possible reason for D to click is that an object is present in arm u and, even more surprising, that the particle have not interacted with it, which means that it has travelled through path v. This is clear because if an interaction occurred, the outcome would have been *i*) instead of *iii*).

Imagine now that we preselect the system in the state just after the interaction with the object is possible, i.e.: $|i\rangle = \frac{1}{\sqrt{2}}(i\,|\gamma\rangle + |v\rangle)$. We intend to measure the arm travelled by the particle (projectors in equation 2.3), or if the particle is scattered, this is, measuring the associated projector $\mathbf{P}_\gamma = |\gamma\rangle\langle\gamma|$. If we postselect the system at $|f\rangle = |c\rangle$, the ABL rule assigns the following probabilities:

$$\bullet\ Prob(u|\,|i\rangle\,,|f\rangle) = 0, \qquad \bullet Prob(v|\,|i\rangle\,,|f\rangle) = 1, \qquad \bullet\ Prob(\gamma|\,|i\rangle\,,|f\rangle = 0, \tag{2.7}$$

meaning that the only possible way the particle can follow, in order to reach detector C, is v.

If the postselection is performed so that the particle reaches the dark output, $|f\rangle = |d\rangle$, the ABL rule returns the same results as in 2.7. This should not surprise us: if there is a detection at the dark output, then, the particle can never be found in the path u, otherwise it can interact with the object making it impossible to reach any of the detectors. Equivalently, the particle cannot interact with the object (leading to $|\gamma\rangle$) if a later measurement founds the particle at the dark output.

This astonishing result shows how nonlocality can be used as a "tool" to determine the properties of a system without interacting with it, which, of course, would be impossible in the classical world. Therefore, in order to discuss other phenomena using this experimental setup, we will use the aforementioned "tool". This will be done, in the spirit of the two state vector formalism, by postselecting the studied systems in the final state corresponding to an interaction free measurement.

2.2 Interaction-Free Collapse

An even more surprising effect that can be seen in the Mach-Zehnder interferometer is the *interaction free collapse*. This phenomenon is a direct consequence of nonlocality and will allow us to answer the question proposed before: *"Is it possible to obtain information about the existence of an object in a given region of space without any prior information about the object?"*[10].

Suppose now that the object, previously placed in one of the interferometer's arms, is now another quantum particle. We do not possess *"a priori"* information on the object's location as it is prepared in a quantum superposition of two states:

$$|\Psi\rangle = \alpha\,|A\rangle + \beta\,|B\rangle\,, \tag{2.8}$$

where $|A\rangle$ is the state corresponding to the particle located in a region of space A, and $|B\rangle$ the state in which the particle is located in a disjoint region of space B, as shown in figure 2.3.

The interferometer is configured so that one of the paths passes through region A. Another particle, prepared in the initial state, $|s\rangle$, is injected in the interferometer. If the particles interact, the system will collapse into the state

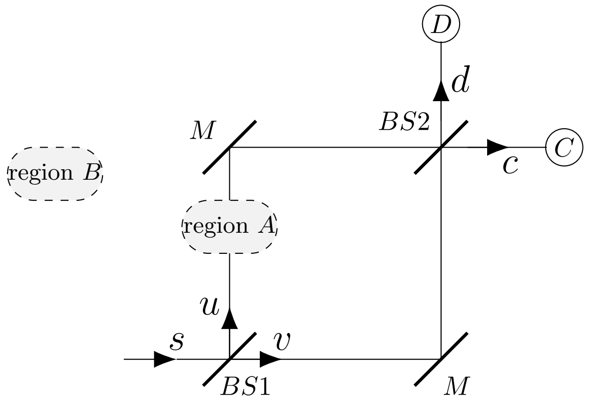


Figure 2.3: Experimental setup: we aim to determine which of the regions is occupied by the object (quantum particle) without directly measuring its position.

describing the scattering process, $|\gamma\rangle$. The initial state of the system is described by the tensor product of the initial states of both systems, i.e.: $|s\rangle \otimes |\Psi\rangle$. During the system's evolution, the particles become an entangled duet:

$$
\begin{aligned}
|s\rangle \otimes |\Psi\rangle &\xrightarrow{BS1} \frac{1}{\sqrt{2}}(i\,|u\rangle + |v\rangle)\,|\Psi\rangle \xrightarrow{Interaction} \frac{\alpha}{\sqrt{2}}(i\,|\gamma\rangle + |v\rangle)\,|A\rangle + \frac{\beta}{\sqrt{2}}(i\,|u\rangle + |v\rangle)\,|B\rangle \\
&\xrightarrow{BS2} \frac{\alpha}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(i\,|c\rangle + |d\rangle) + i\,|\gamma\rangle\right)|A\rangle + \frac{\beta}{\sqrt{2}}\left(\frac{i}{\sqrt{2}}(|c\rangle + i\,|d\rangle) + \frac{1}{\sqrt{2}}(i\,|c\rangle + |d\rangle)\right)|B\rangle \\
&= \alpha\left\{\frac{1}{2}(i\,|c\rangle + |d\rangle) + \frac{i}{\sqrt{2}}\,|\gamma\rangle\right\}|A\rangle + i\beta\,|c\rangle\,|B\rangle\,.
\end{aligned}
\tag{2.9}
$$

By applying Von Neumann's protocol, we can compute the reduced state vector of every possible outcome of a projective measurement and its associated probability. We obtain three possible outcomes:

$$
\begin{cases}
i) & |\gamma\rangle \otimes |A\rangle \to \text{no detector clicks}, \ Prob = \frac{\alpha^2}{2}. \\
ii) & |c\rangle \otimes \left\{\frac{2\beta}{\sqrt{\alpha^2+4\beta^2}}\,|B\rangle + \frac{\alpha}{\sqrt{\alpha^2+4\beta^2}}\,|A\rangle\right\} \to C \text{ clicks}, \ Prob = \frac{\alpha^2}{4} + \beta^2. \\
iii) & |d\rangle \otimes |A\rangle \to D \text{ clicks}, \ Prob = \frac{\alpha^2}{4}.
\end{cases}
\tag{2.10}
$$

Let us now analyse all possible outcomes:

i) The first outcome corresponds to the situation where the particles indeed interact and the injected particle does not reach any of the detectors. Of course, the state of the non traveling particle corresponds to being in the interferometer ($|A\rangle$), otherwise, no scattering would occur.

ii) The second outcome is the most probable. If detector C clicks, the object remains in the superposition (a linear combination of $|A\rangle$ and $|B\rangle$).

iii) The third outcome is the most interesting of the three. When D clicks, the state of the non traveling particle collapses into $|A\rangle$, implying that it is indeed in the interferometer. The injected particle's state collapses into $|d\rangle$, which indicates that there is no interaction between particles, meaning that the travelling particle should have gone through arm v. This last case is an interaction free collapse! We have been able to determine the object's presence in a given region of space without any prior information about it and "without having interacted with it". The detection of the travelling particle at D produces the collapse of the other particle's state while no interaction on it has been made.

An analysis of the system using the ABL rule is analogous to the results from the previous section (see equations 2.4 and 2.7). It is also worth mentioning that this description is only possible if both particles form an entangled system, meaning that the state of one particle cannot be described independently of the other.

As an extension to the interaction free collapse of a wave-function, L. Hardy proposed an alternative view of the wave-particle duality in quantum systems, in which the concept of empty wave is introduced [12].

2.2.1 Existence of the empty wave

The wave-particle duality is a well known debate dating from the very first works of our discipline. De Broglie's wave-particle duality seemed to give an apparent solution for the interpretation on light's nature. By extension, all quantum particles are subject to this duality, meaning that it is not possible to describe their behaviour using exclusively the "wave picture" or the "particle picture". In Hardy's words: *"It seems that the same quantum system will sometimes behave as a particle and sometimes behave as a wave"* [12]. Alternatively, De Broglie-Bohm's model proposes an interpretation where the wave-function describing a quantum object indeed has a physical reality (in the same way that the object is real). On the contrary, in the standard interpretation, the wave function is a mere, though fundamental, mathematical tool.

Hardy proposes a possible interpretation for the duality based on the assumption that the incident particle is accompanied by a wave and that, both, particle and wave, exist objectively. When the combination of particle and wave reach the first beam splitter, as shown in figure 2.4, the wave is divided and goes in both directions. However, the particle will go in one of the directions accompanied by a wave. The wave following the other path is called the *empty wave*.

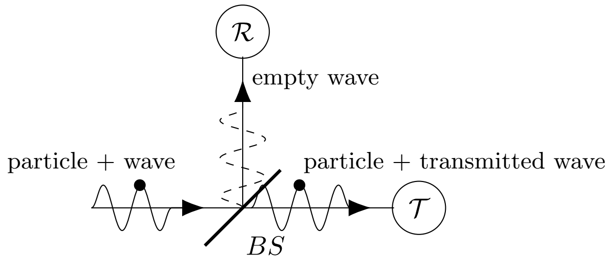


Figure 2.4: In terms of Hardy's description, when the particle impinges on the beam splitter, some part of the wave is reflected and some part is transmitted, while the particle "chooses" one direction leaving the wave taking the other direction "empty".

We now aim to show the physical reality of empty waves using Elitzur's approach to the interaction free collapse. Hardy proposes a method where the reality of the empty wave can be manifested when it is still empty. In order to accept the physical reality of an empty wave, a minimum of two sufficient conditions must be satisfied. These are the following:

1. We know which path the particle goes along.
2. Some measurable property of a system placed in the path followed by the empty wave is changed.

