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Topologías no conmutativas y haces

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Contents

Resumen

A mediados de los años 50, Serre traslada la noción de haz sobre un espacio topológico al contexto de variedades algebraicas, reemplazando la topología usual por la de Zariski. Posteriormente, A. Grothendieck generaliza esta idea e introduce los esquemas afines que sustituyen a las variedades algebraicas. Esto supone una revolución en los métodos y conceptos clásicos de la geometría algebraica, y es a partir de entonces cuando se desarrolla una geometría algebraica moderna mediante el lenguaje de haces y esquemas.

El hecho de poder asociar a todo anillo conmutativo un esquema afín que recupera información del anillo, ha permitido elaborar un diccionario geométrico-algebraico completo con el que podemos traducir los conceptos del álgebra conmutativa al lenguaje de la geometría algebraica y viceversa. La potencia de esta herramienta hace que posteriormente, muchos autores intenten obtener un diccionario análogo para el caso de anillos no conmutativos. Esto da lugar a diferentes propuestas de esquemas afines no conmutativos que plantean como espectro no conmutativo generalizaciones directas de la topología de Zariski. En dichos espectros, sólo se obtiene un haz de estructura bajo ciertas condiciones del anillo. Sin embargo, para anillos arbitrarios estas construcciones no devuelven en general un haz sino un prehaz, y tampoco la hacificación resuelve el problema pues al haz obtenido por hacificación generalmente le corresponde un anillo mayor que el de partida.

Más recientemente varios autores se proponen resolver este problema desde un punto de vista diferente, planteando un nuevo concepto donde el espacio está dotado de una estructura no conmutativa, mediante un operador que juega el rol de la intersección de abiertos. Este es el caso de autores tales como García Román, Van Oystaeyen o Borceux-Van den Boscche (cf. [9, 11, 12, 15, 16, 28, 36]).

Bajo este nuevo enfoque, y para cada una de estas topologías no conmutativas, se puede introducir una noción de prehaz o incluso de prehaz de

A mi madre y a la
memoria de mi abuela.

estructura. Sin embargo, la noción de haz no es tan evidente, y aun menos la cuestión de la hacificación, que en estos espacios se plantea como un problema a resolver.

Dentro de este marco, planteamos esta tesis con el objetivo primordial de estudiar el problema de la hacificación en tales topologías no conmutativas. En este sentido, mediante localización abstracta en la categoría de prehaces sobre un conjunto parcialmente ordenado, encontramos haces definidos en topologías no conmutativas, y un funtor hacificación asociado S. Por otro lado, definimos un nuevo tipo de espacio no conmutativo (inspirado en los modelos anteriores), donde además de garantizar la existencia de S, construimos también otro funtor hacificación de manera más topológica y directa, y probamos que éste coincide con S.

Capítulo 1. Categorías de Grothendieck

En un primer capítulo introductorio, con la intención de dotar a esta memoria de cierto grado de auto-contenido, presentamos una recapitulación de los conceptos básicos y principales resultados conocidos sobre teoría general de localización en categorías de Grothendieck. Esta teoría será una herramienta fundamental para el desarrollo de esta memoria, y en particular prestaremos especial atención al caso de las categorías de R-módulos y prehaces, dando en ellas la mayoría de los ejemplos del capítulo.

En la primera sección, comenzamos pues con unas definiciones preliminares y algunos ejemplos que nos permiten asentar los conceptos de *local*, *envolvente inyectiva* y *categoría de Grothendieck*. También enunciamos ciertos resultados relacionados con estos conceptos que utilizaremos con posterioridad.

En la siguiente sección nos planteamos enunciar el *teorema de Gabriel-Popescu* y establecer sus principales consecuencias. Es este un teorema de máxima relevancia dentro del marco de las categorías de Grothendieck, puesto que *grosso modo*, establece que toda categoría de Grothendieck es una *categoría cociente* de $\mathbf{mod}\text{-}R$. Comprender el mencionado enunciado nos obliga a manejar primero los conceptos de *teoría de torsión* y *radical*, así como la relación existente entre ambos.

Las categorías de Grothendieck también se pueden caracterizar como subcategorías de Giraud estrictas de una de **mod**-R, como establece la versión equivalente del teorema de Gabriel-Popescu (cf. 1.3). Es por ello que dedicamos una sección al concepto de *subcategoría de Giraud*, íntimamente ligado al concepto de *reflector*.

Un ejemplo de reflector fundamental para nuestros objetivos es el *funtor hacificación*, que nos permite pasar de la categoría de prehaces a la categoría de haces. En la sección 1.4, repasamos su construcción geométrica en un espacio topológico arbitrario X, según la cual, dicho funtor viene definido como la composición de dos funtores a través de la categoría de *haces concretos* en X.

Las últimas secciones del capítulo las dedicamos al estudio del *funtor localización*, que permite establecer una biyección entre las subcategorías de Giraud y las teorías de torsión. Dicho funtor permite así mismo probar que toda categoría cociente de una categoría de Grothendieck $\mathcal{C}$ es una subcategoría de Giraud de $\mathcal{C}$, y en consecuencia, es también de Grothendieck.

Finalizamos el capítulo describiendo las propiedades del funtor localización en la categoría R-**mod** que serán necesarias para ciertos resultados posteriores. Señalamos que en esta categoría los *filtros de Gabriel* del anillo juegan un rol destacado, pues también están en correspondencia biyectiva con las subcategorías de Giraud de R-**mod**, lo que permite reformular el teorema de Gabriel-Popescu en términos de filtros de Gabriel.

Capítulo 2. Prehaces

El objetivo principal de este capítulo es introducir y estudiar la categoría de prehaces, considerada como una categoría funtorial donde la categoría inicial no es la que tiene por espacio de base el conjunto de abiertos de un espacio topológico ordinario X, sino un conjunto parcialmente ordenado arbitrario E, denominado *poset* (del inglés *parcial ordered set*). Este enfoque generaliza el caso clásico y también nos permite considerar espacios topológicos no necesariamente conmutativos. Así, dada una categoría arbitraria $\mathcal{C}$, definimos la categoría de prehaces en E con valores en $\mathcal{C}$ como la categoría funtorial $\mathrm{Fun}(\mathcal{E}^{opp}, \mathcal{C})$, y la denotamos por ${}_{\mathcal{C}}\mathcal{P}(E)$. Por ser funtorial hereda muchas de las propiedades de $\mathcal{C}$. De este modo, en 2.1.7 probamos que la categoría de prehaces es de Grothendieck si $\mathcal{C}$ es de Grothendieck, y que si U es un generador para $\mathcal{C}$, entonces el prehaz G definido en cada $b \in E$ por

$$G(b) = U^{(E_b)},$$

es un generador para ${}_{\mathcal{C}}\mathcal{P}(E)$, donde $E_b = \{a \in E \ / \ b \leq a\}$.

Por otro lado, estudiamos la categoría de *prehaces de R-módulos a izquierda* en E, denotada por R-**pre-Mod**, donde R es un prehaz de anillos (no necesariamente conmutativos) definido sobre el poset E. Aunque se trata ahora de una categoría que no es funtorial, con un tratamiento diferente probamos en 2.2.5 que también tiene estructura de categoría de Grothendieck.

Una vez que hemos garantizado dicha estructura, estudiamos el funtor localización en R-**pre-Mod** haciendo uso de los resultados contenidos en el capítulo anterior. Así, en la sección 2.3 demostramos que si M es un prehaz de R-módulos a izquierda y σ un radical en R-**pre-Mod**, entonces, bajo ciertas condiciones de R y σ, la localización de M en σ es un prehaz de $Q_\sigma R$-módulos a izquierda dado para todo $a \in E$ por

$$Q_\sigma M(a) = Q_{\sigma(a)} M(a);$$

es decir, demostramos que el funtor localización en R-**pre-Mod** actúa *localmente* como el funtor localización en R-**mod**.
Un estudio sobre localización en la categoría de prehaces de R-módulos a izquierda sobre un espacio topológico ordinario puede encontrarse en [37]. Los resultados que presentamos en este capítulo no sólo generalizan el estudio anterior sino que desde nuestro punto de vista lo mejoran colocándolo en el marco correcto, pues en algunos resultados de [37] faltan o redundan hipótesis.

Capítulo 3. Topologías no conmutativas

Como hemos señalado, un prehaz es un funtor, y por tanto puede tratarse como un objeto estrictamente algebraico, olvidando así las propiedades topológicas del espacio sobre el que está definido. Sin embargo, no es este el caso de un haz, para cuyo manejo la topología se vuelve indispensable desde la propia definición del objeto. De hecho, esencialmente un haz no es más que un prehaz separado que además *pega bien* en las intersecciones de los recubrimientos.

Dado que uno de los principales objetivos de esta memoria es definir la categoría de haces en espacios topológicos no necesariamente conmutativos, el primer paso es pues estudiar cómo son dichos espacios. Por este motivo, nos planteamos en este capítulo hacer una recapitulación de algunos de los diferentes tipos de espacios topológicos no conmutativos que pueden encontrarse en la literatura, así como también proponer un nuevo modelo de sitio no conmutativo, que sea más adecuado para construir el funtor hacificación en un contexto no necesariamente conmutativo.
Como se observará, estos modelos de topologías no conmutativas poseen al menos una característica básica común: sobre el conjunto de *abiertos* de la topología está definida una operación *intersección* que no es necesariamente conmutativa.

Comenzamos por recordar el concepto de *sitio* según la filosofía de las *topologías de Grothendieck*, como una buena aproximación categórica al concepto de espacio topológico, y además porque desde este punto de vista podemos acercarnos a los diferentes ejemplos que tratamos.

El primer ejemplo que abordamos es el *sitio no conmutativo à la* García Román, que basa su construcción en el empleo de los *filtros de Gabriel* de un anillo asociativo y unitario, y donde la *composición de filtros* juega el papel de la intersección de abiertos. Sobre este ejemplo cabe destacar que generaliza la topología de Zariski sobre el espectro primo de un anillo conmutativo.

