Sección de Física
Universidad de La Laguna

Daniel Galán Morales

On Seiberg-Witten monopole theory

Sobre la teoría de monopolos de Seiberg-Witten

Trabajo Fin de Grado
Grado en Física
La Laguna, Julio de 2024

DIRIGIDO POR
Manuel T. Flores Mederos
Alba Crespo Pérez

Manuel T. Flores Mederos
Facultad de Ciencias
Departamento de Análisis Matemático
Universidad de La Laguna
38200 La Laguna, Tenerife

Alba Crespo Pérez
Facultad de Ciencias
Departamento de Física
Instituto Astrofísico de Canarias
38200 La Laguna, Tenerife

Acknowledgements

In the first place, and mainly, I would like to thank my sister Nerea for her existence and ability to motivate me even when she does not notice. Perhaps she has not computed too many integrals for me, but without her I would have certainly computed them in a sadder mood.

I would also like to thank the friends and colleagues that have accompanied me throughout these four years: my best friend Ángel Caputo, for always knowing how to turn the most challenging times into moments filled with joy and those who are not into memories I will always treasure; my friend Daniel Hernández, may he rest in peace, for those afternoons in the engineering cafeteria and in classroom 2.1; Sofía Ortega, for all the books and doodles and LaTeX drawing competitions that made me the person I am today and the physicist I hope to be; Pablo Avero, Daniel 'Tuti' Kronewitter and Mónica Marín, for really turning these university years into some of the best of my life; to my dearest anonymous friend, for being one of my biggest professional inspirations, reading the first drafts of these memoir so generously and for his valuable comments on QFT. I would like to thank in general every person that has accompanied me during these years and whose participation in my life has made this text and his author who they are.

I also thank my parents for their never ending support and for always believing in me even when I could not. I would have not gotten here the way I have without these people. Thank you all.

Daniel Galán Morales
La Laguna, 9th July, 2024

Abstract

The second Maxwell equation states that there can be no magnetic field sources. However, for the last 130 years, physicists have speculated about the possibility that these magnetic field sources—the magnetic monopoles—actually exist.

In this memoir we study and contextualize three of the most important magnetic monopoles: the Dirac monopole, the $\mathrm{U}(1)$-Seiberg-Witten monopole and the $\mathrm{SU}(2)$-Seiberg-Witten monopole, often called 't Hooft-Polyakov monopole.

We also include a chapter on soliton solutions to Euler-Lagrange equations and their interpretation in field theories and a comment on experimental research for monopoles carried out by different experiments and collaborations in the past 50 years and on the physical implications of the existence of magnetic monopoles.

Keywords: *Magnetic Monopoles – Charge Quantization – Gauge Theories – Gauge Groups – Field Theories – Seiberg-Witten Equations – Symmetry.*

Resumen

La segunda ecuación de Maxwell enuncia que no pueden existir fuentes de campo magnético. Sin embargo, durante los últimos 130 años los físicos han especulado con la posibilidad de que estas fuentes de campo magnético—los monopolos magnéticos—en realidad existan.

En esta memoria estudiamos y contextualizamos tres de los monopolos magnéticos más importante: el monopolo de Dirac, el $\mathrm{U}(1)$-monopolo de Seiberg-Witten y el $\mathrm{SU}(2)$-monopolo de Seiberg-Witten, a menudo llamado monopolo de 't Hooft-Polyakov.

Además, incluimos un capítulo sobre las soluciones solitónicas a ecuaciones de Euler-Lagrange y su interpretación en teorías de campos, y un comentario acerca de la búsqueda experimental de monopolos realizada por diferentes experimentos y colaboraciones en los últimos 50 años y sobre las implicaciones físicas de la existencia de monopolos magnéticos.

Palabras clave: *Monopolos Magnéticos – Cuantización de la Carga – Teorías Gauge – Grupos Gauge – Teorías de Campos – Ecuaciones de Seiberg-Witten – Simetría.*

Contents

Introduction

Magnetic monopoles are one of the safest bets one can make about physics not yet seen.

J. Pochilsky

Physics have historically tried to unite what at first glance may look as different phenomena into the same theory. These phenomena turn out to be different sides of the same dice. The first unification in physics was performed by James Clerck Maxwell, who brought together electricity and magnetism into a single, elegant theory: electromagnetism. Furthermore, electromagnetism was able to also explain the behaviour of light, which turned out to be a perturbation of the electromagnetic field: an electromagnetic wave.

Maxwell's electromagnetism assumes that no magnetic charges exist, but it does not explain why, neither does it forbid them. Isolated magnetic charges were sometimes considered as a useful mathematical tool, but never as real physical objects. A exception to this was Pierre Curie, who speculated on the possibility of free magnetic charges in his 2-page 1894 article *Sur la possibilité d'existence de la conductibilité magnétique et du magnétisme libre*: "Est-il absurde de supposer qu'il existe des corps conducteurs du magnétisme, des courants magnétiques, du magnétisme libre?" (1) [1].

When quantum mechanics arised, it first appeared to forbid isolated magnetic charges. However, in 1931 P.A.M. Dirac showed that they allow certain quantised magnetic charges [2]. Although quantization is something to be expected in quantum mechanics, the surprise was that this quantization was rooted in deep topological considerations, not in the spectrum of any *magnetic charge operator*. Furthermore, the existence of magnetic charges would explain the quantization of the electric charge.

In 1974, Gerard 't Hooft and Alexander Polyakov showed that magnetic monopoles are actually predicted by all GUTs (Grand Unification Theories) and TOEs (Theories Of Everything) [11] such as superstring theory, which also includes gravitation.

(1) "Is it absurd to suppose that there exist bodies conducting magnetism, magnetic currents, free magnetism?"

It is because of this that we would expect magnetic monopoles to exist. Their discovery would go down on the history of physics as and incredible breakthrough, and would allow scientist to design experiments to test GUTs and TOEs directly, which with the current technology is impossible to do. Unfortunately, all attempts of findin them have been in vain, but they are still paving the way to the future of theoretical physics.

1

Getting Acquainted with Magnetic Monopoles

Resumen

En este primer capítulo partiremos de las famosas ecuaciones de Maxwell y veremos que para preservar su invarianza bajo transformaciones del grupo $\mathrm{U}(1)$, deberían existir fuentes de campo magnético: monopolos magnéticos. Asumiendo que generan un campo Coulombiano, estudiamos su topología para ver cómo sería el potencial que los define.

1.1 Classical Electrodynamics as a starting point

It all starts, of course, with Maxwell's Equations. And in $\mathbb{R}^3$, at least by now. This familiar vector space will be equipped with its familiar Euclidean structure: an inner product $\langle\ ,\ \rangle\colon \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$. Vectors in $\mathbb{R}^3$ will be denoted by an arrow on top of them (or a hat if they happen to be unitary), and the elements of the tangent bundle $T\mathbb{R}^3 = \bigsqcup_{p\in\mathbb{R}^3} T_p\mathbb{R}^3$ will be denoted by the same letter but without the arrow.

For the no-source scenario, one can write the Maxwell equations as these two complex expressions:

$$\vec{\nabla} \times (\vec{E} + i\vec{B}) - i\frac{\partial}{\partial t}(\vec{E} + i\vec{B}) = 0$$
$$\vec{\nabla} \cdot (\vec{E} + i\vec{B}) = 0$$

These equations are famously Lorentz invariant—actually, up to a certain point they inspired the Special Relativity Theory—, gauge invariant and conformally invariant, but they are also invariant under transformations of the group U(1), i.e. if $(\vec{E} + i\vec{B})$ is a solution to these equations, $\mathrm{e}^{i\varphi}(\vec{E} + i\vec{B})$ is also one. This is what we refer to as *duality symmetry*. In particular, we can see that when $\varphi = \pi/2$, we may obtain a solution from our previous solution simply by substituting $\vec{E} \to -\vec{B}$ and $\vec{B} \to \vec{E}$.

This symmetry is of course lost when we add field sources; but Dirac [2] noticed that if it were possible for a magnetic charge to exist, it would be restored.

Following the foundations of electrostatics, Dirac assumed that this magnetic charge—the magnetic monopole–would generate a central, Coulomb-like field similar to the one generated by an electric monopole, i.e.

$$\vec{B} = \frac{g}{\rho^2}\hat{e}_\rho, \tag{1.1}$$

where g would be the *charge of the monopole*, whose unit in the SI is the Weber (Wb) and $\hat{e}_\rho$ is the unit vector in the radial direction in spherical coordinates. In cartesian coordinates,

$$\vec{B} = \frac{g}{\rho^3}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Since $\mathbb{R}^3$ is flat, there exists a natural isomorphism between $\mathbb{R}^3$ and $T\mathbb{R}^3$, i.e. $\mathbb{R}^3 \simeq T\mathbb{R}^3$, and because $T\mathbb{R}^3 \simeq \Omega^1(\mathbb{R}^3)$, the space of 1-forms in $\mathbb{R}^3$, we may relate each vector $\vec{p} \in \mathbb{R}^3$ with a 1-form p. Via the natural—sometimes called *musical*—isomorphism we can *forget* about $\vec{B}$ and work with the 1-form

$$\bar{B} = \frac{g}{\rho^3}(x\ dx + y\ dy + z\ dz) \tag{1.2}$$

Since $\vec{B}$ is normally defined as a cross product (it is the curl of the spatial part of the 4-potential, to be precise), it is only natural to define the magnetic field as a 2-form. In the n-dimensional Euclidean space $\mathbb{R}^n$, $\Omega^k(\mathbb{R}^n) \simeq \Omega^{n-k}(\mathbb{R}^n)$, where $0 \leq k \leq n$, via the Hodge star operator.

The *Hodge star operator* $\star\colon \Omega^k(V) \to \Omega^{n-k}(V)$ is defined in spaces equipped with an inner product by the following property:

$$\alpha \wedge (\star\beta) = \langle \alpha, \beta \rangle \omega \tag{1.3}$$

for all $\alpha, \beta \in \Omega^k(V)$ and $\omega = e_1 \wedge \cdots \wedge e_n$. This way, for instance, $\star dx = dy \wedge dz$, since $dx \wedge (\star dx) = \langle dx, dx \rangle \omega = dx \wedge dy \wedge dz$. It is trivial to see that $\star\star\sigma = \sigma$, where $\sigma \in \Omega^k(V)$, i.e. $\star^2 = \mathbb{1}$.