If we are able to find a situation where both conditions are proven true, then, one must accept the physical reality of the empty waves. A projective measurement during the particle's path aiming to determine which alternative the particle actually took would change the outcomes of the experiment. Nevertheless, condition 1 calls for the need of some sort of "quantum trajectory" at least in a situation where there is only one possible path. In the light of Elitzur and Vaidman's experiment, Hardy points out that it should not be too difficult to assume that:

(i) *"If a particle can only reach a detector by one path and the particle is detected by this detector, then, the particle actually took that path"* [12].

Evidently, by using a Mach-Zehnder interferometer, we will be able to know with certainty the path that the particle went along without the need of performing a direct measurement during the particle's "travel". Additionally, condition 2 needs to be proven true in order for empty waves to manifest their physical reality. We will need an ancillary quantum system with some measurable property, so that, if it is placed in the empty wave's path, the interaction between the system and the empty wave causes a change of that property. This system is placed in one of the interferometer's arms. In our case, the quantum object consists on an atom in a superposition of spin states located inside a box, placed in one of the arms of the interferometer.

First, the atom is prepared with a $S_x = +1/2$ spin on the x-axis, so the spin state in terms of its z-axis components is:

$$|atom\rangle = |+\rangle_x = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle). \tag{2.11}$$

Next, a non uniform magnetic field, $\vec{M}$, is placed along the length of the box aligned with the z-axis, similarly to a Stern-Gerlach apparatus which splits the state of the atom between the upper and lower parts of the box containing the states $|+\rangle$ and $|-\rangle$, respectively (see figure 2.5(a)). Then, dividing walls are placed creating box A and B and the non-uniform magnetic field is removed (figure 2.5(b)). The state of the system is now described as:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|A,+\rangle + |B,-\rangle), \tag{2.12}$$

where the states $|A,\pm\rangle$ and $|B,\pm\rangle$ represent the states of the system when the atom is in box A or B, respectively and has spin projection $\pm 1/2$. Finally, the boxes are brought back together as in figure 2.5(c) and a measurement on the spin along the x-axis, S_x, is performed.

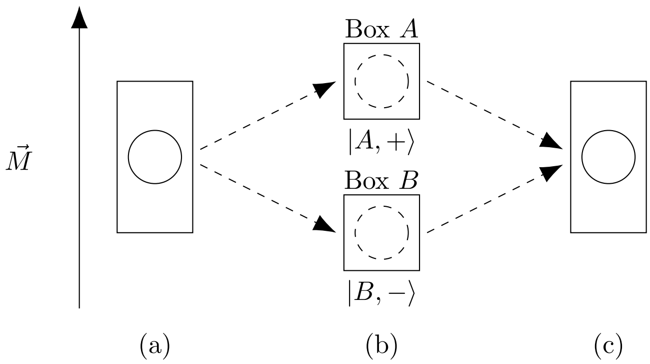


Figure 2.5: (a) An atom, with spin +1/2 along the x-axis, is placed in a box and a non-uniform magnetic field M is applied along the z-axis. (b) Dividing walls are placed creating two different boxes A and B. (c) The boxes are brought back together, and the dividing walls removed.

As already mentioned, we continue the experiment placing the above described system in the interferometer in such a way that box A is in path v as in figure 2.6. The boxes are constructed such that they are transparent to the particles travelling along the interferometer but cannot be traversed by the atom, i.e.: the box represents an infinite potential barrier to the atom but zero potential to the travelling particle. Consequently, if the particle takes path v, and, if the atom is in box A, it will absorb the particle and reach an exited state:

$$|v\rangle\,|A,+\rangle \rightarrow |A,+\rangle_{ex}\,, \tag{2.13}$$

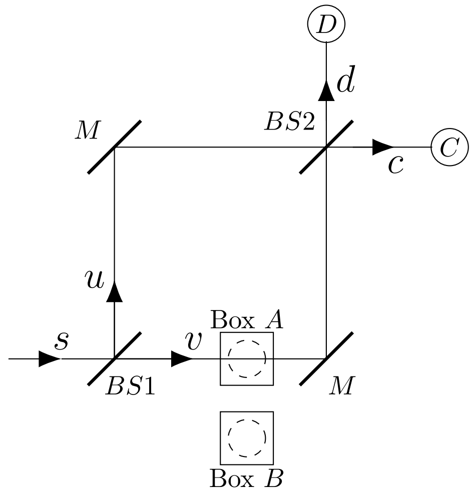


Figure 2.6: Experimental setup. An interferometer is prepared so that, when there is no object in either of the paths, no particle will be detected by D. Box A may contain a 1/2-spin atom.

where $|A,+\rangle_{ex}$ is the state of the atom if it is excited by the travelling particle. If the particle does not interact with the atom, it can be assumed that it travels along the interferometer through the other path, until a detection is made. Then, the two boxes will be brought back together and the dividing walls removed. Finally, a measurement of spin along the x-axis is performed.

Following the procedure from the previous sections, we set up the Mach-Zehnder interferometer for a single particle. The device is prepared in such a way that if there is no object in the particle's path, the particle reaches detector C with probability 100%, leaving detector D as a dark output due to destructive interference. The travelling particle is prepared in the initial state $|s\rangle$. The state of the system evolves during the particle's travel as follows:

$$\begin{aligned}
&\frac{1}{\sqrt{2}}\,|s\rangle\otimes(|A,+\rangle+|B,-\rangle)\xrightarrow{BS1}\frac{1}{2}(i\,|u\rangle+|v\rangle)(|A,+\rangle+|B,-\rangle)\rightarrow\\
&\rightarrow\frac{1}{2}\,|A,+\rangle_{ex}+\frac{i}{2}\,|u\rangle\,(|A,+\rangle+|B,-\rangle)+\frac{1}{2}\,|v\rangle\,|B,-\rangle\xrightarrow{BS2}\\
&\xrightarrow{BS2}\frac{i}{2}\,|A,+\rangle_{ex}-\frac{1}{2\sqrt{2}}\,|c\rangle\,|A,+\rangle-\frac{1}{\sqrt{2}}\,|c\rangle\,|B,-\rangle-\frac{i}{2\sqrt{2}}\,|d\rangle\,|A,+\rangle\,.
\end{aligned}\tag{2.14}$$

From the resulting state vector, after reaching $BS2$, we can conclude that a detection can only result in three possible outcomes:

$$\begin{cases}
i) & |A,+\rangle_{ex}\rightarrow\text{no detector clicks, } Prob=1/4.\\
ii) & \frac{\sqrt{5}}{5}\,|c\rangle\,(|A,+\rangle+2\,|B,-\rangle)\rightarrow\text{detector } C \text{ clicks, } Prob=5/8.\\
iii) & |d\rangle\,|A,+\rangle\rightarrow\text{detector } D \text{ clicks, } Prob=1/8.
\end{cases}\tag{2.15}$$

We will be interested in the third outcome which represents the interaction free measurement and collapse[2]. Just as in Elitzur's experiment, this result enabled us to predict with certainty the presence of the object without interacting with it, which will allow us to obtain information of the "quantum trajectory" previously mentioned. Accordingly, we will only select the runs of the experiment resulting in a detection in the dark output. This means that we will postselect the system in the final state corresponding to "D clicks", i.e.: $|f\rangle=|d\rangle\,|A,+\rangle$.

[2]It is worth noticing that these results are analogous to the previous experiment (see results 2.10).

As the system is postselected so that the particle is detected in the dark output, then, following the results discussed in section 2.10 we would be certain that: (a) there must be an object in path v and, (b) that the particle must have taken the other path.

We will now show that conditions 1 and 2 can be satisfied if a postselection on state $|d\rangle |A,+\rangle$ is performed. Let us remember the timeline of the experiment: at time t_0 an atom is placed inside the box with the magnetic field $\vec{M}$ activated (figure 2.5(a)). At time t_1 the box is closed and the dividing walls are put in place creating box A and B (figures 2.5(b) and 2.6). At a later time, t_2, detector D clicks collapsing the state of the atom into $|A,+\rangle$. Finally, at time t_3 the boxes are brought back together (figure 2.5(c)) and a measurement of the spin along the x-axis, S_x, is performed.

In order to continue our argumentation, Hardy makes two more assumptions [12]:

(ii) *"If, at any time t, the state of the atom is $|A,+\rangle$ we will assume that the atom is actually in box A, at this time, without the need of any measurement."*

(iii) *"If the box is closed at time t_1 and at later time t_2 it is established that the atom is in the box, then, we will assume that the atom is in the box during the times t_1 till t_3, when the box is opened."*

Returning to the experiment, let us show how condition 1 can be satisfied. If the final state of the system is chosen to be $|d\rangle |A,+\rangle$, using assumption (ii), we can be certain that from t_2 to t_3, the state of the atom is actually $|A,+\rangle$ and using assumption (iii), we conclude that the atom is inside box A from t_1 till t_3. Once that we have established that the atom is in box A, which was placed in the path v, we can be sure that the particle must have travelled path u before reaching $BS2$, otherwise, no detector would have clicked. This satisfies condition 1, i.e.: we know which path the particle goes along. Evidently, the last statement implies that the empty wave goes along path v.

We now need to show that this outcome also satisfies condition 2. In order to do so, we will perform two experiments. **Experiment 1:** imagine a run where no particle is sent through the interferometer. If a measurement of the spin of the atom along the x-axis is performed, the outcome will be $S_x = +1/2$ with probability 100% as the atom was initially prepared in this state. Thus, the spin along the x-axis, S_x, is a measurable quantity that will always take the value $S_x = +1/2$ unless "something" affects this property of the atom. **Experiment 2:** imagine now that a particle is injected in the interferometer and that it is detected at the dark output. In this case, the state of the atom is $|A,+\rangle$, i.e.: spin along the z-axis is $S_z = +1/2$. When the boxes are brought back together and the spin along the x-axis is measured, there is a 50% probability that it takes either value $S_x = +1/2$ or $S_x = -1/2$.

