UNIVERSIDAD DE LA LAGUNA

Introduction to String Theory. Bosonic Strings

by

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Abstract

Este trabajo de Fin De Grado que se presenta versa sobre una de las teorías que actualmente más interés suscitan, tanto entre los profesionales de la ciencia como entre aquellos otros que sin dedicarse a la Física son amantes de la misma. Se trata de la *Teoía de Cuerdas*.

Esta teoría es realmente atractiva pues predice hechos que en principio resultan contradictorios con los hechos que se observan en el mundo físico, como esel caso de la predicción de 26 dimensiones. Este trabajo es caraćter bibliográfico y tiene como por finalidad realizar una introducción a esta famosa teoría. Por tanto nada de lo que se expone ha sido desarrollado por el alumno salvo contadas excepciones como pueden ser algunas demostraciones.

El trabajo ha consistido por tanto, entender esta introducción a la teoría para posteriormente tratar de explicarla. Para dicha tarea se han utilizado principalmente el libro *A First Course in String Theory*,escrito por Barton Zwiebach [1], y las las notas de los profesores *Kevin Wray* [2] y *David Tong* [3]

Este trabajo se centra exclusivamente en la teoría de cuerdas bosónicas, y como se explicará posteriormente, esta debe ser ampliada por la teoría de super-cuerdas.

El proposito de este trabajo es predecir las 26 dimensiones y llegar al espectro de masa de las cuerdas bosónicas, donde se conjeturan partículas tan interesantes como el *gravitón* o el *taquión*.

Para llegar a estos resultados haremos un estudio que comienza por tratar la acción de una cuerda clásica,estudiando sus simetrías hasta llegar a su cuantización, donde encontraremos algunos inconvenientes que debemos resolver para llegar a nuestro ya comentado objetivo que es predicir las 26 dimensiones ya comentadas así como el espectro de masas.

Agradecimientos

En primer lugar agradecer a mi tutor del trabajo por la ayuda proporcionada y por esas largas horas de tutorías, así como a todos los docentes del grado de Física de la Universidad de La Laguna por el conocimiento y competencias adquiridas. También agradecer a mis personas más cercanas como son mis padres y mis tíos por haberme apoyado siempre, y especialmente durante este periodo que ha sido mi formación. No quiero olvidarme de dar las gracias a esos amigos que han estado ahí cuando pude ver las cosas con un tono *negro*, pues supieron estar allí firmes a mi lado como un *castillo*. Y como no agradecer a esa persona especial que ha estado aguantándome en mis momentos de estrés estos ultimos meses y que ha estado ahí en todo momento, muchas gracias.

Hasta ahora había decido no decir nombres para no dejarme ninguno en el tintero, pero creo que las siguientes personas merecen ser la excepción. Muchas gracias a mi cuñado James, a mis dos sobrinos Raphael y Lila y a mi hermana Nuria, por entender que no he podido dedicarle todo el tiempo que hubiera querido y que ellos merecían.

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Dedicado a mis sobrinos Lila y Raphael.

Chapter 1

A Brief Introduction

1.0 Resumen.

En este capítulo nos centraremos en hacer una introducción a la teoría de cuerdas, donde hablaremos no sólo de la cuerdas bosónicas sino además de la teoría que la continúa. Haremos también un breve repaso histórico e introduciremos conceptos como Supercuerdas, Teoría M y compactificación.

In this chapter we will focus on an introduction to string theory, where we will talk not only about the bosonic strings but also about the theory that continues it. We will also do a brief historical review and introduce concepts such as Super Strings, M-Theory and compactnification.

1.1 What is String Theory?

String theory is a very peculiar and popular theory, so it predicts $9+1$ dimensions instead of the $3+1$ which we see, also it introduces the possible quantum of the gravity, the graviton. All these things are very interesting, but What is String Theory? What role do the strings play? Over the years the human race has tried to understand how the matter works, from the Greeks to the present days. One idea which has survived from the Greeks is the indivisibility of matter, and here is where the String Theory appears. The String Theory affirms that all the particles, either fermions or bosons, are (surprise!) strings which oscillate instead of point particles.

This idea appears when we try to obtain a quantum gravity theory, that unlike the other forces we do not understand. The problem of the quantum gravity that it is not renormalizable, then when we try to study the energy of some particle we must introduce infinite parameters to absorb the divergences that appear. Introducing an infinite number of parameters is not a good idea if we want a consistent theory, so we have to search for another solution. The idea to resolve this problem is to abandon the point particle picture. Now all the particle are strings which have a given length l_s. In this case, the divergences do not appear in our quantum gravity theory and that is the reason why the String Theory is considered one candidate to the quantum gravity theory. Another reason that supports this idea is the possibility to obtain the Einstein equation from the String Theory. In addition, as the theory has evolved, it has been postulated as a theory capable of unifying all the forces.

1.2 String Theory over time

Even though the string theory has a important role at the present time, this theory was born the last century. In 1968, the physicist Gabriele Veneziano [4] was studying the strong nuclear force and discovered that the *Euler beta function* can been used to describe some scattering processes. The only problem is that Veneziano was not able to explain the reason why this happens.

Later, in 1970 Yoichiro Nambu [5] and Leonard Susskind [6] proved ,each on his own , that if the particles were vibrating strings their nuclear interaction can been described using the *Euler beta function*. They supposed that the length of the strings was so small that the particles could still look as points. However, this explanation was contradicted by the experiment. String Theory lost interest when, in 1973, quantum chromodynamics managed to explain the strong nuclear force.

In 1974, Schwarz and Scherk [7] observed that there were some vibrations of the string corresponding to a boson whose properties are equal to the graviton, the gravitational force's hypothetical messenger particle. This meant the origin of String Theory.

Problems came up soon, the ground state of the theory (the tachyon) was unstable, the critical dimension was $D = 26$ and the theory only predicted bosons. In order to obtain a String Theory that would predict fermions, it was necessary to include the *supersymmetry*, which give us a relation between bosons and fermions. The String Theories that have included the supersymmetry are known as *Superstring Theories*. In 1977, Ferdinando Gliozzi, Jöel Scherk, and David I. Olive [8] obtained a tachyon-free String Theory, which is considered the first consistent Superstring Theory.

1.2.1 First Superstring Revolution

Between 1984 and 1994 the first superstring revolution took place. The physicist understood that the String Theory was able to describe the fundamental particles and the interaction between them. This implies that String Theory became a candidate to unify all the forces. Also, the critical dimension for Superstring Theory was limited to $D = 10$. But in 1985 one more problem appeared, there were five Superstring Theories instead of one.

1.2.2 Second Superstring Revolution.

The second superstring revolution began in 1994 and ended in 2003. In the early 90's the Edward Witten[9] discovered that the different Superstring Theories can be studied as different limits of an 11-dimensional theory. This theory was called *M-Theory*, which unifies the different String Theories and Supergravity by new equivalences. Supergravity is a field theory that combines the principle of supersymmetry and general relativity.

In 1995, Joseph Polchinski discovered the D-branes, which are higher-dimensional objects required to introduce cosmology in the String Theory.

1.3 Different String Theories and M-Theory

As we have seen, there are five different String Theories which include fermions, i.e, use supersymmetry. These are

- **Type I**: This type involve open and closed strings, and the spinor fields have the same chirality. This type is described by $N = 1$ supergravity in ten dimension. These strings are connected to D-branes.
- **Type II A**: This type only involve closed string. This type is described by $N = 2$ supergravity in ten dimension and where the vibrational patterns are symmetric, i.e is a non-chiral theory.
- **Type II B**: This strings are equal to Type II A, the only difference is that the vibrational patterns are antysimmetric, i.e is a chiral theory

There are another two types of Superstring Theories, the heterotic Superstring Theories which only include closed strings. The heterotic strings are hybrid of a superstring Type I and a bosonic string. The left-moving and the right moving of the heterotic string are detached and correspond to the Superstring Theory Type I and to the bosonic String Theory respectively. Observe that then there is a difference between the dimension of the left-moving ($D = 26$) and right moving ($D = 10$). This mismatched of 16 dimensions must be compactified,and there are only two ways to do this, which implies two different types of heterotic strings.

- **Heterotic SO(32)**: its gauge group is SO(32), which corresponds to the special orthogonal group of rotations in 32 internal dimensions
- **Heterotic $E_8 \times E_8$**: its gauge group is $E_8 \times E_8$, this group is described by the product of the group E_8 with itself. The group E_8 is Euclidean group of rotations and translations in 8 internal dimensions.

These Superstring theories live in $D = 10$ dimensional background spacetime . Furthermore, these 5 theories are related to each other by dualities. These findings gave rise to the second Superstring revolution.

As we discused in (1.2.2) M-Theory is a candidate to unify all the forces and combine the five Super String Theories with the supergravity, the problem is that M-Theory is not complete. This

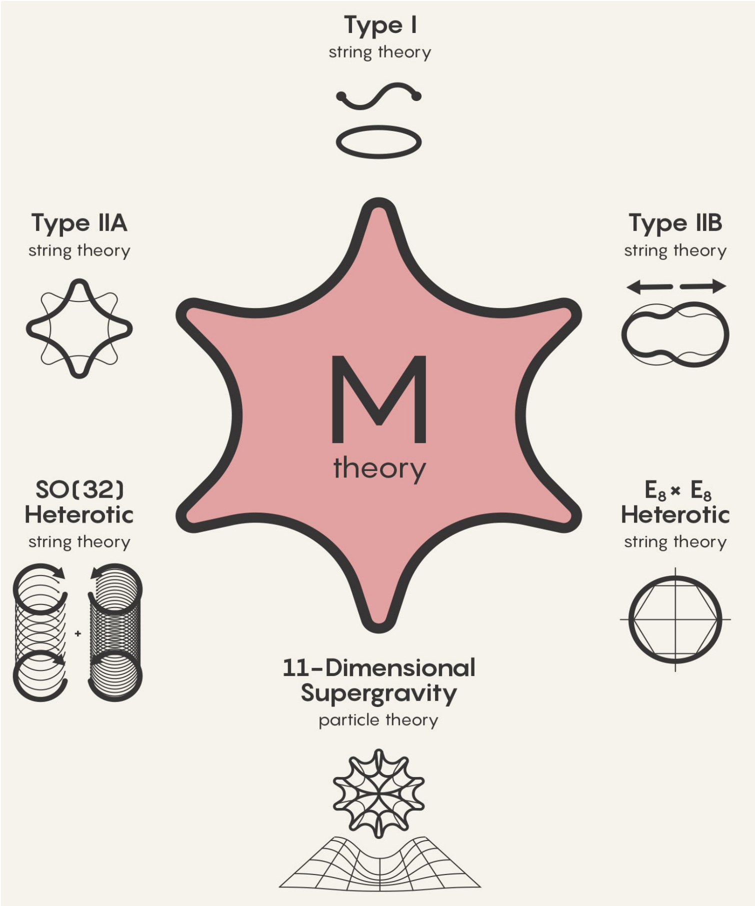

FIGURE 1.1: Relations between the different Superstring Theories and supergravity

theory unify the different String Theories using transformations called *S-Duality* and *T-Duality*. The *S-Duality* relates the Type I Superstring Theory with Heterotic SO(32) Superstring Theory and and type IIB theory with itself., while *T-Duality* relates type II A Superstring Theory with type II B Superstring Theory. In figure (1.1) we can see a diagram of these dualities.

1.4 Compactification

Every Superstring Theory needs 10 dimensions to obtain consistent theories (11 in the case of M-theory), but we only observe three space dimensions and one time dimension, How do these Superstring Theories explain this? The answer is that these extra dimensions are compactified. The compactification is the idea that these extra dimensions close up on themselves with very small scales, so we can not observe these extra dimensions.

This idea is not original to the Superstring Theory, the original one is the Kaluza-Klein[10] theory which imposes this compactification in order to unify the electromagnetism and the gravity. In fact this solution for the extra dimension is a extension of Kalusa theory.

Using this idea we can obtain models in which spacetime is 4-dimensional, but not every compactification procedure is valid. In order to obtain a consistent model of particle it is necessary to use a compactification according to Calabi-Yau manifold.

Chapter 2

The Bosonic String Action

2.0 Resumen

En este capítulo empezamos a tratar ya con la teoría de cuerdas bosónicas, donde empezamos considerando la acción de una partícula puntual relativista viendo bajo que reparametrizaciones es invariante. Para terminar acabaremos deduciendo la acción para una cuerda.

In this chapter we begin to deal with the theory of bosonic strings. We will start by considering the action of a relativistic point particle and we will study under which reparameterizations is this action invariant. Finally, we will expand to the string action.

2.1 Classical Action for Relativistic Point Particles

Suppose we have a relativistic point particle moving in a D dimensional spacetime, its action is given by the integral of the infinitesimal invariant length ds [11]

$$S_0 = -\alpha \int ds \tag{2.1}$$

where α is an arbitrary constant and we are working with natural units ($\hbar = c = 1$). In order to find the equations motion for the particle we must set the variation of the action equal to zero.