In this manner we may establish the isomorphism $\Omega^1(\mathbb{R}^3) \simeq \Omega^2(\mathbb{R}^3)$, which allows us to indeed write the magnetic field as a 2-form. Naming $B = \star\bar{B}$, we see that

$$B = \frac{g}{\rho^3}(x\ dy \wedge dz + y\ dz \wedge dx + z\ dx \wedge dy) \tag{1.4}$$

This way, we have written our magnetic field in the most natural way. It is in this spirit that we not only write the magnetic field in the language of differential forms, but rather the whole theory of electrodynamics.

1.2 Electrodynamics and Differential Forms

Now allowing the existence of field sources—a non-null current density 4-vector $(\rho, \vec{j})$, ρ being the charge density and $\vec{j}$ the current density—, one can write Maxwell's equations in vacuum in its standard, vector-calculus fashion[(1)].

MAXWELL'S EQUATIONS

Vector-Calculus Version

$$\vec{\nabla} \cdot \vec{E} = \rho \qquad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{B} = \vec{j} + \frac{\partial \vec{E}}{\partial t}$$

Since $\vec{E}$ is usually integrated over one-dimensional paths, we could write it as the 1-form $E = E_i dx^i$[(2)]. The exterior derivative dE of this 1-form will be a 2-form related to $\vec{\nabla} \times \vec{E}$, and $\vec{\nabla} \cdot \vec{E}$ will be associated to the 3-form given by $d \star E$.

As stated earlier, the magnetic field vector will be replaced by a 2-form. In a similar fashion, dB corresponds to the divergence of the magnetic field and $d \star B$—where the 1-form $\star B$ is what we previously denoted as $\bar{B}$—is the tangent bundle homologous of the curl of $\vec{B}$. This way, one can write Maxwell's Equations in the language of differential forms (exterior algebra):

MAXWELL'S EQUATIONS

Exterior Algebra Version

$$d \star E = \rho \qquad dE = -\frac{\partial B}{\partial t}$$

$$dB = 0 \qquad d \star B = j + \frac{\partial \star E}{\partial t}$$

Here, j is a 2-form and ρ a 3-form in order for the equations to make sense.

One could even be more ambitious and try to make the whole theory what is called manifestly Lorentz covariant, i.e. write the theory in terms of Lorentz invariants, such as scalars and dot products.

From special relativity we already know that we can join the scalar potential ϕ and the vector potential $\vec{A}$ in a mathematical object: the 4-potential. Using the $(- + ++)$ signature for the Minkowski metric, one can write this 4-potential as

(1) We will be setting $\varepsilon_0 = \mu_0 = 1$. This also implies $c = 1$.

(2) Einstein's summation convention is assumed from now on unless otherwise stated.

$$A = \eta_{\mu\nu} A^\mu dx^\nu = -\phi \, dt + \vec{A} \cdot d\vec{x}$$

We now define the exterior derivative of this 1-form as the *Maxwell field strength tensor*, i.e.

$$F = dA = \frac{1}{2} F_{\mu\nu} \, dx^\mu \wedge dx^\nu \tag{1.5}$$

One can easily see that F is gauge invariant, i.e. if one performs the transformation $A \mapsto A + df$, where f is any differentiable function in $\mathbb{R}^4$[(3)], then $F = d(A + df) = dA + d^2 f = dA = F$, since d is linear and $d^2 = 0$. Componentwise, the field strength tensor is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{1.6}$$

where the covariant ∂_μ operator is given by $\partial / \partial x^\mu$. It can be easily verified that this tensor is antisymmetric[(4)]. With this 2-form, F, Maxwell Equations may be reduced in number to two, and they can be written in a beautiful, index-free fashion.

MAXWELL'S EQUATIONS

Relativistic Exterior Algebra Version

$$\begin{aligned} d \star F &= j \\ dF &= 0 \end{aligned}$$

where j is now a 3-form.

So far, we have taken a 19th–century theory and written it in more modern, geometry based terms. If we wanted to restore the duality symmetry we discussed in the beginning by introducing a fundamental magnetic charged particle—the magnetic monopole—, the magnetic field would sieze to be divergenceless, and this global potential A will no longer exist.

1.3 The Monopole's Topology

If we take a look at (1.4), we can easily check that it has a singularity in the origin, i.e. the domain of the magnetic field is $\mathbb{R}^3 \setminus \{\vec{0}\}$, which is of the same homotopy type as the unit spherical shell $\mathbb{S}^2$.

$$\mathbb{S}^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$$

(3) The exterior derivative of a 0-form (function) is just its gradient $\vec{\nabla} f$.

(4) Actually, every tensor that can be written in terms of a 2-form is antisymmetric, since $dx^\nu \wedge dx^\mu = -dx^\mu \wedge dx^\nu$

This is manifestly true when we write (1.4) in spherical coordinates[(5)]:

$$B = g \sin\phi \, d\phi \wedge d\theta, \tag{1.7}$$

which does not depend on ρ whatsoever[(6)]. For this reason, the magnetic field can be though as a 2-form living in $\mathbb{S}^2$. For the case when $g \neq 0$, which is the case we will always be interested in, we can see that, although closed, B is not exact in $\mathbb{R} \setminus \{\vec{0}\}$, i.e. there is no 1-form potential A[(7)] such that $B = dA$ in $\mathbb{R} \setminus \{\vec{0}\}$.

We shall quickly prove this statement by way of contradiction. Let us suppose that there exists a 1-form potential A such that $B = dA$. Using Stoke's Theorem, it can be easily integrated over a 2-dimensional sphere of radius r centered at the origin, $\mathbb{B}^2_r(0)$, which is—as all balls are—with no boundary.

$$\int_{\mathbb{B}^2_r(0)} B = \int_{\mathbb{B}^2_r(0)} dA = \int_{\partial\mathbb{B}^2_r(0)} A = \int_{\varnothing} A = 0.$$

On the other hand, integrating directly, which we can do since we know the explicit form of B, we arrive at the following:

$$\int_{\mathbb{B}^2_r(0)} B = g\left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi} \sin\phi \, d\phi\right) = 4\pi g,$$

a result that, except for the case when $g = 0$—the only one in which we are not interested—is in contradiction with the Stokes Theorem, thus proving our initial statement.

Dirac was obviously aware of this, and he knew—although he did not phrase it the way we are going to—that if, instead of removing the origin from $\mathbb{R}^3$, we removed a ray extending form the origin to infinity—a *Dirac String*—one would obtain a subspace of $\mathbb{R}^3$ homotopically equivalent to the punctured plane $\mathbb{R}^2 \setminus \{0\}$, whose second De Rham Cohomology group is trivial.

$$H^2_{\text{De Rham}}(\mathbb{R}^3 \setminus \text{Dirac String}) = 0.$$

On this manifold, every closed 2-form is also exact, and so would B. For instance, on $U_- = \mathbb{R}^3 \setminus \{(0,0,z) \mid z \geq 0\}$, the 1-form

$$A_- = \frac{g}{\rho(\rho - z)}(y\,dx - x\,dy) = -g(1 + \cos\phi)\,d\theta \tag{1.8}$$

fulfills $dA_- = B|_{U_-}$. Analogously, on $U_+ = \mathbb{R}^3 \setminus \{(0,0,z) \mid z \leq 0\}$, $dA_+ = B|_{U_+}$ holds for the 1-form

(5) Although tedious, the result is simply obtained by performing a change of variables from Cartesian to spherical coordinates.

(6) We will take ϕ to be the polar angle in spherical coordinates.

(7) It might be important to point out that this potential is again the usual vector potential, not the 4-potential we worked with in the previous section.

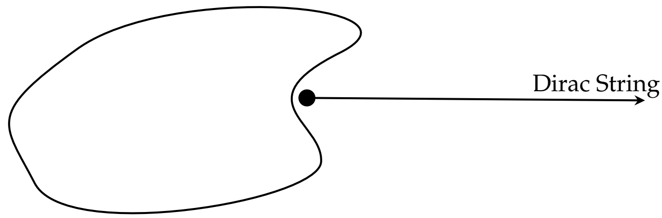

Fig. 1.1. *A more visual approach to the second De Rham Cohomology group being trivial is that, in* $\mathbb{R}^3 \setminus$ Dirac String, *there is no way to have a closed 2-surface enclosing the origin: the point where the magnetic field is singular.*

$$A_+ = -\frac{g}{\rho(\rho+z)}(y\,dx - x\,dy) = g(1-\cos\phi)\,d\theta. \tag{1.9}$$

These potentials define a monopole: the so-called Dirac magnetic monopole. With them we may get the form of the magnetic field generated by the monopole(8). We can see that we have covered the entire domain of B with two open sets in which we can define a potential 1-form(9). Note that for the case when $\phi = \pi/2$, i.e. the curve that encloses the singularity, denoted as $U_- \cap U_+$,

$$A_+ - A_- = 2g\,d\theta \tag{1.10}$$

This result, that at the moment may look somehow arbitrary, is actually one of the most fundamental ones in the magnetic monopole theory: it leads to the quantization of the magnetic charge.

1.4 Dirac's Quantization Condition in two different ways

Dirac was the first to derive the quantization condition that bears his name, but we will not deal with his derivation here. Instead, we will present two proofs based on more modern arguments. The first one, by Wu & Yang, is rooted in gauge theory, and uses the results derived in the previous section [5].

The second one, however, is based in Quantum Field Theory (QFT), particularly in the Aharanov-Bohm effect, and in this sense is probably more intuitive [6, pp. 81-84].

(8) One may point out, and rightfully so, that we found this potential because we *forced* the potential to be Coulomb-like. This is true, but in chapter 3 we will obtain a potential through the Seiberg-Witten equations, and with it, we will be able to find the magnetic field, until then unknown.

