Decoherence Reduction via Continuous Dynamical Decoupling

The role of the noise spectrum

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Abstract

The decoherence of quantum systems, i.e., the loss of purity generated by the coupling of the systems to uncontrollable environments, represents one of the main challenges in the creation and implementation of quantum technologies. This work focuses on a method recently developed for addressing this problem and extending the coherence times: the Continuous Dynamical Decoupling (CDD) method. This technique has been proven useful in numerous contexts: its applicability to reduce the effects of magnetic noise in atomic multiplets, to extend coherence in trapped-ion systems, or to perform noise spectroscopy is continuously being reported.

The study begins with a review of general aspects of stochastic processes. A detailed characterization of three types of noise, white noise, the Ornstein-Uhlenbeck process, and $1/f$ fluctuations, which are particularly relevant to quantum information protocols, will be provided. Subsequently, the introduction of generic time-dependent fluctuations into the description of the dynamics of a simple quantum system will be tackled. In this line, the dependence of the consequent dephasing effect on the correlation time of the fluctuations will evaluated. Particular attention will be paid to the dephasing resulting from the previously mentioned three types of noise. The central part of the work deals with the basic component of the CDD method, i.e., with the application of a conveniently chosen driving field that allows relegating the effect of noise to a secondary (perturbative) role. Working in the basis of eigenstates of the driving term, the effect of noise on the populations of dressed states will be described. Actually, the magnitude of the noise-induced transfer of populations will be used as an indicator of the method's efficiency. The emergence of features specifically associated to the noise spectrum will be uncovered. Moreover, it will be shown that the system responses to the considered three types of noise can be regarded as illustrative examples of the diverse phenomenology that can be found in stochastic environments. The obtained results will allow discussing the applicability and efficiency of the method for various experimental conditions. Finally, the basis of the concatenation schemes commonly employed in experimental setups will be analyzed.

Resumen

La decoherencia de los sistemas cuánticos, i.e., la pérdida del carácter de estado puro generada por el acoplamiento de los sistemas a entornos incontrolables, representa uno de los principales desafíos en la creación e implementación de tecnologías cuánticas. Este trabajo se centra en un método desarrollado recientemente para abordar este problema y extender los tiempos de coherencia: el método de Desacoplamiento Dinámico Continuo (CDD). Esta técnica ha demostrado ser especialmente útil en numerosos contextos: su aplicabilidad para reducir los efectos del ruido magnético en multipletes atómicos, para extender tiempos de coherencia en sistemas de iones atrapados, o para realizar espectroscopía de ruido se sigue documentando continuamente.

El estudio comienza con una revisión de los aspectos generales de los procesos estocásticos. Se proporcionará una caracterización detallada de tres tipos de ruido: ruido blanco, el proceso Ornstein-Uhlenbeck, y las fluctuaciones $1/f$, particularmente relevantes para los protocolos de información cuántica. Posteriormente, se abordará la introducción de fluctuaciones genéricas dependientes del tiempo en la descripción de la dinámica de un sistema cuántico simple. En la misma línea, se evaluará la dependencia con el tiempo de correlación del desfase surgido. Se prestará especial atención al desfase resultante de los tres tipos de ruido mencionados anteriormente. La parte central del trabajo trata con los componentes básicos del método CDD, i.e., con la aplicación de un campo de control (*driving field*) convenientemente elegido que permita relegar el efecto del ruido a un papel secundario (perturbativo). Trabajando en la base de autoestados del término de driving, se describirá el efecto del ruido en las poblaciones de los estados vestidos. De hecho, la magnitud de la transferencia de poblaciones inducida por el ruido se utilizará como un indicador de la eficiencia del método. Se descubrirán características específicamente asociadas al espectro del ruido. Además, se mostrará que las respuestas del sistema a los tres tipos de ruidos considerados pueden ser vistas como ejemplos ilustrativos de la diversa fenomenología que se puede encontrar en entornos estocásticos. Los resultados obtenidos permitirán discutir la aplicabilidad y eficiencia del método para diversas condiciones experimentales. Finalmente, se analizará la base de los esquemas de concatenación comúnmente empleados experimentalmente.

Contents

1 Introduction

En esta sección se introducirá el problema de la decoherencia y se discutirá su importancia en el desarrollo de tecnologías cuánticas. Asimismo se mostrará el surgimiento de la decoherencia a partir de un modelo sencillo y se discutirán brevemente las principales técnicas para su reducción.

In this Section, general aspects of the decoherence problem will be addressed. First, the crucial relevance of the problem to the realization of quantum technologies will be discussed. Then, through the use of a basic model system, the emergence of decoherence in a general context will be traced. Finally, a brief account of the methods of Dynamical Decoupling, proposed for decoherence reduction, will be given.

Quantum technologies are playing a leading role in the fast development of fields like information processing, metrology, or sensing. As they are based on resources specifically associated to the quantum character of the dynamics, their potential depends crucially on maintaining the quantum signatures in the employed setups [1]. A central objective of the research in these fields is the implementation of technical schemes that allow controlling the dynamics while preserving the quantum features. Particularly important in this context is the decoherence problem: the loss of purity generated by the coupling of the primary system to non-controllable environments is a fundamental difficulty in the realization of intrinsically quantum effects. For instance, the relevance of the decoherence issues to the feasibility of the quantum-computing proposals is evident: the power associated to the possibility of working with a superposition of states disappears when the information on the relative phases is lost, i.e., when decoherence turns up. Hence, it is understood that curbing the effect of fluctuations rooted in the interactions with the environments, and, consequently, extending the coherence times is a basic requirement for advancing in the implementation of quantum technologies [2], [3]. Apart from technical importance, preserving the coherence has central relevance to fundamental areas of research. In this sense, it is worth pointing out its crucial role in the realization of fundamental effects with ultracold atoms [4].

The general aim of this work is to evaluate how the efficiency of the method of Continuous Dynamical Decoupling, proposed to reduce decoherence effects, depends on the spectral characteristics of the fluctuations. In order to achieve this objective, the analysis of the decoherence mechanisms and the precise characterization of the sources of noise are required.

Tracing the emergence of decoherence

Valuable insight into the emergence of decoherence can be obtained from the study of a basic model where fundamental aspects of open quantum systems can be traced [5]. Specifically, let us consider the system formed by two interacting qubits, one of them playing the role of *primary system* and the other standing for the *environment*. The dynamics are assumed to be governed by the Hamiltonian

$$H = \frac{\hbar}{2}\omega_1\sigma_{z,1} + \frac{\hbar}{2}\omega_2\sigma_{z,2} + A\sigma_{z,1}\sigma_{z,2}, \tag{1}$$

whose eigenstates are given by

$$|\pm_1\rangle \otimes |\pm_2\rangle\,, \tag{2}$$

where $\sigma_{z,1}\,|\pm_i\rangle = \pm\,|\pm_i\rangle\,, i = 1, 2$. In the interaction picture, the time-evolution operator reads

$$U(t) = e^{itA\sigma_{z,1}\sigma_{z,2}/\hbar}. \tag{3}$$

Now, let us assume that the system is prepared in the state

$$|\Psi(t=0)\rangle = |\Phi_{S,1}\rangle \otimes |\Phi_{S,2}\rangle\,, \tag{4}$$

with $|\Phi_{S,i}\rangle = \frac{1}{\sqrt{2}}(|+_i\rangle + |-_i\rangle), i = 1, 2$. The evolution is straightforwardly derived

$$|\Psi(t)\rangle = \frac{1}{2}(|+_1, +_2\rangle + e^{-\frac{i}{\hbar}2At}\,|+_1, -_2\rangle + e^{-\frac{i}{\hbar}2At}\,|-_1, +_2\rangle + |-_1, -_2\rangle), \tag{5}$$

where the compact notation $|\pm_i, \pm_j\rangle$ has been used.

However, this expression is not applicable if the second qubit is regarded as an *environment*. On this point, one must recall that, in the standard procedure to deal with open quantum systems, the partial trace of the density matrix over the bath degrees of freedom is carried out since there is no access to information on the reservoir dynamics. Therefore, in the present case, we must partially trace over the degrees of freedom associated to the second qubit. Consequently, we obtain for the (reduced) density operator of the first qubit the expression

$$\rho(t) = \mathrm{Tr}_2[|\Psi(t)\rangle\langle\Psi(t)|] = \frac{1}{2}\begin{pmatrix} 1 & \cos(2At) \\ \cos(2At) & 1 \end{pmatrix}. \tag{6}$$

From it, one can single out some important features resulting from the combined effect of coupling to the second qubit and partial tracing. First, the evaluation of $\mathrm{Tr}\rho^2$ uncovers a cyclic process of loss and revival of the pure-state character. Second, as the decoherences are not observed to eventually decay, (the off-diagonal elements of the reduced density matrix experience a mere cyclic evolution with no damping), one can infer that some fundamental components of the decoherence scenario are missed in the model. In fact, in order to simulate an environment, the model must be generalized: the coupling of the primary system to a set of (bath) qubits, with continuously distributed characteristics, must be incorporated. In that framework, the coherences can be shown to be given by a sum of oscillating functions $\cos(2At)$, the associated frequencies $2A$ varying continuously. Specifically, assuming that the amplitudes of coupling of the basic system to the different qubits are distributed according to the function $W_D(A)$, the coherence is found to evolve as

$$\rho_{1,2}(t) \propto \int dA W_D(A)\cos(2At), \tag{7}$$

which can be shown to correspond to a time decaying function. In particular, when the coupling amplitude A is normally distributed, namely, for

$$W_D(A) = \frac{1}{\sqrt{2\pi \mathrm{var}[A]}} \exp\left\{ -\frac{(A - \langle A \rangle)^2}{2\mathrm{var}[A]} \right\}, \tag{8}$$

a Gaussian decay is observed:

$$\rho_{1,2}(t) \propto e^{-\frac{1}{2}\mathrm{var}[A]t^2} \cos(2\langle A \rangle t). \tag{9}$$

Remarkably, it is the functional form of the distribution of coupling amplitudes that determines the form of the decay. It is also pertinent to stress that the considered model actually emulates the specific type of decoherence process known as *dephasing*: the populations maintain their initial values and the coherences decay and eventually disappear. This behavior is rooted in the structure of the coupling terms, which, as can be seen in 1 do commute with the uncoupled Hamiltonian. The scenario changes significantly if the coupling is, for instance, considered to have the form $A\sigma_{x,1}\sigma_{x,2}$. Then, a relaxation process with evolution of populations and coherences can be shown to occur. We emphasize that the considered dephasing process can also be reproduced via a Hamiltonian where the global effect of the environment is incorporated through a stochastic driving term. Specifically, the observed fixed values of the populations and the decay of the coherences are also found when the Hamiltonian is assumed to have the form

$$H = \frac{\hbar}{2}\omega_1 \sigma_{z,1} + A\zeta(t)\sigma_{z,1}, \tag{10}$$

where $\zeta(t)$ is a random signal. In this case, as we will show further in our work, it is the spectral density of noise that determines the form of the coherence decay. This dual picture of the dephasing mechanism is an illustration of a general parallelism existent in the study of open quantum systems. Indeed, there are two general alternative approaches to describe dissipation. In the first line, as in the model previously considered, the environment is explicitly incorporated into the quantum formalism [6]. Subsequently, a reduced description of the primary system is achieved via a partial tracing, and a master equation is derived. In the second line, the role of the environment is described as a random driving that enters directly the primary system. In this case, the procedure applied incorporates as a first step the evaluation of the (unitary) evolution for each noisy trajectory, i.e., for each set of values realized by the random variable along a time sequence. Subsequently, the statistical average over noise realizations is carried out. The choice of one of the alternatives to deal with a specific problem depends on the environment characteristics, in particular, on the potential requirement of a quantum treatment of the bath.