If we only consider the runs of the experiment where $S_x = -1/2$, which can be done by posteselecting the system, we observe a change in the value of a measurable property, even though only the empty wave has interacted with the atom. This satisfies condition 2, i.e.: some measurable property of a system placed in the path followed by the empty wave has changed! Therefore, both sufficient conditions for the existence of empty waves are satisfied when the system is postselected in a state where an interaction free measurement is achieved in Hardy's experiment. Nevertheless, the topic calls for a deeper discussion.

An important remark on the nature of the empty waves is made by Hardy himself: *"We have not demonstrated that it has wave properties and it might be better called an empty "something"."* [12]. Hardy's experiment do not prove the physical nature of the empty wave and must be understood as an analogy that is particularly useful in the De Broglie-Bohm interpretation of quantum mechanics and provides an intuitive understanding of the wave-particle duality. Therefore, for a mind closer to the orthodox interpretation, the physical reality of empty waves is non-existent. It should be considered as nothing more than a mental tool used for a better (alternative) understanding of the manifestation of nonlocality and entangled states.

If we turn again the attention towards our *gedankenexperiment*, some other aspects can be highlighted. We clearly see that the empty wave is necessary for the change in S_x. Otherwise, if nothing interacts with the system, the atom's state would not collapse into $|A,+\rangle$ and a measurement of S_x would always result in $S_x = +1/2$, as the atom was initially prepared in that state. Nevertheless, the interaction with the empty wave is not the only reason for a change in the value of the spin along the x-axis. This phenomenon also depends, in a critical way, on the detection in the dark output. Imagine now that the particle travels through u but a detector is placed before the second beam splitter $BS2$. Then, the particle would not reach the state $|d\rangle$, and from equation 2.14, we see that state would collapse into $\frac{i}{2}|u\rangle(|A,+\rangle + |B,-\rangle)$ and S_x would never change its value. Not even if the empty wave travelled along the region of space where the "box + atom" system is placed. In other words, if the particle is detected travelling along u before achieving an interaction free measurement (and collapse), the empty wave would not manifest its reality. Even further, we see that the detection of the particle and the measurement on the spin of the atom can be performed in two distant regions of space and the results of the experiment would not change. Then again, this is a manifestation of quantum nonlocality in the form of entangled states, as the effect of changing or not the state of spin of the atom depends on the collapse of the atom's state into $|A,+\rangle$, which is linked to the collapse of the travelling particle into $|d\rangle$. In simpler words, a measurement on the particle's state instantly changes the atom's state and the atom and particle duet do not need to be in contact for this effect to occur. A measurement of the system formed by a duet of systems whose states form an entangled state cannot be performed on one system independently of the other. In fact, the description of one part of the system cannot be complete without the other part. This shows a very deep consequence of entanglement and nonlocality: in the framework of quantum mechanics, "the whole is not the sum of its parts".

Whether empty waves possesses physical reality or not, or even the nature of this reality, is still an open question. Be that as it may, for the purpose of our work, we will use the term *empty wave* as a linguistic tool for the description of "quantum trajectories" in Hardy's paradox. However, we need to keep in mind that in the orthodox interpretation, this object does not exist. Hardy employs it as mental representation of the effects of nonlocality, or even more, just as an analogy of the effects of the typical interference terms between state vectors.

Chapter 3

Hardy's Paradox

Resumen

En este capítulo analizaremos la conocida paradoja de Hardy usando el formalismo de sistemas preseleccionados y postseleccionados. Utilizando la interpretación de Feynman de la mecánica cuántica, comprobamos que la situación paradójica es solo un fruto de una interpretación contrafactual del formalismo de dos vectores de estado y de la regla ABL. El dispositivo de Hardy se puede entender, desde la perspectiva de Feynman, como un sistema preparado en un estado inicial que puede alcanzar un estado postseleccionado mediante varios caminos alternativos. Veremos que el uso de la regla ABL para asignar probabilidades puede resultar bastante peligroso si no es interpretado cuidadosamente, evitando un razonamiento *contrafactual*. En esta sección abordaremos la paradoja de Hardy como un producto de este tipo de razonamiento.

Abstract

In this chapter, we analyse Hardy's paradox using the formalism of preselected and postselected systems. By means of the application of Feynman's interpretation of quantum mechanics, we prove that the paradoxical situation results from a counterfactual interpretation of the two state vector formalism and the ABL rule. Hardy's device can be understood in the context of Feynman's formalism as a system prepared in an initial state that can reach a postselected state through a number of alternative paths. We will see that the use of the ABL rule to assign probabilities may become quite dangerous if not interpreted carefully, avoiding *counterfactual* reasoning. In this section we shall tackle Hardy's paradox as a result of this kind of reasoning.

3.1 Hardy's Paradox

Hardy's paradox arises from a *gedankenexperiment* proposed by L. Hardy [9], which highlights some of the "strange" behaviour that a duet of particles present when analysed in a certain way. The proposed experimental setup consists of two Mach-Zehnder interferometers, one for positrons, labelled as MZ^+, and one for electrons, labelled as MZ^-, arranged in such a way that two paths overlap at some point, P (see figure 3.1). As always, each interferometer is prepared so that they have a dark output, labelled as $D^\pm$. The interferometers arms and detectors are labelled in the same manner from the previous chapters, with the difference that we will now distinguish the states describing the positron and electron by the $+$ or $-$ superscripts, respectively. It should not be too hard to see that this experiment is a modification of the three previous experiments where interaction free measurements are possible, the difference being that, instead of an object in quantum superposition of position states or spin states, we have replaced the object (or atom) by another Mach-Zehnder interferometer. Nevertheless, in all cases we have an *EPR situation*[1] with a set of possible outcomes. Postselection of these possible outcomes can be used to interpret the particular physical situation.

[1] The experiment is only possible if the system is prepared in an entangled state of two quantum systems (particles).

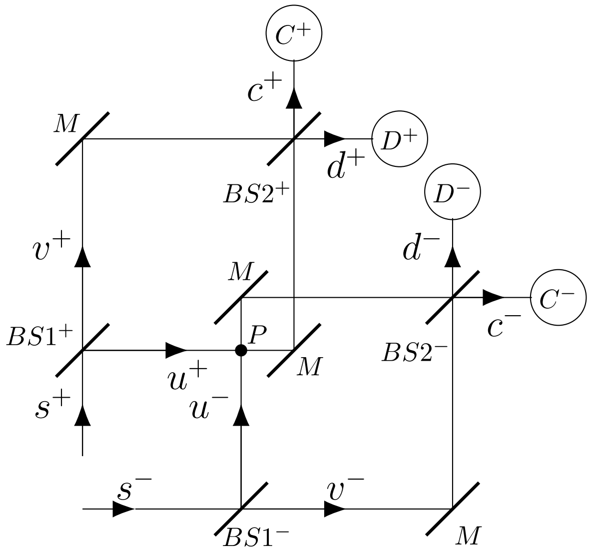


Figure 3.1: Hardy's setup: two Mach-Zehnder interferometers with overlapping arms travelled by a particle and its antiparticle.

When matter and antimatter establish contact, they annihilate each other emitting electromagnetic radiation. Taking this into account, the main particularity of this setup is that if the positron takes path u^+, inside MZ^+, and the electron takes path u^-, inside MZ^-, then, the two particles will meet at point P and annihilate one another with certainty[2] emitting radiation in the state $|\gamma\rangle$. Mathematically, we can express this phenomenon as:

$$\left|u^+\right\rangle \left|u^-\right\rangle \rightarrow |\gamma\rangle . \tag{3.1}$$

Similarly to the previous cases, the operations of the optical devices are described by:

$$\begin{aligned} &\bullet \left|s^\pm\right\rangle \xrightarrow{BS1^\pm} \frac{1}{\sqrt{2}}(i\left|u^\pm\right\rangle + \left|v^\pm\right\rangle). \\ &\bullet \left|u^\pm\right\rangle \xrightarrow{BS2^\pm} \frac{1}{\sqrt{2}}(\left|c^\pm\right\rangle + i\left|d^\pm\right\rangle). \\ &\bullet \left|v^\pm\right\rangle \xrightarrow{BS2^\pm} \frac{1}{\sqrt{2}}(i\left|c^\pm\right\rangle + \left|d^\pm\right\rangle). \end{aligned} \tag{3.2}$$

The initial state of the system is $|s^+\rangle\,|s^-\rangle$. Let us now follow the state evolution of the particles travelling through the interferometer. The state of the system, once the first beam splitters ($BS1^\pm$) are reached, is:

$$\left|s^+\right\rangle \left|s^-\right\rangle \xrightarrow{BS1^\pm} \frac{1}{2}(i\left|u^+\right\rangle + \left|v^+\right\rangle)(i\left|u^-\right\rangle + \left|v^-\right\rangle). \tag{3.3}$$

After reaching point P, we obtain a term representing the emission of radiation, and after passing through $BS2^\pm$, the final state is:

$$\begin{aligned} &\xrightarrow{P} \frac{1}{2}(-|\gamma\rangle + i\left|u^+\right\rangle\left|v^-\right\rangle + i\left|v^+\right\rangle\left|u^-\right\rangle + \left|v^+\right\rangle\left|v^-\right\rangle) \xrightarrow{BS2} \\ &\xrightarrow{BS2} \frac{1}{4}(-2\,|\gamma\rangle - 3\left|c^+\right\rangle\left|c^-\right\rangle + i\left|c^+\right\rangle\left|d^-\right\rangle + i\left|d^+\right\rangle\left|c^-\right\rangle - \left|d^+\right\rangle\left|d^-\right\rangle). \end{aligned} \tag{3.4}$$

[2] 100% probability

When the run of the experiment finishes, we can see that five outcomes are possible:

$$\begin{cases} i) & |\gamma\rangle \rightarrow \text{no detector clicks, } Prob = 1/4. \\ ii) & |c^+\rangle\,|c^-\rangle \rightarrow \text{detector } C^+ \text{ and } C^- \text{ click, } Prob = 9/16. \\ iii) & |c^+\rangle\,|d^-\rangle \rightarrow \text{detector } C^+ \text{ and } D^- \text{ clicks, } Prob = 1/16. \\ iv) & |d^+\rangle\,|c^-\rangle \rightarrow \text{detector } D^+ \text{ and } C^- \text{ click, } Prob = 1/16. \\ v) & |d^+\rangle\,|d^-\rangle \rightarrow \text{detector } D^+ \text{ and } D^- \text{ click, } Prob = 1/16. \end{cases} \tag{3.5}$$

The analysis of the outcomes is analogous to the experiments discussed in the previous chapter: either no detector clicks or two of the four detectors click simultaneously.