A continuación, en la sección 3.3 presentamos la topología no conmutativa propuesta por F. Van Oystaeyen en [36] y [28], que consiste en un sistema axiomático definido sobre un poset Λ con dos operaciones $\vee$, $\wedge$. Además de la presupuesta no conmutatividad de la operación $\wedge$, esta topología se distingue del resto porque no todos los elementos $\lambda \in \Lambda$ satisfacen la propiedad de *idempotencia* $\lambda \wedge \lambda = \lambda$. Posteriormente, vemos cómo el autor define en Λ una noción de *topología de Grothendieck no conmutativa de un sitio no conmutativo*, de manera similar a como ocurre en el caso clásico.

La tercera propuesta la constituyen los *espacios cuánticos* de Borceux-Van den Bossche (cf. 3.4), inspirada en la filosofía de *cuantales*. En este caso, el papel de la intersección clásica lo juega una operación no necesariamente conmutativa denominada *multiplicación* y denotada por &. De hecho, en los casos en los que & es conmutativa el espacio cuántico se convierte en un espacio topológico ordinario. A grandes rasgos, los espacios cuánticos satisfacen todas las propiedades de un espacio topológico clásico salvo la referida a la intersección finita de abiertos. En concreto, la intersección de dos abiertos no es necesariamente un abierto de la topología sino que está contenida en el abierto formado por la multiplicación de ambos.

En la sección 3.5 presentamos finalmente nuestra propuesta, que denominamos *Q-sitio*, cuya construcción también basamos en la estructura de los *cuantales*.
Así como la noción de *local* se acepta como aproximación algebraica a espacios topológicos, aceptamos su análogo no conmutativo, esto es, el *cuantal* (traducción del término *quantale*), como aproximación al de espacio topológic no conmutativo. Un cuantal es un retículo con una operación multiplicación no necesariamente conmutativa, que denotamos por &, satisfaciendo ciertos axiomas. Dado un cuantal $\mathcal{Q}$, definimos para cada $U \in \mathcal{Q}$ los *Q-recubrimientos* de U como las familias $\{U_i\}_{i \in I}$ en $\mathcal{Q}$ que verifican

C1) $U = \bigvee_{i \in I} U_i$;

C2) para cada $i \in I$, $U_i = U \,\&\, U_i$;

y denotamos el conjunto de Q-recubrimientos por $\mathrm{Cov}(U)$. De este modo, un *Q-sitio* es un par de la forma $(\mathcal{Q}, \{\mathrm{Cov}(U)\}_{U \in \mathcal{Q}})$.
A continuación describimos algunas de las propiedades de los Q-recubrimientos (cf. 3.5.9), definimos en cada $\mathrm{Cov}(U)$ un orden, que denotamos por $\preceq$, y probamos en 3.5.11 que $\mathrm{Cov}(U)$ *es un conjunto dirigido considerando en él el orden inverso de* $\preceq$. Finalmente, en 3.5.13 probamos que un Q-sitio es en efecto un sitio en el sentido estricto, pues verifica los tres axiomas de la topología de Grothendieck.
Todos estos resultados nos serán de gran utilidad en el capítulo 5, que dedicamos a la construcción de haces y hacificación en un Q-sitio.

Capítulo 4. Haces y hacificación

Uno de los objetivos de este capítulo es dar una definición de haces lo más general posible, de manera que sea válida tanto en el caso clásico como también para los diferentes sitios no conmutativos, es decir, para una topología no necesariamente conmutativa. Así mismo, nos planteamos el objetivo de encontrar un funtor hacificación de la categoría de prehaces en esta nueva categoría de haces. Haciendo uso de la teoría general de localización en categorías de Grothendieck, alcanzamos ambos objetivos al mismo tiempo y además evitamos el uso de fibras, que sin embargo era indispensable en la construcción de hacificación clásica descrita en 1.4.

Retomando el enfoque del capítulo 2, el primer paso es decidir cuáles son los requerimientos mínimos para que se pueda definir una categoría de *prehaces separados* sobre un poset, teniendo en cuenta que nos interesa la *mínima estructura topológica* que permita una definición ambivalente. En este sentido, nos basta con considerar un poset T tal que cada $a \in T$ tenga asignado un conjunto de *cuasi-recubrimientos*, que denotamos por $C(a)$, esto es, un conjunto de subconjuntos $\{a_i\}_{i\in I}$ de T satisfaciendo para todo $i \in I$ la condición $a_i \leq a$.
Este es obviamente el caso del poset de un espacio topológico ordinario, así como también el de una topología no conmutativa *à la* Van Oystaeyen, o el de un Q-sitio.
Para un poset T que cumpla dicha condición, definimos pues la *categoría de prehaces separados en T con valores en $\mathcal{C}$*, denotada por ${}_{\mathcal{C}}\mathcal{F}(T)$, como la subcategoría plena de ${}_{\mathcal{C}}\mathcal{P}(T)$ cuyos objetos son los prehaces P que verifican, para cada $a \in T$ y cada cuasi-recubrimiento $\{a_i\}_{i\in I}$ de a, que la aplicación

$$\xi : P(a) \longrightarrow \prod_{i\in I} P(a_i); \quad s \mapsto (s|_{a_i})_{i\in I},$$

es inyectiva. Así, esta definición de prehaces separados es válida para cualquiera de los ejemplos recién mencionados, en un contexto no necesariamente conmutativo.
En esta situación general, encontramos en 4.1.10 un generador para ${}_{\mathcal{C}}\mathcal{F}(T)$, y probamos en 4.1.7 que la categoría es completa si lo es $\mathcal{C}$; más aún, probamos que la clase de objetos de ${}_{\mathcal{C}}\mathcal{F}(T)$ forman una clase libre de torsión para alguna teoría de torsión de ${}_{\mathcal{C}}\mathcal{P}(T)$ (cf. 4.1.9).

El objetivo principal del capítulo lo alcanzamos en la sección 4.2, donde finalmente obtenemos la categoría de haces en una topología no necesariamente conmutativa, junto con un funtor hacificación asociado.
La clave consiste en demostrar que la clase de objetos de la categoría ${}_{\mathcal{C}}\mathcal{F}(T)$ es cerrada para extensiones esenciales (cf. 4.2.4), y así, en particular, cerrada para envolventes inyectivas. Para ello necesitamos imponer una condición extra sobre los cuasi-recubrimientos del poset:

(C) para todo $b \leq a$ en T, si $\{a_i\}_{i\in I}$ es un cuasi-recubrimiento de a entonces existe un cuasi-recubrimiento $\{b_i\}_{i\in I}$ de b tal que para todo $i \in I$ tenemos $b_i \leq a_i$.

El hecho de que la clase de objetos de ${}_{\mathcal{C}}\mathcal{F}(T)$ sea cerrada para envolventes inyectivas, unido a que en la sección anterior hemos probado que dicha clase es una clase libre de torsión de una teoría de torsión, nos garantiza que estamos ante una teoría de torsión hereditaria, y que por tanto, tiene asociado un único radical, que denotamos por τ_S. Según la teoría general de localización, podemos definir los haces en T como los objetos de la categoría cociente de ${}_{\mathcal{C}}\mathcal{P}(T)$ con respecto a τ_S, y la denotamos por ${}_{\mathcal{C}}\mathcal{S}(T)$. Además esta categoría es de Grothendieck, por ser subcategoría de Giraud de una de Grothendieck. A su vez, el reflector asociado a τ_S,

$$S:\ {}_{\mathcal{C}}\mathcal{P}(T) \longrightarrow\ {}_{\mathcal{C}}\mathcal{S}(T),$$

es el *funtor hacificación* buscado.
Como casos particulares, concluimos igualmente que *existe hacificación no conmutativa en cada topología no conmutativa* Λ à la *Van Oystaeyen, y Q-hacificación en cada Q-sitio.*

Por último, dedicamos una sección a estudiar la hacificación en la categoría R-**pre-Mod**, para R un haz de anillos (no necesariamente conmutativos) definido sobre un poset. Concluimos que bajo las hipótesis de que el poset tenga cuasi-recubrimientos satisfaciendo la condición (C) y de que R sea sobreyectivo (*flabby*), entonces podemos obtener la categoría de haces de R-módulos a izquierda, que denotamos por R-**Mod**, junto con un funtor hacificación

$$S_R : R\text{-}\mathbf{pre\text{-}Mod} \longrightarrow R\text{-}\mathbf{Mod},$$

de manera similar a como lo obtenemos para la categoría ${}_{\mathcal{C}}\mathcal{P}(T)$ en las secciones anteriores (cf. 4.3.5).

Capítulo 5. Haces y hacificación en Q-sitios

En el último capítulo de esta memoria nos centramos en los Q-sitios. En el capítulo 4 definimos la categoría de Q-haces via localización en la categoría de Q-prehaces, un procedimiento que generaliza naturalmente el punto de vista de las subcategorías de Giraud del caso conmutativo ordinario. Lo que hacemos ahora en el contexto de Q-sitios, desde un planteamiento completamente diferente, es introducir directamente una definición de Q-haz más natural e intuitiva, y construir después un funtor Q-hacificación asociado. Sorprendentemente, lo que resulta es que en este caso particular ambas construcciones coinciden.

En primer lugar, definimos la categoría de haces sobre un Q-sitio arbitrario $(\mathcal{Q}, \{\mathrm{Cov}(U)\}_{U\in\mathcal{Q}})$, que denominamos *categoría de Q-haces*, y que denotamos por ${}_{\mathcal{C}}\mathcal{S}h(\mathcal{Q})$, como la subcategoría plena de ${}_{\mathcal{C}}\mathcal{P}(\mathcal{Q})$ cuyos objetos son los Q-prehaces separados P que satisfacen además la *condición de pegado*:

Sh2) si $U \in \mathcal{Q}$, si $\{U_i\}_{i\in I} \in \mathrm{Cov}(U)$, y si para cada $i \in I$ existe un $s_i \in P(U_i)$ tal que para todo $i, j \in I$ se verifica que $s_i|_{U_i \,\&\, U_j} = s_j|_{U_i \,\&\, U_j}$, entonces existe un $s \in P(U)$ satisfaciendo para todo $i \in I$ que $s|_{U_i} = s_i$.