In the present work, we will be focused on decoherence originated by fluctuations present in the intensities and frequencies of electromagnetic fields which do not require a quantum treatment. Consequently, we will be using the second (stochastic) approach to describe decoherence. Here, it is worth mentioning that, in standard setups, when both decoherence mechanisms (the one associated to a quantum bath and that corresponding to classical noise) coexist, it is the second one that usually dominates, so their effects must be addressed first.

Methods for decoherence reduction

In the last decades, different methods for decoherence reduction have been proposed and applied. Indeed, a variety of strategies have been designed to cope with the specific characteristics of the different sources of noise. Significant objectives have been achieved: in some cases, the coherence times have been enlarged by orders of magnitude [7]. Among the methods applied, the techniques of dynamical decoupling stand out as particularly effective. They basically consist in strategies to effectively disconnect the system from the environment that generates the fluctuations. Their original design incorporated sequences of pulses of control intended to average out the effect of noise [8], [9]. In order to facilitate the integration of the information protocols and aiming at simplifying the experimental realization, the pulses were replaced by continuous-wave driving fields in subsequent variations of the original proposals [10], [11]. For those methods to be operative it is necessary to minimize their (unavoidable) invasive effect on the system whose control is intended. In this sense, concatenation schemes set up to deal with the extra noise introduced by the auxiliary fields have been proposed. The applicability of those techniques to qubits realized with trapped ions and atoms, nitrogen vacancies (NV) centers in diamond, or quantum dots has been extensively reported [12]-[15].

Here, it is worth stressing that it is in slow-noise setups where the performance of the CDD techniques have been mainly evaluated. Moreover, in the studies where non-static noise has been contemplated, its effect has been frequently analyzed via numerical simulation or through approximations valid only in specific regimes [16] (the limit of large observation times or the adiabatic scenario have been usually tackled). It is pertinent to add that the majority of those studies have dealt with the pulsed variant of the dynamical-decoupling technique. In the present work, we will go beyond that scenario: the potential applicability of the CDD method to deal with generic fluctuations will be analytically evaluated. Actually, given the variety of sources of noise that can be relevant to the experimental setups, it is sensible to go beyond a scenario where all the fluctuations (the original input and those resulting from random variations of the different auxiliary fields) are considered to be static. Indeed, a realistic consideration of the applicability of CDD methods should contemplate the potential role of finite correlation times.

The outline of this report is as follows. In Sec. II, I will make a review of general aspects of the characterization of stochastic variables. Moreover, I will review the properties of three specific types of noise, potentially relevant to different experimental setups. Specifically, we will deal with white noise, an Ornstein-Uhlenbeck process, and $1/f$ fluctuations. The effects of those random processes on the qubits dynamics will analyzed along the work. In Sec. III, I will describe some fundamental properties of the decohering effects of generic noise. An approach of complete validity will allow us to trace general dephasing features emergent in the asymptotic regimes. Additionally, we will describe the loss of *purity* in a general time regime for the three previously introduced stochastic processes. In Sec. IV, the system dynamics in the CDD scheme will be tackled. The analytical characterization of the noise-induced transfer of population between dressed states will be used to scrutinize the efficiency of the CDD method. Again, asymptotic expressions valid for generic noise and results for specific types of noise in a general time regime will be given. The inclusion of concatenation schemes will be evaluated. Finally, the general conclusions are summarized in Sec. V.

2 Noise

En esta sección se dará una breve introducción al estudio de procesos estocásticos y se presentarán las principales herramientas para su caracterización. Asimismo, se expondrán algunos de los ruidos más relevantes para el estudio de protocolos de información cuántica y se comentarán sus características fundamentales.

In this Section I present first a short review of the primary tools used in the general characterization of stochastic processes. Then, that basic methodology will be applied to the description of different types of noise relevant to the systems considered in the realization of quantum-information protocols.

2.1 Stochastic Processes

In nature, systems do not behave in an entirely deterministic way. In fact, in the description of any system we must cope with the problem of *limited predictability*: for sufficiently precise measurements, one can observe fluctuations emerging from a generally well-defined deterministic global behavior. The origin of the fluctuations can be traced back to the coupling of the system to uncontrollable environments. Moreover, even if the system can be considered to be highly isolated, the lack of predictability can enter the description via fundamental aspects of the dynamics like the chaotic character of the classical dynamics or the lack of determinism intrinsic to the quantum evolution. Understanding the different sources of fluctuations is therefore fundamental to a wide range of fields. Much effort has been dedicated towards this goal, in this sense, we must recall the pioneering work on Brownian Motion independently developed by Einstein [17] and Smoluchowski [18].

Two equivalent approaches are standardly applied to describe a stochastic process. The first one consists in using a stochastic differential equation to characterize the evolution of the random variable $x(t)$. Specifically, the dynamical equations corresponding to the primary deterministic process are modified to incorporate random terms (*random forces*) that account for the presence of fluctuations. The noisy terms are modeled according to the properties of the environment that generate them. In this approach, frequently denominated a *Langevin-type* description, the stochastic variable can potentially follow a variety of *noisy trajectories*, associated to the different realizations of the random forces. The global effect of noise on the system is obtained by averaging over all trajectories.

The second (counterpart) approach deals with obtaining the distribution function $W(x,t)$ that gives the probability for the stochastic variable to reach a specific value. A partial differential equation, usually called a Fokker-Planck equation, is set up for $W(x,t)$. This differential equation incorporates both the deterministic system-components and the characteristics of the fluctuations [19].

Any of the two approaches can provide the information required for describing the stochastic process. In our analysis of the efficiency of the technique of Continuous Dynamical Decoupling to curb the decohering effects of noise we will use elements from both approaches.

The above general arguments are clearly illustrated by the study of Brownian motion, i.e., the diffusion process experienced by a macroscopic particle immersed in a fluid due to the unpredictable kicks generated by the fluid molecules. The two lines originally followed in the description of Brownian motion correspond to the two previously referred standard approaches. Indeed, the explanation of that process set the basis for the development of stochastic analysis. The first (Langevin) line is based on setting up a differential equation where the effect of the fluid enters via both a deterministic friction force and a rapidly-fluctuating force with zero-mean value. The alternative approach is the (original) Einstein derivation of the probability-distribution function: from general considerations on the effect of the fluid, a diffusion (partial-differential) equation is built up and solved. In both lines, the properties of the environment correspond to the fluctuations presently known as white noise. In the following, we will precisely define the characteristics of this and other types of fluctuations.

2.2 General Characteristics of Noise

There is a variety of sources of fluctuations that can be relevant to decoherence in Quantum Mechanics. To deal with the associated broad range of noise properties, an operative approach is needed. The usual procedure consists in setting up models where different stages of increasing complexity are gradually incorporated. In the present work, we will restrict to the basic scenario, i.e., we will deal with random variables $x(t)$ which are assumed to be completely characterized by two properties, namely, their mean value and their autocorrelation function. Those variables are termed Gaussian variables. One basic objective of the present work is to account for the effect of Gaussian fluctuations on the Quantum Dynamics. The implications of going beyond the *Gaussianity* assumption will be punctually discussed.

The mean value of $x(t)$, denoted as $\langle x(t)\rangle$, is trivially defined as the average over stochastic realizations. (Here, it is important to realize that it is always possible to work with zero-mean random variables since a non-zero mean value can be straightforwardly incorporated into the deterministic dynamics). The second fundamental characteristic is the autocorrelation function, denoted by $G(\tau)$, and given by

$$G(\tau) = \langle x(t)\ x(t+\tau)\rangle. \tag{11}$$

It is apparent that $G(\tau)$ is an indicator of how a particular noise realization affects the consecutive ones. From the form of $G(\tau)$, two additional (secondary) parameters are derived. The first one is the variance, which measures the magnitude of the noisy dispersion, is defined by $\mathrm{var}[x(t)] = \langle x^2(t)\rangle - \langle x(t)\rangle^2$ and can be rewritten as:

$$\mathrm{var}[x(t)] = G(0) - \langle x(t)\rangle^2. \tag{12}$$

Hence, in the case of a zero-mean variable, we simply have:

$$\mathrm{var}[x(t)] = G(0). \tag{13}$$

This expression will be frequently used in the present work as we will generally work with zero-mean value random variables.

Another parameter derived from $G(\tau)$ is the correlation time τ_c, defined by

$$\tau_c = \frac{1}{\mathrm{var}[x(t)]} \int_0^\infty G(\tau) d\tau, \tag{14}$$

which gives the magnitude of the time interval for the decay of the correlation.

Importantly, in our basic model, the random variables, apart from Gaussian, are assumed to be stationary, i.e., the mean value $\langle x(t) \rangle$ is considered to be the same at any time (actually, there is no time dependence in the mean value), and the autocorrelation function is assumed to depend on the time interval τ, but not on the absolute time t.

Very convenient to the identification of the different types of noise is the use of the spectral density $S(\omega)$ defined as

$$S(\omega) = \lim_{T \to \infty} \frac{1}{2\pi T} \left| \int_0^T dt e^{-i\omega t} x(t) \right|^2. \tag{15}$$

In Appendix I, I also present the derivation of the Wiener-Khinchin-Theorem, i.e., of the relationship

$$S(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\tau e^{-i\omega\tau} G(\tau), \tag{16}$$

between the spectral density and the Fourier transform of the autocorrelation function. $S(\omega)$, which can be interpreted as an indicator of the weight that each Fourier component of noise has in the total random signal, is an operative tool in the experimental characterization of the fluctuations. Moreover, we will see that $S(\omega)$ plays a key role in our objective of understanding the relevance of the noise spectrum to the efficiency of the CDD method. As a preamble to achieving that goal, three types of noise with widely different spectra are tackled in the following.

2.3 White noise

The term white noise refers to an *idealized* delta-correlated stochastic process: the autocorrelation function is given by

$$G(\tau) = 2\pi C \delta(\tau), \tag{17}$$

and, correspondingly, the spectrum is constant

$$S(\omega) = C, \tag{18}$$

i.e., all the Fourier components of noise have equal weight.

White noise is used to model stochastic setups where an almost sudden complete loss of correlation can be assumed. Here, we use the term *idealized* as in practice there is always a finite smallest time-scale for observing the system. Hence, a sudden loss of correlation must necessarily be an idealization. Notice that, formally, for a white-noise process the variance is infinite and the correlation time is zero. The divergence disappears as soon as a coarse-graining over the smallest time-scale for observing the system is carried out. Apart from directly modeling noisy environments, white noise enters as a random force in stochastic differential equations used to generate finite correlation-time (colored-noise) processes. This is illustrated by the build up of an Ornstein-Uhlenbeck process presented in the next Section.

2.4 Ornstein-Uhlenbeck process

This type of stochastic process, which was first described by Leonard Ornstein and George E. Uhlenbeck in 1930 [20], is widely applied in Physics.

Formally, a stochastic variable $x(t)$ is said to describe an Orntein-Uhlenbeck process when it obeys the following stochastic differential equation

$$dx = -\alpha x dt + \Gamma(t) dt, \tag{19}$$

where α is a positive real constant, and $\Gamma(t)$ denotes a stochastic force with white-noise characteristics, i.e.,

$$\langle \Gamma(t)\Gamma(t') \rangle = \sqrt{D}\delta(t-t'), \tag{20}$$

with D being a positive real constant. Although the case of a nonzero mean-value of $\Gamma(t)$ can be straightforwardly incorporated in this framework, here, for simplicity, a zero mean-value, $\langle \Gamma(t) \rangle = 0$, is taken.