For S_0 to have dimension of an action α must have units of $Length^{-1}$, which means, in our set of units, that α is proportional to the mass of the particle, and we can, without loss generality, take this constant to be unity. Also, as we are working whit relativistic point particle the invariant distance square is given by

$$ds^2 = -g_{\mu\nu}(X) dX^\mu dX^\nu \tag{2.2}$$

where $g_{\mu\nu}$ $(\nu, \mu = 0, 1, ...D-1)$ is the metric tensor defining the geometry of the background spacetime in which our theory is defined. In this expression, and henceforth, we are using the Einstein notation. The minus sign in the expression (2.1) makes the extremum of S_0 a minimum.

Coming back to the metric, if we consider a $4D$ flat background of the spacetime, we obtain the Minkowski metric. Then our metric can be written as:

$$g_{\mu\nu} \to \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.3}$$

So the (2.1) action becomes

$$S_0 = -m \int \sqrt{-(dX^0)^2 + (dX^1)^2 + (dX^2)^2 + (dX^3)^2} \tag{2.4}$$

If we choose to parameterize the worldline of the particle $X^\mu(\tau)$ by some real parameter τ, we can rewrite (2.2) as

$$-g_{\mu\nu}(X) dX^\mu dX^\nu = -g_{\mu\nu}(X) \frac{dX^\mu(\tau)}{d\tau} \frac{dX^\mu(\tau)}{d\tau} d\tau^2 \tag{2.5}$$

Then, our action can be written also as

$$S_0 = -m \int d\tau \sqrt{-g_{\mu\nu}(X) \dot{X}^\mu \dot{X}^\nu} \tag{2.6}$$

where we used the notation $\dot{X}^\mu = \frac{dX^\mu}{d\tau}$.

This action is parametrization invariant, because the invariant length ds between two points on a particle's wordline does not depend on how the path is parameterized. The proof can be seen in appendix A.1.

We would like to construct an equivalent action which does not include a square root in its argument and valid for masless particles (notice that the action (2.6) is equal to zero for a massless particle). To solve this problem we must add an auxiliary field to the expression (2.6) and thereby construct an equivalent action which is simpler in nature. So, consider the action

$$\tilde{S}_0 = \frac{1}{2} \int d\tau \left[e(\tau)^{-1} \dot{X}^2 - m^2 e(\tau) \right] \tag{2.7}$$

where $\dot{X}^2 \equiv g_{\mu\nu} d\dot{X}^\mu d\dot{X}^\nu$ and $e(\tau)$ is a arbitrary auxiliary field. Note that this action is different from zero for a massless particle. We prove that this expression is equivalent to (2.6) in appendix A.2.

2.1.1 Reparameterization invariance of $\tilde{S}_0$

As S_0 in (2.6), the action $\tilde{S}_0$ is also invariant under a reparameterization of τ. Therfore, we have to prove that $\tilde{S}_0$ is invariant under the transformation

$$\tau' = f(\tau) \tag{2.8}$$

where $f(\tau)$ and $f^{-1}(\tau)$ are continuous and derivable functions. Then, we can affirm that

$$d\tau' = \frac{df}{d\tau} d\tau \tag{2.9}$$

Now, we start from $\tilde{S}'_0$ written in terms of τ' as

$$\tilde{S}'_0 = \frac{1}{2} \int d\tau' \left[e'(\tau')^{-1} (\dot{X}')^2 - m^2 e'(\tau') \right] \tag{2.10}$$

Taking into account eq.(2.9) we can calculate the changes in the fields

$$\dot{X}'^{\mu} = \frac{dX^{\mu}}{d\tau'} = \frac{dX^{\mu}}{d\tau} \left(\frac{d\tau}{d\tau'} \right)^{-1} = \dot{X}^2 \left(\frac{df}{d\tau} \right)^{-2} \tag{2.11}$$

And now we define $e(\tau)$ as

$$e(\tau) \equiv e'[f(\tau)] \left(\frac{df}{d\tau} \right) \tag{2.12}$$

Plugging eq.(2.11) and eq.(2.11) into the action (2.10) we obtain

$$\tilde{S}'_0 = \frac{1}{2} \int d\tau \left[e(\tau)^{-1} (\dot{X})^2 - m^2 e(\tau) \right] = S_0$$

Then, the action does not change under reparameterization. We use this invariance to set the auxiliary field equal to unity.

2.1.2 Variation of $\tilde{S}_0$ in an arbitrary background

As we have seen in the last section, we can choose the parameter τ in such a way that the auxiliary field $e(\tau)$ is equal to unity. Then we obtain that our action is given by

$$\tilde{S}_0 = \frac{1}{2} \int d\tau \left(g_{\mu\nu}(X) \dot{X}^{\mu} \dot{X}^{\nu} - m^2 \right) \tag{2.13}$$

The variation of $\tilde{S}_0$ with respect to $X^\mu(\tau)$ for a general metric (non-flat metric) is given by

$$\delta\tilde{S}_0 = \frac{1}{2}\int d\tau \left(g_{\mu\nu}(X)\delta\dot{X}^\mu\dot{X}^\nu + \partial_k g_{\mu\nu}(X)\dot{X}^\mu\dot{X}^\nu\delta\dot{X}^k\right)$$
$$\delta\tilde{S}_0 = \frac{1}{2}\int d\tau \left(-2\dot{X}^2\partial_k g_{\mu\nu}(X)\delta X^\mu\dot{X}^\nu - 2g_{\mu\nu}(X)\delta X\ddot{X}^\nu + \delta\dot{X}^k\partial_k g_{\mu\nu}(X)\dot{X}^\mu\dot{X}^\nu\right)$$
$$\delta\tilde{S}_0 = \frac{1}{2}\int d\tau \left(-2\ddot{X}^\nu g_{\mu\nu}(X) - 2\partial_k g_{\mu\nu}(X)\dot{X}^k\dot{X}^\nu + \partial_k g_{\mu\nu}(X)\dot{X}^\mu\dot{X}^\nu\right)\delta X^\mu \tag{2.14}$$

We take an arbitrary variation in the field $X^\mu(\tau)$, thus the term between brakets must be equal to zero

$$\left(-2\ddot{X}^\nu g_{\mu\nu}(X) - 2\partial_k g_{\mu\nu}(X)\dot{X}^k\dot{X}^\nu + \partial_k g_{\mu\nu}(X)\dot{X}^\mu\dot{X}^\nu\right) = 0 \tag{2.15}$$

Using the Christoffel symbols (Γ^μ_{kl}) the last expression can be rewritten as

$$\ddot{X}^\mu + \Gamma^\mu_{kl}\cdot X^k \cdot X^l = 0 \tag{2.16}$$

This equation describes the motion of a free particle moving in arbitrary background geometry,namely, this is the general equation for a relativistic free particle. In the flat Miknowski space the Christoffel symbols are equal to zero and we have the classic equation of a free particle ($\ddot{X} = 0$)

2.2 The String action

In the last section we have studied the action for a relativistic point particle, and now we are interested in the string action. This action is given by

$$S_1 = -T\int d\mu \tag{2.17}$$

where T is the tension and it units are $mass/length$ and μ is the surface element

$$d\mu = \sqrt{-det(G_{\alpha\beta}(X))}d\sigma d\tau \tag{2.18}$$

Where $G_{\alpha\beta}$ is the induced metric on the worldsurface or worldsheet, we can understand it as the projection of the background metric into the worldsurface.

$$G_{\alpha\beta}(X) = \frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X^\nu}{\partial\sigma^\beta}g_{\mu\nu} \qquad\qquad \alpha,\beta = 0,1. \tag{2.19}$$

with $\sigma^0 \equiv \tau$ while σ is the spacelike coordinate. The string action describes the motion of the string through a D dimensional spacetime. The embedding of the string into the D dimensional background spacetime is given by the fields $X^\mu(\tau,\sigma)$ (see figure 2.1). In the case where σ is periodic the embedding gives us a closed string. Those fields $X^\mu(\tau,\sigma)$, which are parameterized by the worldsheet coordinates, must tell us how the string propagates and oscillates through the background spacetime and this propagation defines the worldsheet.

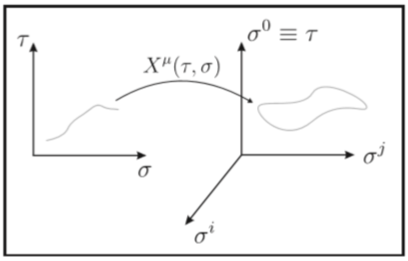

FIGURE 2.1: Embedding of a string into a background spacetime [2]

Now suppouse that our background space time corresponds to the Minkowski metric, then the elements of the induced metric are given by

$$
\begin{aligned}
G_{00} &= \frac{\partial X^\mu}{\partial \tau}\frac{\partial X^\nu}{\partial \tau}\eta_{\mu\nu} \equiv \dot{X}^2 \\
G_{11} &= \frac{\partial X^\mu}{\partial \sigma}\frac{\partial X^\nu}{\partial \sigma}\eta_{\mu\nu} \equiv X'^2 \\
G_{01} &= G_{10} = \frac{\partial X^\mu}{\partial \tau}\frac{\partial X^\nu}{\partial \sigma}\eta_{\mu\nu}
\end{aligned}
\tag{2.20}
$$

So, the induced metric can be written in matrix form as

$$
G_{\alpha\beta} = \begin{pmatrix} \dot{X}^2 & \dot{X}X' \\ \dot{X}X' & X'^2 \end{pmatrix} \tag{2.21}
$$

and its determinant is

$$
det(G_{\alpha\beta}) = (\dot{X}^2)(X'^2) - (\dot{X}X')^2 \tag{2.22}
$$

Therefore, using the expression (2.17) and (2.22), the action (2.18) becomes

$$
S_{NG} = -T\int d\tau d\sigma \sqrt{(\dot{X}^2)(X'^2) - (\dot{X}X')^2} \tag{2.23}
$$

which is known as the *Nambu-Goto* action. This action can be interpreted as giving the area of the worldsheet mapped out by the string spacetime. In order to obtain the equations of motion of the string we must minimize this action.

As we did before with the field $e(\tau)$, we introduce an auxiliary field $h_{\alpha\beta}(\tau,\sigma)$ in order to get rid of the square root. The resulting action is called Polyakov action and is given by

$$
S_\sigma = -\frac{T}{2}\int d\tau d\sigma \sqrt{-h}h^{\alpha\beta}\frac{\partial X^\mu}{\partial \alpha}\frac{\partial X^\nu}{\partial \beta}g_{\mu\nu} \tag{2.24}
$$

where $h = det(h_{\alpha\beta})$. Note that this expression holds for a general background space-time, since we have not reduced $G_{\alpha\beta}$ for a Minkowski spacetime. Also note that at the classical level, the Polyakov and the Nambu-Goto action are the same, but the first is better suited for quantization. The proof of the equality between Polyakov and Nambu-Goto actions is developed in appendix

A.3

Chapter 3

Symmetries and Field Equations for Bosonic Strings

3.0 Resumen

Una vez obtenida la acción de la cuerda estudiaremos sus simetras. Adems trataremos de obtener las ecuaciones de sus campos para su posterior resolución. Esto dejará ya a la teoría en disposición de ser cuantizada.

Already knowing the string action, we will study its symmetries. We will also try to obtain the field equations of the string for later resolve them. This leaves the theory in a position to be quantified.

3.1 Symmetries of the Polyakov action

In the last chapter, we have seen that the action which describes the string propagation in a D dimensional background space-time, with a given metric $g_{\mu\nu}$, can we written as

$$S_\sigma = -\frac{T}{2}\int d\tau d\sigma \sqrt{-h} h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu} \tag{3.1}$$

Now, in this section, we will look for some symmetries of the action (2.6). In general we can find two types of symmetries, global or local symmetries.

We can think that if Nambu-Goto and Polyakov actions are equivalent their symmetries will be the same, but this is not true. When we study the symmetries of the Polyakov action we observe extra symmetries compared to Nambu-Goto due to the presence of the auxilary field. The symmetries of the Polyakov action are:

- **Poincaré invariance** This is a global symmetry on the worldsheet. Poincaré symmetry is the full symmetry of special relativity, it includes invariance under space and time translation and under Lorentz transformations . Then, the action (3.1) is invariant under transformations with the form

$$X^\mu(\tau,\sigma) \to a^\mu_\nu X^\mu + b^\mu \tag{3.2}$$

where α^μ_ν correspond to spatial rotations and boosts.

- **Reparameterization invariance or diffeomorphism** This is a gauge symmetry on the worldsheet. The action (4.57) is invarant under the reparemeterization of the worldsheet coordinates as

$$\sigma \to \sigma'(\sigma) \tag{3.3}$$

and the fields and the metric transform respectively as

$$X^\mu(\tau,\sigma) \to X'^\mu(\tau,\sigma') = X^\mu(\tau,\sigma) \tag{3.4}$$

$$h_{\alpha\beta}(\tau,\sigma) \to h'_{\alpha\beta}(\tau,\sigma') = \frac{\partial\sigma^\gamma}{\partial\sigma'^\alpha}\frac{\partial\sigma^\delta}{\partial\sigma'^\beta}h_{\gamma\delta}(\tau,\sigma) \tag{3.5}$$

Sometime it is useful to work with infinitesimal transformations. In this case the coordinates change as

$$\sigma \to \sigma'(\sigma) = \sigma - \eta(\tau,\sigma) \tag{3.6}$$

for a small η. In this case the changes in the fields are

$$\delta X^\mu = \eta^\alpha \partial_\alpha X^\mu \tag{3.7}$$

$$\delta h_{\alpha\beta} = \nabla_\alpha \eta_\beta + \nabla_\beta \eta_\alpha \tag{3.8}$$

where the covariant derivative is defined by

$$\nabla_\alpha \eta_\beta = \partial_\alpha \eta_\beta - \Gamma^\sigma_{\alpha\beta}\eta_\alpha \tag{3.9}$$

And $\Gamma^\sigma_{\alpha\beta}$ are the Crhistoffel symbols

These symmetries are shared by Polyakov and Nambu-Goto actions, but there is also another symmetry which is exclusive to the Polyakov action.