A continuación para cada Q-prehaz P construimos en 5.2.4 un nuevo Q-prehaz que denotamos por LP, sobre el que basaremos la nueva noción de *Q-hacificación*. En este caso no son las fibras la herramienta utilizada sino los límites y sus bien conocidas propiedades universales (límites directos sobre los conjuntos de índices de los Q-recubrimientos, y límites inversos sobre los conjuntos dirigidos de Q-recubrimientos). Generalizando propiedades del caso clásico, probamos dos resultados fundamentales para la construcción de la Q-hacificación: *para todo Q-prehaz P, el Q-prehaz LP es separado* (cf. 5.2.5); *si además P es separado entonces LP es un Q-haz* (cf. 5.2.6).

En la sección 5.3 describimos el *funtor Q-hacificación* en la nueva categoría de Q-haces:

$$a : {}_{\mathcal{C}}\mathcal{P}(\mathcal{Q}) \longrightarrow {}_{\mathcal{C}}\mathcal{S}h(\mathcal{Q}) .$$

Definimos dicho funtor sobre cada $P \in {}_{\mathcal{C}}\mathcal{P}(\mathcal{Q})$ como el Q-haz aP que viene dado para todo $U \in \mathcal{Q}$ por

$$aP(U) = \varinjlim (LP)^U = \varinjlim{}_{\mathcal{U}\in \mathrm{Cov}(U)}(\varprojlim (LP)\mathcal{U});$$

sobre un morfismo arbitrario $f : P \to P'$ de Q-prehaces, el morfismo de Q-haces $a(f) : aP \to aP'$ consiste en la colección de morfismos de $\mathcal{C}$ dada para cada $U \in \mathcal{Q}$ por

$$a(f)(U) : aP(U) \longrightarrow aP'(U); \quad s \mapsto (\eta')^2_{\mathcal{U}}((L(f)(U_i)(x_i))_{i\in I}).$$

El hecho de que a sea un reflector es lo que nos permite denominarlo *funtor Q-hacificación*. Esto es lo que probamos en el teorema 5.5.1. Previamente, dedicamos la sección 5.4 a desarrollar los resultados técnicos necesarios para simplificar la demostración de dicho teorema. Así mismo, en la sección 5.5 establecemos algunas de sus consecuencias, como por ejemplo, que la categoría de Q-haces es de Grothendieck (siempre que tome valores en una categoría de Grothendieck).

Finalmente en la sección 5.6 resolvemos la cuestión que surge de manera natural tras haber obtenido dos categorías de Q-haces y dos funtores Q-hacificación a priori distintos. Como es de esperar, en 5.6.6 probamos que en efecto ${}_{\mathcal{C}}\mathcal{S}h(\mathcal{Q})$ coincide con ${}_{\mathcal{C}}\mathcal{S}(\mathcal{Q})$ y que a es naturalmente equivalente a $S_{\mathcal{Q}}$.

El estudio de la funtorialidad de nuestra construcción de Q-hacificación lo realizamos en la sección 5.7, donde probamos cómo funciona dicho funtor a través de dos Q-sitios entre cuyos respectivos cuantales hay definido un morfismo estricto. En primer lugar, para cada morfismo $f : \mathcal{Q}_1 \to \mathcal{Q}_2$ definimos un *funtor imagen directa*

$$f_* : {}_{\mathcal{C}}\mathcal{P}(\mathcal{Q}_2) \longrightarrow {}_{\mathcal{C}}\mathcal{P}(\mathcal{Q}_1),$$

que relaciona Q-prehaces entre los correspondientes Q-sitios. Tras comprobar que f_* restringe bien a las subcategorías plenas de Q-prehaces separados y de Q-haces respectivamente, concluimos en 5.7.5 que además conmuta con el funtor Q-hacificación cuando f es biyectiva.

En la sección 5.8 y cerrando la memoria, comprobamos que los objetivos alcanzados para la categoría ${}_{\mathcal{C}}\mathcal{P}(\mathcal{Q})$ en las secciones anteriores, pueden obtenerse de manera similar para la categoría de Grothendieck R-**pre**-${}_{\mathcal{Q}}$**Mod**. Es decir, en primer lugar definimos explícitamente la categoría de haces de R-módulos sobre un Q-sitio, para R un Q-haz de anillos no necesariamente conmutativos, que denotamos por R-${}_{\mathcal{Q}}$**Mod**. A continuación construimos un funtor Q-hacificación

$$a_R : R\text{-}\mathbf{pre}\text{-}{}_{\mathcal{Q}}\mathbf{Mod} \longrightarrow R\text{-}{}_{\mathcal{Q}}\mathbf{Mod};$$

y finalmente, probamos que si R es un prehaz sobreyectivo entonces a_R es naturalmente equivalente al funtor hacificación obtenido en la sección 4.3.

Introduction

Modern algebraic geometry essentially originated, both in concepts and methods, with the introduction of *sheaves* by J. Leray in 1945 and their application to *abstract varieties* by Serre in 1954. Later on, in 1957, Grothendieck generalized algebraic geometry even more, by extending the notion of *affine varieties* to that of *affine schemes*.

Associating to any commutative ring R its affine scheme Spec R defines a dictionary between commutative algebra and algebraic geometry, which has proven to be of extremely high benefit to both branches of mathematics. Indeed, on the one hand, translating algebraic problems to their geometric counterpart allows for geometric intuition and geometric tools to help and solve the original problem, whereas, conversely, many a geometric problem may be solved by applying purely algebraic tools.

It thus hardly came as a surprise that algebraists and geometers intended to define an analogous dictionary in the noncommutative case, i.e., for arbitrary, not necessarily commutative rings. Several constructions of an *affine scheme* associated to a noncommutative ring R may thus be found in the literature, based on different variants of the notion of prime ideal, ranging from two-sided, left or strong prime ideals to primitive ideals, indecomposable injectives or prime torsion theories, –we refer to the references and the literature for details. In general, these constructions only appear to work optimally for particular, relative broad ranges of rings (e.g., constructions with two-sided prime ideals work, of course, best when the ring has many of these, such as is the case for fully bounded noetherian rings or, in particular, rings with polynomial identities).

In most of these constructions a reasonable analog of the usual Zariski topology may be defined, allowing for the introduction of a structure sheaf over this *spectrum* associated to the base ring R or modules over it. Let us already point out here that, in almost all known constructions, general localization theory in the sense of Gabriel and others plays a fundamental role. Unfortunately, in the noncommutative case, this structure sheaf is usually not a sheaf, except if the base ring has some extra, limiting properties, such as being prime, for example. Applying sheafification to the structure presheaf is, in general, not a solution, even if the presheaf is separated. Indeed, whereas for most constructions the base ring R may be recovered from the structure presheaf by taking global sections, the ring of global sections of the associated sheaf will usually be larger than the original ring, i.e., we only obtain a one way dictionary, translating from algebra to geometry, without being able to go back without losing information.

Rather recently, several authors [9, 11, 12, 15, 16, 28, 36] took a rather different point of view: instead of studying generalizations of the prime spectrum of a commutative ring with a straightforward generalization of the Zariski topology, they considered spaces endowed with a noncommutative, topology-like structure, where the noncommutativity essentially amounts to working with an *intersection* operator & for which we do not necessarily have that $U\&V = V\&U$ or even $U\&U = U$! The *open sets* U one considers in this set-up may vary from ordinary open sets to elements of particular lattices or even equivalence classes of words based on Gabriel filters - details and examples will be given in the text. For each of these *noncommutative topologies* a reasonable notion of presheaf or even structure presheaf may be introduced, the idea of an associated sheaf (or even a suitable notion of a sheaf!) still lacking.

The main purpose of this thesis is to study the notion of a sheaf on such noncommutative topologies and to obtain a *sheafification functor* in this noncommutative context. We do find a sheafification functor S through abstract localization in a category of presheaves on a *poset* (partially ordered set) T. In particular, in 4.2.9 we prove that *for every Grothendieck category* $\mathcal{C}$ *there exists a functor* $S : {}_{\mathcal{C}}\mathcal{P}(T) \longrightarrow {}_{\mathcal{C}}\mathcal{S}(T)$, *left adjoint of the inclusion functor* i_{τ_S}, *and such that*

$${}_{\mathcal{C}}\mathcal{S}(T) = \{P \in {}_{\mathcal{C}}\mathcal{P}(T) \mid P = i_{\tau_S}SP = Q_{\tau_S}P\}.$$

In this way we reach our aim, since this result may be particularized for some of the examples of noncommutative topologies. Moreover, working over what we called *Q-sites* (we refer to the text for a precise definition of this particular kind of noncommutative spaces), we are able to introduce another sheafification functor a, this time in a more constructive, topological way. We prove in 5.5.1 that a is indeed a reflector, and show in 5.6.5 that S and a actually

coincide.
All of these results are also considered into the context of presheaves of left R-modules, where R is a presheaf of rings on T. Along the way, we will show how some results in the literature dealing with the abstract localization theory of presheaves may be strengthened (or even corrected).

Before presenting some details about the contents of this text, let us stress that several of our results could have been developed in the wider context of *topos theory* (as kindly pointed out by F. Borceux). The main reason why we preferred to stick to a more limited, down to earth approach is that on the one hand some noncommutative topologies *à la* Van Oystaeyen do not use classical pullbacks and thus seem to fall outside of the scope of classical (Grothendieck) topos theory while on the other hand we hope that our "elementary" approach would make our results more accessible and transparent to the non-specialized reader. Taking a more detailed look at the "topos theory approach" remains part of our *to-do list*.

Let us now take a somewhat more detailed look at the contents of this thesis.

In a first, introductory chapter (which we made relatively self-contained for the reader's convenience) we present the background on Grothendieck categories which will be needed throughout this text, emphasizing classical results as the Gabriel-Popescu theorem and the use of Giraud subcategories in the context of localization and (classical!) sheafification. There is essentially nothing new in this chapter, we just wanted to explicitly link the notions of torsion, localization and reflectors.