Applying standard techniques to solve stochastic differential equations, it is shown that, in the stationary regime, the mean value and the correlation function of $x(t)$ are respectively given by

$$\langle x(t) \rangle = 0, \tag{21}$$

and

$$\langle x(t)x(t+\tau) \rangle = \frac{D}{2\alpha} e^{-\alpha|\tau|}. \tag{22}$$

Accordingly, the correlation time is $\tau_c = \alpha^{-1}$.

It is equally shown that the spectral density of the process has a Lorentzian functional-form, namely,

$$S(\omega) = \frac{D}{2\pi(\alpha^2+\omega^2)}. \tag{23}$$

The Ornstein-Uhlenbeck process is widely used in different scientific contexts to emulate finite correlation-time random inputs, from Physics [21], [22] to Financial Mathematics [23] and even in Evolutionary Biology [24]. Actually, the main components of the fluctuations emerging in a variety of environments seem to be well modeled by the exponential decay of the correlation and the associated Lorentzian spectrum of an Ornstein-Uhlenbeck process.

2.5 $1/f$ noise

This type of noise, is found widely in nature, from Physics to Music, passing through Biology, Economics, Psychology or even in Language [25]-[27]. Firstly discovered by Johnson [28] in 1925, it has been shown to appear in the annual flood levels of various rivers, in electronic devices, or to even be related to ion channels in brains (in this last case, deviations from $1/f$ noise can even be used to identify epilepsy in clinical EEGs [29], [30]).

Formally, the term $1/f$ noise refers to fluctuations with a spectral density of the form

$$S(\omega) = \begin{cases} \frac{A}{\omega} \ , \ \omega \in (\omega_1, \omega_2) \\ 0 \ \ , \ \omega \notin (\omega_1, \omega_2) \end{cases} \tag{24}$$

where ω_1 y ω_2 are the limits of the frequency range on which the system is accessible. Correspondingly, $1/\omega_2$ y $1/\omega_1$ define the bounds of the time interval where the system is observable. In particular, the smallest time-scale to extract information from the system is given by $1/\omega_2$ and the asymptotic limit corresponds to the time $1/\omega_1$.

From the spectral density, we can obtain the autocorrelation function applying the Wiener-Khinchin Theorem as follows

$$G(\tau) = \int_{-\infty}^{\infty} d\omega e^{i\omega\tau} S(\omega) = 2A \int_{\omega_1}^{\omega_2} d\omega \frac{\cos(\omega\tau)}{\omega}, \tag{25}$$

and, to solve the integral, we evaluate first the time derivative and then make a coarse-graining over the smallest time-scale $1/\omega_2$, namely,

$$\begin{aligned} \frac{dG(\tau)}{d\tau} &= -2A \int_{\omega_1}^{\omega_2} d\omega \sin(\omega\tau) = \frac{4A}{\tau} \sin\left(\frac{(\omega_2-\omega_1)\tau}{2}\right) \sin\left(\frac{(\omega_1-\omega_2)\tau}{2}\right) \\ &\simeq -\frac{4A}{\tau} \sin^2\left(\frac{\omega_2\tau}{2}\right) \simeq -\frac{2A}{\tau}, \end{aligned} \tag{26}$$

where we have also considered $\omega_2 \gg \omega_1$. Now, integration is straightforward. We can adjust the integration constant to cover the value of the variance at time $1/\omega_2$

$$var[x(t)] = G(0) = 2 \int_{\omega_1}^{\omega_2} d\omega \frac{A}{\omega} = 2A \ln\left(\frac{\omega_2}{\omega_1}\right), \tag{27}$$

and therefore

$$G(|\tau|) = 2A \ln\left(\frac{1}{\omega_1|\tau|}\right), \tag{28}$$

which reflects that, consistently, the correlation decays to zero at the asymptotic value of time $1/\omega_1$. Fluctuations with a $1/f$ spectrum are ubiquitous in Physics. They are particularly relevant to the field of Quantum Information as they are known to be the main source of decoherence in solid-state devices proposed to implement qubits. In quite generally accepted theoretical approaches, the emergence of $1/f$ noise is traced to the presence of impurities in the solid substrate which can be modeled as two-level systems. In this framework, each two-level unit is considered as a fluctuator: the system can randomly jump between the two states. It is shown that, for specific forms of the distribution of impurities, the combined action of the stochastic transitions of the system in the different fluctuators leads to a global noisy output with a $1/f$ spectral density [31]. Important for the objectives of our study is to identify the features of the decoherence process which are linked to specific characteristics of the spectrum. As we will see, differential aspects associated to the $1/f$ spectrum can be singled out.

3 Noise-induced decoherence: Analytical study of dephasing processes

Esta sección se centrará en la introducción de fluctuaciones genéricas en sistemas de relevancia para protocolos de información cuántica. Se analizarán sus efectos analíticamente, primero desde un enfoque general aplicable a cualquier tiempo para fluctuaciones genéricas. Tras ello, se hará un análisis en profundidad donde se estudiarán diferentes límites y tipos de ruido concretos.

In this Section, I present a general analytical approach to incorporate fluctuations into the evolution of systems relevant to Quantum Information techniques. First, a general description, valid at any time regime and applicable to generic fluctuations, will be given. Subsequently, an in-depth analysis of different asymptotic regimes and types of noise will be presented.

3.1 The model system

As a starting point, a simple scenario where the fluctuations enter the system as a classical driving field. Specifically, we deal with the Hamiltonian

$$H = (\omega_0 + \delta\omega_0(t))\, F_z, \tag{29}$$

where F_z is an Angular Momentum operator, ω_0 represents a characteristic system-frequency, and $\delta\omega_0(t)$ accounts for random variations in the frequency. Note that no restrictions on the number of involved states are assumed as the quantum number F is not specified. Although the used notation specifically refers to a hyperfine Zeeman multiplet, as considered in [16], the above Hamiltonian can actually be regarded as a model system of wide applicability: it is relevant to any context where noisy changes of the characteristic frequencies can occur.

It is worth stressing that this kind of processes, associated to a stochastic modulation of the system frequencies, are usually termed as *dephasing*. In the application of the method of Continuous Dynamical Decoupling, we will also face (alternative) *dissipation* processes induced by fluctuations entering the system via a nondiagonal term (i.e., a term which does not commute with the undriven Hamiltonian). We will see that the evolution of populations and coherences presents differential aspects depending on how stochasticity enters the system.

Convenient for a compact characterization of the dynamics is changing to the rotating frame defined by the unitary transformation

$$U(t) = e^{i\omega_0 t F_z/\hbar}. \tag{30}$$

The transformed Hamiltonian is given by the expression

$$\widetilde{H} = UHU^\dagger + i\hbar\dot{U}U^\dagger = (\omega_0 + \delta\omega_0(t))F_z + i\hbar\frac{i\omega_0}{\hbar}F_z = \delta\omega_0(t)F_z, \tag{31}$$

which is derived making use of the Baker-Campbell-Hausdorff formula (see Appendix II).

In turn, the Schrödinger equation is solved in the form

$$\partial_t |\Psi(t)\rangle = -\frac{i}{\hbar}\delta\omega_0(t)F_z |\Psi(t)\rangle \implies |\Psi(t)\rangle = e^{-iF_z\xi(t)/\hbar} |\Psi(0)\rangle , \tag{32}$$

where we have introduced the (random) phase shift $\xi(t)$ given by

$$\xi(t) = \int_0^t \delta\omega_0(t')dt'. \tag{33}$$

Obviously, this procedure allows also solving for the time evolution of the density operator, obtained as

$$\rho(t) = |\Psi(t)\rangle \langle\Psi(t)| . \tag{34}$$

Moreover, in the representation of states $\{k; m\}$ (m being the quantum number associated to F_z, and k accounting for additionally required quantum numbers), the density-matrix elements are given by

$$\rho_{m,m'}(t) = \rho_{m,m'}(0)e^{i(m-m')\xi(t)}. \tag{35}$$

Now, in order to completely characterize the system dynamics, the average over noise realizations must be carried out. Notice that this last step parallels the derivation of the reduced master equation in a framework based on a complete quantum treatment of the fluctuations. Indeed, in that scenario [5], the analog procedure is tracing over the environment degrees of freedom, which leads to the reduced description. In our approach, the (reduced) density-matrix elements read

$$\rho_{m,m'}(t) = \rho_{m,m'}(0) \langle e^{i(m-m')\xi(t)}\rangle. \tag{36}$$

From this expression, it is apparent that no changes in the populations take place in the considered *dephasing* setup. On the contrary, the coherences can be predicted to evolve, the specific form of their time evolution being determined by the noise characteristics. In the following, we will show that it is possible to go further analytically at specific time regimes and for particular noise properties.

3.2 Dephasing induced by generic fluctuations: Analysis of the asymptotic time regimes

Even when a complete characterization of the random input $\delta\omega_0(t)$ is not accessible, and, consequently, when the properties of the random term $\xi(t)$ are not completely known, it is possible to identify two distinct regimes where analytical conclusions on the evolution of the density-matrix elements can be drawn. Depending on the magnitude of the correlation time τ_c, (i.e. of the time scale over which the random variable remains correlated), it is possible to identify the following distinct regimes:

The limit of large correlation times

At times much smaller than the correlation time, i.e., for $t \ll \tau_c$, the phase shift can be approximated as

$$\xi(t) = \int_0^t \delta\omega_0(t')dt' \simeq \delta\omega_0(0)t. \tag{37}$$

Therefore, we can calculate the average of the density-matrix elements from the probability distribution-function for $\delta\omega_0(t)$, $W_D[\delta\omega_0(t)]$, namely,

$$\rho_{m,m'}(t) = \rho_{m,m'}(0) \int_{-\infty}^{\infty} d(\delta\omega_0) W_D(\delta\omega_0) e^{i(m-m')\delta\omega_0 t}. \tag{38}$$

Hence, this approximation allows us drawing the functional form of the coherence decay from the mere knowledge of general characteristics of the fluctuations, specifically, from the magnitude of the correlation time and from the probability distribution-function.

The predictive power of having analytical results in this regime is illustrated by considering the case of Gaussian noise. For a Gaussian input $\delta\omega_0(t)$ with mean value $\langle\delta\omega_0(t)\rangle = 0$ and variance $\mathrm{var}[\delta\omega_0(t)]$, and, consequently, with the distribution function being given by

$$W_D(\delta\omega_0) = \frac{1}{\sqrt{2\pi \mathrm{var}[\delta\omega_0]}} \exp\left[-\frac{(\delta\omega_0)^2}{2\mathrm{var}[\delta\omega_0]}\right], \tag{39}$$

the averaging in Eq.(38) is straightforwardly carried out to obtain

$$\rho_{m,m'}(t) \propto e^{-\frac{1}{2}\mathrm{var}[\delta\omega_0]t^2(m-m')^2}, \tag{40}$$

which corresponds to Gaussian decay with characteristic time

$$\tau_d = \frac{\sqrt{2}}{(m-m')\sqrt{\mathrm{var}[\delta\omega_0]}}. \tag{41}$$

Note that for $m = m'$, one finds $\tau_d \to \infty$, as corresponds to the previously remarked lack of evolution of the populations.