- **Weyl Invariance**. Weyl transfromations are transformations that change the scale of the metric

$$h_{\alpha\beta}(\tau,\sigma) \to h'_{\alpha\beta}(\tau,\sigma) = e^{2\phi(\sigma)}h_{\alpha\beta}(\tau,\sigma) \tag{3.10}$$

while the fields $X^\mu(\tau,\sigma)$ are invariant under a Weyl transformation, it means, that $\delta X^\mu(\tau,\sigma) = 0$. Note that this is a local transformation since the parameter $\phi(\sigma)$ depends on the wolrdsheet coordinates. The figure (3.1) gives us a view of the Weyl transformation. To see

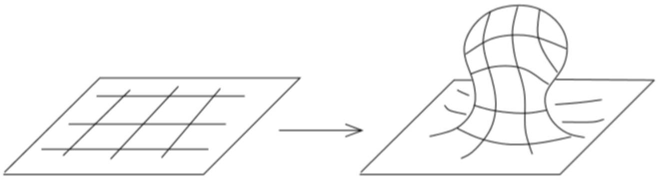

FIGURE 3.1: An example of a Weyl transfromation [2]

whether the Polyakov action is invariant under a Weyl tansfromation we need to know how the quantity $\sqrt{-h}h^{\alpha\beta}$ changes whit it . The transformation of $\sqrt{-h}$ is given by

$$\begin{aligned} \sqrt{-h'} &= \sqrt{-det(h_{\alpha\beta'})} \\ &= \sqrt{-det(h_{\alpha\beta})e^{2(2\phi(\sigma))}} \\ &= e^{2\phi(\sigma)}\sqrt{-det(h_{\alpha\beta})} \end{aligned} \tag{3.11}$$

If we raise both indexes in expression (3.10), the transformation of $h^{\alpha\beta}$ has the form

$$h'^{\alpha\beta}(\tau,\sigma) = e^{-2\phi(\sigma)}h^{\alpha\beta}(\tau,\sigma) \tag{3.12}$$

Then

$$\sqrt{-h'}h'^{\alpha\beta} = e^{-2\phi(\sigma)}e^{2\phi(\sigma)}\sqrt{-h}h^{\alpha\beta} = \sqrt{-h}h^{\alpha\beta} \tag{3.13}$$

Thus, under a Weyl transformations S_σ, does not change, so our bosonic string theory is invariant under Weyl transformation. This means that two metrics which are related by a Weyl transformation can be considered the same physical state. As our bosonic string theory is invariant under Weyl transformations, this implies that the stress-energy tensor associated with this theory is traceless. In appendix B.1 it is proved that the stress-energy tensor is not only traceless but also has all components equal to zero.

Now we know that our theory has local symmetries, and this implies that the theory has a redundancy in its degrees of freedom. We will use this symmetries to cope with this redundancy this is known as gauge fixing. As our theory is invariant under diffeomorphisms and Weyl transformations we can fix a gauge in which our metric $(h_{\alpha\beta})$ becomes flat. Recall that our metric is given by

$$h_{\mu\nu} = \begin{pmatrix} h_{00} & h_{01} \\ h_{10} & h_{11} \end{pmatrix} \tag{3.14}$$

and is symmetric, so it only has three independent components $(h_{00}, h_{11}, h_{10} = h_{01})$. We can do a reparameterization in the components by using two worldsheet coordinate transformations , $f_1(X)$ and $f_2(X)$, to set $h_{10} = 0 = h_{01}$ and $h_{11} = h_{22}$. Thus, through diffeomorphisms our metric can be written as $h_{\alpha\beta}(X) = h(X)\eta_{\alpha\beta}$. Now, we apply a Weyl transformation to remove this function to obtain

$$h_{\alpha\beta} = \eta_{\alpha\beta} \tag{3.15}$$

The combination of diffeomorphisms and Weyl transformations is called *conformal transformation*, and our theory is invariant under them, therefore our metric is flat, i.e,

$$h_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.16}$$

Thereupon, the Polyakov action becomes

$$S_\sigma = \frac{T}{2} \int d\tau d\sigma \left((\dot{X})^2 - (X')^2 \right) \tag{3.17}$$

where $\dot{X} \equiv dX^\mu/d\tau$ and $X' \equiv dX^\mu/d\sigma$. Note that transforming the metric $h_{\alpha\beta}$ into a flat metric is in general possible only locally. Only in the case that the Euler characteristic of the worldsheet is equal to zero the metric $h_{\alpha\beta}$ can be extended to a global flat metric on the worldsheet.

3.2 Field Equations for the Polyakov Action

At the moment, suppose that we are in the case where the flat metric $h_{\alpha\beta}$ can be extended globally. Now, in order to get the field equations for $X^\mu(\tau,\sigma)$, we have to impose that the variation of S_σ with respect to $X^\mu \to X^\mu + \delta X^\mu$ be equal to zero. This variation is given by

$$\delta S_\sigma = \frac{T}{2} \int d\tau d\sigma \left(2\dot{X}\delta\dot{X} - 2X'\delta X' \right) \tag{3.18}$$

Integrating by parts and setting to zero

$$T \int d\tau d\sigma \left(-\partial_\tau^2 + \partial_\sigma^2\right) X^\mu \delta X^\mu + T \int d\sigma \dot{X}\delta\dot{X}^\mu \mid_{\delta\tau} - \left[T \int d\tau X'\delta X' \mid_{\sigma=\pi} - T \int d\tau d\sigma \mid_{\sigma=0} \right] = 0 \tag{3.19}$$

We set the variation of X^μ at the boundary of τ to be zero $(\delta X^\mu)\mid_{\delta\tau} = 0$, so we have

$$\left(-\partial_\tau^2 + \partial_\sigma^2\right) X^\mu - T \int d\tau \left[X'\delta X^\mu \mid_{\sigma=\pi} - X'\delta X^\mu \mid_{\sigma=0} \right] = 0 \tag{3.20}$$

There are two types of strings depending on the way of cancellation of the σ boundary terms, closed and open strings. For the open strings we will have two different cases.

- **Closed Strings**: In this case, we impose to the fields $X^\mu(\tau,\sigma)$ the following σ periodic boundary conditions

$$X^\mu(\tau, \sigma + n) = X^\mu(\tau, \sigma) \tag{3.21}$$

This boundary conditions implies that $\delta X(\tau, \sigma = 0) = \delta X(\tau, \sigma = n)$, and since these terms are being subtracted in (3.20) the resultant field equation for closed string is

$$(-\partial_\tau^2 + \partial_\sigma^2) X^\mu(\tau, \sigma) = 0 \tag{3.22}$$

- **Open Strings (Neumann Boundary Conditions):** These boundary conditions based o the idea that both endpoints of the string move equals, then we set the derivative of the field X^μ with respect to σ must satisfy

$$\partial_\sigma X^\mu(\tau, \sigma) = \partial_\sigma X^\mu(\tau, \sigma + n) \tag{3.23}$$

Under this boundary conditions the boundary terms over σ also vanish and we get again

$$(-\partial_\tau^2 + \partial_\sigma^2) X^\mu(\tau, \sigma) = 0 \tag{3.24}$$

In the figure (3.2) we can see more easily the meaning of the Neumann boundary conditions

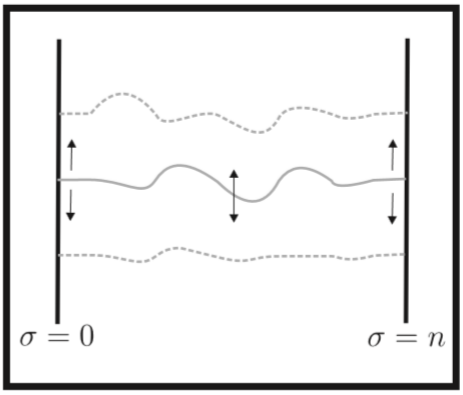

FIGURE 3.2: Neumann conditions: the string can oscillate while its endpoints move along the boundaries as long as their derivatives vanish at the boundaries [2]

- **Open Strings (Dirichlet Boundary Conditions):** In this case we set that the values of $X^\mu(\tau, \sigma)$ and $X^\mu(\tau, \sigma + n)$ are constant as we can see in the figure (3.3) This implies that

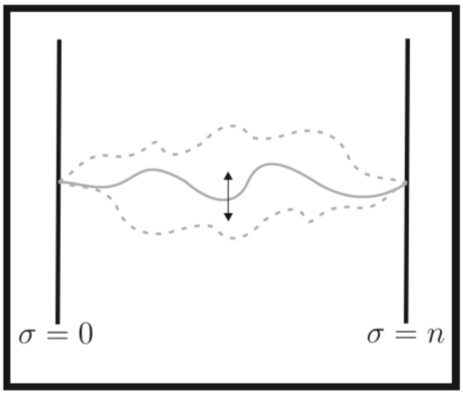

FIGURE 3.3: Dirichlet conditions: the string can oscillate while its endpoints are fixed at the boundary [2]

the σ boundary terms vanish again and we obtain

$$(-\partial_\tau^2 + \partial_\sigma^2)X^\mu(\tau,\sigma) = 0 \tag{3.25}$$

Thus, we observe that in the three cases we obtain the same field equations which is a free wave equation.

$$(-\partial_\tau^2 + \partial_\sigma^2)X^\mu(\tau,\sigma) = 0 \tag{3.26}$$

The only differences between the three cases are the boundary conditions. In addition to this expression, we can impose that our action S_σ be invariant under variation of $h^{\alpha\beta}$ and this invariance imposes that all the components of the energy-stress tensor are equal to zero (see (B.1))

$$T_{\alpha\beta} = 0 \tag{3.27}$$

The energy-stress tensor can be written as

$$T_{\alpha\beta} = \partial_\alpha X \cdot \partial_\beta X - \frac{1}{2}h_{\alpha\beta}h^{\gamma\delta}\partial_\gamma X \cdot \partial_\delta X \tag{3.28}$$

And since we have chosen a gauge where our metric is flat (3.16), the field equations imply

$$T_{00} = T_{11} = \frac{1}{2}(\dot{X}^2 + X'^2) = 0 \tag{3.29}$$

$$T_{10} = T_{01} = \dot{X} \cdot X' = 0 \tag{3.30}$$

3.3 Solving the Field Equations

Now that we know the field equations let us solve them. For this purpose (assuming that we can extend the local flat metric to a global flat metric on the worldsheet) we introduce the the light cone coordinates

$$\sigma^\pm = \tau \pm \sigma \tag{3.31}$$

So our coordinates σ and τ can be expressed as function of our light-cone coordinates

$$\sigma = \frac{1}{2}(\sigma^+ - \sigma^-) \tag{3.32}$$

$$\tau = \frac{1}{2}(\sigma^+ + \sigma^-) \tag{3.33}$$

Thus, the derivatives with respect to light-cone coordinates are given by

$$\partial_+ \equiv \frac{\partial}{\partial\sigma^+} = \frac{\partial\tau}{\partial\sigma^+}\frac{\partial}{\partial\tau} + \frac{\partial\sigma}{\partial\sigma^+}\frac{\partial}{\partial\sigma} = \frac{1}{2}(\partial_\tau + \partial_\sigma) \tag{3.34}$$

$$\partial_- \equiv \frac{\partial}{\partial\sigma^-} = \frac{\partial\tau}{\partial\sigma^-}\frac{\partial}{\partial\tau} + \frac{\partial\sigma}{\partial\sigma^-}\frac{\partial}{\partial\sigma} = \frac{1}{2}(\partial_\tau - \partial_\sigma) \tag{3.35}$$

and since the metric changes as

$$\eta'_{\alpha'\beta'} = \frac{\partial \sigma^{\gamma}}{\partial \sigma^{\alpha'}} \frac{\partial \sigma^{\delta}}{\partial \sigma^{\beta'}} \eta_{\gamma\delta} \tag{3.36}$$

then we have

$$\eta_{++} = -\left(\frac{\partial \tau}{\partial \sigma^{+}}\right)^2 + \left(\frac{\partial \sigma}{\partial \sigma^{+}}\right)^2 = -\frac{1}{4} + \frac{1}{4} = 0 \tag{3.37}$$

$$\eta_{--} = -\left(\frac{\partial \tau}{\partial \sigma^{-}}\right)^2 + \left(\frac{\partial \sigma}{\partial \sigma^{-}}\right)^2 = -\frac{1}{4} + \frac{1}{4} = 0 \tag{3.38}$$

$$\eta_{-+} = -\frac{\partial \tau}{\partial \sigma^{-}} \frac{\partial \tau}{\partial \sigma^{+}} + \frac{\partial \sigma}{\partial \sigma^{-}} \frac{\partial \sigma}{\partial \sigma^{+}} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2} \tag{3.39}$$

$$\eta_{+-} = -\frac{\partial \tau}{\partial \sigma^{+}} \frac{\partial \tau}{\partial \sigma^{-}} + \frac{\partial \sigma}{\partial \sigma^{+}} \frac{\partial \sigma}{\partial \sigma^{-}} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2} \tag{3.40}$$

In terms of light-cone coordinates the metric is given by

$$\eta_{\alpha\beta} \mid_{l.cc} = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.41}$$

and the field equations (3.26) become

$$\partial_{+} \partial_{-} X^{\mu} = 0 \tag{3.42}$$

while the field equations for the intrinsic wolrdsheet metric $h_{\alpha\beta}$ become

$$T_{++} = \partial_{+} X^{\mu} \partial_{+} X_{\mu} = 0 \tag{3.43}$$

$$T_{--} = \partial_{-} X^{\mu} \partial_{-} X_{\mu} = 0 \tag{3.44}$$

The solution to equation (3.42) can we expressed as a linear combination of two functions which depend only in one of the light-cone coordinate

$$X^{\mu}(\sigma^{+}, \sigma^{-}) = X^{\mu}_{R}(\sigma^{-}) + X^{\mu}_{L}(\sigma^{+}) \tag{3.45}$$

Coming back to the worldsheet coordinates, this solution is given by

$$X^{\mu}(\tau, \sigma) = X^{\mu}_{R}(\tau - \sigma) + X^{\mu}_{L}(\tau + \sigma) \tag{3.46}$$

Note that the arbitrary functions X^{μ}_{R} and X^{μ}_{L} correspond to right and left moving waves which propagate through space at the speed of light. Now we apply the boundary conditions.