(9) $U_- \cup U_+ = \mathbb{R}^3 \setminus \{0\}$.

What Dirac's Quantization Condition states is that, if magnetic monopoles were to exist, then their magnetic charge g would be given in terms of the one of the electric monopole, e[10], in the following way:

$$2ge \in \mathbb{Z}, \tag{1.11}$$

i.e. $g = n/2e$, with n an integer. It is conventional to set $e = 1$, therefore leaving g to be a semi-integer. This allows us to write the potentials as

$$A_{\pm} = \mp\frac{n/2}{\rho(\rho \pm z)}(y\,dx - x\,dy) = \pm\frac{n}{2}(1 \mp \cos\phi)\,d\theta \;\text{ in }\; U_{\pm} \tag{1.12}$$

1.4.1 Quantization à la Wu & Yang

Since we are now assuming that magnetic monopoles exist, $\vec{\nabla} \cdot \vec{B} = g\delta(\vec{r})$, we know that the potential we are working with is not global. However, the different regions in which the potential is defined can be *glued* together via gauge transformations. We have already discussed that Maxwell equations are gauge invariant, i.e. if we perform the transformation $A \mapsto A' = A + d\chi$, the observables (the EM fields) remain invariant. When applying such transformations, a field of charge e changes according to

$$\psi \mapsto \psi' = \mathrm{e}^{ie\chi}\psi$$

For this to be gauge, i.e. for the change to be unobservable, we need the gauge $\mathrm{e}^{ie\chi}$ to be single-valued. Since we are talking about actual physical points in space, we will take every point at an angle χ to be equivalent to that at an angle $\chi + 2\pi n$, with n an integer. Therefore, we need to make the gauge fulfill the condition

$$\mathrm{e}^{ie\chi} = \mathrm{e}^{ie\chi + 2\pi ni} = \mathrm{e}^{ie(\chi + 2\pi)} \quad , \quad n \in \mathbb{Z}.$$

from which we deduce that e must actually be an integer, i.e. $e \in \mathbb{Z}$. In our particular case, we would like

$$A_+ = A_- + d(2g\theta)$$

to be a gauge transformation, in order for the magnetic field to be consistent. In this situation, $\chi = 2g\theta$ and the gauge would be $\mathrm{e}^{i2eg\theta}$, and we know that for it to be single valued,

$$2ge \in \mathbb{Z},$$

as we wanted to show.

---

[10] The elementary electric charge is $e \simeq 1.6 \times 10^{-19}$ C. Electrons have charge $-e$ and protons have charge e.

1.4.2 Quantization for the field enthusiast

In order to to derive Dirac's Quantization Condition from a QFT standpoint, we first need to introduce the Ahronov-Bohm effect, which is a quantum phenomenon by which the presence of a magnetic field $\vec{B}$ affects the propagation of a charged particle even when it is travelling through regions where the magnetic field is zero.

In Quantum Mechanics, the evolution of a particle of charge e under the influence of a magnetic field $\vec{B} = \vec{\nabla} \times \vec{A}$ is given by the Schrödinger Equation[(11)]:

$$\left[i \frac{\partial}{\partial t} + \frac{1}{2m} D_k D^k \right] \psi = 0,$$

where $D_\mu = \partial_\mu - ieA_\mu$ is the gauge-covariant derivative operator.

The fact that $\vec{B} = \vec{\nabla} \times \vec{A} = \vec{0}$ does not imply that $\vec{A}$ itself is null, which means that the potential can affect the evolution of the particle even in zero-magnetic field regions, as we stated before.

In the path integral formalism, the probability amplitude of a particle travelling from $\vec{a}$ to $\vec{b}$ in a time $\Delta t = t_2 - t_1$ is given by

$$\langle \vec{a}, t_1 | \vec{b}, t_2 \rangle = \int \mathcal{D}[\vec{x}(t)] \exp \left\{ iS[\vec{x}(t)] \right\}, \tag{1.13}$$

where $\int \mathcal{D}[\vec{x}(t)]$ means that one is integrating over all the possible curves $\vec{x}(t)$ that conect $\vec{a}$ to $\vec{b}$ and

$$S[\vec{x}(t)] = \int_{t_1}^{t_2} L(\vec{x}, \dot{\vec{x}}; t)\, dt \tag{1.14}$$

is the action of the particle in each of the paths, with L being the Lagrangian of the path. When there is a magnetic field present, the action of each one of the paths is transformed in the following way [6, pp. 81-84]:

$$S[x(t)] \mapsto S'[x(t)] = S[x(t)] + e \int_{\vec{a}}^{\vec{b}} \vec{A} \cdot d\vec{\ell}, \tag{1.15}$$

so that each of the the paths receive a phase shift that will in general be different, and when summed up, will change the wavefunction.

Let us now calculate this phase shift for the case of a particle orbiting around the Dirac String[(12)], i.e.

---

(11) We will also set $\hbar = 1$.

(12) Strictly speaking, we should take into account all closed curves around the Dirac String, but because of the topology of the manifold we find ourselves in, it can be proven that the result actually does not depend on the shape of the curve, but rather

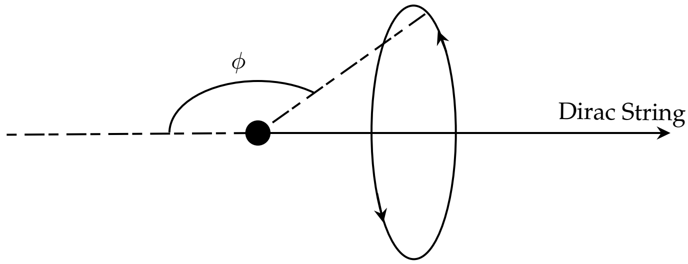

Fig. 1.2. *A loop around the Dirac String. When $\phi \to \pi$, it turns into an infinitesimal loop. It would be equivalent to the particle sitting in the string.*

$$U[C] = \exp\left\{ie \int_C \vec{A} \cdot d\vec{\ell}\right\} \tag{1.16}$$

Since we know the explicit form of the potential—in the region where we are working it will be A_-—, we can integrate directly:

$$\int_C \vec{A}_- \cdot d\vec{\ell} = \int_0^{2\pi} A_-^\theta \, d\theta = \int_0^{2\pi} -g(1 - \cos\phi) d\theta = -2\pi g(1 - \cos\phi)$$

When $\phi \to \pi$, we approach the limit of the particle just *sitting* on the Dirac String. The integral turns into $-4\pi g$, and the phase factor into $\mathrm{e}^{-4\pi i e g}$. Since we want the string not to change our system, we will impose $U[C]$ to be equal to 1, which forces $-4\pi e g$ to be a multiple of 2π. This leads, once again, to the quantization of the magnetic charge:

$$2eg \in \mathbb{Z}.$$

1.5 Letting magnetic monopoles sink in

In this section, the last one of this first chapter, we are going to explore some of the ground-breaking implications about the existence of magnetic monopoles [6, pp. 81-84].

- Since the electric charge in the previous derivations could have been a generic one, say q, we see that if magnetic monopoles were to exist, the electric charge would always have a magnetic counterpart.

---

on the fact that it encloses a singularity, and how many times the curve goes around it. If the phase factor for a curve going around the Dirac string once is λ, the factor for a curve going around the singularity n times will be $n\lambda$. Since at the end we get a multiple of λ, the complex exponential always treats the phase factor as if $n = 1$ when fixing it to be 1. It is because of this, for the sake of simplicity, that we only take into account in the discussion those curves that go around the Dirac string once.

- Furthermore, since we would have obtained a similar constraint by considering a loop around the Dirac string of *any* other magnetic monopole, the quantization condition must hold for every magnetic charge.
- For this reason, every possible choice of q and g will satisfy the quantization condition. The only way this could happen is if $q = mq_{\min}$ and $g = kg_{\min}$, where m, k are integers and $2q_{\min}g_{\min} = 1$. Thus, the detection of even just one magnetic monopole would explain the quantization of the electric charge!
- In the previous derivations, we explicitly took the monopole to have no electric charge. If it were a *dyon*—a particle that would carry both electric and magnetic charge—, it would have its own Dirac String, that would get wound around the monopole. If this monopole was stationary, the discussion would not change!

Besides, if monopoles were to exist, Maxwell equations with field sources—the subindex e refers to *electric* and m to *magnetic*—would take the following form [4]:

$$\vec{\nabla} \times (\vec{E} + i\vec{B}) - i\frac{\partial}{\partial t}(\vec{E} + i\vec{B}) = i(\vec{j}_e + i\vec{j}_m)$$
$$\vec{\nabla} \cdot (\vec{E} + i\vec{B}) = \rho_e + i\rho_m,$$

which are clearly invariant under U(1) transformations(13). We can check that when $(\rho_m, \vec{j}_m) = 0$, we retrieve the original Maxwell equations. Since the magnetic field is no longer divergenceless, Helmholtz's theorem does no longer apply and $\vec{B}$ cannot be expressed only as the curl of a vector potential anymore. We now have to add to the original 4-potential $(\phi, \vec{A})$ another 4-potential $(\varphi, \vec{a})$ to define the EM fields in the presence of both electric and magnetic monopoles. The relations are the following [4]:

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} - \vec{\nabla} \times \vec{a}$$
$$\vec{B} = -\vec{\nabla}\varphi - \frac{\partial \vec{a}}{\partial t} - \vec{\nabla} \times \vec{A}$$

It gets even more beautiful, since this added symmetry allows us to write the Maxwell equations—the explanation for all electromagnetic phenomena in the Universe—in one single equation, in which $\alpha = 1, 2$:

$$\partial_\mu F_\alpha^{\mu\nu} = j_\alpha^\nu, \tag{1.17}$$

where $F_1^{\mu\nu}$ are the components of the Maxwell tensor, $F_2^{\mu\nu}$ are the components of its Hodge dual, j_1^ν are the components of the electric current 4-vector and j_2^ν are the components of the magnetic current 4-vector $(\rho_m, \vec{j}_m)$.