The limit of short correlation times

In the opposite limit, namely, for $t \gg \tau_c$, it is possible to rewrite $\xi(t)$ in the form

$$\xi(t) = \int_0^{\Delta t} \delta\omega_0(t)dt + \int_{\Delta t}^{2\Delta t} \delta\omega_0(t)dt + ... + \int_{(n-1)\Delta t}^{t} \delta\omega_0(t)dt, \tag{42}$$

i.e., as the sum of n integrals each of them covering a time interval $\Delta t = \frac{t}{n}$. In the considered limit, it is feasible to make compatible a large number of terms in the sum (a high value of n) with Δt being much larger than the correlation time. Hence, from the application of the Central Limit Theorem [19], one concludes that, since $\xi(t)$ can be expressed as the sum of a large number of statistically independent variables, it presents an approximate normal distribution.

Therefore, one simply needs to evaluate the mean $\langle \xi(t) \rangle$ and the variance $\langle \xi^2(t) \rangle - \langle \xi(t) \rangle^2$ to completely characterize $\xi(t)$. Accordingly, we proceed as

$$\langle \xi(t) \rangle = \left\langle \int_0^t \delta\omega_0(t')dt' \right\rangle = n \int_0^{\Delta t} \langle \delta\omega_0(t') \rangle = \langle \delta\omega_0 \rangle t = 0, \tag{43}$$

where we have assumed an stationary process (i.e., $\langle \delta\omega_0(t') \rangle = \langle \delta\omega_0 \rangle$).

Additionally, we can calculate the variance

$$\text{var}[\xi(t)] = \langle \xi^2(t) \rangle = \left\langle \left(\int_0^t d\tau \delta\omega_0(\tau) \right) \left(\int_0^t d\tau' \delta\omega_0(\tau') \right) \right\rangle. \tag{44}$$

Solving this integral, as detailed in Appendix III, shows that in the considered limit,

$$\text{var}[\xi(t)] \simeq 2\pi S(0)t. \tag{45}$$

It is possible to make an averaging as in Eq. (38). Indeed, from Eq.(36), one finds

$$\begin{aligned}
\rho_{m,m'}(t) &= \rho_{m,m'}(0) \int d(\xi(t)) W_D[\xi(t)] e^{i(m-m')\xi(t)} \\
&= \rho_{m,m'}(0) \int_{-\infty}^{\infty} d(\xi(t)) \frac{1}{\sqrt{2\pi \text{var}[\xi(t)]}} \exp\left[-\frac{\xi^2(t)}{2\text{var}[\xi(t)]} \right] e^{i(m-m')\xi(t)} \\
&= \rho_{m,m'}(0) e^{-\frac{1}{2}(m-m')\langle \xi(t) \rangle}.
\end{aligned} \tag{46}$$

It is then concluded that, in the regime considered, the coherences present an exponential decay,

$$\rho_{m,m'}(t) \propto \exp\left[-(m-m')^2 \pi S(0) t \right], \tag{47}$$

the $1/e$ scaling time being,

$$\tau_d = [(m-m')^2 \pi S(0)]^{-1}. \tag{48}$$

Hence, it is the noise spectrum at zero frequency that determines the magnitude of the dephasing time. The emergence, irrespective of the noise properties, of a universal exponential-decay regime in the limit of long observation times has been observed in studies on dephasing in different physical contexts. At this point, it is worth emphasizing that a different functional form is associated to processes induced by $1/f$ noise, as we will see in the next Section.

3.3 Dephasing induced by specific types of noise: Analysis of the general time regime

In order to identify the type of noise present in a particular setup, the results extracted from the above analysis of the asymptotic regimes are not sufficient. Advances in tracking the fluctuations demand a more complete description of the coherence decay. A detailed modeling of the noise characteristics is needed to establish the origin of features emergent in the decoherence process. In the following, we will proceed along this line by identifying differential properties of the dephasing process associated to the three different types of noise characterized in Sec. 2. Accordingly, the random input into the system, i.e., the stochastic variable $\delta\omega_0(t)$, will be consecutively assumed to correspond to white noise, to an Ornstein-Uhlenbeck process, and to $1/f$ fluctuations. In the three cases, the Gaussian character of the variables will be considered. Moreover, without loss of generality, we will deal with zero mean-value variables, $\langle\delta\omega_o\rangle = 0$. It follows that $\xi(t)$, given by the time integral of $\delta\omega_0(t)$, is also a Gaussian variable. Moreover, it is trivially found that, for stationary processes, it has zero mean-value $\langle\xi(t)\rangle = 0$.

Therefore, only the variance $\langle\xi^2(t)\rangle$ is required for a complete characterization of the probability distribution-function of $\xi(t)$, $W_D[\xi(t)]$, and, in turn, for obtaining the time-evolution of the density-matrix elements as shown in Eq.(46).

Correspondingly, we address now the evaluation of the variance of $\xi(t)$ for the three considered types of noise using the associated autocorrelation functions presented in the previous Section:

For white noise, we obtain

$$\langle\xi^2(t)\rangle = \int_0^t d\tau \int_0^t d\tau' \langle\delta\omega_0(\tau)\delta\omega_0(\tau)\rangle = \int_0^t d\tau \int_0^t d\tau' 2\pi C\delta(\tau-\tau') = 2\pi S(0)t. \tag{49}$$

For an Ornstein-Uhlenbeck process,

$$\begin{aligned}\langle\xi^2(t)\rangle &= \int_0^t d\tau \int_0^t d\tau' \frac{D}{2\alpha} e^{-\alpha|\tau-\tau'|} = \frac{D}{2\alpha}\int_{-t}^t d\tau (t-|\tau|) e^{-\alpha|\tau-\tau'|} \\ &= \frac{D}{\alpha^2}\left(t + \frac{e^{-\alpha t}-1}{\alpha}\right) = 2\pi S(0)\left(t + \frac{e^{-\alpha t}-1}{\alpha}\right).\end{aligned} \tag{50}$$

And for a $1/f$ noise

$$\begin{aligned}\langle\xi^2(t)\rangle &= \int_0^t d\tau \int_0^t d\tau' (-2A)\ln(\omega_1|\tau|) = -4A\int_0^t d\tau (t-|\tau|)\ln(\omega_1|\tau|) \\ &= At^2\left(3 - 2\ln(\omega_1 t)\right) + \mathcal{O}\left[\frac{4A}{\omega_1\omega_2}\ln\left(\frac{\omega_1}{\omega_2}\right)\right].\end{aligned} \tag{51}$$

From the above results some preliminary conclusions can be drawn:

i) For a noisy input with white-noise characteristics, the decay of the coherences is exponential in the whole temporal range. Note that the previously analyzed limit of large correlation times, $t \ll \tau_c$, cannot be reached as the fluctuations (formally) have zero correlation time. The system is permanently in the other asymptotic regime studied, i.e., in the limit of short correlation times. Indeed, there is complete agreement between the exact value of the variance given by Eq.(49) and that obtained through the approximate analysis of the asymptotic regime. Notice that, in order to emphasize that agreement, in Eq.(49), we have (formally) written the variance in terms of the zero-frequency value of the spectral density $S(0)$. In fact, as shown in the former Section, the white-noise spectrum is flat, so there is no frequency dependence in the spectral density.

ii) For the case of $\delta\omega_0(t)$ corresponding to an Ornstein-Uhlenbeck process, the results obtained in the two asymptotic regimes are consistently recovered from Eq.(50): by fixing α, (we recall that the correlation time in this case is $\tau_c = \alpha^{-1}$), and taking the limits $t \to 0$ and $t \to \infty$ in that equation, we find the results previously derived using general arguments in the limits of large correlation time ($t \ll \alpha^{-1}$) and small correlation time ($t \gg \alpha^{-1}$). In particular, it is shown that, at large times, $\langle \xi^2(t) \rangle$ is correctly expressed as a function of the zero-frequency value of the spectrum $S(0)$. On the other hand, for large correlation times, a faster exponential decay is observed. (Indeed, a Gaussian decay is found, i.e. $\langle \xi^2(t) \rangle \simeq 2\pi S(0)\alpha t^2$). In the crossover between the two asymptotic regimes, a complex time-dependence, determined by the specific value of the correlation time, is observed.

iii) Special care is needed in the analysis of the results for $1/f$ fluctuations. Actually, important specific features can be observed. First, one finds that the results obtained in the study of the limit of large correlation times are consistently recovered, as we had assumed Gaussian variables. However, a functional form different from the exponential decay previously identified as typical of the asymptotic limit of short correlation times is found. Indeed, a Gaussian decay is predicted from Eq.(51). This disagreement is understood taking into account that the analysis made in the limit $t \gg \tau_c$ is not applicable to $1/f$ noise.

That analysis was based on approximating the integrand in Eq.(44) in terms of the Dirac-delta function (See Appendix III). That approximation is based on regarding the integrand as the product of a highly peaked function $\left[\frac{\sin(\omega t/2)}{\omega/2}\right]^2$ and a smooth function $S(\omega)$. However, in the case of $1/f$ fluctuations that approximation is not valid as the function $S(\omega)$ does not vary smoothly at $\omega = 0$; in fact, it formally diverges. Useful to clarify this point is comparing with the behavior at $\omega = 0$ of the (smooth) spectral density (a Lorentzian function) corresponding to an Ornstein-Uhlenbeck process. (Eq.(22)).

4 The Method of Continuous Dynamical Decoupling

Esta sección estará centrada en el desarrollo de la base teórica básica del esquema de concatenación del método de Desacoplamiento Dinámico Continuo. Asimismo se expondrá su aplicabilidad en distintos tipos de ruido y se estudiará su eficacia en cada caso.

This Section will be focused on the theoretical basis of the Method of Continuous Dynamical Decoupling (CDD). The fundamental scheme will be tackled first. Then, the applicability of the method to different types of noise will be analyzed. Finally, some conclusions on the relevance of the noise spectrum to the efficiency of the CDD method will be drawn.

4.1 Basic components of the CDD method

The CDD method is based on using driving fields with characteristics appropriate to force the fluctuations present in the original system to play a secondary (irrelevant) role in the dynamics. This general principle is illustrated by its application to the case considered in [16]. There the original system is a hyperfine Zeeman multiplet. In that setup, the fluctuations present in the Zeeman field, being diagonal in the Hamiltonian-eigenstate representation, have a strong dephasing effect on the system's evolution. In order to mitigate that effect, the CDD method incorporates an orthogonal driving field with a frequency resonant or quasiresonant with the multiplet splitting and with an amplitude sufficiently large to force the noise to have a perturbative character in the driven dynamics. Moreover, to deal with the noise introduced into the system via the driving-field intensity, the scheme includes a second field of control orthogonal to the first one and with frequency and amplitude conveniently chosen. This concatenation scheme can be continued until the required precision is reached. The whole mechanism can then be characterized as a *continuous* adjustment of the dynamics to reduce the effect of noise (i.e., to decouple the original system from the fluctuations).

Here, we will consider the setup realized in [16] as a prototype model system where the components and requirements for the CDD to be operative can be appropriately studied. Specifically, we focus on the Hamiltonian given by Eq.(29) in the previous Section, where now a first driving field along the OX-axis is incorporated,

$$H = (\omega_0 + \delta\omega_0(t))\, F_z + 2\Omega_d \cos(\omega_d t) F_x. \tag{52}$$

Note that the control-field parameters, i.e., the frequency ω_d and the amplitude, incorporated into Ω_d, can be tuned as can be demanded by the operativity of the CDD method.