- Closed Strings: Applying the boundary conditions $X^\mu(\tau,\sigma) = X^\mu(\tau,\sigma+n)$, the particular solutions for the left and the right movers are given by

$$X_R^\mu = \frac{1}{2}x^\mu + \frac{1}{2}l_s^2(\tau-\sigma)p^\mu + \frac{i}{2}l_s\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-2in(\tau-\sigma)} \tag{3.47}$$

$$X_L^\mu = \frac{1}{2}x^\mu + \frac{1}{2}l_s^2(\tau+\sigma)p^\mu + \frac{i}{2}l_s\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^\mu e^{-2in(\tau+\sigma)} \tag{3.48}$$

where x^μ is a constant called the center of mass of the string, p^μ is a constant which corresponds to the total momentum of the string and l_s is the length of the string, also a constant. The tension T of the string can be expressed as an function of the length l_s ($T = 1/\pi l_s^2$). Then the general solution X^μ is given by an expansion in Fourier modes

$$X^\mu = \underbrace{x^\mu + \frac{1}{2}l_s^2 p^\mu\tau}_{\substack{\text{center of mass}\\ \text{motion of the string}}} + \underbrace{\frac{i}{2}l_s\sum_{n\neq 0}\frac{1}{n}(\alpha_n^\mu e^{2in\sigma} + \tilde{\alpha}_n^\mu e^{-2in\sigma})e^{-2in\tau}}_{\text{oscillations of the string}} \tag{3.49}$$

and satisfies the boundary conditions since the first two terms do not depend on σ and the second part is periodic in σ. We Also, note that the two first terms look like the trajectory of a relativistic free point particle satisfying

$$\frac{d^2}{d\tau^2}X^\mu = 0 \to X^\mu = x^\mu + p^\mu\tau \tag{3.50}$$

which we will call the center of mass (c.o.m) of the string, while the summation part looks like an oscillatory term due to the modes α and $\tilde{\alpha}$. So, the string moves throughout space-time via the first part and oscillates via the second part.

The field X^μ must be real. This means that $X^\mu = (X^\mu)^*$, then necessarily x^μ and p^μ are real and the oscillators must fulfill

$$\alpha_{-n}^\mu = (\alpha_n^\mu)^* \tag{3.51}$$

$$\tilde{\alpha}_{-n}^\mu = (\tilde{\alpha}_n^\mu)^* \tag{3.52}$$

In addition, from the definition of the canonical momentum

$$P^\mu(\tau) = \frac{\partial L}{\partial \dot{X}^\mu} \tag{3.53}$$

we can see that the mode expansion of the canonical momentum of the worldsheet is given by

$$\begin{aligned} P^\mu(\tau,\sigma) &= T\dot{X}^\mu = \frac{\dot{X}^\mu}{\pi l_s^2} \\ &= \frac{p^\mu}{\pi} + \frac{1}{\pi l_s}\sum_{n\neq 0}(\alpha_n^\mu e^{-2in(\tau-\sigma)} + \tilde{\alpha}_n^\mu e^{-2in(\tau+\sigma)}) \end{aligned} \tag{3.54}$$

Later on, we will see that the canonical momentum is the 0th component of the conserved current corresponding to a translational symmetry. The field and its canonical momentum satisfy the following Poission bracket relations.

$$\left\{P^\mu(\tau,\sigma), P^\nu(\tau,\sigma')\right\}_{P.B} = 0 \tag{3.55}$$
$$\left\{X^\mu(\tau,\sigma), X^\nu(\tau,\sigma')\right\}_{P.B} = 0 \tag{3.56}$$
$$\left\{P^\mu(\tau,\sigma), X^\nu(\tau,\sigma')\right\}_{P.B} = \eta^{\mu\nu}\delta(\sigma-\sigma') \tag{3.57}$$
$$\tag{3.58}$$

It can be shown that the terms α_n^μ, $\tilde{\alpha}_n^\mu$, x^μ and p^μ satisfy also a Possion bracket relations.

$$\{\alpha_m^\mu, \alpha_n^\nu\}_{P.B} = \{\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu\}_{P.B} = imn\eta^{\mu\nu}\delta_{m,-n} \tag{3.59}$$
$$\{p^\mu, x^\mu\}_{P.B} = \eta^{\mu\nu} \tag{3.60}$$

We have solved the field equations for the closed string boundary conditions, now we must do the same for the two types of open string boundary conditions.

- Open String (Neumann Boundary Conditions): in this case, recall that the σ boundary condition is $\partial_\sigma X^\mu(\tau,\sigma)\mid_{\sigma=0,\pi} = 0$. Then the general solution to the field equations is

$$X^\mu(\tau,\sigma) = a_0 + a_1\sigma + a_2\tau + a_3\sigma\tau + \sum_{k\neq 0}\left(b_k^\mu e^{ik\sigma}\tilde{b}_k^\mu e^{-ik\sigma}\right)e^{-ik\tau} \tag{3.61}$$

  where $a_i(i=1,2,3)$, b_k and $\tilde{b}_k$ are constant and the only constraint on k in the summation is that it cannot be zero. Now when we apply the Neumman boundary conditions we get the following specific solution

$$X^\mu(\tau,\sigma) = x^\mu + l_s\tau p^\mu + il_s\sum_{m\neq 0}\frac{1}{m}\alpha_m^\mu e^{-im\tau}cos(m\sigma) \tag{3.62}$$

  where we have introduced new constants to make it look like the closed string mode expansion

- Open string (Dirichlet Boundary Conditions): This boundary conditions assume a constant value of the field at the σ boundary, it means that $X^\mu(\tau,\sigma=0)$ and $X^\mu(\tau,\sigma=\pi)$ are constant. The solution to the field equation obeying the Dirichlet boundary conditions is given by

$$X^\mu(\tau,\sigma) = x\mu_0 + \frac{\sigma}{\pi}(x_\pi^\mu - x_0^\mu) + \sum_{m\neq 0}\frac{1}{m}\alpha_m^\mu e^{im\tau}sin(m\sigma) \tag{3.63}$$

Chapter 4

Canonical Quantization

4.0 Resumen

Sabiendo ya como son las ecuaciones del campo de la cuerda prodecemos a cuantizar la teroía utilizando la cuantización canónica. En esta parte del trabajo encontraremos algunos inconvenientes que serán necesarios resolver para obetener una teoría consistente, como lo son los estados de norma negativa y la dificultad de eliminar estos de la teoría. Veremos además por primera vez la relación de la teoría con las dimensiones extras.

Knowing the field equation, we are able to quantize the theory using canonical quantization method. Here we will find some problems which must be solved in order to obtain a consistent theory, such as the negative norm states and the difficulty of removing them from the theory. Also, we will see that the solutions to these problems impose extra dimensions in our theory.

4.1 The Classical Hamiltonian and Energy-Momentum Tensor

The worldsheet time evolution of our theory is generated by the following Hamiltonian [12]

$$H = \int_{\sigma=0}^{\sigma=\pi} d\sigma \left(\dot{X}^\mu P_\mu - \mathcal{L} \right) \tag{4.1}$$

where P^μ is the canonical momentum defined in the equation (3.53) and $\mathcal{L}$ is the Lagrangian. In our case, the bosonic string theory, we have that the canonical momentum is given by $P^\mu = T\dot{X}^\mu$ while the Lagrangian is $\mathcal{L} = \frac{1}{2}T(\dot{X}^2 - X'^2)$. In consideration of these expression, is easy to get the following form to our Hamiltonian

$$H = \frac{T}{2} \int_0^\pi d\sigma \left(\dot{X}^2 + X'^2 \right) \tag{4.2}$$

This is the bosonic string theory Hamiltonian, this is valid for open and closed strings. By substituting the corresponding mode expansions for the fields we obtain in the case of closed strings the

Hamiltonian

$$H = \sum_{n=\infty}^{\infty} (\alpha_{-n} \cdot \alpha_n + \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n) \tag{4.3}$$

Observe that we do not have super-indices since we are dealing with dot products and also we have defined $\alpha_0^\mu = \tilde{\alpha}_0^\mu = (1/2) l_s p^\mu$. While, for open string we have

$$H = \frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha_{-n} \cdot \alpha_n \tag{4.4}$$

where $\alpha_0^\mu = \tilde{\alpha}_0^\mu = l_s p^\mu$. In both cases our Hamiltonian is conserved since it does not depend on time explicitly.

We have seen that it is possible to write our Hamiltonian as a mode expansion. Now we will see how to do the same thing with the stress-energy tensor. We are only going to do so for closed string because for open string the procedure is analogous. We already know that the components of the stress-energy tensor, in light-cone coordinates, are given by

$$T_{\pm\pm} = \partial_\pm X^\mu \partial_\pm X_\mu \tag{4.5}$$
$$T_{+-} = T_{-+} = 0 \tag{4.6}$$

If we take the mode expansion for a closed string and we plug it in eq.(4.5) we obtain

$$\begin{aligned} T_{--} &= \partial_- X^\mu \partial_- X_\mu = \partial_- X^\mu_{(R)} \partial_- X_{\mu(R)} = \\ &= l_s^2 \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \alpha_{m-n} \cdot \alpha_n e^{-2im(\tau-\sigma)} = 2 l_s^2 \sum_{m=-\infty} L_m e^{-2im(\tau-\sigma)} \end{aligned} \tag{4.7}$$

In the same way, for T_{++} we obtain

$$T_{++} = 2 l_s^2 \sum_{m=-\infty} \tilde{L}_m e^{-2im(\tau-\sigma)} \tag{4.8}$$

where we have defined L_m and $\tilde{L}_m$ as

$$L_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha_{m-n} \cdot \alpha_n \quad ; \quad \tilde{L}_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} \tilde{\alpha}_{m-n} \cdot \tilde{\alpha}_n \tag{4.9}$$

Note that we can use this new defined quantities L_m and $\tilde{L}_m$ to rewrite our Hamiltonian. For closed strings we then have

$$H = 2(L_0 + \tilde{L}_0) \tag{4.10}$$

while for an open string we get

$$H = L_0 \tag{4.11}$$

The set of these quantities $\{L_m\}$ forms an algebra, the Witt algebra. In section (4.4) we will study the Virasoro algebra, which is the quantum analogue.
Recall that all these expressions are classical ones and they must be modified when we quantize our bosonic string theory.

4.2 Classical Mass Formula for Bosonic Strings

We have studied in earlier sections that classically all the components of the stress-energy tensor vanish which implies $L_m = \tilde{L}_m = 0$ for all possible values of m. As is known, the mass-energy relation is given by [13]

$$M^2 = -p^\mu p_\mu \tag{4.12}$$

For our bosonic string theory we have

$$p^\mu = \int_0^\pi d\sigma P^\mu = T\int_0^\pi \dot{X}^\mu = \begin{cases} \frac{2\alpha_0^\mu}{l_s}\text{for a closed string} \\ \frac{\alpha_0^\mu}{l_s}\text{for an open string} \end{cases} \tag{4.13}$$

Observe that only the zero mode gives a non vanishing integral (recall that we defined $\alpha_0^\mu = \tilde{\alpha}_0^\mu = \frac{1}{2}l_s p^\mu$ for closed string and $\alpha_0^\mu = \tilde{\alpha}_0^\mu = l_s p^\mu$ for open string) . Then

$$p^\mu p_\mu = \begin{cases} \frac{2\alpha_0^2}{\alpha'} \\ \frac{\alpha_0^2}{2\alpha'} \end{cases} \tag{4.14}$$

where $\alpha' = l_s^2/2$. Using this, for open strings we have

$$L_0 = \frac{1}{2}\sum_{n=-\infty}^{\infty} \alpha_{-n}\cdot\alpha_n = \frac{1}{2}\left(\sum_{n=-\infty}^{n=-1} \alpha_{-n}\cdot\alpha_n + \sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n\right) + \underbrace{\frac{1}{2}\alpha_0^2}_{=\alpha' p^\mu p_\mu} =$$

$$= \frac{1}{2}\left(\sum_{m=1}^{m=\infty}\alpha_m\cdot\alpha_{-m} + \sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n\right) + \alpha' \underbrace{p^\mu p_\mu}_{-M^2} \frac{1}{2}\left(\sum_{n=1}^{n=\infty}\alpha_{-n}\cdot\alpha_n + \sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n\right) - \alpha' M^2 \rightarrow$$

$$L_0 = \left(\sum_{n=1}^{n=\infty}\alpha_{-n}\cdot\alpha_n\right) - \alpha' M^2 \tag{4.15}$$

As we have seen $L_0 = 0$, then for an open string we have from eq.(4.15)

$$M^2 = \frac{1}{\alpha'}\sum_{n=1}^{n=\infty}\alpha_{-n}\cdot\alpha_n \tag{4.16}$$

Recall that in the closed string case we have left and right movers, then the condition to take into account is $L_m = \tilde{L}_m = 0$. Therefore, for closed string the mass formula is

$$M^2 = \frac{2}{\alpha'} \sum_{n=1}^{n=\infty} \alpha_{-n} \cdot \alpha_n + \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n \tag{4.17}$$

All this expressions are only valid classically, then in the quantization they could change.