The most surprising outcome has a probability of 1/16 and corresponds to the situation where both particles are detected in their respective dark outputs, that we have labelled as (D^+, D^-). Let us now postselect the system in the state $|f\rangle = |d^+\rangle\,|d^-\rangle$. For a simpler understanding of what the paradox involves, we will now assume the existence of *empty waves*. Let us consider two frames of reference, one for the positron and another one for the electron, that we shall label as: F^+ and F^-. In the spirit of the two state vector formalism, we now have a preselected and postselected system, in addition, we are able to discuss the particles' "quantum trajectories" from both frames of reference:

- F^+: In this frame, detector D^+ clicks, then, following the results from section 2.1, the positron went through v^+ and its empty wave must have "interacted with something" while travelling through u^+, at point P. The interaction could only occur with the electron, which, consequently, must have travelled through u^-. Otherwise, both particles would have met at point P and no detection would take place.
- F^-: Applying symmetrical reasoning: in this frame, D^- clicks, then, the electron must have gone through v^- and its empty wave must have "interacted" with the positron while travelling through u^-. Therefore, the positron went through u^+. Otherwise, they would annihilate each other. Just as before, the detection in the dark output is due to the "interaction" between the electron's empty wave and the positron.

Therefore, these results are contradictory. In consequence, we must assume that: if both particles are detected in the dark output, this is, the state collapses to $|d^+\rangle\,|d^-\rangle$, both particles should have interacted with each other at point P, but this should have led to the annihilation of the particles and not to their detection. We have found a paradox!

There exist multiple ways out of this famous paradoxical situation. One that is particularly interesting is based on the analysis of the paradox when the particle's "quantum trajectories" are seen in terms of Feynman's path integral formalism as it was done by D. Sokolovski, I. Puerto Giménez and R. Sala Mayato [13].

3.2 Hardy's Paradox and the Path Integral Formalism.

3.2.1 Measurements in Feynman's Formalism

Feynman's picture is based on two main axioms, which applies to any system capable of reaching final states via a number of alternative routes [16]:

(I) *(Uncertainty Principle)* [16]: "Any determination of the alternative taken by a process capable of following more than one alternative destroys the interference between alternatives", from which it follows that interfering alternatives cannot be told apart and form, therefore, a single indivisible pathway.

(II) *(Recipe for Assigning Probabilities)* [2]: "When an event can occur in several alternative ways, the probability amplitude for the event is the sum of the probability amplitudes for each way considered separately. If an experiment is performed which is capable of determining when one or another alternative is actually taken, the probability of the event is the sum of the probabilities of each alternative".

The basic idea behind the uncertainty principle, in the context of Feynman's path integral formalism, is that we can define a **measurement** as the act of destruction of interference between virtual paths, allowing to differentiate a virtual path from a real one. In this context, a *virtual path* is a possible alternative whose interference has not been destroyed by the act of measurement. On the other hand, a *real path* is created by the act of measurement. This allows us to know which of the alternative interfering paths the system actually took to evolve from one state to another. A particularly interesting example of the application of this formalism is the analysis of the so called "three box paradox" (which we will analyse later in this chapter), where counterfactual reasoning also plays a crucial role. It is worth mentioning that these so called "paths" are not "real" in the same sense that classical trajectories, followed by macroscopic particles that obey the classical laws of movement, are real.

Let us now see how we can assign the probabilities associated to the outcomes of measurements in this context. Suppose a quantum system in an N-dimensional Hilbert space with an orthonormal basis $\{|n\rangle\}, n = 1, 2, ..., N$ formed by the eigenvalues of the "path operator[3]", $\mathbf{N} \equiv \sum_{n=1}^{N} |n\rangle\, n\, \langle n|$. The system is preselected in the initial state $|i\rangle$ at time t_i and postselected in the final state $|f\rangle$ at time t_f. The system can reach the final state departing from the initial state via a number N of possible interfering (virtual) paths. Then, the **transition amplitude** between arbitrary initial and final states, $|i\rangle$ and $|f\rangle$, can be written as a sum over the virtual (interfering) paths:

$$\langle f|i\rangle = \sum_{n=1}^{N} \langle f|n\rangle \langle n|i\rangle \equiv \sum_{n=1}^{N} \phi_n^{f\leftarrow i}, \tag{3.6}$$

where $\phi_n^{f\leftarrow i} = \langle f|n\rangle \langle n|i\rangle$ gives the **probability amplitude**, whose modulus squared corresponds to the probability of reaching state $|f\rangle$ from state $|i\rangle$ through each possible path[4]. The squared modulus of equation 3.6, gives the total probability[5] of reaching a final state $|f\rangle$ when the initial state is $|i\rangle$.

3.2.2 Virtual paths in Hardy's Setup

Let us now return to Hardy's setup. The result of the experiment will always be two simultaneous detections, leading to four possible outcomes, and an additional outcome where no detection is achieved, which happens when the particles travel the overlapping arms. Each particle is able to "choose" through which of the two arms travels. As there are four possible alternatives, we can assign each combination to a virtual Feynman path. We, therefore, have only four possible detections: $(D+, D-)$, $(C+, D-)$, $(D+, C-)$ and $(C+, C-)$. For an easier understanding, one might think of Hardy's setup as equivalent to a four slit Young experiment where each slit represents one of Feynman's paths. The virtual paths are labelled as follows:

- $1 \rightarrow |v^+\rangle\, |v^-\rangle$.
- $2 \rightarrow |u^+\rangle\, |v^-\rangle$.
- $3 \rightarrow |v^+\rangle\, |u^-\rangle$.
- $4 \rightarrow |u^+\rangle\, |u^-\rangle$.

(3.7)

[3]This is analogous to the position operator for a particle in one special dimension with a continuous spectrum, $\hat{x} = \int |x\rangle\, x\, \langle x|\, dx$.

[4]Note that, since $\mathbf{H} = 0$, the N paths are constant.

[5]An example of a path sum such as this is the Feynman path integral over the paths defined in the coordinate space of a point particle.

Let us now recall that the state of the system just after passing point P was given by the first line in equation 3.4. We will preselect the system in this state:

$$|i\rangle = \frac{1}{2}(-|\gamma\rangle + i|u^+\rangle|v^-\rangle + i|v^+\rangle|u^-\rangle + |v^+\rangle|v^-\rangle). \tag{3.8}$$

The outcomes described in 3.5 can be written in the basis describing the paths. Solving for $|c^\pm\rangle$ and $|d^\pm\rangle$ (taking 3.2 as a change of basis relation) and substituting in each final state, we obtain the possible final states:

$$\begin{aligned}
&\bullet\ (D+,D-) : |f\rangle = |d^+\rangle|d^-\rangle = \frac{1}{2}(-i|u^+\rangle + |v^+\rangle)(-i|u^-\rangle + |v^-\rangle).\\
&\bullet\ (D+,C-) : |g\rangle = |d^+\rangle|c-\rangle = \frac{1}{2}(-i|u^+\rangle + |v^+\rangle)(|u^-\rangle - i|v^-\rangle).\\
&\bullet\ (C+,D-) : |h\rangle = |c^+\rangle|d-\rangle = \frac{1}{2}(|u^+\rangle - i|v^+\rangle)(-i|u^-\rangle + |v^-\rangle).\\
&\bullet\ (C+,C-) : |j\rangle = |c^+\rangle|c^-\rangle = \frac{1}{2}(|u^+\rangle - i|v^+\rangle)(|u^-\rangle - i|v^-\rangle).
\end{aligned} \tag{3.9}$$

Notice how only one virtual path ($|u^+\rangle|u^-\rangle$) connects $|i\rangle$ to $|\gamma\rangle$. We have labelled this path as 4, while all four final states $|f\rangle$, $|g\rangle$, $|h\rangle$, $|j\rangle$ can be reached by any of the other three paths; 1, 2 and 3 (see paths in 3.7). Figure 3.2 presents an illustrative representation.

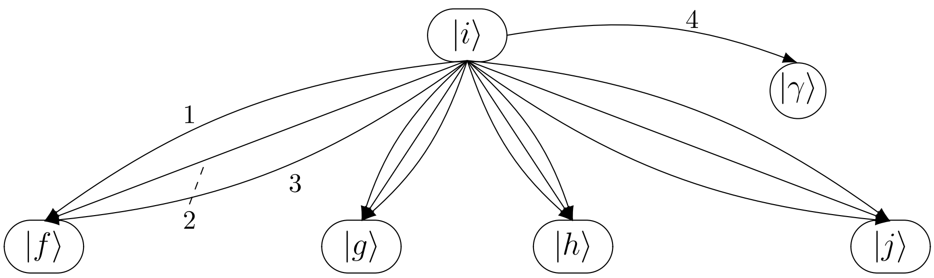


Figure 3.2: Feynman paths 1, 2, and 3 connect the preselected state to each of the four final states. Path 4 connects the initial state to the annihilation state.