It is also important to remark that, whereas in the model used to analyze the results of [16], only static (i.e., time-independent) fluctuations were considered, here, the situation of having nonstatic noise is also evaluated. Actually, this is not a secondary aspect of the problem since the experimental setup is known to present time-dependent fluctuations. Going further in this line, our study will be centered on revealing how the efficiency of the CDD method depends on the noise spectrum.

In our procedure to solve for the dynamics governed by the Hamiltonian given by Eq.(52), we start by choosing the driving frequency as $\omega_d = \omega_0$, and applying the following rotating-frame transformation

$$U_1(t) = e^{i\omega_d F_z t/\hbar}. \tag{53}$$

For simplicity, we will keep on using the same notation, H, for the transformed Hamiltonian, which, accordingly, is written as

$$H = \Omega_d F_x + \delta\omega_0(t) F_z, \tag{54}$$

where we have made the Rotating Wave Approximation, i.e., we maintain the secular terms and neglect the oscillations with frequency $2\omega_d$ (See Appendix IV). This approximation is justified as we assume that the restriction $\omega_d \gg \Omega_d$ is fulfilled. It is apparent in Eq.(54) that, in the case of static fluctuations, the Hamiltonian eigenstates can be straightforwardly obtained. In contrast, we must face a non- trivial problem in the case of nonstatic noise. Let us see how that problem can be approximately solved using time-dependent Perturbation Theory. In order to precisely discuss our approach, it is convenient to make a second unitary transformation, namely,

$$U_2(t) = e^{i\frac{\pi}{2} F_y t/\hbar}, \tag{55}$$

which amounts to rotate the system a $\pi/2$ angle around the OY-axis. Consequently, the Hamiltonian is rewritten as

$$H = \Omega_d F_z - \delta\omega_0(t) F_x. \tag{56}$$

(We recall that the technical procedure applied in the consecutive changes in the Hamiltonian incorporates the use of the BCH formula, see Appendix II). From the above equation, the feasibility of applying Perturbation Theory becomes evident: for driving-field intensities much larger than the magnitude of the fluctuations, i.e., for $\Omega_d \gg |\delta\omega_0(t)|$, the Hamiltonian can be regarded as composed by a zero- order term (the time-independent contribution $H_0 \equiv \Omega_d F_z$) and a perturbative random part, $W(t) \equiv \delta\omega_0(t) F_x$. Consequently, we rewrite Eq.(56) as

$$H = H_0 + W = \Omega_d F_z - \delta\omega_0(t) F_x. \tag{57}$$

Hence, writing the eigenstates of H_0 as $|k; m\rangle$ (the associated eigenvalues being $m\Omega_d\hbar$), and using time-dependent Perturbation Theory [33], one can evaluate the probability of having a noise-induced transition between the states m and m'. First, we formally consider a particular stochastic trajectory, i.e., a specific noise realization, and write

$$\begin{aligned} P_{m,m'}(t) &= \frac{1}{\hbar^2}\left|\int_0^t dt' W_{m,m'}(t') e^{i(E_{m'}-E_m)t'/\hbar}\right|^2 \\ &= \frac{|(F_x)_{m,m'}|^2}{\hbar^2}\left|\int_0^t dt' \delta\omega_0(t') e^{i(m-m')\Omega_d t'}\right|^2, \end{aligned} \tag{58}$$

where we have incorporated the matrix elements:

$$W_{m',m}(t') = \delta\omega_0(t')(F_x)_{m',m}, \tag{59}$$

and have introduced the eigenvalues $E_{m'} - E_m = (m' - m)\Omega_d\hbar$.

Second, we proceed on averaging over stochastic realizations. Accordingly, we must evaluate

$$\begin{aligned}
\langle P_{m,m'}(t)\rangle &= \frac{\left|(F_x)_{m,m'}\right|^2}{\hbar^2}\left\langle\left|\int_0^t dt'\delta\omega_0(t')e^{i(m'-m)\Omega_d t'}\right|^2\right\rangle \\
&= \frac{\left|(F_x)_{m,m'}\right|^2}{\hbar^2}\int_0^t\int_0^t d\tau d\tau'\langle\delta\omega_0(\tau)\delta\omega_0(\tau')\rangle e^{i\Omega_e(\tau-\tau')} \\
&= \mathcal{F}_{m,m'}\int_0^t\int_0^t d\tau d\tau' G(\tau-\tau')e^{-i\Omega_e(\tau'-\tau)}
\end{aligned} \tag{60}$$

Observe that the notation has been simplified by introducing the effective frequency $\Omega_e = (m' - m)\Omega_d$ and using $\mathcal{F}_{m,m'}$ for the global multiplicative factor, i.e., $\mathcal{F}_{m,m'} = \frac{\left|(F_x)_{m,m'}\right|^2}{\hbar^2}$. Additionally, through an appropriate change of variables (94), assuming a stationary process and using the Wiener-Khinchin theorem, the probability of population transfer can be cast into the form

$$\begin{aligned}
\langle P_{m,m'}(t)\rangle &= \mathcal{F}_{m,m'}\int_{-t}^t d\tau_D G(\tau_D)e^{-i\Omega_e\tau_D}(t-|\tau_D|) \\
&= \mathcal{F}_{m,m'}\int_{-\infty}^{\infty} d\omega\int_{-t}^t d\tau(t-|\tau|)e^{-i\Omega_e\tau}e^{i\omega\tau}S(\omega) \\
&= \mathcal{F}_{m,m'}\int_{\infty}^{\infty} d\omega\ S(\omega)\left(\frac{\sin[(\omega-\Omega_e)t/2]}{(\omega-\Omega_e)/2}\right)^2.
\end{aligned} \tag{61}$$

Hence, we have derived two expressions, Eqs. 60 and 61, which can alternatively be used to evaluate the efficiency of the CDD method. It is worth stressing that the probability of population transfer provides a good measure of the effect of noise: for small values of the transition probabilities, the dressed states, obtained through the applied sequence of unitary transformations, can be considered as closely approaching the eigenstates of the complete Hamiltonian, i.e., to approximately be noise immune.

4.2 Dressed-state population transfer induced by generic fluctuations: Analysis of the limit of short correlation times

In the limit $t \gg \tau_c$, it is possible to go further analytically in the evaluation of the population transfer. The analysis of the integral in Eq.(61) gives the clues to proceed: it can be shown that for t being much larger than the inverse width of the spectral density (τ_c), the integrand can be approximated in the form (see Appendix III)

$$S(\omega)\left(\frac{\sin[(\omega-\Omega_e)t/2]}{(\omega-\Omega_e)/2}\right)^2 \sim 2\pi t S(\omega)\delta(\omega-\Omega_e). \tag{62}$$

Actually, in that limit, $S(\omega)$ varies smoothly in the range where $\left(\frac{\sin[(\omega-\Omega_e)t/2]}{(\omega-\Omega_e)/2}\right)^2$ is a nonzero and highly-peaked function. Consequently, the integral in Eq.(61) is directly evaluated to obtain

$$\langle P_{m,m'}(t)\rangle \simeq \mathcal{F}_{m,m'} 2\pi t S(\Omega_e). \tag{63}$$

This result provides a clear insight into the mechanism of noise reduction incorporated by the CDD method. More specifically,

i) It is apparent that it is the spectral density at the effective frequency Ω_e that determines the magnitude of the noise effect, (i.e., of the noise-induced population transfer). This preliminary conclusion will be generalized later on: we will introduce the Fourier components of noise and will conclusively show that the decoherence process is determined by the components which are quasiresonant with the effective frequency.

ii) A strategy for noise reduction is uncovered: by controlling the effective frequency Ω_e, which can be achieved by modifying the intensity of the driving field Ω_d, it is possible to displace the characteristic frequency of the system to a region of reduced spectral density. For instance, for a decaying spectral density, an increase in Ω_e leads to noise reduction.

iii) It is also clear that this strategy does not work for white noise: in that case, the spectral density is the same in the whole range of frequencies. We will see that the above arguments, which have been shown to be applicable only in the limit of short correlation times, can, in large extent, be extrapolated to a general time regime for the types of noise that are usually found in experimental setups.

4.3 Dressed-state population transfer induced by specific types of noise: Analysis of the general time regime

The functional form of the probability for noise-induced transitions can be analytically characterized with no restrictions on the time regime in the following three cases, which actually correspond to fluctuations standardly used to emulate random environments. In our calculation procedure, we have introduced the autocorrelation function for each kind of noise in Eq.(60), and have subsequently evaluated the integral through appropriate changes of variables. A summary of the results is presented:

White noise

$$\langle P_{m,m'}(t)\rangle = \mathcal{F}_{m,m'} \int_0^t d\tau \int_0^t d\tau' 2\pi C\delta(\tau-\tau') e^{-i\Omega_e(\tau'-\tau)} = \mathcal{F}_{m,m'} 2\pi S(0) t. \tag{64}$$

This result makes it apparent that there is no possibility of control in the case of white noise. Because of the at-spectrum characteristics, the transition probability does not change when Ω_e is modified.

The Ornstein-Uhlenbeck process

$$\begin{aligned}\langle P_{m,m'}(t)\rangle &= \mathcal{F}_{m,m'} \int_0^t d\tau \int_0^t d\tau' \frac{D}{2\alpha} e^{-\alpha|\tau-\tau'|} e^{-i\Omega_e(\tau'-\tau)} \\ &= \mathcal{F}_{m,m'} 2\pi S(\Omega_e)\left(t + \frac{1}{\alpha}\frac{\Omega_e^2-\alpha^2}{\Omega_e^2+\alpha^2}[1-e^{-\alpha t}\cos(\Omega_e t)] - \frac{2\Omega_e e^{-\alpha t}}{\Omega_e^2+\alpha^2}\sin(\Omega_e t)\right),\end{aligned} \tag{65}$$

where we have used the form of the autocorrelation function for this type of noise.

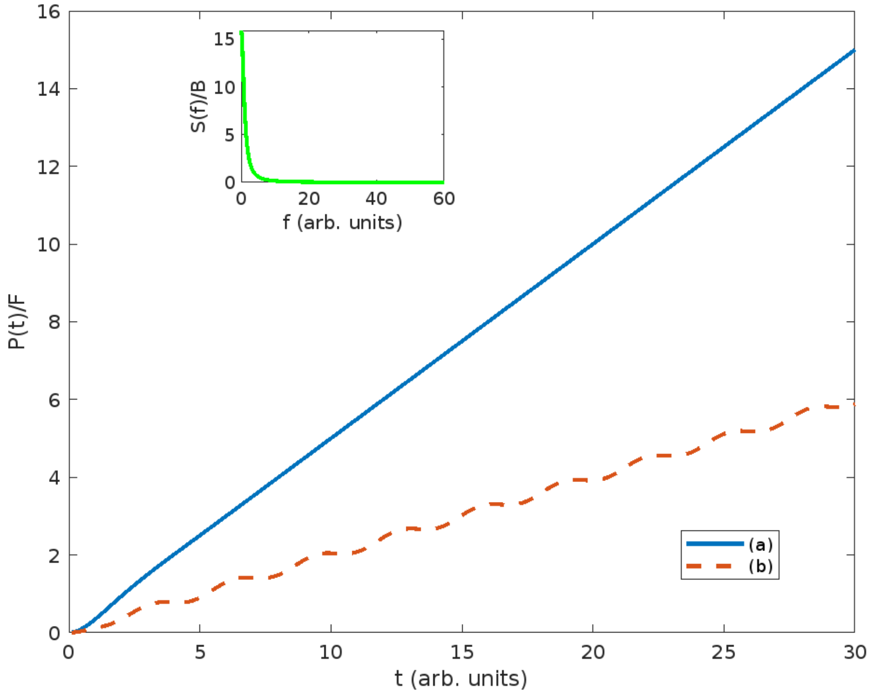

Figure 1: Probability as a function of time (in arbitrary units) for dressed-state population transfer induced by an Ornstein-Uhlenbeck process. ($F = \mathcal{F}_{m,m'}2\pi$). The used parameters are $\alpha = 1$ and $\Omega_e = 1$ (a); and, $\alpha = 1$ and $\Omega_e = 2$ (b). In the inset, the spectral density is represented as a function of the frequency $f = \omega/2\pi$ in arbitrary units. (B is a scale factor proportional to the noise variance).