4.3 Canonical Quantization of Bosonic Strings

There are two methods to quantize our bosonic string theory, using the *canonical quantization* or the *light-cone gauge quantization*, in this section we will use the first one. In the canonical quantization procedure, we quantize the theory by changing Possion brackets into commutators.

$$\{\cdot, \cdot\}_{P.B} \to i[\cdot, \cdot]$$

Also we promote the fields X^μ to operators in our corresponding Hilbert space, or, in other words, promote the modes α, the constant x^μ and the total momentum p^μ to operators. In particular, for the modes α_m^μ we have

$$\begin{aligned}
[\hat{\alpha}_m^\mu, \hat{\alpha}_n^\nu] &= im\eta^{\mu\nu}\delta_{m,-n} \\
[\hat{\tilde{\alpha}}_m^\mu, \hat{\tilde{\alpha}}_n^\nu] &= im\eta^{\mu\nu}\delta_{m,-n} \\
[\hat{\alpha}_m^\mu, \hat{\tilde{\alpha}}_n^\nu] &= 0
\end{aligned} \tag{4.18}$$

where the $\hat{\alpha}'s$ are operators in the Hilbert space. We can redefine these operators as $\hat{a}_m^\mu \equiv \frac{1}{\sqrt{m}}\hat{\alpha}_m^\mu$ and $\hat{a}_m^{\mu\dagger} \equiv \frac{1}{\sqrt{m}}\hat{\alpha}_{-m}^\mu$, so the commutators can we written as

$$[\hat{a}_m^\mu, \hat{a}_n^{\nu\dagger}] = [\hat{\tilde{a}}_m^\mu, \hat{\tilde{a}}_n^{\nu\dagger}] = \eta^{\mu\nu}\delta_{m,n} \tag{4.19}$$

This commutators remind us of the algebra constructed by the creation and annihilation operators of the harmonic oscillator, except that for $\mu = \nu = 0$ we obtain a negative sign

$$[\hat{a}_m^0, \hat{a}_n^{0\dagger}] = -\delta_{m,n} \tag{4.20}$$

This negative sign in the commutators predicts physical states with negative norm, or ghost states, which have no sense.

To get started, we define the ground state $|0\rangle$ of our Hilbert space as the state which is annihilated by all operators $\hat{\alpha}_m^\mu$

$$\hat{\alpha}_m^\mu |0\rangle = 0 \qquad \forall\ m\ > 0 \tag{4.21}$$

The states of our Hilbert space can be expressed by acting on the ground state with creation operators $\hat{\alpha}_m^{\mu\dagger}$

$$|\phi\rangle = \hat{\alpha}_{m_1}^{\mu_1\dagger}\hat{\alpha}_{m_2}^{\mu_2\dagger}\dots\hat{\alpha}_{m_n}^{\mu_n\dagger}|0;k^\mu\rangle \tag{4.22}$$

which are also eigenstates of the momentum operator $\hat{p}^\mu$

$$\hat{p}^\mu|\phi\rangle = k^\mu|\phi\rangle \tag{4.23}$$

Now, according to our definition of physical state, we can consider the physical state $|\varphi\rangle = \hat{\alpha}_m^{0\dagger}|0;k^\mu\rangle$ $(m > 0)$, then its norm is given by

$$\begin{aligned}\langle\varphi|\varphi\rangle &= \langle 0;k^\mu|\,\hat{\alpha}_m^0\hat{\alpha}_m^{0\dagger}\,|0;k^\mu\rangle = \langle 0;k^\mu|\,\hat{\alpha}_m^{0\dagger}\hat{\alpha}_m^0 + [\hat{\alpha}_m^0,\hat{\alpha}_m^{0\dagger}]\,|0;k^\mu\rangle \\ &= \langle 0;k^\mu|\,\hat{\alpha}_m^{0\dagger}\hat{\alpha}_m^0\,|0;k^\mu\rangle + \langle 0;k^\mu|\,[\hat{\alpha}_m^0,\hat{\alpha}_m^{0\dagger}]\,|0;k^\mu\rangle\end{aligned} \tag{4.24}$$

We have seen that for any value $m \geq 0$ it is fulfilled that $\hat{\alpha}_m^\mu|0\rangle = 0$ and that $[\hat{\alpha}_m^0,\hat{\alpha}_n^{0\dagger}] = -\delta_{m,n}$, so

$$\langle\varphi|\varphi\rangle = -\langle 0|0\rangle \tag{4.25}$$

This relation implies the existence of negative norm states, either $|\varphi\rangle$ or $|0\rangle$. These states with negative norm have not physical sense and must be eliminated from our theory. We will see that those unphysical states can be removed, but this will put a constraint on the dimensions of the background spacetime in which our theory is defined.

4.4 Virasoro Algebra

When we quantize our theory the modes α become operators and this implies that the quantities L_m will also become operators. These operators will conform an algebra, the Virasoro algebra. This algebra is a complex Lie algebra and the unique central extension of the Witt algebra. When we write the corresponding operator we must be careful, because now we are working with operators and it is important take into account the commutator relations. This means that we must choose an operator ordering to define $\hat{L}_m$. Our order will be normal ordering, i.e,

$$\hat{L}_m = \frac{1}{2}\sum_{n=-\infty}^{\infty} :\hat{\alpha}_{m-n}\cdot\hat{\alpha}_n: \tag{4.26}$$

where " : ... : " means normal ordering, i.e all creation operator are written to the right of all annihilation operators. For $\hat{L}_0$ we readily obtain

$$L_0 = \frac{1}{2}\hat{\alpha}_0^2 + \sum_{n=1}^{\infty}\hat{\alpha}_{-n}\cdot\hat{\alpha}_n \tag{4.27}$$

Note that $\hat{L}_0$ is the only operator where the normal ordering is relevant. Then, due to normal ordering, we expect to pick up an extra constant due to moving creation modes to the left of annihilation modes. Therefore, $\hat{L}_0$ is defined by equation (4.27) except for this extra constant.

It can be shown that these operators satisfy the following hermiticity property (see appendix (C.1))

$$\hat{L}_m^\dagger = \hat{L}_{-m} \tag{4.28}$$

Knowing the commutator relations for the operators $\hat{\alpha}$, and since $\hat{L}_m$ are defined as functions of these operators, we can calculate the corresponding commutator relations for $\hat{L}_m$. This commutator relations are given by [1]

$$[\hat{L}_m, \hat{L}_n] = (m-n)\hat{L}_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m,-n} \tag{4.29}$$

where c is called the central charge, which is equal to the dimension (D) of the spacetime where the theory lives. Observe that for $m = 0, \pm 1$, the term proportional to c vanishes. Then the set $\{\hat{L}_{-1}, \hat{L}_0, \hat{L}_1\}$ defines a subalgebra with commutator relations

$$[\hat{L}_m, \hat{L}_n] = (m-n)\hat{L}_{m+n} \quad ; m, n = 0, \pm 1 \tag{4.30}$$

4.5 Physical States

We know that classically $L_m = 0$ since the stress-energy tensor vanishes, but when we quantize the theory we cannot confirm that the equivalent expression $\hat{L}\,|\phi\rangle = 0$ is satisfied for all physical states. At the moment of quantizing the theory we have taken the normal order, so the operator $\hat{L}_0$ could have some arbitrary constant due to this normal ordering. Then, we can say that for open string the vanishing of L_0 constraint transform to

$$(\hat{L}_0 - a)\,|\phi\rangle = 0 \tag{4.31}$$

where a is an arbitrary constant. This equation is known as the mass-shell condition for the open string. Analogously, for closed strings we have $\hat{L}_0$ and $\hat{\tilde{L}}_0$, therefore

$$(\hat{L}_0 - a)\,|\phi\rangle = 0 \tag{4.32}$$

$$(\hat{\tilde{L}}_0 - a)\,|\phi\rangle = 0 \tag{4.33}$$

where $\hat{\tilde{L}}_0$ is the operator corresponding to the classical generator $\tilde{L}$. The normal order also might add correction terms to the mass formula, for open strings

$$\alpha' M^2 = \sum_{n=1}^{\infty} :\hat{\alpha}_{-n}\cdot\hat{\alpha}_n: - a = \hat{N} - a \tag{4.34}$$

where $\hat{N}$ is the number operator defined as

$$\hat{N} = \sum_{n=1}^{\infty} : \hat{\alpha}_{-n} \cdot \hat{\alpha}_n : = \sum_{n=1}^{\infty} n \hat{a}_{-n}^{\dagger} \cdot \hat{a}_n \tag{4.35}$$

This number operator is useful to calculate the mass spectrum

$$\begin{aligned} \alpha' M^2 &= -a && \text{(ground state)} \\ \alpha' M^2 &= -a + 1 && \text{(first excited state)} \\ \alpha' M^2 &= -a + 2 && \text{(second excited state)} \\ &\vdots \end{aligned}$$

For a closed string the mass formula is given by

$$\frac{\alpha'}{4} M^2 = \sum_{n=1}^{\infty} : \hat{\alpha}_{-n} \cdot \hat{\alpha}_n : -a = \sum_{n=1}^{\infty} : \hat{\tilde{\alpha}}_{-n} \cdot \hat{\tilde{\alpha}}_n : -a \tag{4.36}$$

with

$$\hat{N} - a = \hat{\tilde{N}} - a \rightarrow \hat{N} = \hat{\tilde{N}} \tag{4.37}$$

This implies

$$(\hat{L}_0 - \hat{\tilde{L}}_0) |\phi\rangle = 0 \tag{4.38}$$

where $\hat{N}$ and $\hat{\tilde{N}}$ are the number operator for right and left movers respectively.

We know that $\hat{L}_0 |\phi\rangle \neq 0$, in contrast with the classical generators L_m, which are equal to zero for all m, including $m = 0$. Now we are interesting in what happens with the operator $\hat{L}_m$ $(m \neq 0)$, if we consider that $\hat{L}_m |\phi\rangle = 0$ for all $(m \neq 0)$ and we take the operators $\hat{L}_m$ and $\hat{L}_{-m}$, we have

$$[\hat{L}_m, \hat{L}_{-m}] |\phi\rangle = \hat{L}_m \underbrace{\hat{L}_{-m} |\phi\rangle}_{=0} - \hat{L}_{-m} \underbrace{\hat{L}_m |\phi\rangle}_{=0} = 0 \tag{4.39}$$

or, from (4.29)

$$\begin{aligned} (m+m) \underbrace{\hat{L}_{m-m}}_{\hat{L}_0} |\phi\rangle + \frac{c}{12} m(m^2-1) \underbrace{\delta_{m,m}}_{=1} |\phi\rangle = 0 \rightarrow \\ 2m \underbrace{\hat{L}_0 |\phi\rangle}_{=a|\phi\rangle} + \frac{c}{12} m(m^2-1) |\phi\rangle = 0 \rightarrow \\ \left(2a + \frac{c}{12}(m^2-1)\right) |\phi\rangle = 0 \rightarrow \\ 2a + \frac{c}{12}(m^2-1) = 0 \end{aligned} \tag{4.40}$$

which does not make sense since a is a constant and comes from L_0, so it cannot depend on m. Then the expression (4.40) cannot be true, i.e,

$$[\hat{L}_m, \hat{L}_{-m}] \left|\phi\right\rangle \neq 0 \tag{4.41}$$

Therefore, instead of supposing that $\hat{L}_{m\neq 0} \left|\phi\right\rangle = 0$ we impose that the physical states verify

$$\hat{L}_{m>0} \left|\phi\right\rangle = 0 = \left\langle\phi\right| \hat{L}^{\dagger}_{m>0} \tag{4.42}$$

and the corresponding mass-shell condition

$$(\hat{L}_0 - a) \left|\phi\right\rangle = 0 \tag{4.43}$$

Observe that equation (4.42) is equivalent to the following expression

$$\left\langle\phi\right| \hat{L}_m = 0 = \hat{L}_{-m} \left|\phi\right\rangle \qquad ; m < 0 \tag{4.44}$$

The Lorentz generators $Q^{\mu\nu}$ can we written as [1]

$$Q^{\mu\nu} = T \int_{\sigma=0}^{\pi} d\sigma \left(X^{\mu} \dot{X}^{\nu} - X^{\nu} \dot{X}^{\mu} \right) \tag{4.45}$$

If we write them in mode expansion

$$Q^{\mu\nu} = (x^{\mu} p^{\nu} - x^{\nu} p^{\mu}) - \sum_{m=1}^{\infty} \frac{1}{m} (\alpha^{\mu}_{-m} \alpha^{\nu}_{m} - \alpha^{\nu}_{-m} \alpha^{\mu}_{m}) \tag{4.46}$$

Here there is not any normal ordering ambiguity, so when we quantize

$$Q^{\mu\nu} = (\hat{x}^{\mu} \hat{p}^{\nu} - \hat{x}^{\nu} \hat{p}^{\mu}) - \sum_{m=1}^{\infty} \frac{1}{m} (\hat{\alpha}^{\mu}_{-m} \hat{\alpha}^{\nu}_{m} - \hat{\alpha}^{\nu}_{-m} \hat{\alpha}^{\mu}_{m}) \tag{4.47}$$

Then we have an operator expression for the Lorentz generators. If we calculate the commutators of these operators with the Virasoro operators we obtain that

$$[\hat{L}_m, \hat{Q}^{\mu\nu}] = 0 \tag{4.48}$$

This commutator relation implies that the physical states defined in terms of the Virasoro generators appear in complete Lorentz multiplets. Therefore, we can find a relation between the physical states of our Virasoro algebra and the particles of the Lorentz group, it means that we can associate to each physical states one type of particle.