If postselection is imposed to one of these final states, that we can label as $|z\rangle$, where $z = f, g, h, j, \gamma$, the probability amplitudes of each path are the following:

$$\begin{aligned}
&\bullet\ \phi_1^{z\leftarrow i} = \langle z|v^+\rangle|v^-\rangle\langle v^-|\langle v^+|i\rangle.\\
&\bullet\ \phi_2^{z\leftarrow i} = \langle z|u^+\rangle|v^-\rangle\langle v^-|\langle u^+|i\rangle.\\
&\bullet\ \phi_3^{z\leftarrow i} = \langle z|v^+\rangle|u^-\rangle\langle u^-|\langle v^+|i\rangle.\\
&\bullet\ \phi_4^{z\leftarrow i} = \langle z|u^+\rangle|u^-\rangle\langle u^-|\langle u^+|i\rangle.
\end{aligned} \tag{3.10}$$

Table 3.1 shows the probability amplitude for each path and final state.

If no intermediate measurement is performed, the pathways connecting the initial state $|i\rangle$ to different final states do not interfere and are mutually exclusive, meaning that only one of the five final states can be reached on each run of the experiment. Hence, if no intermediate measurement is performed on the system, virtual paths reaching the same final state interfere and must be treated as an indivisible real path. From table 3.1 it can be seen that the probability amplitude for a given path is not necessarily the same for different final states. Even more, the transition amplitude between the initial and the final states are, in general, different for each final state. Evidently, they match with the coefficients from the state just before BS2 given in equation 3.4. Additionally, as it is illustrated in figure 3.2, the three paths connecting the initial state to the final state are alternative interfering paths. Therefore, as it

Path	$\phi_n^{f\leftarrow i}$	$\phi_n^{g\leftarrow i}$	$\phi_n^{h\leftarrow i}$	$\phi_n^{j\leftarrow i}$
1	$+1/4$	$+i/4$	$+i/4$	$-1/4$
2	$-1/4$	$-i/4$	$+i/4$	$-1/4$
3	$-1/4$	$+i/4$	$-i/4$	$-1/4$
4	0	0	0	0
Transition amplitude: $\langle z\vert i\rangle$	$-1/4$	$+i/4$	$+i/4$	$-3/4$

Table 3.1: Probability amplitudes for virtual paths in Hardy's setup.

was already mentioned, these interfering paths form an indivisible path, $\{1+2+3\}$, with an associated transition amplitude. Hence, if we postselect the system in a given state, say, for example $|f\rangle$, and no intermediate measurement is performed, only one real pathway, $\{1+2+3\}$, connects the preselected state $|i\rangle$ to $|f\rangle$. If we aim to get some understanding on how the particle behaves inside the interferometer, we may ask ourselves if we are able to determine the path chosen by the particles.

3.3 Intermediate Measurements and Counterfactual Interpretation

We aim to determine the "occupation" of each arm of the interferometer, i.e.: we will ascertain through which path the particles travel. Let us now postselect the system in the state describing a detection at the dark output, $|f\rangle$, which is the one producing the paradox. We define the "path operators" (or "occupation operator") in the orthonormal basis $\{|v^+\rangle|v^-\rangle, |u^+\rangle|v^-\rangle, |v^+\rangle|u^-\rangle, |u^+\rangle|u^-\rangle\}$ as the following projector operators:

$$\begin{aligned}
&\bullet\, \mathbf{N}(v^+|v^-) = diag(1,0,0,0).\\
&\bullet\, \mathbf{N}(u^+|v^-) = diag(0,1,0,0).\\
&\bullet\, \mathbf{N}(v^+|u^-) = diag(0,0,1,0).\\
&\bullet\, \mathbf{N}(u^+|u^-) = diag(0,0,0,1),
\end{aligned} \tag{3.11}$$

where *diag* represents a diagonal matrix. Of course, these projectors only have two eigenvalues, the eigenvalue "1" is obtained when the particles indeed travel the measured paths, and "0" when the paths are not occupied.

We will also need to build single particle projectors to determine the position of one of the particles inside its own interferometer, while the other is left undetermined:

$$\begin{aligned}
&\bullet\, \mathbf{N}(v^-) = \mathbf{N}(v^+|v^-) + \mathbf{N}(u^+|v^-) = \mathbb{1} - \mathbf{N}(u^-) = diag(1,1,0,0).\\
&\bullet\, \mathbf{N}(v^+) = \mathbf{N}(u^+|v^-) + \mathbf{N}(v^+|u^-) = \mathbb{1} - \mathbf{N}(u^+) = diag(1,0,1,0).\\
&\bullet\, \mathbf{N}(u^-) = \mathbf{N}(v^+|u^-) + \mathbf{N}(u^+|u-) = \mathbb{1} - \mathbf{N}(v^-) = diag(0,0,1,1).\\
&\bullet\, \mathbf{N}(u^+) = \mathbf{N}(u^+|v^-) + \mathbf{N}(u^+|u^-) = \mathbb{1} - \mathbf{N}(v^+) = diag(0,1,0,1),
\end{aligned} \tag{3.12}$$

where $\mathbb{1}$ represents the identity matrix. If we now perform an intermediate measurement of each of these operators, the associated probability of such measurement can be computed using the ABL rule. For example, if we measure the occupation of the paths v^+ and v^-, using equation 1.7 we get:

$$\bullet\ Prob(N(v^+|v^-)=1|\,|i\rangle\,,|f\rangle) = \frac{|\,\langle f|v^+\rangle\,|v^-\rangle\,\langle v^-|\,\langle v^+|i\rangle\,|^2}{|\,\langle f|v^+\rangle\,|v^-\rangle\,\langle v^-|\,\langle v^+|i\rangle\,|^2 + |\,\langle f|u^+\rangle\,|v^-\rangle\,\langle v^-|\,\langle u^+|i\rangle + \langle f|v^+\rangle\,|u^-\rangle\,\langle u^-|\,\langle v^+|i\rangle\,|^2} = \frac{1}{5},$$
$$\bullet\ Prob(N(v^+|v^-)=0|\,|i\rangle\,,|f\rangle) = \frac{|\,\langle f|u^+\rangle\,|v^-\rangle\,\langle v^-|\,\langle u_+|i\rangle + \langle f|v^+\rangle\,|u^-\rangle\,\langle u^-|\,\langle v^+|i\rangle\,|^2}{|\,\langle f|v^+\rangle\,|v^-\rangle\,\langle v^-|\,\langle v^+|i\rangle\,|^2 + |\,\langle f|u^+\rangle\,|v^-\rangle\,\langle v^-|\,\langle u^+|i\rangle + \langle f|v^+\rangle\,|u^-\rangle\,\langle u^-|\,\langle v^+|i\rangle\,|^2} = \frac{4}{5}. \tag{3.13}$$

It is not too difficult to see that the ABL rule can be derived using Feynman's recipe to assigned probabilities, as it is shown in [7]. In Feynman's formalism, when we have defined the initial and final states, the square of the transition amplitude gives the probability of reaching the final state. However, if we perform an intermediate measurement of an operator with a number M of eigenvalues, *"the transition probabilities are, in general, altered by the measurement"* (see [13]). In an N-dimensional system with N virtual interfering paths that connect the initial to the final states, the act of measurement destroys interference between paths, creating $M < N$ real pathways associated to each of the possible eigenvalues. The probability associated to a real pathway $\{m\}$ is then given by:

$$P^{f\leftarrow i}_{\{m\}} = \left|\sum_{n'} \phi^{f\leftarrow i}_{n'}\right|^2, \tag{3.14}$$

where n' represent the interfering paths forming the indivisible pathway $\{m\}$. The transition probability, $P^{f\leftarrow i} = \sum_{\{m\}} P^{f\leftarrow i}_{\{m\}}$, is computed by adding the probabilities associated to each real pathway. Notice that, while $P^{f\leftarrow i}_{\{m\}}$ is the probability of measuring $|f\rangle$, after having measured m; the ABL rule (equation 1.8) gives the probability of measuring m, given the postselected state $|f\rangle$.

In Hardy's setup, three interfering virtual paths connect $|i\rangle$ to $|f\rangle$. When an intermediate measurement of any of the projectors is made, two real pathways are created, corresponding to the particles actually travelling the measured paths or, on the contrary, not travelling said path. If the associated probability of one of the paths is nonzero, then, we can conclude that there is nonzero probability for the particles to be in the measured paths. In this sense, we can see that Feynman's rule to assign probabilities is analogous to the ABL rule. Again, we may think of this is terms of Young's setup, a measurement of the "path operator" is equivalent to the *which slit?* question.

Let us now measure the projector $\mathbf{N}(v^+|v^-)$, which will lead to the paradoxical situation. The measurement destroys interference between the three possible paths connecting $|i\rangle$ to $|f\rangle$ creating two real pathways, $\{1\}$, formed by a single path, and $\{2+3\}$ formed by two interfering paths. The probability associated to each path is computed by squaring the probability amplitude associated to the pathway. The probabilities associated to reaching $|f\rangle$ from $|i\rangle$ through each pathway are:

$$\bullet\ P^{f\leftarrow i}_{\{1\}} = ||\phi^{f\leftarrow i}_1|^2 = 1/16. \qquad \bullet\ P^{f\leftarrow i}_{\{2+3\}} = ||\phi^{f\leftarrow i}_2 + \phi^{f\leftarrow i}_3|^2 = 1/4. \tag{3.15}$$

The ABL probabilities are also computed and shown in table 3.2.