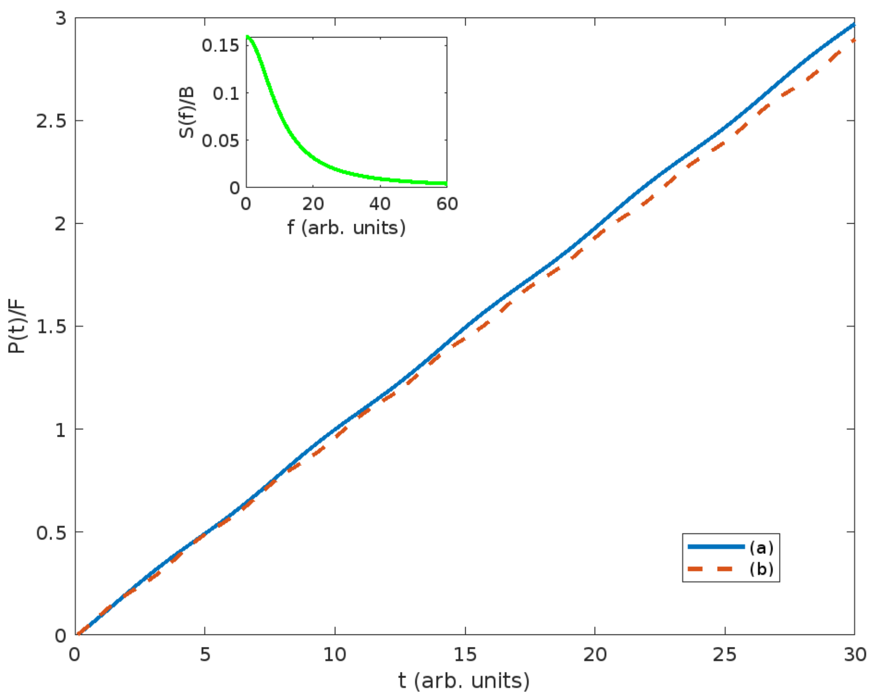

Figure 2: Same as Figure 1, with $\alpha = 10$ and $\Omega_e = 1$ (a); and, $\alpha = 10$ and $\Omega_e = 2$ (b).

Notice that the general result obtained formerly in the limit $t \gg \tau_c$ (Eq.(63)) is consistently recovered here. Another proof of consistency of the whole approach is obtained by checking that the results corresponding to static fluctuations are recovered in the limit of large correlation times: for $\tau_c \to \infty$ ($\alpha \to 0$), the transfer of population matches the average over noise realizations of the probability of transition between two dressed states induced by a static random perturbation $W = \delta\omega_0 F_x$ [16].

In order to illustrate how the efficiency of the CDD method depends on the noise spectrum, we represent in Figs. 1 and 2, the transition probability as a function of time for two sets of parameters that differ in the spectral width ($\alpha = 1$ in Fig. 1, and $\alpha = 10$ in Fig. 2). Hence, as can be seen comparing the insets, the spectrum is much broader in the case depicted in Fig. 2. In both cases, the effective frequency Ω_e has been considered to take the values $\Omega_e = 1$ and $\Omega_e = 2$. It is observed that, as predicted from general considerations, the efficiency of the CDD method declines as fluctuations with broader spectra are confronted. In Fig. 2, the spectral density slightly diminishes as the effective frequency is varied from $\Omega_e = 1$ to $\Omega_e = 2$. As a consequence, only a small reduction of the population transfer is achieved through the effective-frequency doubling. In contrast, for the narrow spectrum corresponding to Fig. 1, a significant reduction of the spectral density, and, in turn, of the transition probability is brought about by doubling the frequency. Consequently, the efficiency of the CDD method significantly improves.

$1/f$ **noise**

$$\begin{aligned} \langle P_{m,m'}(t)\rangle &= -\mathcal{F}_{m,m'}\int_0^t d\tau \int_0^t d\tau' 2A\ln(\omega_1|\tau-\tau'|)e^{-i\Omega_e(\tau'-\tau)} \\ &= \mathcal{F}_{m,m'}2\pi S(\Omega_e)\Bigg\{t\int_0^t d\tau\frac{2\sin(\Omega_e\tau)}{\pi\tau}+ \\ &\left[(\ln[\omega_1\tau]+1)\left(\frac{2\cos(\Omega_e\tau)}{\pi\Omega_e}\right)\right]_0^t - \int_0^t d\tau\frac{2\cos(\Omega_e\tau)}{\pi\tau\Omega_e}\Bigg\}, \end{aligned} \tag{66}$$

where we have again employed the specific form of the autocorrelation function.

It is important to emphasize that the divergences that appear in Eq.(66) are merely formal. The smallest time scale relevant to the case of $1/f$ noise is the inverse of the cutoff frequency $1/\omega_2$. (See the characterization of 1/f fluctuations in Section 2.5). Therefore, in Eq.(66), $t = 0$ actually refers to $t = 1/\omega_2$.

Also in this case, the general result obtained formerly in the limit $t \gg \tau_c$ is consistently recovered. In fact, the analysis of the recovery of that limit is particularly interesting in this case. Actually, the approximation of the Dirac delta function (see Eq. (63)) applied to analyze that regime is valid for $1/f$ noise except for $\omega = 0$, where the spectral density (formally) diverges. This is the reason why the results from the asymptotic regime are not recovered for $1/f$ noise in the absence of the driving field. (See the discussion after Eq.(51) in the previous Section).

Fig. 3 illustrates how increasing the effective frequency, and, in turn, reducing the value of the associated spectral density, a noise reduction can be achieved. Here, as in Figs. 1 and 2, a damped oscillation with the effective frequency is added to the linear asymptotic increase. In the case of Ornstein-Uhlenbeck fluctuations the damping rate is given by α, the inverse of the correlation time. For $1/f$ noise, as shown in Eq.(66), a different form is observed: the damping is dependent on the effective frequency.

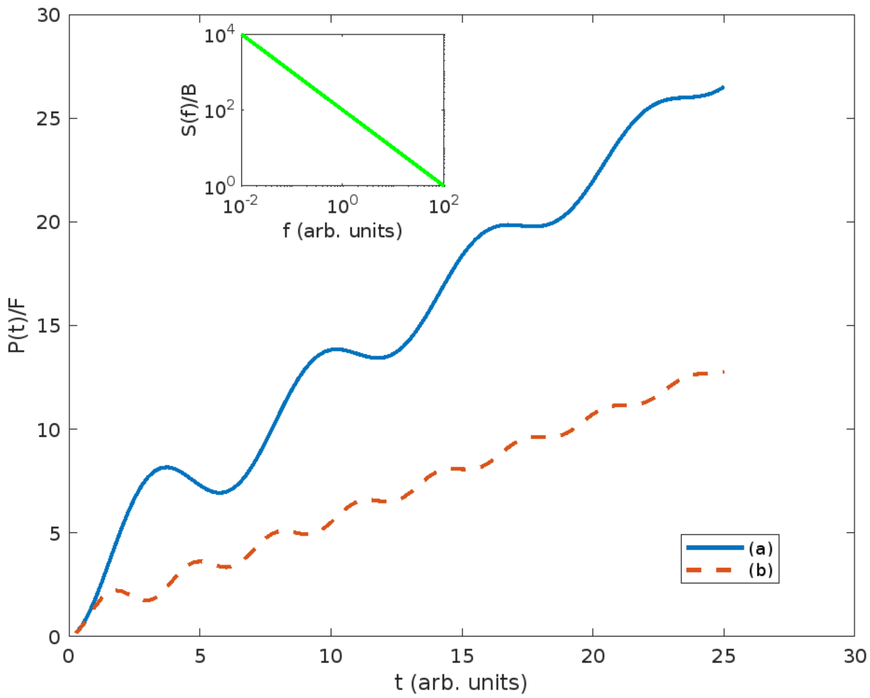

Figure 3: Probability as a function of time (in arbitrary units) for dressed-state population transfer induced by $1/f$ noise. ($F = \mathcal{F}_{m,m'}2\pi$). The used parameters are $\omega_1 = 0.01$, $\omega_2 = 60$, and $\Omega_e = 1$ (a); and, $\omega_1 = 0.01$, $\omega_2 = 60$, and $\Omega_e = 2$ (b). In the inset, the spectral density is represented as a function of the frequency $f = \omega/2\pi$ in arbitrary units. (A $\log - \log$ scale is used). (B is a scale factor proportional to the noise variance).

Some general comments are pertinent:

i) Notice that the three final expressions contain the factor $S(\Omega_e)$. Then, the validity of the strategy for controlling the noise effect proposed from the results corresponding to the asymptotic regime $t \gg \tau_c$ is confirmed by the findings for a general time regime.

ii) We emphasize the relevance of having analytical results in the whole time regime. Recent experimental studies on noise spectroscopy indicate that, in particular experimental setups, as some system parameters are modified, the fluctuations that dominate the decoherence process display changing characteristics. Specifically, transitions between Ornstein-Uhlenbeck and $1/f$ properties were detected [32]. The differential features of the population transfer obtained here for those types of fluctuations can be an additional tool to clarify that issue.

iii) It is worth pointing out that, since the application of time-dependent perturbation theory to first-order requires only up to the second moment of noise, the used framework embodies in fact a Gaussian approximation for the fluctuations.

iv)It is interesting to consider, even as a thought experiment, the situation corresponding to work with an effective frequency Ω_e that overcomes the whole spectral range of noise. The developed theory predicts an almost complete cancellation of the effect of noise. Indeed, this prediction is confirmed by our numerical calculations. It is also worthwhile to trace a parallelism between this scenario and that corresponding to the suppression of the effect of static noise via the CDD method. In both cases, the cancellation of the fluctuations is achieved by shifting the effective system frequency outside the relevant spectral range. Whereas that effect has been realized in the case of static noise, its implementation in the case of $1/f$ noise requires a careful evaluation of potential setups.