4.6 Elimination of Ghost States

In order to have a consistent theory we have to eliminate from it the negative norm states, for this work we need to introduce the spurious states.

4.6.1 Spurious States.

We define a spurious state $|\varphi\rangle$ as the state which satisfies the mass-shell condition and is orthogonal to all physical states.

$$(\hat{L}_0 - a)\,|\varphi\rangle = 0 \tag{4.49}$$

$$\langle\varphi|\phi\rangle = 0 \qquad \forall \text{ physical states } |\phi\rangle \tag{4.50}$$

The set of all spurious states can be understood as the vacuum state because this second one is an orthogonal subspace to the space of all physical states. In general, the spurious states are given by [14]

$$|\varphi\rangle = \sum_{n=1}^{\infty} \hat{L}_{-n}\,|\chi_n\rangle \tag{4.51}$$

where $|\chi_n\rangle$ verify

$$\hat{L}_m\,|\chi_n\rangle = 0 \quad ; \quad m \geq 0 \tag{4.52}$$

From eq.(4.50) and eq.(4.51) we obtain (see appendix (C.2))

$$(\hat{L}_0 - a + n)\,|\chi_n\rangle = 0 \tag{4.53}$$

It can be proven that any state $\hat{L}_{-n}\,|\chi_n\rangle$ (for $n \geq 1$) can be written as a combination of $\hat{L}_{-1}{-}n\,|\chi_{-1}\rangle$ and $\hat{L}_{-2}\,|\chi_{-2}\rangle$. Then any spurious state can be written as

$$|\chi\rangle = \hat{L}_{-1}\,|\chi_1\rangle + \hat{L}_{-2}\,|\chi_2\rangle \tag{4.54}$$

where we will refer to $|\chi_1\rangle$ and $|\chi_2\rangle$ as level 1 and level 2 states respectively. We will prove (4.54) for $n = 3$, $|\varphi\rangle = \hat{L}_{-3}\,|\chi_3\rangle$ and recall that (using the commutator (4.29)) $\hat{L}_{-3} = [\hat{L}_{-1}, \hat{L}_{-2}]$, then

$$\begin{aligned}\hat{L}_{-3}\,|\chi_3\rangle &= [\hat{L}_{-1}, \hat{L}_{-2}]\,|\chi_3\rangle \\ &= (\hat{L}_{-1}\hat{L}_{-2} - \hat{L}_{-2}\hat{L}_{-1})\,|\chi_3\rangle = \hat{L}_{-1}(\hat{L}_{-2}\,|\chi_3\rangle) - \hat{L}_{-2}(\hat{L}_{-1}\,|\chi_3\rangle)\end{aligned} \tag{4.55}$$

From (4.53) and (4.52) we know that

$$\hat{L}_0\,|m\rangle = (a - m)\,|m\rangle \tag{4.56}$$

where we use the notation $|m\rangle \equiv |\chi_m\rangle$. Observe that

$$\hat{L}_0\hat{L}_{-n}|m\rangle = \underbrace{[\hat{L}_0, \hat{L}_{-n}]}_{n\hat{L}_{-n}}|m\rangle + \hat{L}_{-n}\hat{L}_0|m\rangle = n\hat{L}_{-n}|m\rangle + (a-m)|m\rangle \rightarrow$$

$$\hat{L}_0\hat{L}_{-n}|m\rangle = (n-m+a)\hat{L}_{-n}|m\rangle \tag{4.57}$$

Note that following the expression (4.53), for the state $|m-n\rangle$ we have the next condition

$$\hat{L}_0|m-n\rangle = (a-(m-n))|m-n\rangle \tag{4.58}$$

If we compare this equation with (4.57) we obtain

$$\hat{L}_{-n}|m\rangle = |m-n\rangle \tag{4.59}$$

Then,

$$\begin{aligned}\hat{L}_{-2}|\chi_3\rangle &= |\chi_1\rangle \\ \hat{L}_{-1}|\chi_3\rangle &= |\chi_2\rangle\end{aligned} \tag{4.60}$$

So, if we include eq.(4.60) into (4.55) we obtain

$$|\varphi\rangle = \hat{L}_{-3}|\chi_3\rangle = \hat{L}_{-1}|1\rangle + \hat{L}_{-2}|2\rangle \tag{4.61}$$

So, we have seen that the state level 3 can be written as a combination of $\hat{L}_{-1}|1\rangle$ and $\hat{L}_{-2}|2\rangle$. This proof can be extended to the level state n using the same arguments.

With our definition of a spurious state $|\varphi\rangle$ in (4.51) it is easy to show that they are orthogonal to any physical state $|\phi\rangle$

$$\langle\phi|\varphi\rangle = \sum_n \langle\phi|\hat{L}_{-n}|\chi_n\rangle = \sum_n(\langle\chi_n|\hat{L}_n|\phi\rangle)^* = \sum_n(\langle\chi_n|0|\phi\rangle)^* = 0 \tag{4.62}$$

where we used the expression (4.44) and the hermiticity property of $\hat{L}_m$ in (4.28). Now, if we want to define the spurious states as physical states, by definition, they must be orthogonal to themselves, it means that they will have zero norm.

$$\langle\varphi|\varphi\rangle = 0 \tag{4.63}$$

Now we have defined physical states with norm equal to zero, and, believe it or not, we will use them to try to remove the negative norm states of our bosonic string theory.

4.6.2 Removing Negative Norm Physical States.

At the moment, we have worked with the constants c (from the commutators relations of the operators) and a (from the mass-shell conditions) but we do not know their values. Also we have in our theory states with negative norm (ghost states) and states with norm equal to zero (spurious states). In order to get a rigorous bosonic string theory we must resolve all this problems. We should start finding the a value, for that purpose we take the level 1 physical spurious state and look fot the condition to convert it into a physical state

$$|\varphi\rangle = \hat{L}_{-1} |\chi_n\rangle \tag{4.64}$$

where $|\chi_1\rangle$ satisfies the conditions (4.53) and (4.52). For it to be a physical state, it must have eigenvalue equal to zero for any operator $\hat{L}_{m>0}$ and for $m = 0$ it must satisfy the mass shell condition

$$\left(\hat{L}_0 - a\right) |\varphi\rangle = 0 \tag{4.65}$$

Taking into account eq.(4.53) we can see that the state (4.64) satisfies (4.65). Now we have to impose the condition $\hat{L}_{m>0} |\chi_1\rangle = 0$, we start whit $m = 1$

$$\begin{gathered}\hat{L}_1 |\varphi\rangle = 0 \rightarrow \hat{L}_1 \left(\hat{L}_{-1} |\chi_1\rangle\right) = 0 \rightarrow \\ ([\hat{L}_1, \hat{L}_{-1}] + \hat{L}_{-1}\hat{L}_1) |\chi_1\rangle = [\hat{L}_1, \hat{L}_{-1}] |\chi_1\rangle = 2\hat{L}_0 |\chi_1\rangle = 2(a-1) |\chi_1\rangle = 0\end{gathered} \tag{4.66}$$

We can repeat this process to show that $L_{m\geq 1} |\phi\rangle = 0$. Then $a = 1$ for $|\varphi\rangle$ to be a physical state. We have seen that the a parameter has an important effect in the number of zero norm states of the theory, which defines the boundary between positive norm (physical) and negative norm (ghost) states.

Once that we fix the value of a, we can see what happens with the other parameter c. In order to do it we take a general level 2 spurious state

$$|\varphi\rangle = (\hat{L}_{-2} + \gamma\hat{L}_{-1}\hat{L}_{-1}) |\chi_2\rangle \tag{4.67}$$

where γ is a constant determine by the zero norm condition of the spurious state. We already know that the level 2 state obeys (4.53) and (4.52). Thus, if $|\varphi\rangle$ is physical, then it must satisfy $\hat{L}_{m>0} |\varphi\rangle = 0$. We start again with $m = 1$

$$\begin{gathered}\hat{L}_1(\hat{L}_{-2} + \gamma\hat{L}_{-1}\hat{L}_{-1}) |\chi_2\rangle = 0 \rightarrow \\ \left([\hat{L}_1, \hat{L}_{-2} + \gamma\hat{L}_{-1}] + (\hat{L}_{-2} + \gamma\hat{L}_{-1}\hat{L}_{-1})\hat{L}_1\right) |\chi_2\rangle = [\hat{L}_1, \hat{L}_{-2} + \gamma\hat{L}_{-1}] |\chi_2\rangle = \\ (3\hat{L}_{-1} + 4\gamma \underbrace{\hat{L}_{-1}\hat{L}_0}_{=-\hat{L}_{-1}} + 2\gamma\hat{L}_{-1}) |\chi_2\rangle = (3 - 4\gamma + 2\gamma)\hat{L}_{-1} |\chi\rangle \rightarrow \\ \rightarrow 3 - 2\gamma = 0 \rightarrow \gamma = \frac{3}{2}\end{gathered} \tag{4.68}$$

Where we take into account that $a = 1$ and the commutator relation of the Virasoro algebra (4.29). So, our level 2 physical spurious state

$$|\varphi\rangle = (\hat{L}_{-2} + \frac{3}{2}\hat{L}_{-1}\hat{L}_{-1})\,|\chi_2\rangle \tag{4.69}$$

Also, since it is a physical state, it satisfies $\hat{L}_{m>0}\,|\varphi\rangle = 0$ in particular for $m = 2$, which implies

$$\begin{aligned}\hat{L}_2(\hat{L}_{-2} + \frac{3}{2}\hat{L}_{-1}\hat{L}_{-1})\,|\chi_2\rangle &= \left([\hat{L}_2, (\hat{L}_{-2} + \frac{3}{2}\hat{L}_{-1}\hat{L}_{-1})] + (\hat{L}_{-2} + \frac{3}{2}\hat{L}_{-1}\hat{L}_{-1})\hat{L}_2\right)|\chi_2\rangle = \\ = \left([\hat{L}_2, (\hat{L}_{-2} + \frac{3}{2}\hat{L}_{-1}\hat{L}_{-1})]\right)|\chi_2\rangle &= \left(13\hat{L}_0 + 9\hat{L}_{-1}\hat{L}_1 + \frac{c}{2}\right)|\chi_2\rangle \rightarrow= (-13 + \frac{c}{2})\,|\chi_2\rangle = 0\end{aligned} \tag{4.70}$$

If eq.(4.70) is satisfied then $\hat{L}_{m\geq 2}\,|\varphi\rangle = 0$ also is automatically satisfied. Therfore, if $|\varphi\rangle$ is a physical spurious state

$$c = 26 \tag{4.71}$$

Compiling all these results

$$\begin{aligned} a &= 1 \\ \gamma &= \frac{3}{2} \\ c &= 26 \end{aligned}$$

Note that these results are not conclusive, i.e we can observe that the elimination of the norm negative states is associated with the values of a and c but we cannot confirm that these values remove all the ghost states of our theory. Then, we must use another quantization method to confirm this supposition and this method will be the light-cone quantization.

Chapter 5

Light-Cone Quantization

Habiendo visto ya que la cuantización canónica no nos resuelve el problema de los estados de norma negativa procederemos en este capítulo a buscar otra solución, que será la cuantización del cono de luz. En este capítulo será donde lleguemos a los resultados deseados que son la determinación de un espacio-tiempo con 26 dimensiones y al espectro de masas.

We have seen that the canonical quantization does not solve the problem of the negative norm states. In this chapter we will use another method in order to find a solution, which will be the quantization in the light-cone gauge. This method includes automatically the constraits of the theory and will lied to the desired results, which are the determination of a space-time with 26 dimensions and the mass spectrum

5.1 Light-Cone Coordinates and dimensions Mass-Shell Condition

We have sees that the canonical quantization is not the best way to quantize our bosonic string theory, then now we will study the light-cone quantization. Using this new method, we will obtain the same values for the constant a and c but in this case the results will be more conclusive.