In the second chapter, we present a detailed study of presheaves over a poset E and with values in an arbitrary category $\mathcal{C}$. We choose the poset point of view, since it generalizes the classical case and also includes the case of the noncommtutative topologies over which we want to consider sheafification. We prove the Grothendieck category structure of the category of presheaves on E with values in a Grothendieck category. Moreover, we also prove that the category of presheaves of left R-modules on E, where R is a presheaf of rings on E, has a Grothendieck category structure, as it will appear later that sheafification may be introduced from within this context too. Special emphasis is put on the localization theory within the latter category. In particular, we generalize and extend several results previously proved over ordinary topological spaces and show how several of these may be strengthened or reformulated with their proper hypotheses.

Our third chapter deals with different types of noncommutative topologies recently introduced in the literature. These are the ones considered by García Roman [15, 17], Van Oystaeyen [36, 28] and Borceux-Van den Bossche [12]. Essentially all of them may be approached from the Grothendieck topology point of view. This implies that other morphisms than just inclusion may occur (as in [15]), and that intersections will necessarily be replaced by pull-back operators. In the noncommutative case, it will also be clear that a noncommutative analog of pull-backs will be needed. In a final section, we will present our own alternative which we will use throughout the rest of the thesis. These are the so-called *Q-sites*, which basically consist of a quantale $\mathcal{Q}$ endowed with a particular family of *Q-coverings* which constitutes a Grothendieck topology on $\mathcal{Q}$. In particular, we remark that the set of Q-coverings is directed, a basic property for the construction of a.

In the fourth chapter, we finally come to the main topic of this text: sheaves and sheafification. We define the category of separated presheaves on a poset T with *quasi-coverings*. Through this definition, which generalizes the classical one, we obtain separated presheaves in the examples of noncommutative topologies we want to consider. The principal result of this chapter is that, under certain condition imposed on the quasi-coverings, the class of separated presheaves on T is a torsion-free class for some hereditary torsion theory in the category of presheaves. Thus, making use of the general theory of localization in Grothendieck categories, we obtain a *sheafification functor* over noncommutative topologies, where our sheaves are exactly the closed objects for this functor. Applying a similar procedure, in a final section we obtain a sheafification functor in the Grothendieck category of presheaves of left R-modules and a category of sheaves of left R-modules, where R is a flabby sheaf of rings on T.

In the last chapter, we focus our attention on a special type of noncommutative topological space, the so-called Q-sites we introduced before. Mimicking the classical definition in this context, we present a more concrete definition of sheaf than that of just being an “object of a particular quotient category”. It appears that a more intuitive construction of sheafification functor may be given in this case. Somewhat surprisingly (at least in the noncommutative context), both approaches turn out to be the same, i.e., we prove that the two sheafification functors introduced in this thesis are naturally equivalent over Q-sites, and that their respectively associated categories of sheaves coincide. We conclude this final chapter with a look at functoria ity aspects of this construction with respect to base change, and a final loo at the behaviour of the *intuitive* sheafification functor over presheaves of le R-modules, where R is a sheaf of rings on a Q-site.

Arrived at this point, we are now ready to consider generalizations large parts of algebraic geometry, such as studying cohomology or invariant like the Picard group. It is our aim to come back to this in future work, jus as the problem of trying to adapt our methods to context of noncommutativ topologies in the sense of Van Oystaeyen, where the lack of the idempotenc property clearly will complicate matters.

Chapter 1

Grothendieck categories

The general theory of localization in Grothendieck categories represents a fundamental tool for the development of this work. Thus, in this chapter we summarize its well-known concepts and main results. In particular, we focus our interest on the categories of presheaves and R-modules, to which are related the main examples of this chapter.

1.1 Preliminary definitions

In order to add some self-content to this work, we start by giving the basic definitions which are going to be used throughout the following chapters. We begin with the concept of *lattice* and continue with the elementary definitions and the most common examples related to the concepts of *Grothendieck category* and *injective hull*.

1.1.1 Definition. A set E with a partial ordering $\leq$ is said to be a *poset*. An element $a \in E$ which satisfies for all $x \in E$ that $a \leq x$, is called a *zero element*; an element $b \in E$ which satisfies for all $x \in E$ that $x \leq b$, is called a *unit element*. A zero (resp. unit) element of E is necessarily unique if it exists; we will then denote it by 0 (resp. 1).
A poset E is said to be *directed* if for every pair of elements $x, y \in E$ there exists an element $z \in E$ such that $z \leq x$ and $z \leq y$.

1.1.2 Definition. A *lattice* is a poset $(L, \leq)$ in which every couple of elements x, y has a least upper bound called the *join* of x and y (written $x \vee y$), and a greatest lower bound called the *meet* of x and y (written $x \wedge y$). It then follows by induction that every nonempty finite set of elements has a join and a meet.

If L and L' are lattices, then a *morphism* $\alpha : L \to L'$ is a map from L to L' satisfying for all $x, y \in L$

$$\alpha(x \vee y) = \alpha(x) \vee \alpha(y) \text{ and } \alpha(x \wedge y) = \alpha(x) \wedge \alpha(y).$$

In this way one obtains the *category* **Lat** *of lattices.*

1.1.3 Definition. A lattice $(L, \leq)$ is called *distributive* if it satisfies the following equivalent conditions

i) $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$;

ii) $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$;

iii) $(x \vee z) \wedge y \leq (x \wedge y) \vee z$.

for all $x, y, z \in L$.

1.1.4 Definition. A lattice $(L, \leq)$ is called *complete* if every subset S of L has a least upper bound, written $\sup S$ or $\bigvee_{s\in S} s$, called the *join* of S, or equivalently, a greatest lower bound, written $\inf S$ or $\bigwedge_{s\in S} s$ and called the *meet* of S (cf. [33, III,prop.1.2]).
From the very definition it follows that in a complete lattice there exists a greatest element $1 = \sup L$ and a smallest element $0 = \inf L$.

1.1.5 Definition. A *locale* is a complete lattice $(L, \leq)$ in which arbitrary joins distribute over finite meets, i.e. the distributivity law

$$a \wedge (\bigvee_{i\in I} b_i) = \bigvee_{i\in I}(a \wedge b_i)$$

holds, where I is an arbitrary indexing set and a, b_i are elements of L.

1.1.6 Example. Let X be an arbitrary set. A basic example of locale is the lattice 2^X of subsets of X ordered by the inclusion, with the meet and the join given by the intersection and the union of subsets respectively, since obviously arbitrary unions of subsets distribute over intersections. In this case X and $\emptyset$ are the greatest and the smallest element respectively.

1.1.7 Example. Another classical example of locale is the lattice open subsets of a topological space X ordered by the inclusion, with the meet and the join given by the intersection and the union of open subsets respectively, and where the greatest and the smallest element are respectively X and $\emptyset$.

1.1.8 Definition. A category is said to be *small* if its class of objects is a set. For example, a poset $(E, \leq)$ may be viewed as a small category $\mathcal{E}$ whose objects are the elements of E and the set $\mathrm{Hom}_{\mathcal{E}}(a, b)$ of morphisms is a singleton when $a \leq b$ and is empty otherwise.

1.1.9 Definition. A category $\mathcal{C}$ is said to be *preadditive* if each set of morphisms $\mathrm{Hom}_{\mathcal{C}}(C_1, C_2)$ is an abelian group and if the composition maps

$$\mathrm{Hom}_{\mathcal{C}}(C_1, C_2) \times \mathrm{Hom}_{\mathcal{C}}(C_2, C_3) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(C_1, C_3); \quad (f, g) \mapsto g \circ f$$

are biadditive, for any $C_1, C_2, C_3 \in \mathcal{C}$.

1.1.10 Definition. A preadditive category $\mathcal{C}$ is said to be *additive* if it has a zero object 0 and a biproduct for each pair of its objects.

1.1.11 Definition. A category $\mathcal{C}$ is said to be *abelian* if

A1) $\mathcal{C}$ is preadditive;

A2) every finite family of objects in $\mathcal{C}$ has a product (and a coproduct);

A3) every morphism $f : C \to C'$ has a kernel $\ker f : \mathrm{Ker}\, f \to C$ and a cokernel $\mathrm{coker}\, f : C' \to \mathrm{Coker}\, f$;

A4) for every morphism $f : C \to C'$ in $\mathcal{C}$, the canonical morphism

$$\bar{f} : \mathrm{Coker}\,(\ker f) \to \mathrm{Ker}\,(\mathrm{coker}\, f)$$

is an isomorphism.

For historical reasons, the last condition A4) is known as the *Ab2* condition, according to Grothendieck's original work (cf. [20]).

1.1.12 Examples.

1. The category **ab** (resp. **gr**) of abelian groups (resp. graded abelian groups) is an abelian category.

2. Let R be an (associative) ring with unity. The category R-**mod** (resp. **mod**-R) of left (resp. right) R-modules is an abelian category.

3. If R is a *graded ring*, i.e. $R = \bigoplus_{n \in \mathbb{Z}} R_n$ where the R_n are additive subgroups of R with the property that $R_m R_n \subseteq R_{m+n}$, for all $m, n \in \mathbb{Z}$, then a graded abelian group $M = \bigoplus_{n \in \mathbb{Z}} M_n$ is said to be a *graded left* (resp. *graded right*) *R-module* if $M \in R$-**mod** (resp. **mod**-R) and

$R_m M_n \subseteq M_{m+n}$ (resp. $M_n R_m \subseteq M_{m+n}$), for all $m, n \in \mathbb{Z}$.
An R-linear map

$$f : M = \bigoplus_{n \in \mathbb{Z}} M_n \longrightarrow N = \bigoplus_{n \in \mathbb{Z}} N_n$$

is said to be *graded* or *homogeneous (of degree 0)*, if for every $n \in \mathbb{Z}$ the map f restricts to $f|_{M_n} : M_n \to N_n$.
The category of graded left (resp. right) R-modules and graded R-linear maps, denoted by R-gr (resp. gr-R), is an abelian category.

4. Let X be a topological space. A *presheaf of abelian groups P on X* consists of the data:

   i) for all U open subset of X, an abelian group $P(U)$;

   ii) for every inclusion $V \subseteq U$ of open subsets in X, a homomorphism of abelian groups

   $$P_{UV} : P(U) \to P(V);\ s \in P(U) \mapsto P_{UV}(s) := s|_V$$

   called the *restriction morphism*,

   subject to the conditions:

   1) for every open subset U of X, $P_{UU} = id_{P(U)}$;

   2) if $W \subseteq V \subseteq U$ are open subsets of X, then $P_{UW} = P_{VW} \circ P_{UV}$.