(13) $(\vec{j}_e + i\vec{j}_m) \mapsto e^{i\theta}(\vec{j}_e + i\vec{j}_m)$, $(\rho_e + i\rho_m) \mapsto e^{i\theta}(\rho_e + i\rho_m)$ as well.

2

Study and Interpretation of Solitons in Field Theory

Resumen

En el tercer capítulo estudiaremos el Monopolo de Seiberg-Witten en detalle, pero desde un punto de vista geométrico y abstracto. No hablaremos (explícitamente) de Lagrangianos, energía u otras cantidades físicas a las que podamos estar acostumbrados, sino de spinores y grupos gauge. Este capítulo pretende ser un puente entre las matemáticas abstractas y la intuición física del monopolo magnético desde una perspectiva moderna. Los resultados derivados aquí serán de campos clásicos, pero la subsiguiente teoría cuántica se puede obtener cuantizando los resultados que presentaremos, aunque no lo haremos, ya que el propósito de este capítulo es puramente ilustrativo.

2.1 From the point particle to the field

Let us assume a point particle in a potential. In classical Lagrangian mechanics, one arrives at the equations of motion via the Euler-Lagrange Equations,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0,$$

which are a consequence of the Principle of Least Action. Here, the $\{(q_k, \dot{q}_k)\}_{k=1}^{N}$ are the generalized coordinates and their time derivatives, that are closely related to the position and velocity coordinates of our particle—and often *are* the position and velocity coordinates—. When one solves this equations, one has an explicit, functional form of the generalized coordinates, which define perfectly the trajectory of the particle.

What happens, however, when we find ourselves in a scale such that the laws of classical mechanics do not apply? We know from quantum mechanics that in this case the *trajectory* of a *particle* is not defined. There is no trajectory to solve any equation for, but rather a *wave function* (a scalar field) Ψ that contains all the information about the particle. This wavefuntion behaves according to the Schrödinger equation:

$$i\frac{\partial}{\partial t}\Psi = \hat{H}\Psi$$

The well-known problem with this equation is that it is not invariant under Lorentz transformations, i.e. not compatible with Special Relativity. A way to construct relativistic field theories is by using the Lagrangian formalism. In this case we work with a local version of the Lagrangian: the Lagrangian density $\mathcal{L}$, defined as follows:

$$L = \int_{\mathbb{R}^3} \mathcal{L}(\phi, \partial_\mu \phi; t)\, d^3x,$$

where the role of the generalized coordinates and their time derivative is taken over by the field ϕ (there could be several, but we will focus on real scalar field theory) and their derivatives $\partial_\mu \phi$. A generalized version of the Principle of Least Action yields this version of the Euler-Lagrange equations:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0.$$

In QFT, every particle has a (quantum and relativistic) field associated. This field, in the same fashion as the wavefunction for the non-relativistic case, contains all the information about the particle; and the perturbations of these fields *are* the particles.

A soliton solution to a field equation is a non-dissipative non-trivial finite energy solution. They are a subset of *kinks*. The particularity is that solitons remain unperturbed in collisions with other solitons, while kinks in general do not. However, the literature usually refers to kinks and solitons as the same thing, as we shall do as well.

In scalar field theory, the standard Lagrangian[(1)] for an arbitrary potential $U(\phi)$ is the following:

$$\mathcal{L} = \frac{1}{2}|\partial_\mu \phi|^2 - U(\phi),$$

where the kinetic term is

$$\frac{1}{2}|\partial_\mu \phi|^2 = \frac{1}{2}|\partial^\mu \phi|^2 = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) = \frac{1}{2}\eta^{\mu\nu}(\partial_\nu \phi)(\partial_\mu \phi) = \frac{1}{2}\eta^{\mu\mu}(\partial_\mu \phi)^2,$$

with $\eta^{\mu\nu}$ being the components of the (diagonal) Minkowski metric tensor. We now compute the derivatives required by the Euler-Lagrange equations,

(1) In this context, it is understood that Lagrangian means Lagrangian density.

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = \partial_\mu(\eta^{\mu\mu}\partial_\mu \phi) = \eta^{\mu\mu}\partial_\mu \partial_\mu \phi = \partial^\mu \partial_\mu \phi,$$

and

$$\frac{\partial \mathcal{L}}{\partial \phi} = -\frac{\partial U}{\partial \phi}$$

Therefore, the equation of motion for a field immersed in a general potential will be the following:

$$\partial^\mu \partial_\mu \phi - \frac{\partial U}{\partial \phi} = 0 \tag{2.1}$$

Note that for the free field case, i.e. $U(\phi) = 0$, we get the wave equation.

2.2 On the ϕ^4-theory

To illustrate how from a Lagrangian one can end up finding actual particles, we will examine the ϕ^4-theory[(2)], which is widely used because of it being renormalizable and a good approximation to more complex potentials.

The Lagrangian of this theory has the following form:

$$\mathcal{L} = \frac{1}{2}|\partial_\mu \phi|^2 - \frac{\lambda}{4}(\phi^2 - v^2)^2 \quad , \quad (v^2 = m^2/\lambda), \tag{2.2}$$

where m, λ are real parameters that define the potential and ϕ is the field. A necessary condition for a potential to have soliton solutions is that it must have at least two degenerate minima (the vacua), fulfilled by the ϕ^4-potential. For this potential, using (2.1), the equation of motion for the field has the following form,

$$\partial^\mu \partial_\mu \phi - \lambda(\phi^2 - v^2)\phi = 0, \tag{2.3}$$

which is clearly non-linear. We can see this equation has two trivial solutions, $\phi = \pm v$, that correspond to the vacua solutions (for $\phi = \pm v$, $U = 0$). We will be interested in finding non-trivial, finite energy solutions to this equation. For simplicity, we will solve for the 1+1 dimensional case. Explicitly, our equation of motion is

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} = -\lambda(\phi^2 - v^2)\phi \tag{2.4}$$

We can also just seek stationary solutions to this equation, since we can trivially find the time-dependent solutions just by performing Lorentz transformations $x^\mu \mapsto \Lambda^\mu{}_\nu x^\nu$ on the stationary ones. Therefore, we set the time derivative of the field to be zero.

(2) The theory gets its name because the potential that defines it is a polynomial of order 4 in the field.

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} - \lambda(\phi^2 - v^2)\phi = 0. \tag{2.5}$$

Two solutions (the soliton and the antisoliton) to this differential equation centered at $x = x_0$ and with the additional condition

$$\lim_{x\to\infty} \phi^2 = v$$

are the functions [6, pp. 6-10] [7]

$$\phi(x) = \pm v \tanh\left(\frac{m}{\sqrt{2}}(x - x_0)\right) \tag{2.6}$$

The most intuitive way of realising that this is a soliton is by inspecting the energy density, which can be computed explicitly. The energy density $\mathscr{E}(x)$ will be given by the sum of the kinetic and potential terms in the Lagrangian, i.e.

$$\mathscr{E}(x) = \frac{1}{2}|\partial_\mu\phi|^2 + \frac{\lambda}{4}(\phi^2 - v^2)^2 \tag{2.7}$$

Let us compute these terms, which will be equal for the soliton and antisoliton solutions:

$$\begin{aligned}\frac{1}{2}|\partial_\mu\phi|^2 = \frac{1}{2}\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)^2 &= \frac{1}{2}\left(\frac{vm}{\sqrt{2}}\operatorname{sech}^2\left(\frac{m}{\sqrt{2}}(x - x_0)\right)\right)^2 \\ &= \frac{v^2m^2}{4}\operatorname{sech}^4\left(\frac{m}{\sqrt{2}}(x - x_0)\right) = \frac{m^4}{4\lambda}\operatorname{sech}^4\left(\frac{m}{\sqrt{2}}(x - x_0)\right)\end{aligned}$$

$$\begin{aligned}\frac{\lambda}{4}\left(\phi^2 - v^2\right)^2 &= \frac{\lambda}{4}\left(v^2\left[\tanh^2\left(\frac{m}{\sqrt{2}}(x - x_0)\right) - 1\right]\right)^2 \\ &= \frac{m^4}{4\lambda}\operatorname{sech}^4\left(\frac{m}{\sqrt{2}}(x - x_0)\right),\end{aligned}$$

recalling that $v^2 = m^2/\lambda$. Therefore,

$$\mathscr{E}(x) = \frac{m^4}{2\lambda}\operatorname{sech}^4\left(\frac{m}{\sqrt{2}}(x - x_0)\right) \tag{2.8}$$

According to this result, we see that the energy of the field is concentrated in a region of width $\sim m^{-1}$ centered at x_0. The field ϕ is indistinguishable from the vacuum far enough from x_0. This localization of the energy may suggest that the soliton could be interpreted as a particle of radius $\sim m^{-1}$ and mass M given by the total energy of the field, i.e.

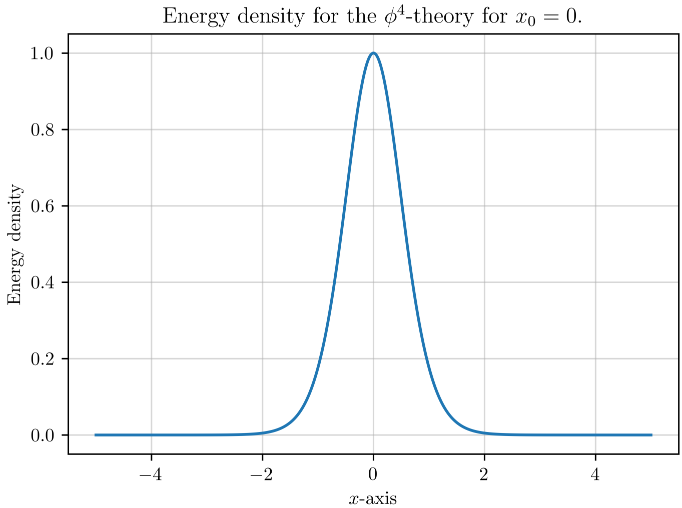

Fig. 2.1. *Representation of the energy density function $\mathscr{E}(x)$ centered at $x_0 = 0$ for the ϕ^4-theory with $m = \sqrt{2}, \lambda = 2$. These values make no physical sense and were only chosen for normalization purposes.*

$$M = \int_{-\infty}^{\infty} \mathscr{E}(x)\, dx = \frac{2\sqrt{2}}{3}\frac{m^3}{\lambda}$$

This integral can be easily solved by considering the well-known relation $\text{sech}^4(u) = \text{sech}^2(u)(1 - \tanh^2(u))$ and performing the change of variables $\tanh^2(u) = t$.