4.4 Reinterpreting the noisy dynamics in the CDD scenario

Useful insight into the dynamical features previously identified is provided by arguments relative to the spectral decomposition of the fluctuations. In this sense, it is convenient to work with the Fourier transform of the stochastic variable $\delta\omega_0(t)$. Accordingly, we write

$$\delta\omega_0(t) = \int d\omega\ c(\omega)\ e^{i\omega t}, \tag{67}$$

where the harmonic components are given by

$$c(\omega) = \frac{1}{2\pi}\int dt\ \delta\omega_0(t) e^{-i\omega t}. \tag{68}$$

As can be seen, assuming a stationary process, $c(\omega)$ is a stochastic variable which fulfills $c(\omega) = c^*(-\omega)$ and which is characterized by its mean value

$$\langle c(\omega)\rangle = \frac{1}{2\pi}\int dt\ \langle\delta\omega_t(t)\rangle\ e^{-i\omega t} = 0, \tag{69}$$

and by its autocorrelation function

$$\begin{aligned}\langle c(\omega)c^*(-\omega')\rangle &= \frac{1}{(2\pi)^2}\int\int dt\ dt'\ e^{i(\omega' t'-\omega t)}\langle\delta\omega_0(t)\delta\omega_0(t')\rangle \\ &= \frac{\delta(\omega-\omega')}{2\pi}\int d\tau G(\tau)e^{i\omega'\tau} = \delta(\omega-\omega')S(\omega').\end{aligned} \tag{70}$$

Now, using Eq.(67), the Hamiltonian in Eq.(57) can be rewritten as

$$H = \Omega_d F_z - \left(\int d\omega\ c(\omega)\ e^{i\omega t}\right) F_x. \tag{71}$$

In this picture, it is apparent that the system can be regarded as corresponding to a multiplet driven by a superposition of harmonic fields with frequencies continuously distributed. Moreover, in the monofrequency reduction, the scenario corresponds to the standard Rabi-oscillation model (ROM). Therefore, it is possible to apply our knowledge of the ROM dynamics to interpret the effects of noise. In this sense, we remark:

i) Only the harmonic signals with frequencies close to the characteristic frequency of the system are able to induce significant changes of population between the states of the used representation. (Note that the system frequency is the effective frequency previously used and that the states are in fact the dressed states obtained via the sequence of unitary transformations).

ii) The probability for a transition between states is proportional to the weights that the quasiresonant components of noise have in the Fourier integral. Hence, the crucial role of the spectral density at the effective frequency $S(\Omega_e)$, previously observed, finds an insightful interpretation here.

iii) Although the continuous character of the distribution of frequencies implies dealing with a generalized version of the ROM, the whole framework provides an operative approach where the effects of noise can be explained and where conjectures on the implications of considering diverse noise properties can be soundly set up.

4.5 Applying the CDD method: the concatenation scheme

In this Section, I will analyze the generalization of the basic version of the CDD method to incorporate a concatenation scheme that allows dealing with the effect of noise introduced into the system by the first driving field. The focus will be put on two schemes frequently used in practice, which, actually, can be shown to be equivalent. A first common variation of the previously studied CDD techniques consists in applying a field orthogonal to the primary driving term. Specifically, The Hamiltonian given by Eq.(52) is modified according to

$$H_{C1} = [\omega_0 + \delta\omega_0(t)]F_z + 2[\Omega_1 + \delta\Omega_1(t)]\cos(\omega_0 t)F_x + 2\Omega_2\cos[(\omega_0+\omega_p)t]F_y, \tag{72}$$

where fluctuations in the amplitude of the first field of control are explicitly taken into account: they are represented by $\delta\Omega_1(t)$. To reduce the effect of those fluctuations is why the second field of control is introduced. Observe that the second driving term, with amplitude $2\Omega_2$ and frequency $\omega_0+\omega_p$, has been considered to be applied along the $OY-$axis. The other standardly used concatenation scheme is based on applying a phase modulation in the primary field of control. Specifically, it corresponds to the Hamiltonian

$$H_{C2} = [\omega_0 + \delta\omega_0(t)]F_z + 2[\Omega_1 + \delta\Omega_1(t)]\cos[\omega_0 t + \Phi(t)]F_x, \tag{73}$$

where the phase modulation has the form

$$\Phi(t) = \frac{2\Omega_2}{\Omega_1}\sin(\omega_p t)\ ,\ \Omega_2 \ll \Omega_1. \tag{74}$$

The application of a convenient sequence of unitary transformations to the Hamiltonians given by Eqs. 72 and 73, transforms them until reaching compact forms where the strategy to reduce the effect of the noisy term $\delta\Omega_1(t)$ can be implemented. Let us illustrate the procedure with the specific form of transforming H_{C2}. We start with the unitary transformation

$$U(t) = e^{i\omega_0 t F_z/\hbar}, \tag{75}$$

which converts H_{C2} into the form

$$H_{C2} =\simeq \delta\omega_0(t)F_z + [\Omega_1 + \delta\Omega_1(t)]F_x + 2\Omega_2\sin(\omega_p t)F_y. \tag{76}$$

At this point, it is important to take into account that, since the parameters of the first driving field have been chosen to approximately suppress the effects of the random term $\delta\omega_0(t)$, we can safely neglect its contribution to the Hamiltonian H_{C2}, which, consequently, can be rewritten as

$$H_{C2} =\simeq [\Omega_1 + \delta\Omega_1(t)]F_x + 2\Omega_2\sin(\omega_p t)F_y. \tag{77}$$

Now, the rotation of the system a $\pi/2$ angle around the $OY-$axis, via $U(t) = e^{i\frac{\pi}{2}F_y/\hbar}$, transforms H_{C2} as

$$H_{C2} = [\Omega_1 + \delta\Omega_1(t)]F_z + 2\Omega_2 \sin(\omega_p t)F_y. \tag{78}$$

Finally, by choosing $\omega_p = \Omega_1$, and going to the rotating frame defined by the unitary transformation

$$U(t) = e^{i\omega_p t F_z/\hbar}, \tag{79}$$

and making the Rotating Wave Approximation once again, we arrive at

$$H_{C2} \simeq \delta\Omega_1(t)F_z + \Omega_2 F_x. \tag{80}$$

Here, it is important to notice that, in this final form, H_{C2} has the same structure as the Hamiltonian given by Eq.(54) (in the preceding Section), whose analysis allowed us to conclude that the effect of the random term $\delta\omega_0(t)$ could be conveniently reduced by appropriately choosing the parameters of the first driving field. A similar analysis can then be carried out to deal with the stochastic term $\delta\Omega_1(t)$. It is then apparent that the applied concatenation scheme can serve the purpose of mitigating the effect of the additional noise introduced into the system through the amplitude of the field of control. Indeed, one can conclude that the noise-induced transitions between dressed states (now doubly dressed-states because of the use of two driving fields) can be inhibited by an appropriate choice of the field-dependent effective frequency of the system Ω_e. Specifically, the effect of the fluctuations can be reduced by shifting the effective system frequency to a range where the spectral density is significantly lower.

The above conclusions, extracted from the study of the CDD method with a concatenation arrangement, i.e., for the scheme incorporating two driving fields or, equivalently, a phase modulated driving field, are straightforwardly extrapolated to more elaborate setups. As, in any stage in the CDD scheme, the last noisy component entering the system is transferred to an off-diagonal term through an appropriate change of representation, its effect on the dynamics can always be characterized in terms of a population transfer between effective zero-order eigenstates similar to that given by Eqs. 66, 65, 64. Therefore, the effectiveness of the decoherence-reduction method is guaranteed provided that the *final* interstate transition frequencies are out of the dominant part of the spectral range of the residual noise. Note that controlling the frequencies of transition, in particular, the effective frequency Ω_e, to avoid the occurrence of resonances with the noise spectral components has the limitations associated to the application of the RWA and to the system reduction employed in the description of the model system. A careful analysis of each experimental setup is needed: since the consecutive application of the RWA as different drivings are incorporated implies a reduction in the magnitude of the splittings, keeping the last Ω_e outside the spectral range of the corresponding final noise is not trivial.

5 Conclusions

En esta sección se presentan las conclusiones de este trabajo. Se expone la efectividad del método CDD y se discuten brevemente las limitaciones a la hora de aplicar el esquema de concatenación planteado.

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In the present work, I have analyzed some fundamental aspects of the applicability of the CDD method to systems of general interest in Quantum Technologies. I have combined the revision of established bibliography with the consideration to some open issues. Specifically, the evaluation of the efficiency of the CDD method to deal with 1/f noise constitutes the original part of the study. The main achieved objectives are summarized in the following points:

i) The physical basis of the CDD techniques has been identified as rooted in making the fluctuations to have a second-order perturbative effect on the dynamics. A key point is to set up a dressed state-basis associated to the applied driving field where the noise becomes off-diagonal. In this framework, the effects of the fluctuations can be mitigated for a sufficiently large separation of the diagonal components. Whereas, previous to the application of the CDD method, it is the zero-frequency value of the noise spectrum that determines the asymptotic dephasing rate, in the CDD setup, the decoherence time is basically determined by the noise spectrum at the final effective frequency Ω_e. Decoherence is significantly reduced if Ω_e does not enter the relevant part of the spectrum. The same mechanism is responsible for the operative character of the concatenation schemes.

ii) The present study allows evaluating previous approaches to the physics of the CDD method based on a static scenario for the fluctuations. The use of simplified time-independent-noise models implies assuming that the original stochastic input and the random variations of the different auxiliary fields have very large correlation times. Those models are found to be appropriate as far as the inter-state transition frequencies are far from the (near-to-zero) spectral range of the fluctuations.

iii) Rigorous formal support has been given to the generally accepted criterion on the inability of the methods of Dynamical Decoupling to deal with white noise. More generally, it has been traced how the performance of the methods decline as the noise correlation-time decreases.

iv) The application of our approach to $1/f$ noise has been carried out using a compact expression for the associated correlation function derived from the spectrum functional form. The efficiency of the CDD method to deal with $1/f$ noise is guaranteed provided that a significant increase in the system effective frequency is implemented.

v) Differential effects associated to the spectral characteristics of the fluctuations have been detected in the analysis of the noise-induced dephasing processes. In contrast with the asymptotic exponential decay of the coherences corresponding to fluctuations with Ornstein-Uhlenbeck characteristics, a Gaussian decay is observed in dephasing induced by $1/f$ noise.

vi) The analytical expressions obtained for the population transfer between dressed states can find applicability as elements of noise identification methods. The high level of control achieved in the experimental setups makes it advisable to employ the techniques proposed for the realization of quantum information protocols in Noise Spectroscopy.

From the present work, a critical evaluation of of the CDD scheme can be made. The following comments are to the point:

vii) A first criticism refers to the changes in the magnitude of the effective energy splitting occurring as the different driving fields are applied. One could think that adjusting the field intensities can be an appropriate strategy to fix the spectral range in the sequence of energy splittings. However, there are serious limits on this plan of action. First, the intensities must be large enough for the noise to be considered as a perturbation. Second, since larger intensities can lead to the transfer of population to states left out of the model, they can put at risk the applicability of the different state-reductions applied in the design of the methods. An additional argument in the same line refers to the Rotating Wave Approximation, implemented in the description of the method procedure. Actually, for the RWA to be applicable, i.e., for soundly neglecting rapidly oscillating terms in the Hamiltonians, the Rabi frequencies, which are proportional to the field intensities, must be much smaller than the original frequencies. Notice that this is actually a fundamental limitation of the CDD schemes which can make them to be inappropriate for some applications. Indeed, the convenience of the imposed variation of the original spectral range must be pondered in each particular case.

viii) Closely related to the previous criticism is the requirement, for the method to be operative, of reaching a point in the sequence of driving fields where the magnitude of the remnant noise, entering the system through the last driving field, can be considered to be negligible compared with the final splitting. One must be aware of the difficulty of reaching that compromise given the diminishing energy separation implemented in the experiments.