Note that we have residual symmetries in our bosonic string theory after fixing the gauge that makes the metric flat

$$h^{\alpha\beta} = \eta^{\alpha\beta} \tag{5.1}$$

We fixed this gauge using Weyl, transformation, i.e

$$h^{\alpha\beta} = e^{2\phi(\sigma)}\eta^{\alpha\beta} \tag{5.2}$$

Now we will see that this metric is invariant under additional transformations. We will use this residual freedom to define the light-cone gauge. Note that if we use any coordinate transformation such that the metric changes by

$$\eta^{\alpha\beta} = e^{2\phi(\sigma)}\eta^{\alpha\beta} \tag{5.3}$$

this changes can be undone by a Weyl Transformation. In order to obtain this coordinate transformation we take the light-cone coordinates

$$\sigma^{\pm} = \tau \pm \sigma \tag{5.4}$$

where the flat metric can be written as

$$ds^2 = d\sigma^+ d\sigma^- \tag{5.5}$$

Observe that in this case, the transformations

$$\sigma^{\pm} \to \tilde{\sigma}^{\pm}(\sigma^{\pm}) \tag{5.6}$$

will only produce a multiplying factor in the metric as in the eq.(5.3), and this factor can be removed using a Weyl Transformation. Then, the theory is invariant under these coordinate transformations. This implies that we are able to add another gauge choice fulfilling such conditions. In order to obtain a Fock space free of ghost states, we will chose a partiular non-covariant gauge. Now, we define the light-cone coordinates for the background space-time in which our theory lives. Usually, this coordinates are defined as a linear combination of the time-like coordinate along with one of the space-like coordinates.

$$X^{\pm} \equiv \frac{1}{\sqrt{2}}(X^0 \pm X^{D-1}) \tag{5.7}$$

Then, the spacetime coordinate set is $\{X^+, X^-, X^1, ..., X^{D-2}\}$. Note that now we refer to c as D. In these coordinates, the Minkowski metric of background space becomes

$$ds^2 = -2dX^+dX^- + \sum_{i=0}^{D-2}(dX^i)^2 \tag{5.8}$$

and the inner product for two vectors A and B is given by

$$A \cdot B = -A^+B^- - A^-B^+ + A^iB^i \tag{5.9}$$

These new coordinates remove the manifest Lorentz invariance of the theory.

Now, note that the residual symmetry implies that we can reparameterize the worldsheet coordinates by

$$\tau \to \tilde{\tau} = \frac{1}{2}\left[\tilde{\sigma}^+ + \tilde{\sigma}^-\right] = \frac{1}{2}\left[\xi^+(\sigma^+) + \xi^-(\sigma^-)\right] \tag{5.10}$$

$$\sigma \to \tilde{\sigma} = \frac{1}{2}\left[\tilde{\sigma}^+ - \tilde{\sigma}^-\right] = \frac{1}{2}\left[\xi^+(\sigma^+) - \xi^-(\sigma^-)\right] \tag{5.11}$$

since they are given by a linear combination of the light-cone coordinates, which can be reparametrized keeping the theory invariant. From (5.10), we can see that $\tilde{\tau}$ obeys

$$\partial_+\partial_-\tilde{\tau} = (\partial_\tau^2 - \partial_\sigma^2)\tilde{\tau} = 0 \tag{5.12}$$

This is the same equation that space-time coordinates $X^\mu(\tau,\sigma)$ satisfy.

$$(\partial_\tau^2 - \partial_\sigma^2)X^\mu(\tau,\sigma) = 0 \tag{5.13}$$

Then $\tilde{\tau}$ can be identify with a linear combination of the space-time coordinates plus an arbitrary. The light-cone gauge choice is

$$\tilde{\tau} = \frac{X^+}{l_s^2 p^+} + x^+ \tag{5.14}$$

which can be written as

$$X^+ = x^+ + l_s^2 p^+ \tilde{\tau} \tag{5.15}$$

Henceforth we will drop the tildes. Observe that this choice fixes the r reparameterization invariance (5.6). Now, knowing the solution for the field X^+ we are able to calculate the field X^-. This field must obey

$$\partial_+\partial_- X^- = 0 \tag{5.16}$$

which can be resolved using

$$X^- = X_L^-(\sigma^+) + X_R^-(\sigma^-) \tag{5.17}$$

We will see now that the solution for X^- is completely determine by the constraints eq.(3.43) and eq.(3.44)

$$\partial_+ X^\mu \partial_+ X_\mu = 0 \tag{5.18}$$

$$\partial_- X^\mu \partial_- X_\mu = 0 \tag{5.19}$$

which means that

$$2\partial_+ X^+ \partial_+ X^- - \sum_{i=1}^{D-2} \partial_+ X^i \partial_+ X^i = 0 \rightarrow$$

$$2\partial_+ X^+ \partial_+ X^- = \sum_{i=1}^{D-2} \partial_+ X^i \partial_+ X^i \tag{5.20}$$

Then, taking into account the solution for X^+ when we fix the gauge, i.e equation (5.15) we obtain

$$\partial_+ X_L^- = \frac{2}{p^+ l_s} \sum_{i=1}^{D-2} \partial_+ X^i \partial_+ X^i \tag{5.21}$$

Equivalent

$$\partial_- X_R^- = \frac{2}{p^+ l_s} \sum_{i=1}^{D-2} \partial_+ X^i \partial_+ X^i \tag{5.22}$$

Then, the field X^- can be expressed in terms of the other fields, and its solution is determined except for an integration constant. We know that the solution of the left and right movers of X^- are given by (see equations (3.47) and (3.48))

$$X_L^-(\sigma^+) = \frac{1}{2}x^- + \frac{1}{2}l_s^2 p^- \sigma^+ + i\frac{l_s}{2}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^- e^{-in\sigma^+} \tag{5.23}$$

$$X_R^-(\sigma^-) = \frac{1}{2}x^- + \frac{1}{2}l_s^2 p^- \sigma^- + i\frac{l_s}{2}\sum_{n\neq 0}\frac{1}{n}\alpha_n^- e^{-in\sigma^-} \tag{5.24}$$

where x^- corresponds to the integration constant. Then, comparing (5.24) to (5.22) we can observe that the modes α_n^- must satisfy

$$\alpha_n^- = \frac{1}{l_s p^+}\left(\frac{1}{2}\sum_{m=-\infty}^{\infty}\sum_{i=1}^{D-2}\alpha_{n-m}^i \alpha_m^i - \underbrace{a\delta_{n,0}}_{\substack{\text{from}\\ \text{n.o}}}\right) \tag{5.25}$$

Now, we are interested in α_0^- which is given by

$$\alpha_0^- = \frac{1}{2}l_s p^- \tag{5.26}$$

Therefore, following the equation (5.25) and taking into account that $\alpha_0^i = \frac{l_s}{2}p^i$ we obtain

$$\begin{gathered}
\alpha_0^- \equiv \frac{1}{2}l_s p^- = \frac{1}{l_s p^+}\sum_{i=1}^{D-2}\left(\frac{ls^2}{4}p^i p^i + \sum_{n\neq 0}\alpha_n^i \alpha_{-n}^i - a\right) \rightarrow \\
\frac{1}{2}l_s^2 p^- p^+ - \frac{ls^2}{4}\sum_{i=1}^{D-2} p^i p^i = \underbrace{\sum_{n>0}\alpha_n^i \alpha_{-n}^i}_{\equiv N} - a \rightarrow \\
2p^+ p^- - \sum_{i}^{D-2} p^i p^i = \frac{2}{l_s^2}(N-a)
\end{gathered} \tag{5.27}$$

Note that in light-cone coordinates

$$M^2 = -p^\mu p_\mu = 2p^+ p^- - \sum_{i}^{D-2} p^i p^i \tag{5.28}$$

Therefore, we obtain

$$M^2 = \frac{4}{l_s^2}(N - a) \tag{5.29}$$

Equivalent

$$M^2 = \frac{4}{l_s^2}(\tilde{N} - a) \tag{5.30}$$

Which look as eq.(4.36), the only difference is that in eq.(5.30) we do not include the oscillators α_n^- and α_n^+. The oscillators α_n^i which appears in this mass-formula are called transverse oscillators and they are the physical excitations of the string. The constraints of the theory remove any longitudinal excitation in the light-cone gauge.

In the same way, using the similar arguments, it can be proved that for open strings we obtain

$$M^2 = \frac{2}{l_s^2}(N - a) \tag{5.31}$$

5.2 The Lorentz invariance and the parameters a and D

Now, will we see that this light-cone gauge eliminates the negative norm states from our theory. Recall that the commutator relations of the oscillators are given by

$$[\alpha_m^\mu, (\alpha_n^\nu)^\dagger] = \eta^{\mu\nu}\delta_{m,n} \tag{5.32}$$

but, the excitations are only generated by the transverse oscillations α_n^i. Then, the negative norm states coming from $[\alpha_m^0, (\alpha_n^0)^\dagger]$ vanish since we only consider the transverse oscillations.

Now, we are interested in determining the values of a and D. For this purpose we consider the first excited states

$$\alpha_{-1}^i \left|0; k^\mu\right\rangle \tag{5.33}$$

Since $i = 1, ..., D-2$ these states correspond to a (D-2)-component vector belonging to the rotation group SO(D-2). The Lorentz invariance implies the following rule for the states corresponding to physical particles [14]

Massive particles $\rightarrow$ SO(D-1) representation

Massless particles $\rightarrow$ SO(D-2) representation

Then, our states $\alpha^i_{-1}|0;k^\mu\rangle$ must correspond to a massless particle in the Lorentz representation. Applying the mass squared operator to this state we obtain

$$\begin{aligned} M^2(\alpha^i_{-1}|0;k^\mu\rangle) &= \frac{2}{l_s}(N-a)\alpha^i_{-1}|0;k^\mu\rangle \\ &= \frac{2}{l_s}(1-a)\alpha^i_{-1}|0;k^\mu\rangle \end{aligned} \tag{5.34}$$

If our states have zero mass, the the constant a should have the value

$$a = 1 \tag{5.35}$$

Recall that a comes from the ambiguity of the normal ordering. If we start with the symmetrically ordered form, we obtain

$$\frac{1}{2}\sum_{i=1}^{D-2}\sum_{m=-\infty}^{\infty}\alpha^i_{-m}\alpha^i_m = \frac{1}{2}\sum_{i=1}^{D-2}\sum_{m=-\infty}^{\infty} :\alpha^i_{-m}\alpha^i_m: + \underbrace{\frac{D-2}{2}\sum_{m=1}^{\infty} m}_{\equiv -a} \tag{5.36}$$

where we have taken into account the oscillator commutation relations ($[\alpha^i_m, \alpha^j_{-m}] = m\delta_{i,j}$). In order to resolve our problem, we must use the Riemann ξ-function since the second term diverges, then we must find its regular part. The Riemann ξ-function is defined as

$$\xi(s) = \sum_{m=1}^{\infty} m^{-s} \tag{5.37}$$

for any $s \in C$. Via analytic continuation to our value $s = -1$,the Riemann ξ-function takes the value $\xi(-1) = -1/12$. Then equation (5.36) after regularization becomes

$$\frac{1}{2}\sum_{i=1}^{D-2}\sum_{m=-\infty}^{\infty} :\alpha^i_{-m}\alpha^i_m: - \frac{D-2}{24} \tag{5.38}$$

Recall that the last term is equal to a, so

$$\begin{aligned} \frac{D-2}{24} &= a = 1 \rightarrow \\ D &= 26 \end{aligned} \tag{5.39}$$

In turns out that these choices for a and D make the theory invariant under Lorentz transformations. This requirement is the starting point of a completely rigorous proof of the previous results.

5.3 Mass spectrum

5.3.1 Open Strings

Now, knowing the parameters a and D we are able to calculate the mass for different physical states. We are going to star with the ground state

$$|0; k^\mu\rangle \tag{5.40}$$

Then, using the mass (squared) operator we obtain

$$M^2 |0; k^\mu\rangle = \frac{2}{l_s}(N-1)|0; k^\mu\rangle \tag{5.41}$$

$$= -\frac{2}{l_s}|0; k^\mu\rangle \tag{5.42}$$

Note that the ground state correspond to a particle, the tachyon, which has imaginary mass, i.e, imaginary energy . This implies that the ground state is not stable and it would decay. This affirmation is not possible if we want a consistent theory. This problem is solved in the *super-string theory*, where we include the fermions in the theory, although that is another story.

In section 5.2 we have studied the first excited states of the theory which are massless states. Now we will go to study the second excited states. These states correspond to the quantum number $N = 2$ and they are the first states with positive mass. The states with $N = 2$ can be written as

$$\begin{gathered} \alpha^i_{-2}|0; k^\mu\rangle \\ \alpha^j_{-1}\alpha^i_{-1}|0; k^\mu\rangle \end{gathered} \tag{5.43}$$

and these have 24 and $24 \times 25/2$. Therefore, the degeneracy of the second excited level is 324.