   A *morphism $f : P \to P'$ of presheaves of abelian groups* consists of a family $\{f(U) : P(U) \to P'(U)\}$ of homomorphisms of abelian groups such that, for every inclusion $V \subseteq U$ of open subsets in X, the following diagram is commutative

$$\begin{tikzcd} P(U) \arrow[r, "f(U)"] \arrow[d, "P_{UV}"'] & P'(U) \arrow[d, "P'_{UV}"] \\ P(V) \arrow[r, "f(V)"'] & P'(V) \end{tikzcd}$$

   where P_{UV} and P'_{UV} are the restriction morphisms of P and P' resp.
   The category ${}_{\mathbf{ab}}\mathcal{P}(X)$ of presheaves of abelian groups on X is an abelian category.
   This category may also be defined over a basis B_X for the topology on X, taking the open subsets only from B_X instead of from the whole

space. In this case we obtain another category of presheaves of abelian groups on B_X, and will be denoted by ${}_{\mathbf{ab}}\mathcal{P}(B_X)$.

More generally, let R be an (associative) ring with unity. If in the previous definition we take the category R-**mod** of left R-modules instead of the category of abelian groups, then we obtain the categories ${}_{R\text{-}\mathbf{mod}}\mathcal{P}(X)$ and ${}_{R\text{-}\mathbf{mod}}\mathcal{P}(B_X)$ of presheaves of left R-modules, which are also abelian categories.

In a similar way we obtain the categories ${}_{\mathbf{Sets}}\mathcal{P}(X)$ and ${}_{\mathbf{Sets}}\mathcal{P}(B_X)$ of presheaves of sets by just taking sets instead of abelian groups. However, Sets does not have enough structure to endow these categories with an abelian structure.

1.1.13 Definition. A (covariant) functor $T : \mathcal{C} \to \mathcal{D}$ between abelian categories is said to be *additive* if for all $f, g \in \operatorname{Hom}_{\mathcal{C}}(C, C')$,

$$T(f + g) = T(f) + T(g).$$

This is equivalent to asserting that T commutes with finite products (or coproducts).

1.1.14 Definition. An additive functor $T : \mathcal{C} \to \mathcal{D}$ is said to be *left exact* if each short exact sequence

$$0 \to C' \to C \to C'' \to 0$$

in $\mathcal{C}$ induces an exact sequence

$$0 \to T(C') \to T(C) \to T(C'')$$

in $\mathcal{D}$. Right exact functors are defined similarly. If T is assumed to be both left and right exact then it is said to be *exact*. In this case, T maps arbitrary exact sequences in $\mathcal{C}$ to exact sequences in $\mathcal{D}$.

1.1.15 Proposition. *Let $T : \mathcal{C} \to \mathcal{D}$ be an additive functor between abelian categories. Then the following are equivalent:*

i) T is left exact;

ii) T preserves kernels;

iii) T preserves finite limits (i.e. projective limits over small categories with finitely many objects and morphisms);

iv) T preserves pullbacks.

Proof. [33, IV,prop.8.6] and [29, 2.1]. ☐

1.1.16 Definition. Let $\mathcal{C}$ and $\mathcal{D}$ be preadditive categories and $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ additive functors. Then G is said to be a *right adjoint* of F, F a *left adjoint* of G and (F, G) an *adjoint pair*, if there exists a natural equivalence

$$\eta : \mathrm{Hom}_{C}(\,\cdot\,, G(\cdot)) \cong \mathrm{Hom}_{D}(F(\cdot), \,\cdot\,)$$

of functors $\mathcal{C}^{opp} \times \mathcal{D} \to$ **ab**, i.e. for any $C \in \mathcal{C}$ and $D \in \mathcal{D}$ there is an isomorphism

$$\eta_{C,D} : \mathrm{Hom}_{\mathcal{C}}(C, G(D)) \cong \mathrm{Hom}_{\mathcal{D}}(F(C), D)$$

which is natural in C and D. We denote this by $F \dashv G$.
When they exist, a right adjoint preserves limits (projective limits) whereas a left adjoint preserves colimits (inductive limits), (cf. [33, IV,prop.9.4]).

1.1.17 Proposition. *Let $\mathcal{C}$ and $\mathcal{D}$ be abelian categories and $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ covariant functors. If $F \dashv G$ is an adjoint pair then*

i) F is right exact;

ii) G is left exact.

Proof. This follows directly from the fact that a right adjoint preserves limits, whereas a left adjoint preserves colimits, since the right adjoint obviously preserves finite limits too, and in abelian categories this is equivalent to being a left exact functor (cf. 1.1.15). Dually a left adjoint is right exact for being finite colimit preserving. ☐

1.1.18 Proposition. *Consider two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$. Then F is a left adjoint of G if and only if there exist natural transformations $\phi : \mathrm{id}_{\mathcal{C}} \to GF$ and $\varphi : FG \to \mathrm{id}_{\mathcal{D}}$ such that the following diagrams are commutative*

$$\begin{tikzcd}
F \arrow[r] \arrow[dr, equal] & FGF \arrow[d] & & G \arrow[r] \arrow[dr, equal] & GFG \arrow[d] \\
& F & & & G
\end{tikzcd}$$

Proof. Cf. [6, thm.3.1.5] ☐

1.1.19 Proposition. *If F_1 and F_2 are both left adjoints of a functor G, then F_1 and F_2 are naturally equivalent.*

Proof. Cf. [34, prop.1.7.5] ☐

1.1.20 Definition. A functor $T : \mathcal{C} \to \mathcal{D}$ between abelian categories is said to be *faithful* if $T(f) \neq 0$ for every nonzero morphism f in $\mathcal{C}$ or, equivalently, if for all objects C, C' in $\mathcal{C}$, the map

$$\mathrm{Hom}_{\mathcal{C}}(C, C') \to \mathrm{Hom}_{\mathcal{D}}(T(C), T(C')); \; f \mapsto T(f),$$

is injective. If all these maps are surjective, the functor T is said to be *full*, and if they are all bijections then T is said to be *fully faithful*.
Note that if T is faithful then, in particular, $T(C) \neq 0$ for every $0 \neq C \in \mathcal{C}$.

1.1.21 Proposition. *Let $\mathcal{C}$ and $\mathcal{D}$ be preadditive categories and $F : \mathcal{C} \to \mathcal{D}$ a left adjoint of $G : \mathcal{D} \to \mathcal{C}$. Then*

i) G is fully faithful $\iff \Psi : FG \to \mathrm{id}_{\mathcal{D}}$ is a natural equivalence;

ii) F is fully faithful $\iff \Phi : \mathrm{id}_{\mathcal{C}} \to GF$ is a natural equivalence.

Proof. Cf. [34, thm.1.7.8,1.7.8^o] □

1.1.22 Definition. Let $\mathcal{C}$ be an abelian category. An object U of $\mathcal{C}$ is said to be a *generator* for $\mathcal{C}$ if $\mathrm{Hom}_{\mathcal{C}}(U, \cdot\,)$ is faithful or, equivalently, if for every nonzero morphism $\alpha : C \to C'$ there exists a morphism $\beta : U \to C$ such that $\alpha \circ \beta \neq 0$.

$$\begin{tikzcd}
U \arrow[r, "\exists\beta"] \arrow[dr, "\alpha\circ\beta\neq0"'] & C \arrow[d, "\alpha"] \\
& C'
\end{tikzcd}$$

More generally, a family of objects $\{U_i\}_{i\in I}$ in $\mathcal{C}$ is a family of generators for $\mathcal{C}$, if for any nonzero morphism $\alpha : C \to C'$ in $\mathcal{C}$, there exists, for some $i \in I$, a morphism $\beta_i : U_i \to C$ such that $\alpha \circ \beta_i \neq 0$; i.e. if the functor

$$\prod_{i\in I} \mathrm{Hom}_{\mathcal{C}}(U_i, \,\cdot\,) : \mathcal{C} \to \mathbf{ab}$$

is faithful. If $\mathcal{C}$ has coproducts this is equivalent to asserting $\bigoplus_{i\in I} U_i$ to be a generator in $\mathcal{C}$.
Dually, an object U of $\mathcal{C}$ is said to be a *cogenerator* if $\mathrm{Hom}_{\mathcal{C}}(\,\cdot\,, U)$ is faithful.

1.1.23 Examples.

1. In R-**mod** it is easy to see that R itself is a generator. In particular, the additive group $\mathbb{Z}$ is a generator for **ab**.

2. $\{\mathbb{Z}(n) \mid n \in \mathbb{Z}\}$ is a family of generators in **gr**, where $\mathbb{Z}(n)$ is the graded abelian group with $\mathbb{Z}(n)_p = 0$ if $p \neq n$, and $\mathbb{Z}(n)_n = \mathbb{Z}$.

3. In R-gr it is easy to verify that $\{R(n) \mid n \in \mathbb{Z}\}$ is a family of generators, where $R(n)$ is the left graded left R-module that coincides with R as an ungraded left R-module and which is graded by $R(n)_m = R_{n+m}$, for all $m \in \mathbb{Z}$.

4. In ${}_{R\text{-}\mathbf{mod}}\mathcal{P}(X)$ the class $\{G_U\}_{U \in O(X)}$ is a family of generators, where $O(X)$ denotes the set of open subsets of X and G_U is the presheaf defined by
$$G_U(V) = \begin{cases} R, & \text{if } V \subseteq U; \\ 0, & \text{otherwise.} \end{cases}$$

1.1.24 Definition. A *concrete category* is a pair $(\mathcal{C}, F)$ where $\mathcal{C}$ is a category and $F : \mathcal{C} \to \mathbf{Sets}$ is a faithful functor. It thus may be described as a category $\mathcal{C}$ in which each object C comes equipped with an underlying set $F(C)$, each arrow $f : C \to D$ is an actual function $F(C) \to F(D)$, and where the composition of arrows is the composition of the corresponding functions. In practice, the faithful functor is usually clear, and we simply do not mention it.