Again, if we wanted to have the soliton moving with a constant velocity u, we would just have to Lorentz-transform the stationary field ϕ, i.e.

$$\phi(x,t) = \pm v \tanh\left(\frac{m}{\sqrt{2}}\frac{x - x_0 - ut}{\sqrt{1-u^2}}\right), \tag{2.9}$$

and in this case the total energy of the field would correspond to that of a Lorentz-contracted soliton of mass M, i.e.

$$\begin{aligned} E &= \int_{-\infty}^{\infty} \left[-\frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 + U(\phi)\right] dx \\ &= \frac{M}{\sqrt{1-u^2}} \end{aligned}$$

Thus, from a quartic potential, we have found a particle, its radius and its mass. It is interesting to notice that the energy density function $\mathscr{E}(x)$ does not dissipate. i.e. the width of the peak does not grow in time. In reality, it does, but we are not accounting for the terms that make this happen in our Lagrangian $\mathcal{L}$.

For instance, in the case of trying to describe a soliton in a water aisle, if we did not take into account the air resistance and the superficial tension of the water, amongst other factors, the soliton we would obtain would not dissipate, when in reality it does. Really slowly, but it does.

This way, the Lagrangian of a theory will yield the equation of motion for the field. Their non-trivial non-dissipative finite energy solutions will define a soliton solution of the field, from which we will obtain the energy density of the field. This, we can interpret as the actual particle: the monopole.

3

The Seiberg-Witten Monopole

Resumen

En este capítulo estudiaremos el monopolo de Seiberg-Witten: un $\mathrm{U}(1)$-monopolo, y esbozaremos su generalización a $\mathrm{SU}(2)$-monopolo, el Monopolo de 't Hooft-Polyakov. Resolveremos para cada caso las Ecuaciones de Seiberg-Witten, distinguiendo para el $\mathrm{U}(1)$-monopolo entre las ecuaciones en $\mathbb{R}^4$ y en el espacio-tiempo de Minkowski $\mathbb{R}^{1,3}$.

3.1 Seiberg-Witten Equations in Flat Spacetime

Although we will not derive them here, the Seiberg-Witten equations describe a coupling between the space(time) 4-manifold (via the connection A and its curvature) and a spinor field ψ associated to the monopole, and they emerge as a consequence of duality arguments in the framework of $N = 2$ supersymmetric Yang-Mills theory. The fact that the monopole is described by a spinor field implies that it is a spin-1/2 particle, and therefore a fermion, just like its electric counterparts. It would also be a fundamental particle, i.e. they would not be formed by more primordial constituents, having its own place in the Standard Model of Particle Physics. Let us write the equations and then dissect them:

<u>SEIBERG-WITTEN EQUATIONS</u>

$$\not{D}_A \psi = 0 \tag{3.1}$$

$$\rho^+(F_A) = (\psi \otimes \psi^\dagger)_0 \tag{3.2}$$

we will denote by R either the Euclidean 4-space $\mathbb{R}^4$ ($g_{\mu\nu} = \delta_{\mu\nu}$) or the Minkowski spacetime $\mathbb{R}^{1,3}$ ($g_{\mu\nu} = \eta_{\mu\nu}$). Let us write $A = A_\alpha dx^\alpha$ for a $\mathrm{U}(1)$-potential and

$$F_A = dA = \sum_{\alpha<\beta} F_{\alpha\beta} dx^\alpha \wedge dx^\beta = \sum_{\alpha<\beta} (\partial_\alpha A_\beta - \partial_\beta A_\alpha) dx^\alpha \wedge dx^\beta \tag{3.3}$$

for its curvature. For any map

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} : R \to \mathbb{C}^2, \tag{3.4}$$

we denote by $\psi \otimes \psi^\dagger$ the following endomorphism of $\mathbb{C}^2$:

$$\psi \otimes \psi^\dagger = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \begin{pmatrix} \bar{\psi}_1 & \bar{\psi}_2 \end{pmatrix} = \begin{pmatrix} |\psi_1|^2 & \psi_1 \bar{\psi}_2 \\ \bar{\psi}_1 \psi_2 & |\psi_2|^2 \end{pmatrix}, \tag{3.5}$$

where $\psi^\dagger$ denotes the congugate transpose of ψ and $\bar{\psi}_i$ the complex conjugate of its components. The RHS of (3.2) refers to the traceless part of $\psi \otimes \psi^\dagger$, i.e.

$$(\psi \otimes \psi^\dagger)_0 = \frac{1}{2} \begin{pmatrix} |\psi_1|^2 - |\psi_2|^2 & 2\psi_1 \bar{\psi}_2 \\ 2\bar{\psi}_1 \psi_2 & |\psi_2|^2 - |\psi_1|^2. \end{pmatrix} \tag{3.6}$$

This can be written in terms of the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

as

$$(\psi \otimes \psi^\dagger)_0 = \frac{1}{2} \sum_{i=1}^{3} (\psi^\dagger \sigma_i \psi) \sigma_i$$

Furthermore, if we recall that the basis quaternions are given by $I_1 \equiv I = i\sigma_1$, $I_2 \equiv J = i\sigma_2$, $I_3 \equiv K = i\sigma_3$, the the RHS of (3.2) can also be written as

$$(\psi \otimes \psi^\dagger)_0 = -\frac{1}{2} \sum_{i=1}^{3} (\psi^\dagger I_i \psi) I_i \in \mathbb{H}. \tag{3.7}$$

To define other objects such as $\not{D}_A$ and ρ^+ in (3.1) and (3.2), we need to take into account the metric. This, of course, gives a distinction between the Euclidean and the Minkowskian scenarios.

3.1.1 Seiberg-Witten Equations in $\mathbb{R}^4$

The reason for this division is that we need precise information about the Clifford algebras of $\mathbb{R}^4$ and $\mathbb{R}^{1,3}$, since they do not coincide. As we know, the standard $\mathbb{R}^4$ scalar product between two vectors $\vec{x}$ and $\vec{y}$ is given by:

$$\langle \vec{x}, \vec{y} \rangle_4 = x^0 y^0 + x^1 y^1 + x^2 y^2 + x^3 y^3 = \delta_{\mu\nu} x^\mu y^\nu$$

Let us now construct a convenient way of computing the scalar product. Let $\mathcal{R}^4$ be the set of all 2×2 complex matrices X such that

$$X = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} = \begin{pmatrix} x^0 + ix^1 & x^2 + ix^3 \\ -x^2 + ix^3 & x^0 - ix^1 \end{pmatrix}$$
$$= x^0 \mathbb{1} + x^1 I + x^2 J + x^3 K,$$

where $\mathbb{1}$ is the identity matrix. Then, one can easily note that $\langle \vec{x}, \vec{x} \rangle_4 = \det(X)$. Let us now notice that for each $\vec{x}$ in $\mathbb{R}^4$ its transpose conjugate has the following form[(1)]:

$$X^\dagger = x^0 \mathbb{1} - x^1 I - x^2 J - x^3 K, \tag{3.8}$$

and define the map $\mathcal{T} \colon \mathcal{R}^4 \to \mathbb{C}^{4\times 4}$ as

$$T(\vec{x}) = \begin{pmatrix} 0 & X \\ -X^\dagger & 0 \end{pmatrix} \tag{3.9}$$

This map $\mathcal{T}$ is linear and injective, which implies that $\mathbb{R}^4 \simeq \mathcal{T}(\mathbb{R}^4) \subset \mathbb{C}^{4\times 4}$. A basis for $\mathcal{T}(\mathbb{R}^4)$ is given by $\{\mathcal{T}(\hat{e}_\alpha)\}_{\alpha=0}^3$, where $\hat{e}_\alpha$, $\alpha = 0, 1, 2, 3$, are the canonical basis vectors in $\mathbb{R}^4$. We shall denote $\{\mathcal{T}(\hat{e}_\alpha)\}_{\alpha=0}^3$ as $\{\gamma_\alpha\}_{\alpha=0}^3$, i.e.

$$\gamma_\alpha = \mathcal{T}(\hat{e}_\alpha) \tag{3.10}$$

These γ-matrices anticommute as follows:

$$\gamma_\alpha \gamma_\beta + \gamma_\beta \gamma_\alpha = -2\delta_{\alpha\beta} \mathbb{1}, \tag{3.11}$$

where $\mathbb{1}$ is still the identity matrix, but in this case in four dimensions. The Clifford algebra $\mathfrak{Cl}(\mathbb{R}^4)$ of $\mathbb{R}^4$ is the real subalgebra of $\mathbb{C}^{4\times 4}$ generated by the γ-matrices. The basis of $\mathfrak{Cl}(\mathbb{R}^4)$ include scalars (the identity), vectors (the generators of the algebra), bivectors (the products $\gamma_\alpha \gamma_\beta$ for $\alpha < \beta$), trivectors (the products $\gamma_\alpha \gamma_\beta \gamma_\mu$ for $\alpha < \beta < \mu$) and a pseudo-scalar (the element $\gamma_0 \gamma_1 \gamma_2 \gamma_3$). Any other product between the gamma matrices can be computed using the elements of the basis of the algebra and the anticommutation relations. That is why they are called elements of the basis. In total there are $2^4 = 16$ of them, meaning that the real dimension of $\mathfrak{Cl}(\mathbb{R}^4)$ is 16.