A measurement of the operator $\mathbf{N}(u^+|v^-)$, creates the real pathways, $\{2\}$ and $\{1+3\}$, with probabilities:

$$\bullet\ P^{f\leftarrow i}_{\{2\}} = ||\phi^{f\leftarrow i}_2|^2 = 1/16. \qquad \bullet\ P^{f\leftarrow i}_{\{1+3\}} = ||\phi^{f\leftarrow i}_1 + \phi^{f\leftarrow i}_3|^2 = 0, \tag{3.16}$$

suggesting that, when the system is postselected in the dark output, as it is the case, (I) the particles always travel path 2 with certainty. This means that the electron always travel through the non overlapping arm, v^-, while the positron always travel through the overlapping arm, u^+.

Then, a measurement of $\mathbf{N}(v^+|u^-)$ gives:

$$\bullet\ P^{f\leftarrow i}_{\{3\}} = ||\phi^{f\leftarrow i}_3|^2 = 1/16. \qquad \bullet\ P^{f\leftarrow i}_{\{1+2\}} = ||\phi^{f\leftarrow i}_1 + \phi^{f\leftarrow i}_2|^2 = 0, \tag{3.17}$$

suggesting that, (II) the particles always travel path 3 with certainty. This means that the electron always travel the overlapping arm, u^-, while the positron travels the non overlapping arm, v^+.

We can clearly see that the statements (I) and (II) contradict each other. Even further, if we measure the single particle operator $\mathbf{N}(v^-)$, we find that:

$$\bullet\ P^{f\leftarrow i}_{\{3\}} = ||\phi^{f\leftarrow i}_3|^2 = 1/16. \qquad \bullet\ P^{f\leftarrow i}_{\{1+2\}} = ||\phi^{f\leftarrow i}_1 + \phi^{f\leftarrow i}_2|^2 = 0, \tag{3.18}$$

meaning that (III) the electron travels the overlapping arm u^- with certainty.

On the other hand, if we measure $\mathbf{N}(v^+)$, we find that:

$$\bullet\ P^{f\leftarrow i}_{\{2\}} = ||\phi^{f\leftarrow i}_2|^2 = 1/16. \qquad \bullet\ P^{f\leftarrow i}_{\{1+3\}} = ||\phi^{f\leftarrow i}_1 + \phi^{f\leftarrow i}_3|^2 = 0, \tag{3.19}$$

leading to (IV) the positron always travel the overlapping arm u^+.

Hardy's paradox arises once again. If statements (III) and (IV) are true, then, the particle and antiparticle would have met at point P, annihilating each other, making a detection at the dark output impossible and contradicting the fact that we have postselected the system in the aforementioned output, $(D+, D-)$. Table 3.2 provides deeper insights.

Measured	Pathways	$Prob(n\| \|i\rangle, \|f\rangle)$ [ABL]	$P^{f\leftarrow i}_{\{m\}}$	$P^{f\leftarrow i}$	$P^{g\leftarrow i}$	$P^{h\leftarrow i}$	$P^{j\leftarrow i}$
Nothing	$\{1+2+3\}$	-	1/16	1/16	1/16	1/16	9/16
$\mathbf{N}(v^+\|v^-)$	$\{1\}$	1/5	1/16	5/16	1/16	1/16	5/16
	$\{2+3\}$	4/5	1/4				
$\mathbf{N}(u^+\|v^-)$	$\{2\}$	1	1/16	1/16	5/16	1/16	5/16
	$\{1+3\}$	0	0				
$\mathbf{N}(v^+\|u^-)$	$\{3\}$	1	1/16	1/16	1/16	5/16	5/16
	$\{1+2\}$	0	0				
$\mathbf{N}(u^-)$	$\{3\}$	1	1/16	1/16	1/16	5/16	5/16
	$\{1+2\}$	0	0				
$\mathbf{N}(v^-)$	$\{3\}$	1	1/16	1/16	1/16	5/16	5/16
	$\{1+2\}$	0	0				
$\mathbf{N}(u^+)$	$\{2\}$	1	1/16	1/16	5/16	1/16	5/16
	$\{1+3\}$	0	0				
$\mathbf{N}(v^+)$	$\{2\}$	1	1/16	1/16	5/16	1/16	5/16
	$\{1+3\}$	0	0				

Table 3.2: Transition probabilities for the final states when intermediate measurements are performed. ABL probabilities and path probabilities for the $|f\rangle$ state are shown in detail. In all cases, annihilation occurs with 1/4 probability.

If all four previous statements refer to the same system, these results would imply that there exists some mechanism that allows the duet of particles to "be" and "not be" in some way, at the same place and at the same time. However, the paradoxical character of Hardy's setup can be removed if we take into account that the four statements refer to different statistical ensembles produced from the same unobserved *parent system*. Therefore, if a measurement of the observable $\mathbf{N}(u^+|v^-)$ takes place, only the statement (I) is true while statements (II), (III) and (IV) refer to unmeasured attributes of the system and must be discarded. The fact that each measurement creates a different statistical ensemble, or different networks of real pathways, can be seen by computing the transition probabilities for each possible choice of intermediate measurement. From the table 3.2 we can clearly see that the probability of arriving at the different final states $|f\rangle, |g\rangle, |h\rangle, |j\rangle$ are different for each choice of measured quantity. This further proves that a different set of probabilities is created by each measurement.

The paradox arises from a counterfactual interpretation of the ABL rule and, equivalently, of the recipe to assign probabilities, when assuming that arguments (I), (II), (III) and (IV) refer to the same system. By *counterfactual* we refer to the argumentation that results in paradoxical situations by treating statements as true facts, even though they have not actually occurred. One might fall under the dangers of counterfactual reasoning by "forgetting" the implications of the measurement postulate. Any quantum measurement is invasive, the projection postulate implies that the state of the system, after a measurement is performed, is collapsed, changing the state of the system from the time immediately after the measurement onwards. Assuming all arguments as true is equivalent to assigning probabilities to unmeasured quantities, which, due to the consequences of the wave-function collapse, is not possible to do in quantum mechanics.

Feynman's approach provides us with deeper insight on the issues of counterfactual reasoning. When we understand a measurement as a destruction of interference between virtual paths, a set of real pathways is created for each measurement. Each choice of intermediate measurement produces a different network of real pathways with their own set of probabilities. Each set is assigned in a way that no properties of one network can be inferred from the other. Consequently, the set of probabilities assigned to one of the networks (statistical ensemble) cannot be assigned to the other. Even more, no property of the unobserved system (when no intermediate measurement is performed) can be inferred from one of the observed systems. In conclusion, assigning the four statements obtained during the previous discussion to the unobserved system is a form of counterfactual reasoning.

Again, as we have discussed at the end of the first chapter, this issue can be reduced to Young's double slit experiment. Using Feynman's language, if we observe the probabilities with which the particle reaches a set a final states, from an initial state, via a single real pathway formed by two interfering paths associated to each slit, an interference pattern is formed. On the contrary, a measurement of the slit aiming to answer the *which slit?* question creates two real pathways, and in consequence, a different system. Therefore, assigning probabilities of the last system, which is measured, to the first one, which is unmeasured, is a counterfactual statement.

3.3.1 Another counterfactual interpretation: the 3 box paradox

Another application of the ABL rule and the TSVF was given by Aharonov and collaborators in a problem that is known as the "three box paradox" [8]. This seemingly "paradoxical" situation is another example of a counterfactual interpretation of the ABL rule.

Suppose an experimental setup where a particle can be inside one out of three boxes with equal probability. The states describing the particle inside each box form an orthonormal basis, labelled as $\{|1\rangle, |2\rangle, |3\rangle\}$. At time $t = t_i$ the state of the particle is selected to be the state:

$$|i\rangle = \frac{1}{\sqrt{3}}(|1\rangle + |2\rangle + |3\rangle). \tag{3.20}$$

Then, at a later time $t = t_f$, such that $t_f > t_i$, the system is postselected at the state given by:

$$|f\rangle = \frac{1}{\sqrt{3}}(|1\rangle + |2\rangle - |3\rangle). \tag{3.21}$$

We now aim to determine in which box the particle is located at an intermediate time. In order to do so, we need to perform a measurement at an intermediate time t_m (such that $t_i < t_m < t_f$), of the "occupation operators" that are constructed, in an analogous way to what we have seen in section 3.3, in the basis $\{|1\rangle, |2\rangle, |3\rangle\}$:

$$\begin{aligned} &\bullet\, \mathbf{N}(1) = diag(1,0,0). \\ &\bullet\, \mathbf{N}(2) = diag(0,1,0). \\ &\bullet\, \mathbf{N}(3) = diag(0,0,1). \end{aligned} \tag{3.22}$$

We now perform the intermediate measurements. If we measure the operator $\mathbf{N}(1)$, which is equivalent to "opening" the first box in order to see if the particle is there, the ABL rule provides the following probability:

$$Prob(N(1) = 1|\,|i\rangle, |f\rangle) = \frac{|\langle f|1\rangle \langle 1|i\rangle|^2}{|\langle f|1\rangle \langle 1|i\rangle|^2 + |\langle f|2\rangle \langle 2|i\rangle + \langle f|3\rangle \langle 3|i\rangle|^2} = 1, \tag{3.23}$$

which suggests that the particle is always found in box 1. Now, notice that if we perform a measurement of $\mathbf{N}(2)$, the ABL rule also indicates that the particle is found inside box 2 with certainty! We have found a paradoxical situation: how can a particle be found at two different places at the same time?