Finally, it is worth depicting some lines of research where the generalization of the theoretical framework developed in the present work can find applicability. Particular interest has the analysis of the role of non-Gaussian fluctuations. Previous effort in this line has been concentrated on the effect of random telegraph noise (RTN), associated to a Poissonian distribution function and to a Lorentzian spectrum. It is apparent that dealing with that kind of fluctuations requires going beyond the perturbative approach: since the application of time-dependent perturbation theory to first-order requires only up to the second moment of noise, the used framework embodies in fact a Gaussian approximation. This problem is actually connected with a second topic of general importance, namely, the emergence of $1/f$ noise from the microscopy of the environment. In a quite generally accepted model, $1/f$ fluctuations are regarded as formed by specific distributions of RTN fluctuators. The connection between the characterized global effect of the distribution and the role of each particular fluctuator is a challenging problem.

6 Appendix I. The Wiener-Khinchin Theorem

En este complemento se presenta el Teorema de Wiener-Khinchin junto con una breve demostración del mismo.

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In this complement we present this famous theorem firstly developed by Norbert Wiener [34] for well-defined functions and later expanded for stochastic processes by A. Khintchine [35].

The Wiener-Khinchin Theorem states that the spectral density of a random variable $x(t)$, defined as

$$S(\omega) = \lim_{T \to \infty} \frac{1}{2\pi T} \left| \int_0^T dt e^{-i\omega t} x(t) \right|^2, \tag{81}$$

and the autocorrelation function, given by

$$G(\tau) = \langle x(t) x(t+\tau) \rangle, \tag{82}$$

($\langle \ \rangle$ denotes the average over stochastic realizations), are connected by the relation

$$S(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\tau e^{-i\omega\tau} G(\tau), \tag{83}$$

i.e., the spectral density is the Fourier transform of the autocorrelation function.

Proof

It is important to remark that in the derivation of Eq. (83) the *ergodic* character of $x(t)$ will be assumed, i.e., the average over stochastic realizations at a particular time will be considered to be equal to the time average in a particular noisy trajectory over a large time interval. Consequently, the autocorrelation function can be expressed as

$$G(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_0^T dt x(t) x(t+\tau). \tag{84}$$

As a first step in our procedure, Eq. (81) is rewritten as

$$\begin{aligned} S(\omega) &= \lim_{T \to \infty} \frac{1}{2\pi T} \left(\int_0^T dt e^{-i\omega t} x(t) \times \int_0^T dt' e^{i\omega t'} x(t') \right) \\ &= \lim_{T \to \infty} \frac{1}{2\pi T} \int_0^T dt \int_0^T dt' e^{-i\omega(t-t')} x(t) x(t'). \end{aligned} \tag{85}$$

Moreover, through an appropriate change of variables, one finds

$$S(\omega) = \lim_{T \to \infty} \frac{1}{2\pi T} \left[\int_{-T}^0 d\tau e^{i\omega\tau} \int_{-\tau}^T dt x(t) x(t+\tau) + \int_0^T d\tau e^{i\omega\tau} \int_0^{T-\tau} dt x(t) x(t+\tau) \right]. \tag{86}$$

Now, additional changes of variables and the explicit consideration of the limit $T \to \infty$, allows one to establish the connection with the autocorrelation function. Specifically, it is shown that

$$S(\omega) = \frac{1}{\pi} \int_0^\infty d\tau \cos(\omega\tau) G(\tau). \tag{87}$$

Finally, taking into account that the autocorrelation function fulfills the relation $G(\tau) = G(-\tau)$, Eq. (87) can be recast into the standard form of the Wiener-Khinchin Theorem, namely,

$$S(\omega) = \frac{1}{2\pi} \int_{-\infty}^\infty d\tau e^{-i\omega\tau} G(\tau). \tag{88}$$

□

7 Appendix II. BCH formula

En este complemento se presenta de forma detallada la aplicación de la fórmula Baker-Cambell-Hausdorff.

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This complement aims to give an overview of the application of the Baker-Cambell-Hausdorff to different Hamiltonians, such as (31) or (52).

The BCH formula, derived from general considerations on the algebra of linear operators, establishes that two generic linear operators A and B satisfy

$$e^{xA}Be^{-xA} = B + x[A,B] + \frac{x^2}{2!}[A,[A,B]] + \frac{x^3}{3!}[A,[A,[A,B]]] + ... \tag{89}$$

The applicability of the above expression in the implementation of unitary transformations in Quantum Mechanics is evident. For instance, in the case of Eq.(31), applying the unitary transformation (30) requires us to evaluate

$$\widetilde{H} = e^{i\omega F_z/\hbar}[(\omega_0 + \delta\omega_0(t))F_z]e^{-i\omega F_z/\hbar} + i\hbar(\frac{i\omega F_z}{\hbar}), \tag{90}$$

for which we must resort to the aforementioned expression (Eq. (89)). Thus, setting $x = i\omega t/\hbar$, $A = F_z$ and $B = F_z$, we trivially get,

$$e^{i\omega F_z t/\hbar}F_z e^{-i\omega F_z t/\hbar} = F_z. \tag{91}$$

And using this result, we have shown how the unitary transformation (30) transforms the operator F_z.

For other cases such as Eq.(52), we may have to consider other operators. In the mentioned case, we would use $B = F_x$, obtaining

$$\begin{aligned} e^{i\omega_d F_z t/\hbar}F_x e^{-i\omega_d F_z t/\hbar} &= F_x - (\omega_d t)F_y - \frac{1}{2!}(\omega_d t)^2 F_x + \frac{1}{3!}(\omega_d t)^3 F_y + \frac{1}{4!}(\omega_d t)^4 F_x + ... \\ &= \cos(\omega_d t)F_x - \sin(\omega_d t)F_y. \end{aligned} \tag{92}$$

8 Appendix III. Characterization of an integrated stochastic variable: limit of short correlation times

En esta sección se tratará la integración de una variable estocástica, necesaria para su caracterización. Asimismo, se contemplará su evaluación en el límite de tiempos de correlación cortos.

The integral expression for a zero-mean variance (Eq.(44)) can be written as

$$\langle \xi^2(t) \rangle = \int_0^t \int_0^t d\tau' d\tau \langle \delta\omega_0(\tau)\delta\omega_0(\tau') \rangle, \tag{93}$$

which can be further simplified employing a suitable change of variables, namely,

$$\begin{cases} \tau_D = \tau' - \tau \\ \tau_S = \frac{1}{2}(\tau' + \tau) \end{cases} \tag{94}$$

resulting in,

$$\langle \xi^2(t) \rangle = \int_{-t}^{t} d\tau_D \int_{\frac{|\tau_D|}{2}}^{t-\frac{|\tau_D|}{2}} d\tau_S \left\langle \delta\omega_0\left(\tau_S - \frac{\tau_D}{2}\right) \delta\omega_0\left(\tau_S + \frac{\tau_D}{2}\right) \right\rangle, \tag{95}$$

and considering a stationary process,

$$\langle \xi^2(t) \rangle = \int_{-t}^{t} d\tau_D G(\tau_D)(t - |\tau_D|). \tag{96}$$

Employing the Wiener-Khinchin theorem (Eq.(16)),

$$\langle \xi^2(t) \rangle = \int_{-\infty}^{\infty} d\omega \int_{-t}^{t} d\tau (t - |\tau|) e^{i\omega\tau} S(\omega), \tag{97}$$

which when integrated can be expressed as

$$\langle \xi^2(t) \rangle = \int_{-\infty}^{\infty} d\omega \; S(\omega) \left(\frac{\sin(\omega t/2)}{\omega/2} \right)^2. \tag{98}$$

Evaluating this expression in the considered limit can be done via the following function

$$\frac{\epsilon}{\pi} \frac{\sin^2(x/\epsilon)}{x^2}, \tag{99}$$

which approaches $\delta(x)$ when the parameter ϵ approaches zero (from the positive side).

In order to arrive to a similar form for integral (98), we can define the parameter $\epsilon = \frac{1}{t}$, make the change of variables $\frac{\omega}{2} = x$, multiply by $\frac{\epsilon}{2\pi}$ and consider said limit for ϵ, i.e.,

$$\lim_{\epsilon \to 0^+} \frac{\epsilon \langle \xi^2(t) \rangle}{2\pi} = \lim_{\epsilon \to 0^+} \int S(2x) \left(\frac{\epsilon}{\pi} \frac{\sin^2(x/\epsilon)}{x^2} \right) dx. \tag{100}$$

Identifying $\delta(x)$, it follows that

$$\lim_{\epsilon \longrightarrow 0^+} \frac{\epsilon \langle \xi^2(t) \rangle}{2\pi} = \int S(2x)\delta(x)dx = S(0) \implies \lim_{t \longrightarrow \infty} \langle \xi^2(t) \rangle = \lim_{t \longrightarrow \infty} 2\pi t S(0). \tag{101}$$

At this point we can also check how, had we considered the limits of Eq.(44) to be Δt instead of t:

$$\langle \xi^2(t) \rangle = \frac{t}{\Delta t} \cdot \left\langle \left(\int_0^{\Delta t} d\tau \delta\omega_0(\tau) \right) \left(\int_0^{\Delta t} d\tau' \delta\omega_0(\tau') \right) \right\rangle, \tag{102}$$

and it is easy to see how in Eq.(98) t would be Δt. With a similar procedure (now defining the parameter $\epsilon = \frac{1}{\Delta t}$), we can show how

$$\begin{aligned} \langle \xi^2(t) \rangle \frac{\epsilon \Delta t}{2\pi t} = \int_{-\infty}^{\infty} d\omega S(\omega) \frac{\sin^2(\omega/(2\epsilon))}{(\omega/2)^2} &\implies \lim_{\epsilon \longrightarrow 0^+} \langle \xi^2(t) \rangle = \lim_{\epsilon \longrightarrow 0^+} 2\pi t S(0) \\ &\implies \lim_{\Delta t \longrightarrow \infty} \langle \xi^2(t) \rangle = \lim_{\Delta t \longrightarrow \infty} 2\pi t S(0). \end{aligned} \tag{103}$$

Therefore obtaining the same result. It is important to note that this treatment just assumes $\Delta t \gg \tau_c$, something previously considered for the use of the Central Limit Theorem.

9 Appendix IV. The Rotating-Wave Approximation

En este complemento se expone la Aproximación de Onda Rotante y se discute brevemente su validez en relación con los Hamiltonianos considerados en el trabajo.

In the context considered in the present study, the Rotating Wave Approximation basically consists in retaining only the secular terms, i.e., the terms with slow time dependence, in the different Hamiltonians obtained after applying the sequence of unitary transformations. Hence, the rapidly oscillating terms (usually with frequencies which duplicate the frequency of the rotating frame defined by the applied unitary transformation) are neglected. A typical example of the application of the RWA is given by the transformation of Hamiltonian (52). Specifically, after applying the specified unitary transformation (30), it is cast into the following form

$$\widetilde{H} = \frac{\Omega_d}{2}\left[(F_x + iF_i)e^{2i\omega_d t} + (F_x - iF_i)e^{-2i\omega_d t} + 2F_x\right] + (\omega_0 + \delta\omega_0(t) - \omega_d)F_z. \tag{104}$$

Note that the corresponding trigonometric functions have been expressed in their respective exponential forms for clarity.

The application of the RWA to the above expression implies neglecting the terms that contain the (rapidly oscillating) functions $e^{\pm 2i\omega_d t}$. Hence, after applying the RWA, one writes

$$\widetilde{H} \approx \Omega_d F_x + \delta\omega_0(t) F_z. \tag{105}$$

This approximation is justified as these terms averaged over a measurable time provide a null contribution, and is valid as long as $\omega_d \gg \Omega_d$. (The approximation fails for $\omega_d \sim \Omega_d$).

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