1.1.25 Definition. An abelian category $\mathcal{C}$ is said to be *complete* if the limit $\varprojlim F$ exists for every functor $F : I \to \mathcal{C}$ when I is a small category or, equivalently, if it has arbitrary products.

1.1.26 Definition. An abelian category $\mathcal{C}$ is said to be *cocomplete* if the colimit $\varinjlim F$ exists for every functor $F : I \to \mathcal{C}$ when I is a small category or, equivalently, if it has arbitrary direct sums (coproducts). This condition, according to Grothendieck's terminology in [20], is known as the *Ab3* condition.

1.1.27 Definition. A *Grothendieck category* is a cocomplete abelian category which has a generator and is such that the colimits over directed families of indices are exact; i.e. if I is a directed set and
$$0 \to A_i \to B_i \to C_i \to 0$$
is an exact sequence for any $i \in I$, then
$$0 \to \varinjlim A_i \to \varinjlim B_i \to \varinjlim C_i \to 0$$
is an exact sequence. This condition is equivalent to the so-called *Ab5* condition, according to Grothendieck's terminology in [20], which states that for any directed family $\{A_i\}_{i \in I}$ of subobjects of A and for any subobject B of A, the following relation holds:
$$(\sum_{i \in I} A_i) \cap B = \sum_{i \in I} (A_i \cap B).$$

1.1.28 Examples. All the examples mentioned in 1.1.12 are well-known examples of Grothendieck categories.

In the following definitions $\mathcal{C}$ is assumed to be an abelian category.

1.1.29 Definition. An object E of $\mathcal{C}$ is said to be *injective* if the functor $\mathrm{Hom}\,(\,\cdot\,, E) : \mathcal{C}^{opp} \to \mathbf{ab}$ is exact, i.e. if for every monomorphism $\alpha : C \to C'$ and for every morphism $\beta : C \to E$, there exists a morphism $\gamma : C' \to E$ such that $\gamma = \beta \circ \alpha$.

$$\begin{tikzcd}
0 \arrow[r] & C \arrow[r, "\alpha"] \arrow[d, "\beta"'] & C' \arrow[dl, "\exists\gamma"] \\
 & E &
\end{tikzcd}$$

$\mathcal{C}$ is said to have *enough injectives* if every object in $\mathcal{C}$ is a subobject of an injective object.
Dually, E is said to be *projective* if the functor $\mathrm{Hom}\,(E, \,\cdot\,) : \mathcal{C} \to \mathbf{ab}$ is exact, and $\mathcal{C}$ is said to have *enough projectives* if every object in $\mathcal{C}$ is a quotient object of a projective object.

The following result, which may be found in [27, II,15.3], is very useful to test whether a projective object is a generator (and dually, whether an injective object is a cogenerator).

1.1.30 Proposition. *Let $\mathcal{C}$ be an abelian category and U a projective object of $\mathcal{C}$. If for every non-zero object C of $\mathcal{C}$ the set of morphisms* $\mathrm{Hom}_{\,\mathcal{C}}(U, C)$ *is non-zero, then U is a generator for $\mathcal{C}$.*

1.1.31 Definition. If C is a subobject of E in $\mathcal{C}$ represented by a monomorphism $i : C \hookrightarrow E$ then we say that E or i is an *extension* of C. The extension is called *essential* if for any nonzero subobject E' of E the intersection $E' \cap C$ is nonzero.
Note that if $\alpha : C \to C'$ and $\beta : C' \to C''$ are monomorphisms, then it is straightforward to prove that $\beta \circ \alpha$ is essential if and only if both α and β are essential.

1.1.32 Definition. An *injective hull* of an object C is an essential extension $C \hookrightarrow E$ with E an injective object. An injective hull is unique up to (non-canonical) isomorphism and will be denoted by $E(C)$ (cf. [33, V,prop.2.3]).

1.1.33 Note. As a consequence of the Gabriel-Popescu theorem (stated in the next section), it may be proved that every Grothendieck category has enough injectives (cf. 1.6.3). Therefore, all the objects in a Grothendieck category have an injective hull.

1.2 The Gabriel-Popescu theorem

The aim of this section is to establish the Gabriel-Popescu theorem and its consequences. Roughly speaking it asserts that every Grothendieck category is a *quotient category* of a module category. To understand its formal statement it is necessary to manage the concepts of *radical* and *torsion theory*, which are linked in what follows.

From hereon until the end of this thesis, $\mathcal{C}$ will always denote a small concrete category. In particular, until the end of this chapter $\mathcal{C}$ will also denote a Grothendieck category.

1.2.1 Definition. A *torsion theory* for $\mathcal{C}$ is a pair $(\mathcal{T}, \mathcal{F})$ of classes of objects in $\mathcal{C}$ such that

i) $\mathrm{Hom}\,(T, F) = 0$ for all $T \in \mathcal{T},\ F \in \mathcal{F}$;

ii) if $\mathrm{Hom}\,(C, F) = 0$ for all $F \in \mathcal{F}$ then $C \in \mathcal{T}$;

iii) if $\mathrm{Hom}\,(T, C) = 0$ for all $T \in \mathcal{T}$ then $C \in \mathcal{F}$.

In particular, it follows that $\mathcal{T} \cap \mathcal{F} = \{0\}$, and that $\mathcal{T}$ and $\mathcal{F}$ determine each other mutually.

$\mathcal{T}$ is called a *torsion class* and its objects are *torsion objects*, while $\mathcal{F}$ is a *torsion-free class* consisting of *torsion-free objects*.

$(\mathcal{T}, \mathcal{F})$ is said to be *hereditary* if $\mathcal{T}$ is closed under subobjects or, equivalently, if $\mathcal{F}$ is closed under injective hulls.

An hereditary torsion theory $(\mathcal{T}, \mathcal{F})$ is said to be *stable* if $\mathcal{T}$ is closed under injective hulls.

1.2.2 Note. Any given class of objects $\mathcal{D}$ in $\mathcal{C}$ generates the torsion theory

$$\begin{aligned} \mathcal{F} &= \{F \mid \mathrm{Hom}\,(D, F) = 0,\ \forall D \in \mathcal{D}\}; \\ \mathcal{T} &= \{T \mid \mathrm{Hom}\,(T, F) = 0,\ \forall F \in \mathcal{F}\}, \end{aligned}$$

where $\mathcal{T}$ is the smallest torsion class containing $\mathcal{D}$. Dually $\mathcal{D}$ cogenerates a torsion theory such that $\mathcal{F}$ is the smallest torsion-free class containing $\mathcal{D}$ (cf. [33, VI,§2]).

1.2.3 Definition. A class of objects $\mathcal{D}$ in $\mathcal{C}$ is said to be *closed under extensions* if for every exact sequence

$$0 \to C' \to C \to C'' \to 0$$

with C', C'' objects of $\mathcal{D}$, it follows that C also belongs to $\mathcal{D}$.

1.2.4 Proposition. *Let $\mathcal{C}$ be an abelian category (which we assume to be complete and cocomplete), and $\mathcal{D}$ a class of objects of $\mathcal{C}$.*

i) $\mathcal{D}$ is a torsion class for some torsion theory if and only if it is closed under quotients, direct sums and extensions.

ii) $\mathcal{D}$ is a torsion-free class for some torsion theory if and only if it is closed under subobjects, products and extensions.

Proof. Cf. [23, (2.4)] or [33, VI,2.1 and 2.2]. ☐

1.2.5 Definition. A *preradical* in $\mathcal{C}$ is a subfunctor of the identity functor, i.e. a functor $\sigma : \mathcal{C} \to \mathcal{C}$ such that σC is a subobject of C, for every object C in $\mathcal{C}$ and if $f : C \to C'$ is a morphism in $\mathcal{C}$ then $\sigma f = f|_{\sigma C} : \sigma C \to \sigma C'$. A preradical σ is *idempotent* if $\sigma \circ \sigma = \sigma$, and is called a *radical* if for all C in $\mathcal{C}$

$$\sigma(C/\sigma C) = 0.$$

Note that in the literature we may also find the term *kernel functor* to refer to a radical and *idempotent kernel functor* to refer to a left exact radical (as in [37], for instance).

1.2.6 Proposition. *A preradical σ in $\mathcal{C}$ is left exact if and only if it satisfies the equality $\sigma D = \sigma C \cap D$, for every subobject D of an object C in $\mathcal{C}$.*

Proof. Cf. [33, VI,prop.1.7]. ☐

We denote by $K(\mathcal{C})$ the class of all left exact radicals in $\mathcal{C}$.

1.2.7 Lattice structure in $K(\mathcal{C})$.

We may put on $K(\mathcal{C})$ the structure of complete distributive lattice with the partial ordering

$$\sigma_1 \leq \sigma_2 \iff \text{ for all } C,\ \sigma_1 C \subseteq \sigma_2 C. \tag{1.1}$$

Any family $\{\sigma_i\}_{i\in I}$ of preradicals has its meet and its join given on all $C \in \mathcal{C}$ by

$$(\bigwedge\nolimits_{i\in I}\sigma_i)C = \bigcap_{i\in I}\sigma_i C; \quad (\bigvee\nolimits_{i\in I}\sigma_i)C = \sum_{i\in I}\sigma_i C,$$

(cf. [33, VI,§1.], e.g.)

1.2.8 Bijection between radicals and torsion theories.

To every left exact radical σ we may associate the pair of classes of objects in $\mathcal{C}$

$$\begin{aligned} \mathcal{T}_\sigma &= \{C \mid \sigma C = C\}; \\ \mathcal{F}_\sigma &= \{C \mid \sigma C = 0\}, \end{aligned}$$

called respectively the *σ-torsion class* consisting of *σ-torsion objects*, and the *σ-torsion-free class* consisting of *σ-torsion-free objects*. This pair is a hereditary torsion theory.

Note that as an immediate consequence, (1.1) is equivalent to

$$\sigma_1 \leq \sigma_2 \Leftrightarrow \mathcal{T}_{\sigma_1} \subseteq \mathcal{T}_{\sigma_2} \Leftrightarrow \mathcal{F}_{\sigma_1} \supseteq \mathcal{F}_{\sigma_2}.$$

Conversely, to any hereditary torsion theory $(\mathcal{T}, \mathcal{F})$ of $\mathcal{C}$ we may associate a left exact radical σ assigning to any object C in $\mathcal{C}$ the largest subobject belonging to $\mathcal{T}$, that is, the sum of all subobjects of C belonging to $\mathcal{T}$.