In what follows, we will write $\mathbb{C}^4$ as the direct sum of two subspaces, each one of them isomorphic to $\mathbb{C}^2$:

$$\mathbb{C}^4 = W^+ \oplus W^-,$$

where

(1) This can be easily seen recalling that the Pauli matrices are Hermitian, and therefore $I_i^\dagger = i^\dagger \sigma_i^\dagger = -i\sigma_i = -I_i$.

$$W^+ = \{(z_1, z_2, 0, 0) \mid z_1, z_2 \in \mathbb{C}\}$$

and

$$W^- = \{(0, 0, z_3, z_4) \mid z_3, z_4 \in \mathbb{C}\}$$

The even elements of $\mathfrak{Cl}(\mathbb{R}^4)$, i.e. those formed by the span the identity, the bivectors and the pseudo-scalar preserve $W^\pm$, whereas the odd ones do not.

Any map $\Psi : \mathbb{R}^4 \to \mathbb{C}^4$ will be called a 4-component spinor field on $\mathbb{R}^4$. A 2-component positive spinor field is a map $\psi : \mathbb{R}^4 \to W^+$ and a 2-component negative spinor field is a map $\phi : \mathbb{R}^4 \to W^-$.

With a little abuse of notation, we will write

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \text{ instead of } \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ 0 \\ 0 \end{pmatrix}$$ [2]

and

$$\Psi = \begin{pmatrix} \psi \\ \phi \end{pmatrix} \text{ instead of } \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \phi_1 \\ \phi_2 \end{pmatrix}$$

This differentiation really responds to the physical interpretation of the spinor fields. In this chapter, Ψ plays the role that $\phi(x)$ played in the previous one: it is the field that describes the particle (a spinor field instead of a scalar one, but a field nonetheless). A spin-1/2 particle—like our monopole—is described by a spinor. Dirac showed this while stating and solving the famous equation that bears his name [9]. The two upper elements of the spinor correspond to the particle associated to the field, while the other two correspond to its antiparticle (from here the notation for the subspaces $W^\pm$ and the $\pm$ signs).

Associated to a $\mathrm{U}(1)$-potential A, we introduce the covariant differential as

$$\nabla\Psi = (\nabla_\alpha \Psi) dx^\alpha = (\partial_\alpha + A_\alpha)\Psi \, dx^\alpha, \tag{3.12}$$

used to define the so-called *"physicist's Dirac operator"* $\mathcal{D}_A$ as

$$\mathcal{D}_A \Psi = \sum_{\alpha=0}^{3} \gamma_\alpha \nabla_\alpha \Psi \tag{3.13}$$

To write this out explicitly we need to recall that $\gamma_\alpha = \mathcal{T}(\hat{e}_\alpha)$, and that $X(\hat{e}_\alpha) = I_\alpha$, if we let I_0 be the 4×4 identity matrix,

$$\gamma_0 = \begin{pmatrix} 0 & I_0 \\ -I_0 & 0 \end{pmatrix} \qquad \gamma_i = \begin{pmatrix} 0 & I_i \\ I_i & 0 \end{pmatrix},$$

(2) Idem for ϕ.

or, to put it in a more explicit way:

$$\gamma_\alpha = \begin{pmatrix} 0 & I_\alpha \\ (-1)^{\delta_{\alpha 0}} I_\alpha & 0 \end{pmatrix} \tag{3.14}$$

Now, we can write $\mathcal{D}_A \Psi$ as follows

$$\begin{aligned}
\mathcal{D}_A \Psi &= \sum_\alpha \begin{pmatrix} 0 & I_\alpha \\ (-1)^{\delta_{\alpha 0}} I_\alpha & 0 \end{pmatrix} \nabla_\alpha \begin{pmatrix} \psi \\ \phi \end{pmatrix} \\
&= \sum_\alpha \begin{pmatrix} 0 & I_\alpha \\ (-1)^{\delta_{\alpha 0}} I_\alpha & 0 \end{pmatrix} \begin{pmatrix} \nabla_\alpha \psi \\ \nabla_\alpha \phi \end{pmatrix} \\
&= \sum_\alpha \begin{pmatrix} I_\alpha \nabla_\alpha \phi \\ (-1)^{\delta_{\alpha 0}} I_\alpha \nabla_\alpha \psi \end{pmatrix} \\
&= \begin{pmatrix} \nabla_0 \phi + I \nabla_1 \phi + J \nabla_2 \phi + K \nabla_3 \phi \\ -\nabla_0 \psi + I \nabla_1 \psi + J \nabla_2 \psi + K \nabla_3 \psi \end{pmatrix}
\end{aligned}$$

It is interesting to note that this operator carries a positive spinor to a negative one, and vice versa.
The second component of this vector is what we define to be the *"mathematician's Dirac operator"* $\not{D}_A$ acting on the 2-component positive spinor field ψ,

$$\not{D}_A \psi = -\nabla_0 \psi + I \nabla_1 \psi + J \nabla_2 \psi + K \nabla_3 \psi, \tag{3.15}$$

which, according to equation (3.1), is known to be zero. This way, we have expressed out the first Seiberg-Witten in a more explicit fashion:

$$\nabla_0 \psi = I \nabla_1 \psi + J \nabla_2 \psi + K \nabla_3 \psi \tag{3.16}$$

If we go further and write out the covariant differentials, identifying $(x^0, x^1, x^2, x^3) \mapsto (t, x, y, z)$, we get the following system of equations:

$$\begin{aligned}
(i\partial_1 - \partial_0 + iA_1 - A_0)\, \psi_1 &= -\,(i\partial_3 + \partial_2 + iA_3 + A_2)\, \psi_2 \\
(i\partial_3 - \partial_2 + iA_3 - A_2)\, \psi_1 &= (i\partial_1 + \partial_0 + iA_1 + A_0)\, \psi_2
\end{aligned}$$

In order to construct the second Seiberg-Witten equation (3.2), we must describe a natural action of complex-valued 2-forms on $\mathbb{C}^4$. If we think of all $\mathbb{C}$-linear endomorphisms of $\mathbb{C}^4$ as a vector space, $\mathrm{End}_{\mathbb{C}}(\mathbb{C}^4)$, we may define the map $\rho\colon \Omega^2(\mathbb{R}^4) \otimes \mathbb{C} \to \mathrm{End}_{\mathbb{C}}(\mathbb{C}^4)$[(3)] as

$$\rho(F) = \rho\left(\sum_{\alpha<\beta} F_{\alpha\beta} e^\alpha \wedge e^\beta \right) = \sum_{\alpha<\beta} F_{\alpha\beta} T(\hat{e}_\alpha) T(\hat{e}_\beta) = \sum_{\alpha<\beta} F_{\alpha\beta} \gamma_\alpha \gamma_\beta$$

(3) $\Omega^2(\mathbb{R}^4) \otimes \mathbb{C}$ is the complexification of $\Omega^2(\mathbb{R}^4)$.

$$
= \begin{pmatrix} \begin{matrix} (F_{01}+F_{23})I \\ +(F_{02}+F_{31})J \\ +(F_{03}+F_{12})K \end{matrix} & \mathbf{0} \\ \mathbf{0} & \begin{matrix} -(F_{01}-F_{23})I \\ -(F_{02}+F_{13})J \\ -(F_{03}-F_{12})K, \end{matrix} \end{pmatrix}
$$

where here $\mathbf{0}$ is the 2×2 zero matrix. Since it is diagonal, ρ preserves $W^{\pm}$, so we can define

$$
\rho^{\pm}(F) = \rho(F)|W^{\pm}
$$

In particular, to write out the LHS of (3.2),

$$
\rho^{+}(F) = (F_{01}+F_{23})I + (F_{02}+F_{31})J + (F_{03}+F_{12})K
$$

In order to achieve (3.2), we compare with expression (3.7) to obtain

$$
\begin{aligned}
F_{01}+F_{23} &= -\frac{1}{2}(\psi^{\dagger} I \psi) \\
F_{02}+F_{31} &= -\frac{1}{2}(\psi^{\dagger} J \psi) \\
F_{03}+F_{12} &= -\frac{1}{2}(\psi^{\dagger} K \psi),
\end{aligned} \tag{3.17}
$$

or, explicitly:

$$
\begin{aligned}
\partial_0 A_1 - \partial_1 A_0 + \partial_2 A_3 - \partial_3 A_2 &= -\frac{i}{2}(|\psi_1|^2 - |\psi_2|^2) \\
\partial_0 A_2 - \partial_1 A_3 - \partial_2 A_0 - \partial_3 A_1 &= -i\mathrm{Im}(\bar{\psi}_1 \psi_2) \\
\partial_0 A_3 + \partial_1 A_2 - \partial_2 A_1 - \partial_3 A_0 &= -i\mathrm{Re}(\bar{\psi}_1 \psi_2).
\end{aligned}
$$

We have dedicated the last few pages to finding out the explicit form of the Seiberg-Witten equations in the hope of being able to solve them, but Witten proved [8] that if a 1-form A and a positive 2-component spinor ψ satisfy (3.16) and (3.17), then $\psi \equiv 0$[(4)], i.e. there are no non-relativistic $\mathrm{U}(1)$-magnetic monopoles. This is why we need to extend our analysis to the Minkowski spacetime.