5.3.2 Closed Strings

For closed strings, the ground states $N = 0$ corresponds to a tachyon too since we obtain again an imaginary mass. However, for the first excited states $N = 1$ we have degeneracy $24^2 = 576$. These states can be expressed in the form

$$|\Omega^{ij}\rangle = \alpha^i_{-1}\tilde{\alpha}^j_{-1}|0; k^\mu\rangle \tag{5.44}$$

which are the tensor product of two massless vectors corresponding to left and right movers. The symmetric and traceless part of $|\Omega^{ij}\rangle$ corresponds in the group SO(24) to an masless particle with spin-2. This is the graviton! On the other hand, the antisymmetric part correspond to the dilaton.

Let us mention that the bosonic string theory is a good theory for the quantum gravity since the Einstein equations can be deduced from the bosionic string theory. The only problem is the existence of the tachyon.

Chapter 6

Conclusions

We started by studying the classical relativistic string action and its symmetries in order to find the equations and the solutions for the fields of the string. From these solutions we introduced the quantities L_m, which are quadratic functions of the oscillation modes of the string.

After that, we tried to quantize our theory using the canonical quantization but we had some problems. The first one was that we obtain states with negative norm and in order to eliminate these we constructed the Virasoro algebra, where the parameters a and D turned up. The operators of this algebra were $\hat{L}_m$, the quantum analogues of the generators L_m. From those operators we constructed the spurious states of our theory, states which were used in order to remove the norm negative states. However the results were not conclusive, but we observed that there was a relation between the numbers of negative norm states and the parameters a and D.

Since we did not obtain conclusive results with the canonical quantization, we tried to quantize our theory using the quantization in the light-cone gauge. From the symmetries of the action we got a residual gauge freedom, which we used to fix the gauge by including all the constraints of the theory, losing its manifest covariant. Now the theory recovers the Lorentz invariance only for given values of a and D, namely $a = 1$ abd $D = 26$. This means that we have obtained the dimension of the background space-time where our theory lives (i.e. 26).

Also we have obtained the discreet mass spectrum for the particles of our theory, which are all bosons. The graviton is one of the particle predicted by this String Theory. This is the hypothetical elementary particle that mediates the force of gravity. Therefore, String Theory includes quantum gravity. The only problem with the bosonic string theory is that its ground state is a tachyon, i.e, a state with imaginary mass and, therefore, supposedly unstable.

In order to eliminate this incongruity we have to include the fermions in our theory. The way to do that is to introduce an additional term in the action, which corresponds to the fermions and has extras symmetries called supersymmetries.

Appendix A

A.1 Relativistic particle invariant action

Take the reparameterization

$$d\tau \to d\tau' = \frac{\partial f}{\partial \tau} d\tau \tag{A.1}$$

and so

$$\frac{dX^\mu(\tau')}{d\tau} = \frac{dX^\mu(\tau')}{d\tau'} \frac{d\tau'}{d\tau} \tag{A.2}$$

Introducing this in the expression 2.6 we obtain

$$\begin{aligned}
S_0' &= -m \int d\tau' \sqrt{-g_{\mu\nu}(X) \frac{X^\mu(\tau')}{d\tau'} \frac{X^\nu(\tau')}{d\tau'}} \\
&= -m \int d\tau \frac{\partial f}{\partial \tau} \sqrt{-g_{\mu\nu}(X) \frac{dX^\mu(\tau')}{d\tau} \frac{dX^\nu(\tau')}{d\tau} \left(\frac{\partial f}{\partial \tau}\right)^{-2}} \\
&= -m \int d\tau \frac{\partial f}{\partial \tau} \left(\frac{\partial f}{\partial \tau}\right)^{-1} \sqrt{-g_{\mu\nu}(X) \frac{dX^\mu(\tau)}{d\tau} \frac{dX^\nu(\tau)}{d\tau}} \\
&= -m \int d\tau' \sqrt{-g_{\mu\nu}(X) \frac{X^\mu(\tau)}{d\tau} \frac{X^\nu(\tau)}{d\tau}} \\
&= S_0
\end{aligned} \tag{A.3}$$

A.2 Equivalence between the actions S_0 and $\tilde{S}_0$

First we consider the variation of $\tilde{S}_0$ with respect to the field $e(\tau)$

$$\begin{aligned}
\delta\tilde{S}_0 &= \delta\left[\frac{1}{2}\int d\tau(e^{-1}\dot{X}^2 - m^2 e)\right] \\
&= \frac{1}{2}\int d\tau\left(-\frac{1}{e^2}\dot{X}^2\delta e - m^2\delta e\right) \\
&= \frac{1}{2}\int d\tau\frac{\delta e}{e^2}\left(-\dot{X}^2 - (me)^2\right)
\end{aligned} \tag{A.4}$$

If S_0 and $\tilde{S}_0$ are the same action then $\delta\tilde{S}_0$ must be zero, and because δe is arbitrary

$$-\dot{X}^2 - (me)^2 = 0 \rightarrow e = \sqrt{-\frac{\dot{X}^2}{m^2}} \tag{A.5}$$

Now, plugging this expression for the field back into the action $\tilde{S}_0$

$$\begin{aligned}
\tilde{S}_0 &= \frac{1}{2}\int d\tau\left[\left(\frac{-\dot{X}^2}{m^2}\right)^{-\frac{1}{2}}\dot{X}^2 - m^2\left(\frac{-\dot{X}^2}{m^2}\right)^{\frac{1}{2}}\right] \\
&= \frac{1}{2}\int d\tau\left[\left(\frac{-\dot{X}^2}{m^2}\right)^{-\frac{1}{2}}\left(\dot{X}^2 - m^2\left(\frac{-\dot{X}^2}{m^2}\right)\right)\right] \\
&= \frac{1}{2}\int d\tau\left(\frac{-\dot{X}^2}{m^2}\right)^{-\frac{1}{2}}\left(\dot{X}^2 + \dot{X}^2\right) \\
&= -\int d\tau\left(\frac{-\dot{X}^2}{m^2}\right)^{-\frac{1}{2}}(-\dot{X}^2)(-1) \\
&= \int -m(\dot{X}^2)^{-\frac{1}{2}}(-\dot{X}^2) \\
&= \int d\tau(-\dot{X}^2)^{-\frac{1}{2}} \\
&= \int d\tau\sqrt{-g_{\mu\nu}\frac{dX^\mu}{d\tau}\frac{dX^\mu}{d\tau}} \\
&= S_0
\end{aligned}$$

So if $e(\tau) = \sqrt{-\frac{\dot{X}^2}{m^2}}$, the actions S_0 and $\tilde{S}_0$ are equals.

A.3 Equality between the Polyakow (S_σ) and the Nambu-Goto (S_{NG}) actions

The stress-energy tensor of the string $T_{\alpha\beta}$ is given by

$$T_{\alpha\beta} = -\frac{2}{T}\frac{1}{\sqrt{-h}}\frac{\delta S}{\delta h^{\alpha\beta}} \tag{A.6}$$

The variation of the action S_σ with respect to $h^{\alpha\beta}$ can be written as

$$\begin{aligned} \delta S_\sigma &\equiv \int \frac{\delta S_\sigma}{\delta h^{\alpha\beta}}\delta h^{\alpha\beta} \\ \delta S_\sigma &= -\frac{T}{2}\int d\tau d\sigma \sqrt{-h}\delta h^{\alpha\beta} T_{\alpha\beta} \end{aligned} \tag{A.7}$$

This variation must to be equal to zero, and the only possibility is that $T_{\alpha\beta} = 0$. The explicit form of this tensor is

$$T_{\alpha\beta} = \left(-\frac{1}{2}h_\beta h^{\gamma\delta}\partial_\gamma X \cdot \partial_\delta X + \partial_\alpha X \cdot \partial_\beta X\right) = 0 \tag{A.8}$$

That implies

$$-\frac{1}{2}h_\beta h^{\gamma\delta}\partial_\gamma X \cdot \partial_\delta X = \underbrace{\partial_\alpha X \cdot \partial_\beta X}_{G_{\alpha\beta}} \tag{A.9}$$

Taking the square root of minus the determinant of both sides

$$\frac{1}{2}\sqrt{-h}\delta h^{\alpha\beta}\partial_\gamma X \cdot \partial_\delta X = \sqrt{-det(G_{\alpha\beta})} \tag{A.10}$$

Now, note that the variation of the auxiliary field δh is defined as

$$\delta h \equiv \delta(det(h_{\alpha\beta})) = -h h_{\alpha\beta}\delta h^{\alpha\beta} \tag{A.11}$$

So, this give us

$$\delta\sqrt{-h} = -\frac{1}{2}\sqrt{-h}\delta h^{\alpha\beta} h_{\alpha\beta} \tag{A.12}$$

Then

$$\begin{aligned} S_\sigma &= -\frac{T}{2}\int d\tau d\sigma \sqrt{-h} h^{\alpha\beta}\partial_\alpha X^\mu \cdot \partial_\beta X^\nu \\ &= -T\int d\tau d\sigma \sqrt{-det(G_{\alpha\beta})} \\ &= S_{NG} \end{aligned} \tag{A.13}$$

Appendix B

B.1 Proof that all the stress-energy tensor are equal to zero

We start from the same for the strees energy tensor $T_{\alpha\beta}$, i.e

$$T_{\alpha\beta} \equiv -\frac{2}{T}\frac{1}{\sqrt{-h}}\frac{\delta S_\sigma}{\delta h_{\alpha\beta}} \tag{B.1}$$

so under a generic transformation of $h^{\alpha\beta}$ the action S_σ changes as

$$\delta S_\sigma \equiv \int \frac{\delta S_\sigma}{\delta h^{\alpha\beta}}\delta h^{\alpha\beta} = -\frac{T}{2}\int d\tau d\sigma \sqrt{-h}\delta h^{\alpha\beta}T_{\alpha\beta} \tag{B.2}$$

The field $h'^{\alpha\beta}$ (3.12) can be expressed as an expansion in ϕ yields. Namely,

$$h'^{\alpha\beta}(\tau,\sigma) = e^{-2\phi(\sigma)}h^{\alpha\beta}(\tau,\sigma) = (1-2\phi+...)h^{\alpha\beta}(\tau,\sigma) \tag{B.3}$$

Thus an infinitesimal variation is given by

$$\delta h^{\alpha\beta} = -2\phi h^{\alpha\beta} \tag{B.4}$$

and we can rewrite the expression (3.1) as

$$S_\sigma = -\frac{T}{2}\int d\tau d\sigma \sqrt{-h}\delta h^{\alpha\beta}T_{\alpha\beta} \tag{B.5}$$

which must be equal to zero because S_σ is invariant under a Weyl transformation and since $\sqrt{-h}$ and ϕ are arbitrary then

$$T_{\alpha\beta} = 0 \tag{B.6}$$

Appendix C

C.1 Hermiticity property of $\hat{L}_m$

By definition $\hat{L}_m^\dagger$ is given by

$$\hat{L}_m^\dagger = \frac{1}{2}\sum_{n=-\infty}^{\infty}(:\hat{\alpha}_{m-n}\cdot\hat{\alpha}_n:)^\dagger = \frac{1}{2}\sum_{n=-\infty}^{\infty}:(\hat{\alpha}_{m-n})\dagger\cdot(\hat{\alpha}_n)^\dagger: = \frac{1}{2}\sum_{n=-\infty}^{\infty}:\hat{\alpha}_{n-m}\cdot\hat{\alpha}_{-n}: \tag{C.1}$$

When we calculate the hermician conjugate of each product we do not care about the order of the product, because we are using the normal order. Then if we do the change $n = -k$

$$\hat{L}_m^\dagger = \frac{1}{2}\sum_{k=-\infty}^{\infty}:\hat{\alpha}_{-k-m}\cdot\hat{\alpha}_{-k}: = \hat{L}_{-m} \tag{C.2}$$

C.2 Modified mass-shell condition.

For a spurious state $|\varphi\rangle$ the following condition should be satisfied

$$\langle\varphi|\phi\rangle = 0 \tag{C.3}$$

for any physical state $|\phi\rangle$. Then,

$$\begin{gathered}\hat{L}_0|\varphi\rangle - a|\varphi\rangle = 0 \rightarrow \\ \hat{L}_0\left(\sum_n \hat{L}_{-n}|\chi_n\rangle\right) - a\sum_n \hat{L}_{-n}|\chi_n\rangle = \left(\sum_n \hat{L}_0\hat{L}_{-n}|\chi_n\rangle\right) - a\sum_n \hat{L}_{-n}|\chi_n\rangle = \\ \sum_n\left(\underbrace{[\hat{L}_0,\hat{L}_{-n}]}_{n\hat{L}_{-n}} + \hat{L}_{-n}\hat{L}_0\right)|\chi_n\rangle - a\sum_n \hat{L}_{-n}|\chi_n\rangle = \sum_n\left(n\hat{L}_{-n} + \hat{L}_{-n}\hat{L}_0 - \hat{L}_{-n}a\right)|\chi_n\rangle =\end{gathered} \tag{C.4}$$

$$= \sum_n \hat{L}_{-n}\left(n + \hat{L}_0 - a\right)|\chi_n\rangle = 0 \tag{C.5}$$

So, for all states $|\chi\rangle$ the mass-shell condition is given by

$$\left(\hat{L}_0 - a + n\right)|\chi_n\rangle = 0 \tag{C.6}$$

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