In this way, one establishes a bijective correspondence between $K(\mathcal{C})$ and the hereditary torsion theories of $\mathcal{C}$ (cf. [23, (2.11)]).

In view of this bijection, the meet and the join of any family of radicals may be determined by

$$\mathcal{T}_{\bigwedge_{i \in I} \sigma_i} = \bigcap_{i \in I} \mathcal{T}_{\sigma_i}, \qquad \mathcal{F}_{\bigvee_{i \in I} \sigma_i} = \bigcup_{i \in I} \mathcal{F}_{\sigma_i}.$$

From hereon, all radicals considered are left exact and all torsion theories are hereditary, so in the sequel we omit these prefixes and shortly refer to them as radicals and torsion theories respectively.

Another basic concept related to the Gabriel-Popescu theorem is the concept of *quotient category*, which is explained in what follows.

Let $\sigma \in K(\mathcal{C})$.

1.2.9 Definition. An object E in $\mathcal{C}$ is said to be *σ-injective* if for any subobject $C' \subseteq C$ in $\mathcal{C}$ with $C/C' \in \mathcal{T}_\sigma$, every morphism $f : C' \to E$ extends to a morphism $\bar{f} : C \to E$, i.e. such that the following diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & C' \arrow[r, hook, "i"] \arrow[rd, "f"'] & C \arrow[r, two heads, "\pi"] \arrow[d, dashed, "\bar{f}"] & C/C' \arrow[r] & 0 \\ & & E & & \end{tikzcd}$$

E is said to be *σ-closed* if $\bar{f}$ is unique or, equivalently, if E is σ-injective and σ-torsion-free (cf. [23, (3.1)]).

Note that if E is an injective object then it is obviously σ-injective as well.

1.2.10 Definition. A morphism $f : C_1 \to C_2$ in $\mathcal{C}$ is said to be a σ-*isomorphism* if both Ker f and Coker f are σ-torsion.

1.2.11 Proposition. *An object E in $\mathcal{C}$ is σ-closed if and only if for every σ-isomorphism $f : C_1 \to C_2$, and for every morphism $g : C_1 \to E$, there exists a unique morphism $\bar{g} : C_2 \to E$ such that the following diagram is commutative*

$$\begin{tikzcd} C_1 \arrow[r, "f"] \arrow[d, "g"'] & C_2 \arrow[dl, "\bar{g}"] \\ E & \end{tikzcd}$$

Proof. Cf. [23, (3.2)]. □

1.2.12 Definition. The class of all σ-closed objects in $\mathcal{C}$ forms a full subcategory of $\mathcal{C}$, denoted by $\mathcal{C}(\sigma)$ and called the *quotient category* of $\mathcal{C}$ with respect to σ. The canonical inclusion $\mathcal{C}(\sigma) \hookrightarrow \mathcal{C}$ is denoted by i_σ.
If $\mathcal{C}$ is the category of right R-modules, we usually write $\mathbf{mod}\text{-}(R, \sigma)$ for the quotient category $(\mathbf{mod}\text{-}R)(\sigma)$. Note that $\mathrm{Hom}_{\mathcal{C}(\sigma)}(C, C') = \mathrm{Hom}_{\mathcal{C}}(C, C')$, for all σ-closed objects C and C'.

At this point all the concepts related to the famous Gabriel-Popescu theorem have been introduced. Let us now formulate it and refer to [33, X,thm.4.1] or [18, thm.4.1] for a complete proof.

1.2.13 Theorem. (Popescu and Gabriel) *Let $\mathcal{C}$ be a Grothendieck category with generator U. Put $R = \mathrm{Hom}_{\mathcal{C}}(U, U)$ and let T denote the functor* $\mathrm{Hom}_{\mathcal{C}}(U, \cdot\,) : \mathcal{C} \to \mathbf{mod}\text{-}R$. *Then,*

i) T is full and faithful.

ii) T induces an equivalence between $\mathcal{C}$ and the category $\mathbf{mod}\text{-}(R, \sigma)$, where σ is the largest radical for which all $T(C)$ are σ-closed.

1.2.14 Note. This associated ring R is not uniquely determined by $\mathcal{C}$, since it also depends on the generator U which is not unique, and thus $\mathcal{C}$ may be isomorphic to different quotient categories $\mathbf{mod}\text{-}(R, \sigma)$, $\mathbf{mod}\text{-}(S, \tau), \ldots$, with different rings R, S, …

1.2.15 Remark. The relevance of this theorem is that it reduces (at least in principle) the study of Grothendieck categories to a study of quotient categories of some module category $\mathbf{mod}\text{-}R$. For instance, to prove that *a Grothendieck category is complete* or that *it has enough injectives*, it is sufficient to check that the category $\mathbf{mod}\text{-}(R, \sigma)$ satisfies these properties (and this, indeed, happens to be the case, as we will see in 1.6.2 and 1.6.3 respectively).

1.3 Giraud subcategories

Another equivalent version of the Gabriel-Popescu theorem states that every Grothendieck category is a strict Giraud subcategory of some **mod**-R. Indeed, every **mod**-(R, σ) is a Giraud subcategory of **mod**-R as we point out in 1.5.8. Let us first define what Giraud subcategories are.

1.3.1 Definition. A class of objects $\mathcal{C}$ in a category $\mathcal{D}$ is called *strict* if it is closed under isomorphisms, i.e. every object in $\mathcal{D}$ isomorphic to an object in $\mathcal{C}$ also belongs to $\mathcal{C}$. A subcategory is called *strict* if it class of objects is strict.

Let $\mathcal{D}$ be a (complete) Grothendieck category and $\mathcal{C}$ a strict full subcategory of $\mathcal{D}$. (We write complete between brackets because here it is redundant, since every Grothendieck category is complete as we already mentioned in 1.2.15, as a consequence of the Gabriel-Popescu theorem.)

1.3.2 Definition. $\mathcal{C}$ is said to be a *(full) reflective subcategory* if the inclusion functor $i : \mathcal{C} \to \mathcal{D}$ has a left adjoint $a : \mathcal{D} \to \mathcal{C}$, which we then refer to as the *reflector*. This means that, for every $C \in \mathcal{C}$ and $D \in \mathcal{D}$, there is an isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(C, i(D)) \cong \mathrm{Hom}_{\mathcal{D}}(a(C), D)$$

which is natural in C and D. Moreover, by 1.1.21 there exists a natural equivalence $\Psi : ai \to \mathrm{id}_{\mathcal{C}}$, so for every $C \in \mathcal{C}$ we have $a(C) \cong C$.
The reflective subcategory $\mathcal{C}$ is called a *Giraud subcategory* if the reflector preserves kernels or, equivalently, if it is exact (since a left adjoint is always right exact, by 1.1.17).

1.3.3 Proposition. *Let $\mathcal{C}$ be a Grothendieck category. If $\mathcal{D}$ is a Giraud subcategory of $\mathcal{C}$ then $\mathcal{D}$ is also a Grothendieck category.*

Proof. Cf. [33, X,§1]. □

1.3.4 Proposition. *Let $\mathcal{C}$ be a Grothendieck category and $\mathcal{D}$ a Giraud subcategory of $\mathcal{C}$. Then $D \in \mathcal{D}$ is an injective object in $\mathcal{D}$ if and only if $i(D)$ is injective in $\mathcal{C}$.*

Proof. Cf. [33, X,prop.1.4]. □

1.3.5 Example. Let R be a commutative ring and S a multiplicatively closed subset of R. The classical functor $S^{-1} : R\text{-}\mathbf{mod} \longrightarrow S^{-1}R\text{-}\mathbf{mod}$ which assigns to every R-module M the module of fractions $S^{-1}M$, and to every homomorphism of R-modules $f : M \to N$ the homomorphism of $S^{-1}R$-modules

$$S^{-1}f : S^{-1}M \longrightarrow S^{-1}N;\ m/s \mapsto f(m)/s,$$

is one of the simplest examples of reflector functor. In this case it is well known that S^{-1} is not only right exact (by 1.1.17) but even exact.

1.4 The reflector "sheafification"

The main example of reflector functor we want to distinguish in this work is the so-called *sheafification functor* which allows to go form the category of presheaves to the category of sheaves. In this section we give its classical construction on an ordinary topological space, defined as the composition of two functors through the category of *concrete sheaves*. First of all let us recall the definition of the involved categories.

Let X be a topological space and B_X a basis for the topology on X.

1.4.1 Definition. A presheaf of sets P on B_X (recall the definition from 1.1.12) is a *sheaf of sets* if it also satisfies:

Sh1) if $U \in B_X$ and $\{U_i\}_{i\in I}$ is an open covering of U in B_X, then for every $s \in P(U)$ we have $s = 0$ whenever $s|_{U_i} = 0$ for all $i \in I$;

Sh2) if $U \in B_X$, if $\{U_i\}_{i\in I}$ is an open covering of U in B_X, and if for all $i \in I$ there is given some element $s_i \in P(U_i)$ satisfying for all $i, j \in I$ that $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$, then there exists some $s \in P(U)$ such that $s|_{U_i} = s_i$, for all $i \in I$.

If P satisfies Sh1), we say that P is *separated*. Condition Sh2) is sometimes referred to as the *gluing condition*.
Note that Sh1) guarantees the uniqueness of the element s claimed to exist in Sh2).
If P and P' are sheaves of sets on B_X, a *morphism* $f : P \to P'$ *of sheaves* is just a morphism of presheaves between them.
We denote this full subcategory of ${}_{\mathbf{Sets}}\mathcal{P}(B_X)$ by ${}_{\mathbf{Sets}}\mathcal{S}(B_X)$.

Let R be an (associative) ring with unity. In a similar way, just by substituting sets by left R-modules, one obtains the definition of the category ${}_{R\text{-}\mathbf{mod}}\mathcal{S}(B_X)$ of *sheaves of left R-modules on B_X.*

We remark that the categories ${}_{\mathbf{Sets}}\mathcal{P}(B_X)$ and ${}_{R\text{-}\mathbf{mod}}\mathcal{S}(B_X)$ are respectively equivalent to ${}_{\mathbf{Sets}}\mathcal{P}(X)$ and ${}_{R\text{-}\mathbf{mod}}\mathcal{S}(X)$ from the very definition of sheaf.