In 2008, D. Sokolovski and collaborators published an study of this paradox using Feynman's formalism [7]. In this case, if no intermediate measurement is performed, 3 virtual interfering paths (labelled as 1, 2 and 3) connect the state $|i\rangle$ to the postselected state $|f\rangle$, forming an indivisible real pathway. The probability amplitudes for each path are:

$$\begin{aligned} &\bullet\;\; \phi_1^{f\leftarrow i} = \langle f|1\rangle \langle 1|i\rangle = \frac{1}{3}, \\ &\bullet\;\; \phi_2^{f\leftarrow i} = \langle f|2\rangle \langle 2|i\rangle = \frac{1}{3}, \\ &\bullet\;\; \phi_3^{f\leftarrow i} = \langle f|3\rangle \langle 3|i\rangle = -\frac{1}{3}. \end{aligned} \tag{3.24}$$

If we perform an intermediate measurement of $\mathbf{N}(1)$, we have seen that this destroys the interference between the three paths, creating two real pathways labelled as $\{1\}$ and $\{2+3\}$. If, for example, we open the first box, Feynman's recipe to assign probabilities provides the probabilities of reaching the final state through each real path:

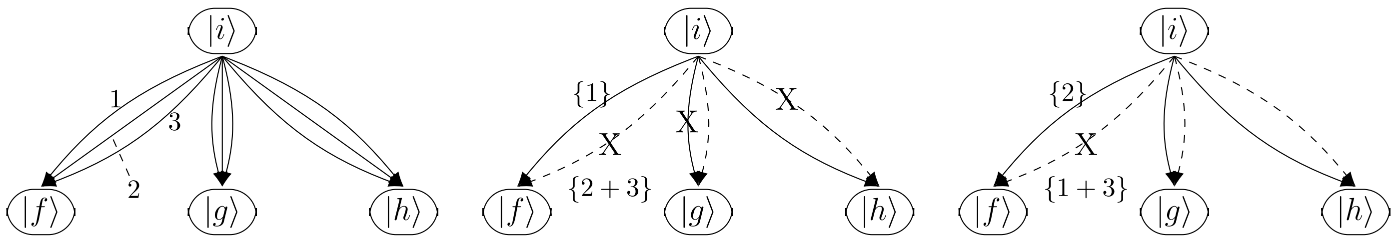
$$\bullet\, P_{\{1\}}^{f\leftarrow i} = ||\phi_1^{f\leftarrow i}|^2 = 1/9. \qquad \bullet\, P_{\{2+3\}}^{f\leftarrow i} = ||\phi_2^{f\leftarrow i} + \phi_3^{f\leftarrow i}|^2 = 0. \tag{3.25}$$

Suggesting that (I) the particle is always found in box 1. Now, a measurement of $\mathbf{N}(2)$ creates two different real pathways, $\{2\}$ and $\{1+3\}$, therefore:

$$\bullet\, P_{\{2\}}^{f\leftarrow i} = ||\phi_2^{f\leftarrow i}|^2 = 1/9. \qquad \bullet\, P_{\{1+3\}}^{f\leftarrow i} = ||\phi_1^{f\leftarrow i} + \phi_3^{f\leftarrow i}|^2 = 0, \tag{3.26}$$

meaning that (II) the particle is always found in box 2.

We seem to fall again into the paradoxical situation. Nevertheless, Feynman's formalism provides a helpful insight on the issue at hand, if we take into account the definition of measurement. We can see how this is just another example



(a) No measurement. (b) $\mathbf{N}(1)$ measured. (c) $\mathbf{N}(2)$ measured.

Figure 3.3: Real pathways. The symbol X indicates that the path is never traveled.

of counterfactual reasoning, where we have assigned probabilities of measurements to unmeasured quantities. This cannot be done, as the choice of measurement creates different networks of real paths. If box 1 was opened, then, we cannot assign a probability of finding the particle in box 2, if it has not been opened. Consequently, the paradox arises from a reasoning where statements (I) and (II) refer to the same system. Suppose now, that instead of postselecting the system at state $|f\rangle$, we analyse the probabilities of reaching other final states forming an orthonormal basis composed by $|f\rangle$ and, for example, $|g\rangle$ and $|h\rangle$. Analogously to what we have seen for Hardy's setup, the three paths connecting $|i\rangle$ to the different final states, $\{|f\rangle, |g\rangle, |g\rangle\}$, form an indistinguishable real pathway if no intermediate measurement is performed, see figure 3.3a. However, if we perform an intermediate measurement, two real pathways are created. From figures 3.3b and 3.3c we can clearly see that the networks of real pathways created by each choice of measurement are different from each other, which shows that they are, indeed, different statistical ensembles.

Chapter 4

Bell's Theorem and Hardy's Setup

Resumen

Uno de los resultados fundamentales sobre el problema de la no localidad en mecánica cuántica es el teorema de Bell, basado una colección de desigualdades que la distinguen de la mecánica clásica. En este capítulo veremos que es posible una demostración de dicho teorema sin necesidad de desigualdades y mediante el uso del interferómetro de Mach-Zehnder.

Abstract

One of the fundamental results on the problem of non locality in quantum mechanics is Bell's theorem, which is based in a collection of inequalities that distinguish it from classical mechanics. In this chapter we shall see that it is possible to obtain a proof of Bell's theorem without the need of using inequalities and by means of the Mach-Zehnder interferometer.

4.1 Local Realism and Hidden Variables

One of the greatest challenges that the establishment of quantum mechanics had to face was the publication of the famous EPR article [3]. Let us now dive deeper into the debate and try to understand what Einstein, Podolsky and Rosen meant by stating that a "*more complete theory is possible*".

EPR defended that the success of a scientific theory can be measured by asking two questions, (1) is the theory correct? and (2) is the theory complete? By correct, the authors mean that the theory is able to predict the experimental results. They conclude that the quantum theory complies with this requirement. Subsequently, in order to understand the meaning of the second question, we need to introduce some philosophical concepts used by Einstein and collaborators.

In the authors' view, reality is formed by *elements of reality* of which we are able to get a picture through the lenses of a physical theory. Therefore, Einstein's *criterion of reality* established that if a theoretical element describes a particular *element of reality*, then, it should be possible to predict its value with certainty. From this, it follows that, a theory can only be considered complete if, for every element of the reality which is studied, the theory provides a concept, complying with the criterion of reality, that describes said element. The correspondence between the elements of reality and the elements of the theory is called the *condition of completeness*.

From the criterion of reality, it follows that two incompatible observables cannot possess simultaneous reality because a certain measurement of one of them destroys the certainty on the other. This made the authors arrive at two statements: either (I), quantum mechanics is not complete or (II) non commuting observables cannot possess simultaneous reality. If both statements are true, then the quantum description is not complete. On the other hand, if both are false, then, the wave function description should provide predictions of non commuting observables with certainty. However, this does not occur, therefore, the statements are mutually exclusive, only one of them can be true. Einstein and collaborators famously used entangled states of position and momentum (EPR states) in order to show the paradox that arises from the properties of entangled states when analysed from the point of view of *local realism.*

Theoretical physicists J.F Clauser and A. Shimony introduced the expression "local realistic theory" and defined realism is a *"philosophical view, according to which external reality is assumed to exist and have definite properties, whether or not they are observed by someone."*[17]. In other words: *local realism* as a philosophical point of view in which all physical systems always have well-defined properties independently if it is observed or not. Additionally, when an observation is carried out, sufficiently distant events[1] should not affect the result of the observation. This is called *locality.*

Using a duet of entangled particles, EPR noticed that the choice of measurement of an observable of one of the particles can leave the other particle in two different states, however, as the particles are sufficiently distant and cannot interact, locality imposes that the measurement of one particle cannot affect the other, therefore concluding that both states describe the same physical reality. Hence, two incompatible observables can possess simultaneous reality. Therefore, (II) is false and (I) must be true, meaning that the description provided by the wave function is not complete. A detailed analysis of EPR's argumentation can be found in the Stanford Encyclopedia of Philosophy [18] and by G. Abad in [19].

The paper ends by hinting the idea that a local and complete description is possible to achieve. This idea was later developed into what is now known as *hidden variable theories.* Given that quantum mechanics is correct (in the sense that it predicts the experimental results), a hidden variable theory should complete the description of the wave function while reproducing its experimental outcomes. This is based under the assumption that the wave function would result from taking the average of an ensemble of values of these hidden variables, in the same sense that thermodynamics can be considered an averaged description of a many particles system described by statistical mechanics. David Bohm's interpretation of quantum mechanics proves that it is possible to construct a nonlocal but realistic interpretation of quantum mechanics. We marginally approached one of Bohm's concepts in Chapter 2, when we discussed the existence of *empty waves* in Elitzur's setup. Nevertheless, one of the main contributors to this debate is J. S. Bell and the contradictions that come to the surface when the hypotheses of local realism are applied to quantum mechanics are the basis of Bell's theorem.

4.2 Bell's Theorem and Hardy's Setup

Bell's theorem is the name given to a collection of results, first developed by J. S. Bell, that answer the question of whether a theory based under local realism is compatible with the results of quantum mechanics. The first Bell inequality [4] is based in the context of Aharonov and Bohm's version of the EPR paradox [20], where, instead of position and momentum operators, different spin directions are measured. Assuming that a local deterministic hidden variable theory is possible, Bell arrives at an inequality that must be satisfied for every hidden variable theory. Then, it is shown that the predictions from quantum mechanics do not satisfy this inequality. Different literature, both theoretical, and experimental, famously, Aspect's experiments, followed the original paper, some examples are: ([21], [22], [23]). These "no-go" results developed around Bell's inequalities can be summarized in a sentence that is commonly known as Bell's theorem:

"No local deterministic theory can reproduce the results of quantum mechanics".

[1]By sufficiently distant, we refer to causally disconnected in the relativistic sense.

The theorem is based on two hypotheses: determinism and locality. The need for additional variables allowing to *"realistically"* describe the observables is fulfilled by a set of hidden variables, λ, that "complete" the quantum theory. This set of variables introduce the notion of realism into the theory. Determinism removes the probabilistic character of quantum measurements which, in consequence, makes all observables comply with the criterion of reality. Additionally, locality is achieved as the results of measurements can only depend on the hidden variables and on the configuration of the measuring device but never on any measurement performed on another system in a distant place.