3.1.2 Seiberg-Witten Equations in $\mathbb{R}^{1,3}$

To write the Seiberg-Witten equations in flat Minkowski spacetime $\mathbb{R}^{1,3}$, the only thing we must take into account is that the Clifford algebra is (slightly) different. In this case, the scalar product of two vectors (what we call *events* in special relativity) $\vec{x}$ and $\vec{y}$ of $\mathbb{R}^{1,3}$ is given by

(4) At least for $\psi \in L^2(\mathbb{R}^4)$.

$$\langle \vec{x}, \vec{y} \rangle_{1,3} = -x^0 y^0 + x^1 y^1 + x^2 y^2 + x^3 y^3 = \eta_{\mu\nu} x^\mu y^\nu$$

If we now let $\mathcal{R}^{1,3}$ be the set of all 2×2 complex matrices X of the form

$$X = x^0\sigma_0 + x^1\sigma_1 + x^2\sigma_2 + x^3\sigma_3 = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix},$$

we note that $-\det(X) = \langle \vec{x}, \vec{y} \rangle_{1,3}$. Now, for every $X = x^\alpha \sigma_\alpha \in \mathbb{R}^{1,3}$, we define $\tilde{X}$ as

$$\tilde{X} = x^0\sigma_0 - x^1\sigma_1 - x^2\sigma_2 - x^3\sigma_3, \tag{3.18}$$

while the map $\mathcal{T}\colon \mathcal{R}^{1,3} \to \mathbb{C}^{4\times 4}$ is defined as

$$\mathcal{T}(\vec{x}) = \begin{pmatrix} 0 & \tilde{X} \\ -\tilde{X} & 0 \end{pmatrix}.$$

Again, $\mathcal{T}$ is linear and injective, so we may identify $\mathbb{R}^{1,3}$ with $\mathcal{T}(\mathbb{R}^{1,3}) \subset \mathbb{C}^{4\times 4}$. A basis of the space will be given by:

$$\gamma_\alpha = \mathcal{T}(\hat{e}_\alpha),$$

and satisfy the anticommutation relations:

$$\gamma_\alpha \gamma_\beta + \gamma_\beta \gamma_\alpha = -2\eta_{\alpha\beta}\mathbb{1} \tag{3.19}$$

This way, the Clifford algebra of $\mathbb{R}^{1,3}$ is the real subalgebra $\mathfrak{Cl}(\mathbb{R}^{1,3})$ generated by these γ-matrices. Now, proceeding in the exact same fashion as in the previous section, one finds the Seiberg-Witten equations can be written as

$$\nabla_0\psi = \sigma_1\nabla_1\psi + \sigma_2\nabla_2\psi + \sigma_3\nabla_3\psi \tag{3.20}$$

$$\begin{aligned} F_{01} + iF_{23} &= \frac{1}{2}(\psi^\dagger \sigma_1 \psi) \\ F_{02} + iF_{31} &= \frac{1}{2}(\psi^\dagger \sigma_2 \psi) \\ F_{03} + iF_{12} &= \frac{1}{2}(\psi^\dagger \sigma_3 \psi) \end{aligned} \tag{3.21}$$

Witten's theorem still holds in $\mathbb{R}^{1,3}$, but physically relevant solutions can be found, unlike in the Euclidean scenario. For these coupled system of PDEs there is a particularly interesting solution, found by P. Freund [8]. Once again we will rename $(x^0, x^1, x^2, x^3) \mapsto (t, x, y, z)$. Then, on $\mathbb{R}^{1,3} \setminus \{(t, 0, 0, z) \mid t \in \mathbb{R},\ z \geq 0\}$, a solution $A = A_\alpha dx^\alpha$, $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ to (3.20) and (3.21)—this is, a time independent solution for the region that in chapter 1 we denoted by U_-—is given by:

$$A_0 = A_3 = 0, \qquad A_1 = i\frac{-y}{2\rho(\rho - z)}, \qquad A_2 = i\frac{x}{2\rho(\rho - z)} \tag{3.22}$$

$$\begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix} = \frac{1}{\rho\sqrt{2\rho(\rho - z)}}\begin{pmatrix}x - iy \\ \rho - z\end{pmatrix}. \tag{3.23}$$

The potential

$$A = -i\frac{y\,dx - x\,dy}{2\rho(\rho - z)} = i(1 + \cos\phi)d\theta = -iA_-$$

is, except for a global phase factor, identical to the potential A_- found in chapter 1 on U_- for a monopole with the least positive magnetic charge. This way, we again meet the Dirac monopole, i.e. doing a thorough examination of equations deeply rooted in symmetry, geometry and topology yields the same result that assuming that the monopole would create a Coulomb-like potential. Quite fascinating! Actually, the RHS of all three equations in (3.21)—known as the *curvature equations*—essentially say that the spinor ψ defines a Coulomb-like field:

$$\frac{1}{2}(\psi^\dagger\sigma_j\psi) = \frac{1}{2}\frac{x^j}{\rho^3}. \tag{3.24}$$

3.2 SU(2) Generalization of the Seiberg-Witten Monopole: the 't Hooft-Polyakov Monopole

There is another important kind of magnetic monopole in $\mathbb{R}^{1,3}$: the SU(2)-monopole, often called 't Hooft-Polyakov monopole. As we will see in the following chapter, 't Hooft-Polyakov monopoles are predicted by all Grand Unification Theories (GUTs) and Theories of Everything (TOEs) [11], so their experimental discovery would open a door to experimental evidence on theories such as superstrings theories. Since the energies required to perform direct observation are of the order of 10^{15} GeV for GUTs and 10^{19} GeV for TOEs and since the Large Hadron Collider (LHC) works, at best, in the order of 10^3 GeV, as for now, one would think that direct experimental confirmation for such theories is impossible.

For this monopole the procedure is essentially the same. We will simply write the Seiberg-Witten equations and present an interesting solution. A sketch of the derivation may be found in [8].

Using the matrices $T_i = \sigma_i/2$ as a basis for $\mathfrak{su}(2)$—the Lie algebra of SU(2)—, the structure constants are given by the Levi-Civita symbols

$$[T_i, T_j] = \varepsilon^{ijk}T_k. \tag{3.25}$$

In this case, our SU(2) gauge potential A is given by:

$$A = A^\alpha T_\alpha = A_\mu dx^\mu = (A^\alpha_\mu T_\alpha)dx^\mu,$$

and our 4-component spinor field may be written in terms of the $\mathfrak{su}(2)$ basis elements as

$$\Psi = \Psi^\alpha T_\alpha = \begin{pmatrix} \psi^\alpha \\ \phi^\alpha \end{pmatrix} T_\alpha = (\Psi^\alpha_\mu T_\alpha)dx^\mu$$

In this scenario, the Seiberg-Witten equations are too complicated to be explicitly written, but when $A^1 = A^2 = A^3_0 = 0$, $\Psi^3 = 0$ and both the potential and the spinor field are independent of x^0, the equations take the form:

$$\begin{aligned}
&\partial_1\Psi^1_2 + \frac{1}{2}A^3_1\Psi^2_2 - i\left(\partial_2\Psi^1_2 + \frac{1}{2}A^3_2\Psi^2_2\right) + \partial_3\Psi^1_1 = 0\\
&\partial_1\Psi^1_1 + \frac{1}{2}A^3_1\Psi^2_1 + i\left(\partial_2\Psi^1_1 + \frac{1}{2}A^3_2\Psi^2_1\right) - \partial_3\Psi^1_2 = 0\\
&\partial_1\Psi^2_2 - \frac{1}{2}A^3_1\Psi^1_2 - i\left(\partial_2\Psi^2_2 - \frac{1}{2}A^3_2\Psi^1_2\right) + \partial_3\Psi^2_1 = 0\\
&\partial_1\Psi^1_2 - \frac{1}{2}A^3_1\Psi^1_1 + i\left(\partial_2\Psi^2_1 - \frac{1}{2}A^3_2\Psi^1_1\right) - \partial_3\Psi^2_2 = 0
\end{aligned} \tag{3.26}$$

$$\begin{aligned}
-\partial_3 A^3_2 &= \mathrm{Im}\left(\bar\Psi^2_1\Psi^1_2 + \bar\Psi^2_2\Psi^1_1\right)\\
\partial_3 A^3_1 &= \mathrm{Re}\left(\bar\Psi^2_2\Psi^1_1 - \bar\Psi^2_1\Psi^1_2\right)\\
\partial_1 A^3_2 - \partial_2 A^3_1 &= \mathrm{Im}\left(\bar\Psi^2_1\Psi^1_1 - \bar\Psi^2_2\Psi^1_2\right),
\end{aligned} \tag{3.27}$$

A solution to this system of coupled PDEs in U_- was found by T. Dereli and M. Tekmen in [10], which we now present ($(x^1, x^2, x^3) \mapsto (x, y, z)$):

$$A^1 = A^2 = 0, \qquad A^3 = -(1+\cos\phi)d\theta = \frac{y\,dx - x\,dy}{\rho(\rho - z)} \tag{3.28}$$

$$\Psi^1 = \frac{1}{\sqrt{2}}\frac{1}{\rho}(\xi + \eta), \qquad \Psi^2 = \frac{1}{\sqrt{2}}\frac{1}{\rho}(-\xi + \eta), \qquad \Psi^3 = 0, \tag{3.29}$$

where

$$\xi = \frac{1}{\sqrt{2\rho(\rho - z)}}\begin{pmatrix} x - iy \\ \rho - z \end{pmatrix} \in W^+$$
$$\eta = \frac{1}{\sqrt{2\rho(\rho - z)}}\begin{pmatrix} \rho - z \\ -x - iy \end{pmatrix} \in W^-$$

This way, we recover the Dirac monopole once more:

$$A = \frac{y\,dx - x\,dy}{\rho(\rho - z)}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.30}$$

Furthermore, we can check that the $\mathrm{U}(1)$-monopole is contained in this solution, which is reasonable, since $\mathrm{U}(1)$ is a subgroup of $\mathrm{SU}(2)$.

The $\mathrm{SU}(2)$ Seiberg-Witten equations also admit a non-Abelian monopole solution singular only at the origin and not in the entire z-axis [10].

Finally, let us mention that Seiberg-Witten equations can also be generalized to any gauge group $\mathrm{SU}(n)$.

4

Research on Magnetic Monopoles and Implications of its Existence

Resumen

En este último capítulo, discutiremos los esfuerzos pasados y presentes más relevantes realizados por la comunidad científica para encontrar monopolos magnéticos experimentalmente y las consecuencias que su existencia tendría en la física.

4.1 Research on Magnetic Monopoles

Historically, the search for magnetic monopoles has been carried out in three different ways:

- Directly producing them in particle accelerators such as the LHC.
- Searching for magnetic monopoles in cosmic rays or trapped in materials.
- Looking for indirect signs of magnetic monopoles in astronomical observations.

4.1.1 Particle accelerator searches

As we advanced in the previous chapter, the energy required to observe GUT-predicted monopoles ('t Hooft-Polyakov monopoles) is a billion[(1)] times higher than the one currently achievable by the LHC, so one would quickly disregard this option. However, we are only familiar with the laws of physics up to the electroweak scale, which is ~ 100 GeV, and it is therefore possible for an unknown, lighter monopole to exist. These are called *intermediate-mass monopoles* (IMM), and if they were to exist, they could very well be produced in LHC experiments.