Let us now return to Hardy's setup (figure 3.1). Consider now that $BS2^{\pm}$ are removable from the device. This will change the state vector evolution given by equation 3.4. If $BS2^{\pm}$ are removed, then:

$$\bullet \left|u^{\pm}\right\rangle \to \left|c^{\pm}\right\rangle, \qquad \bullet \left|v^{\pm}\right\rangle \to \left|d^{\pm}\right\rangle, \tag{4.1}$$

from which, it follows that:

$$\begin{cases} i) & \text{If } BS2^{\pm} \text{ are removed: } \to \frac{1}{2}(-\left|\gamma\right\rangle + i\left|c^{+}\right\rangle\left|d^{-}\right\rangle + i\left|d^{+}\right\rangle\left|c^{-}\right\rangle + \left|d^{+}\right\rangle\left|d^{-}\right\rangle). \\ ii) & \text{With } BS2^{+} \text{ in place and } BS2^{-} \text{ removed: } \to \frac{1}{2\sqrt{2}}(-\sqrt{2}\left|\gamma\right\rangle - \left|c^{+}\right\rangle\left|c^{-}\right\rangle + 2i\left|c^{+}\right\rangle\left|d^{-}\right\rangle + i\left|d^{+}\right\rangle\left|c^{-}\right\rangle). \\ iii) & \text{With } BS2^{+} \text{ removed and } BS2^{-} \text{ in place: } \to \frac{1}{2\sqrt{2}}(-\sqrt{2}\left|\gamma\right\rangle - \left|c^{+}\right\rangle\left|c^{-}\right\rangle + i\left|c^{+}\right\rangle\left|d^{-}\right\rangle + 2i\left|d^{+}\right\rangle\left|c^{-}\right\rangle). \\ iv) & \text{With } BS2^{\pm} \text{in place: } \to \frac{1}{4}(-2\left|\gamma\right\rangle - 3\left|c^{+}\right\rangle\left|c^{-}\right\rangle + i\left|c^{+}\right\rangle\left|d^{-}\right\rangle + i\left|d^{+}\right\rangle\left|c^{-}\right\rangle - \left|d^{+}\right\rangle\left|d^{-}\right\rangle). \end{cases} \tag{4.2}$$

Local realism is imposed by describing the state of the duet of particles, before the measurements, by the hidden variables, λ. The hidden variables can take different values each time the experiment is run. There are two possible choices of measurement for each particle, either $BS2^{\pm}$ are in place or removed. We will note by a 0 the results obtained with the beam splitter in place and by ∞ the results obtained with the beam splitter removed. Therefore, the outcomes of the measurements can be expressed as $X^{\pm}(0;\lambda)$ or $X^{\pm}(\infty;\lambda)$, with $X = C, D$. **If our hidden variable theory is to be local, it is required that the result of a measurement of one particle does not depend on the choice of measurement of the other particle**. For example, if a particle is detected at the dark output ($D^{\pm}$), with the beam splitter in place, we will write this outcome as $D^{\pm}(0;\lambda) = 1$. If it is not detected, then, $D^{\pm}(0;\lambda) = 0$. We can see that the notation assumes locality as the results of a measurement on one of the particles cannot depend on the choice of measurement of the other. For example, $D^{+}(0;\lambda)$, cannot depend on whether $BS2^{-}$ is in place or not and only depends on the configuration of its Mach-Zehnder device, i.e: by adopting this notation, we adopt the hypothesis of locality.

We are now able to describe the possible outcomes of the experiment in the situations described by the equations 4.2 using the hidden variables theory. From (4.2 *i)*) we see that:

$$C^{+}(\infty;\lambda)C^{-}(\infty;\lambda) = 0, \tag{4.3}$$

which is always true because the term $\left|c^{+}\right\rangle\left|c^{-}\right\rangle$ is not present in (4.2 *i)*). From (4.2 *ii)*) we get that,

$$\text{if } D^{+}(0;\lambda) = 1, \text{ then, } C^{-}(\infty;\lambda) = 1, \tag{4.4}$$

due to the state collapsing into the term $\left|d^{+}\right\rangle\left|c^{-}\right\rangle$. Analogously, from (4.2 *iii)*):

$$\text{if } D^{-}(0;\lambda) = 1, \text{ then, } C^{+}(\infty;\lambda) = 1. \tag{4.5}$$

Lastly, from (4.2 *iv)*) we see that:

$$D^{+}(\infty;\lambda)D^{-}(\infty;\lambda)=1, \tag{4.6}$$

which is true for 1/16th of the runs of the experiment ($Prob = 1/16$).

Now, we will only consider the runs of the experiment where a detection at the dark output is achieved, i.e.: we postselect[2] the system in the outcome (D^{+}, D^{-}), so equation 4.6 is always true. From 4.4 and 4.5, we see that this implies:

$$C^{+}(\infty;\lambda)C^{-}(\infty;\lambda)=1, \tag{4.7}$$

but this contradicts 4.3 ($C^{+}(\infty;\lambda)C^{-}(\infty;\lambda)=0$), which is true for all experiments. Therefore, we have reached a contradiction between our local realistic hidden variable theory and the results from quantum mechanics, thus proving Bell's theorem.

Hardy's paradox is proof of both irrealism and non locality and shows the "strange behaviour" of quantum systems, which do not obey our classical reasoning. We might get some sense of the result if we abandon the idea of locality. If locality is no longer a restriction, we can understand this contradiction as the result of some sort of "communication" between particles where, if a detection at the dark output is made, then the system changes in order to satisfy equation 4.7. Then again, this also proves Bell's theorem as even if the theory retains realism (or determinism) it must be non local.

We find again another paradoxical situation when a detection at the dark output is achieved. Moreover, it is worth noticing that this paradox is not due to counterfactual reasoning but due to the discrepancy between local realistic theories and quantum mechanics, which is described by Bell's theorem. We have seen how Hardy's paradox arises when realism is imposed by the use of "quantum trajectories" and the "empty waves", leading to contradictory trajectories. On the one hand, the use of these so called "quantum trajectories" satisfies realism as the position of the particle posses reality even if it is not observed by our intermediate measurements, on the other hand, the trajectories contradict each other. Consequently, one might be tempted to abandon realism. Nevertheless, even if we impose realism, we must accept nonlocality. These arguments should serve as a convincing insight on the acceptance of Bell's theorem.

[2] In the original paper Hardy does not impose this, in fact, he accepts that this result can only be achieved 1/16th of the time.

Chapter 5

Conclusions

The *gedankenexperiments* have been, since the early stages of our discipline, one of the most important tools used for the understanding of the physical theories. This Bachelor Thesis has been a first approach into the challenges posed by the interpretation of quantum mechanics since it first appeared in the scientific community. The project has served as a bibliographical review of the literature that lead up to Hardy's paradox, developed during the last decade of the 20th century.

Starting from a historical contextualisation of two of the main issues concerning the understanding of the theory at hand, we aimed to examine some of the fundamental characteristics of the quantum theory, as well as complementary formalisms, using Hardy's paradox as our beacon. We approach the wave-particle duality and nonlocality through the experimental device proposed by A. C. Elitzur and L. Vaidman, the Mach-Zehnder interferometer, which served as a guiding thread towards our objectives. This setup is able to show the peculiarities of the quantum behaviour in a simple and elegant way that can be easily understood at this level. The *gendankenexperiment* proposed by the aforementioned authors placed the focus on nonlocality, which is manifested in the interaction free measurements. Following the publication of Elitzur and Vaidman's paper, L. Hardy establishes a new timeline of publications that followed this first steps. It seemed almost natural to use the method proposed by Elitzur and Vaidman in order to continue the prolonged discussion on the interpretation of quantum mechanics. This was done by the publication of the two renowned articles that we have studied: [9] and [12]. Following this, in the early two thousands, Hardy's paradox served as the perfect workbench for the analysis of the much criticised two state vector formalism: [7], [13].

The main effort of this work was to unify all these publications into a single notation while studying the different variations of the Mach-Zehnder device using the TSVF and Feynman's interpretation. Interaction free measurements were studied through the ABL rule, which provided no paradoxical results. Following the guiding thread, Hardy's paradox, a situation that appears as a modification of Elitzur's setup by implementing two Mach-Zehnder interferometers, was analysed, in the same way that was proposed by Hardy, by using the concept of the empty wave. Additionally, it was shown that the paradox can also appear when studied using the TSVF. However, the implementation of Feynman's formalism allowed us to understand that the paradoxical situation is a product of a counterfactual interpretation of the ABL rule. In the context of "quantum paradoxes", Feynman himself states that: *"The "paradox" is only a conflict between reality and your feeling of what reality "ought to be.""*[2]. Finally, we presented a proof of Bell's theorem using Hardy's setup, with the particularity that Bell's inequalities were not used. This allowed us to obtain deeper insights on the effects of nonlocality, in addition to what is studied during the Physics Degree.

The treatment and understanding of wide-open research paths such as the counterfactual interpretation of the ABL rule, the use of Feynman's formalism and the discussion on Bell's theorem, not only served to obtain new insights on the fundamentals of modern physics but as a thoroughly comprehensive experience of the activities carried out in the field of theoretical physics. This work has served as a first experience in the implementation of methods and skills used at the level of top tier investigation. We have found this topic quite fruitful, which may suggest a continuation of this work using more advanced concepts. For example, the idea of weak measurements in the context of the TSVF may provide deeper insight in the use of the ABL rule and in the analysis of interference phenomena in quantum mechanics.

In conclusion, we have successfully elaborated an integrated work that unifies this specific research path, departing from the fundamental ideas that were already studied during the Physics Degree and arriving at a profound understanding of the fundamental concepts of quantum mechanics.

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