[(1)] A million times a million.

Unlike most of the particles searched at particle accelerators, magnetic monopoles are stable and can only be annihilated by encountering an anti-monopole. This way, once a monopole is created, it will not decay. This is the inspiration behind the experiment MoEDAL (Monopole and Exotics Detector at the LHC), a project consisting on plastic nuclear track detector sheets placed around the LHCb (LHC-beauty) experiment. This way, if monopoles are produced during LHCb collisions, they will fly through the sheets, leaving marks on them that can be detected when the sheets are removed and analysed [13]. Other accelerators like Tevatron, LEP and HERA have also tried to find magnetic monopoles in the past, but they were not successful [12]. Not finding monopoles, however, sets a lower bound for their masses: if magnetic monopoles existed their masses could not be lower than roughly 10^3 GeV [11].

This means that monopoles would be really massive particles. Primordial GUT monopoles—sometimes called *supermassive monopoles*—, possibly formed in the early stages of the Universe, such as the 't Hooft-Polyakov monopoles, would have masses in the order of $10^{16} - 10^{17}$ GeV, which is 14 to 15 orders of magnitude greater than the mass of a proton!

The intermediate mass monopoles would still be very massive, having masses in range of $10^5 - 10^{13}$ GeV [12]. However, the uncertainty in this data is quite large.

4.1.2 Direct Searches

Since magnetic monopoles would be very stable, one could try, instead of creating them, search for the ones that already exist. The level of difficulty of this task is determined by how many of them there are; or more precisely, by their flux F. This quantity does not only depend on the number of monopoles but also on their velocities.

Magnetic monopoles have long been searched for in cosmic rays. Obviously, the higher the flux of monopoles, the more often we will find them in cosmic rays. In the 1970s and the 1980s several detections compatible with magnetic monopoles were made, but since later experiments have not been able to reproduce these results, they are believed to be caused by some other effects [11]. Furthermore, these experiments were also able to search for different particles in cosmic rays, inspiring the way we search for dark matter nowadays.

The strongest bounds for GUT monopoles were found by the MACRO (Monopole, Astrophysics and Cosmic Ray Observatory) experiment, which operated from 1989 until the year 2000. It had a total effective area of approximately 10^4 m^2, and since it did not detect any monopoles, they were able to conclude that $F \lesssim 10^{-16}$ cm^{-2}s^{-1} sr^{-1}. Experiments such as AMANDA (Antarctic Muon And Neutrino Detector Array) and Baikal later confirmed this bound.

Apart from cosmic rays, attempts have also been made to detect magnetic monopoles in meteorites, moon rocks and sea water, since they could be trapped

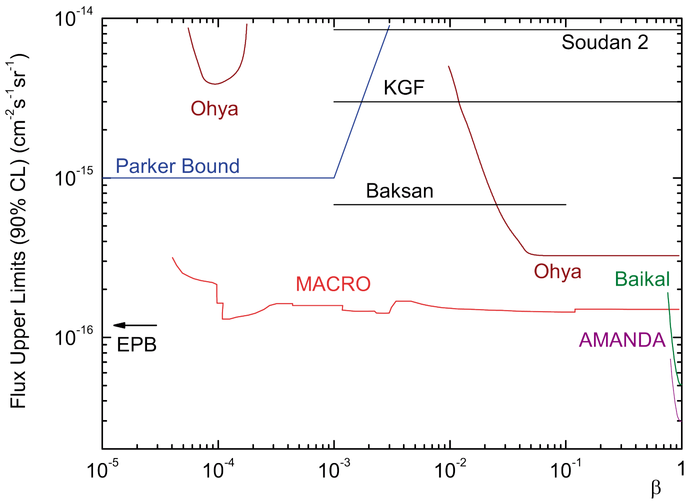

Fig. 4.1. *The 90% confidence level (CL) lower limits for a flux of cosmic GUT monopoles with magnetic charge $g = g_D = \hbar c/2e$ given by different experiments as function of $\beta = v/c$ the velocity of the monopole.* ***Source:*** [12]

---

in there, because of them being very stable. Once again, there has been no success so far [14].

4.1.3 Astrophysical Bounds

If magnetic monopoles were real, they would certainly produce astrophysical effects, which we might be able to measure. Perhaps, one of the most significant ones would be that they would be accelerated by magnetic fields, generating currents that would drain energy from the field [11] . The magnetic flux would tell us how the energy of the magnetic field decreases.

If the flux were very high, there would not be any large-scale magnetic fields in space, since they would have dissipated. However, it is known that in the Milky Way there is a magnetic field of $\sim 3\ \mu$G, which means that the monopole flux cannot be very high. This is known as the *Parker Bound*: $F \lesssim 10^{-15}$ cm^{-2}s^{-1} sr^{-1}. This limit was then extended to the EPB (Extended Parker Bound) by taking into account that this effect is mass dependent: $F \lesssim m_{17} 10^{-16}$ cm^{-2}s^{-1} sr^{-1}, where $m_{17} = m_M \times 10^{-17}$, with m_M being the mass of the monopole [12].

4.2 Implications of the Existence of Magnetic Monopoles

Although—as we have seen in the previous section—magnetic monopoles have not yet been discovered, even as a theoretical chimera they have proven to be an extremely useful tool in theoretical physics.

They, for instance, are compatible with Kaluza-Klein theories, which aim to generalize General Relativity by interpreting gauge fields as components of a higher-dimensional geometry.

It spacetime were 5-dimensional, adding a periodic x^4 dimension, then this 5-dimensional gravity gives rise to gravity and Maxwell field in 4 dimensions. The metric components $g_{\mu 4}$ are given by the Maxwell potential, and gauge invariance arises from reparametrizations of x^4. What we would see as electric charge would actually be momentum in x^4, and it would be quantized because of it being periodic [3].

Including magnetic monopoles in QFT has forced physicists to develop new calculation techniques.The standard way to do calculations in QFT is perturbation theory, which cannot be used for topological solitons such as magnetic monopoles. Besides, perturbation theory assumes that the particles interact weakly with one another. The strength of the interaction between electrons is given by the fine structure constant $\alpha \simeq 1/137$, but for the magnetic monopoles it would be given by its inverse, because of Dirac's Quantization Condition, $\alpha_M = 1/\alpha \simeq 137$, which is most definitely not a small number. This way, the interaction between magnetic monopoles is actually very strong, and perturbation theory is not generally applicable.

All GUTs and TOEs predict magnetic monopoles [11]. Therefore, if we detected magnetic monopoles, we could have direct experimental evidence to contrast the predictions of theories such as string theory or supergravity theories, for which we have no experimental support yet.

As a last example, magnetic monopoles could also help us to understand Quantum Chromodynamics (QCD) better. The theory is written in terms of quarks and gluons, which have never been seen as free particles because of a phenomenon called *asymptotic confinement*. In 1976, G. 't Hooft and S. Mandelstam suggested that this confinement would be explained by magnetic monopoles [11].

This way, as we can see, the experimental discovery of magnetic monopoles would explain many unkwown phenomena and pave the way for the future of theoretical physics.

5

Conclusions

In the same way as in a significant part of theoretical physics, we have seen that not having experimental evidence on magnetic monopoles has not stopped scientists from developing both a mathematically consistent theory for monopoles and experiments to search for them both in the smallest and largest scales of the Universe.

As we have seen, magnetic monopoles are of great relevance in the modern theoretical physics landscape: there are plenty of phenomena this spin-1/2 particle could be involved in, and its existence would prove that the Standard Model of Particle Physics is at least incomplete, since it would be a *new* fundamental particle. Even if it is not real after all, the questions they have raised and the challenges they have thrown at the scientific community have been of paramount importance in the development of modern particle and high energy physics.

There are however many symmetry-based arguments in favor of the existence of magnetic monopoles that would imply a total change in how we see the world and a very beautiful way to turn physics upside down.

References

[1] CURIE, P. *Sur la possibilité d'existence de la conductibilité magnétique et du magnétisme libre.* Journal de Physique, 3^e série, t.III, p. 415 (1894).
[2] DIRAC, P.A.M. *Quantised singularities in the electromagnetic field.* Proc. Roy. Soc. Lond., A133, 60–72 (1931).
[3] POLCHINSKI, J. *Monopoles, Duality and String Theory.* International Journal of Modern Physics A19, 145-154 (2004).
[4] KÜHNE, R. *A Model for Magnetic Monopoles.* arXiv:hep-ph/9708394 (1997)
[5] MIHAI, M. *A study of the Dirac Monopole.* (2018)
[6] WEINBERG, ERICK J. *Classical Solutions in Quantum Field Theory. Solitons and Instantons in High Energy Physics.* Cambridge University Press (2012)
[7] LAUREL, H. *A summary on Solitons in Quantum Field Theory.* Uppsala Universitet, 10-14 (2016)
[8] NABER, GERGORY L. *Dirac and Seiberg-Witten Monopoles.* Coral Press (1999)
[9] DIRAC, P.A.M. *The quantum theory of the electron.* Proc. R. Soc. Lond. A117, 610–624 (1928)
[10] DERELI, T., TEKMEN, M. *Wu-Yang Monopoles and Non-Abelian Seiberg-Witten Monopole Equations.* Mod.Phys.Lett.A13, 1803-1808 (1998)
[11] RAJANTIE, A. *Introduction to Magnetic Monopoles.* Department of Physics, Imperial College London (2012)
[12] GIACOMELLI, G., PATRIZII, L., SAHNOUN, Z. *Searches for Magnetic Monopoles and ... beyond.* arXiv:1105.2724 (2011)
[13] PINFOLD, J. *MoEDAL becomes LHC's magnificent seventh.* CERN Cour. 50N4 (2010)
[14] JEON, H., LONGO, M.J. *Search for magnetic monopoles trapped in matter.* Phys.Rev.Lett.75, 1443-1446 (1995)