1.4.2 Definition. Let $P \in {}_{\mathcal{C}}\mathcal{P}(B_X)$ and $x \in X$. The *stalk of P in x* is defined as the inductive limit $P_x = \varinjlim_{U \in V_{B_X}(x)} P(U)$, where

$$\{P_{UV} : P(U) \to P(V)\}_{V \subseteq U \text{ in } V_{B_X}(x)}$$

is the directed system with direct set

$$V_{B_X}(x) = \{U \in B_X \mid U \text{ open neighborhood of } x\}$$

ordered by $U \leq V$ if $V \subseteq U$. The elements of P_x are called *germs of sections of P at x.* This limit comes equipped with maps

$$\eta_U : P(U) \to P_x;\ s \mapsto s_x,$$

for every $U \in V_{B_X}(x)$, such that

i) for each germ $e \in P_x$, there exists $s \in P(U)$ for some $U \in V_{B_X}(x)$ with $e = s_x$;

ii) two germs $s_x, t_x \in P_x$ with $s \in P(U)$ and $t \in P(V)$, for $U, V \in V_{B_X}(x)$, are equal if and only if there exists $W \in V_{B_X}(x)$ with $W \subseteq U \cap V$ and $s|_W = t|_W$.

(Cf. [35, 1,prop.4.2]).

1.4.3 Note. If $f : P \to P'$ is a morphism of presheaves on B_X then we obtain an induced morphism on the stalks

$$f_x : P_x \to P'_x;\ s_x \mapsto (f(U)(s))_x, \text{ with } s \in P(U),$$

for every $x \in X$.

The following proposition, –which may be found in [21, II,prop.1.1]–, is really fundamental for lots of proofs, in particular it will be used in this section to obtain the sheafification functor on an ordinary topology.

1.4.4 Proposition. *Let $f : P \to P'$ be a morphism of sheaves on B_X. Then f is an isomorphism if and only if $f_x : P_x \to P'_x$ is an isomorphism, for all $x \in X$.*

1.4.5 Definition. A *concrete sheaf* or *sheaf space* (or *espace étalé* in the French literature), is a triple $\mathbb{E} = \ulcorner E, \pi, X \urcorner$ where E and X are topological spaces, called respectively the *total space* and the *base space*, and $\pi : E \to X$ is a local homeomorphism called the *projection*, that is, for all $e \in E$ there exists an open neighbourhood O_e of e and an open neighbourhood U_x of $x = \pi(e)$ such that $\pi|_{O_e} : O_e \to U_x$ is a homeomorphism. Since locally its inverse is continuous, π is also an open mapping.
For any $x \in X$, the set $E_x = \pi^{-1}(x)$ is called the *stalk* of E at x.
A *section* over an open subset U of X is a continuous map $s : U \to E$ such that $\pi \circ s = id_X|_U$.
The collection of all sections over U is denoted by $\Gamma(U, E)$. In particular, the elements of $\Gamma(X, E)$ are called *global sections.*

1.4.6 Definition. Let $\mathbb{E}_1 = \ulcorner E_1, \pi, X \urcorner$ and $\mathbb{E}_2 = \ulcorner E_2, \pi, X \urcorner$ be two concrete sheaves on X. A *morphism* $f : \mathbb{E}_1 \to \mathbb{E}_2$ *of concrete sheaves* is a continuous map $f : E_1 \to E_2$ such that $\pi_2 \circ f = \pi_1$.
By $CSh(X)$ we denote the *category of concrete sheaves* on X with these morphisms.
From the characteristics of the projections follows immediately that the morphisms on $CSh(X)$ are also open and locally injective maps.

1.4.7 Properties of concrete sheaves.

Given a concrete sheaf $\mathbb{E} = \ulcorner E, \pi, X \urcorner$, it is very easy to check that it satisfies the following properties:

i) $E = \bigcup_{x \in X} E_x$;

ii) if $s : U \to E$ is a section and V is an open subset of U then the restriction $s|_V$ is a section;

iii) sections that coincide in some point coincide on an open set containing that point;

iv) for any section $s : U \to E$ the set $s(U)$ is open in E and homeomorphic to U;

v) the collection $\{s(U) \mid U \text{ is open in } X \text{ and } s \text{ is a section on } U\}$ is a basis for the open sets of E.

In what follows we just give the description of the functors involved in the sheafification functor for sets and for left R-modules. We refer to [34, 4.4] or [45, 1] for the proofs.

1.4.8 From concrete sheaves on X to presheaves on B_X.

The functor

$$T : CSh(X) \to {}_{\mathbf{Sets}}\mathcal{P}(B_X)$$

is described on the concrete sheaves and on the morphisms of concrete sheaves as follows:
Let $\mathbb{E} = \ulcorner E, \pi, X \urcorner$ be a concrete sheaf on X. The functor T on $\mathbb{E}$ is the presheaf $T\mathbb{E} = \Gamma(\,\cdot\,, E)$, called the *presheaf of sections of* $\mathbb{E}$ *on* B_X, which assigns to each $U \in B_X$ the set of sections over U, i.e. for all $U \in B_X$

$$T\mathbb{E}(U) = \Gamma(U, E),$$

and to every inclusion $V \subseteq U$ in B_X, the restriction morphism

$$(T\mathbb{E})_{UV} : \Gamma(U, E) \to \Gamma(V, E);\ s \mapsto s|_V.$$

Let $f : \mathbb{E}_1 \to \mathbb{E}_2$ be a morphism of concrete sheaves. $Tf : T\mathbb{E}_1 \to T\mathbb{E}_2$ is the natural transformation given for every U in B_X by

$$Tf_U : \Gamma(U, E_1) \to \Gamma(U, E_2);\ s \mapsto f \circ s.$$

1.4.9 Remark. Let $\mathbb{E} = \ulcorner E, \pi, X \urcorner$ be a concrete sheaf. The functor T transforms the stalks of $\mathbb{E}$ into the stalks of the presheaf $T\mathbb{E}$, i.e. for every $x \in X$ we obtain $(T\mathbb{E})_x \cong E_x$ (cf. [34, 4.3]).

1.4.10 From presheaves on B_X to concrete sheaves on X.

The functor

$$S : {}_{\mathbf{Sets}}\mathcal{P}(B_X) \to CSh(X)$$

is given for every $P \in {}_{\mathbf{Sets}}\mathcal{P}(B_X)$ by the concrete sheaf $SP = \ulcorner E, \pi, X \urcorner$, where the total space is the disjoint union of stalks $E = \bigsqcup_{x \in X} P_x$ with the topology generated by the basis

$$B = \{\tilde{s}(U) \mid U \in B_X, s \in P(U)\},$$

where

$$\tilde{s} : U \to E;\ x \mapsto s_x;$$

and the projection is the local homeomorphism

$$\pi : E \to X;\ e \mapsto x \text{ if } e \in P_x.$$

Therefore, for all $x \in X$,

$$(SP)_x = P_x. \tag{1.2}$$

The functor S on the morphisms is obtained by gluing together the induced maps on the stalks, i.e. for every $f : P \to P'$, it is given by the morphism of concrete sheaves

$$Sf : SP \to SP';\ e \mapsto f_x(e) \text{ if } e \in P_x.$$

1.4.11 Proposition. *Let P be a presheaf on B_X. Then the following assertions are equivalent:*

i) P is a sheaf;

ii) there exists a concrete sheaf $\mathbb{E}$ on X with $P \cong T\mathbb{E}$ in ${}_{\mathbf{Sets}}\mathcal{P}(B_X)$.

Proof. Cf. [34, thm.4.5.3]. □

As an immediate consequence, for every concrete sheaf $\mathbb{E}$ we may thus construct a sheaf $T\mathbb{E}$. Hence, the functor T factors as $i \circ T'$, where i denotes the inclusion functor ${}_{\mathbf{Sets}}\mathcal{S}(B_X) \hookrightarrow {}_{\mathbf{Sets}}\mathcal{P}(B_X)$ and

$$T' : CSh(X) \to {}_{\mathbf{Sets}}\mathcal{S}(B_X);\ \mathbb{E} \mapsto T\mathbb{E}.$$

1.4.12 Definition. The *sheafification functor*, denoted by S', is defined to be the composition

$${}_{\mathbf{Sets}}\mathcal{P}(B_X) \xrightarrow{S} CSh(X) \xrightarrow{T'} {}_{\mathbf{Sets}}\mathcal{S}(B_X)\,.$$

1.4.13 Remark. The previous construction may be extended to the Grothendieck category ${}_{R\text{-}\mathbf{mod}}\mathcal{P}(B_X)$ if we define a *concrete sheaf of left R-modules* to be a concrete sheaf $\mathbb{E} = \ulcorner E, \pi, X \urcorner$ such that

i) for every $x \in X$, the stalk E_x is a left R-module;

ii) the addition defined on the stalks is continuous;

iii) for each $r \in R$ the scalar multiplication with r is continuous,

and a *morphism $f : \mathbb{E}_1 \to \mathbb{E}_2$ of concrete sheaves of left R-modules* to be a morphism of concrete sheaves such that its restrictions to the stalks are homomorphisms of left R-modules. This new category is denoted by ${}_RCSh(X)$. With this definition, the functor $T : {}_RCSh(X) \to {}_{R\text{-}\mathbf{mod}}\mathcal{P}(B_X)$ may be constructed by defining for any $U \in B_X$ a left R-module structure on each $T\mathbb{E}(U) = \Gamma(U, E)$, and the functor $S : {}_{R\text{-}\mathbf{mod}}\mathcal{P}(B_X) \to {}_RCSh(X)$ taking into account the induced left R-module structure on the stalks (cf. [34, 4.8]).

The following result is well-known. We include its proof for completeness' sake since some of the *noncommutative* results below are inspired by it.

1.4.14 Proposition. *The sheafification functor*

$$S' : {}_{\mathbf{Sets}}\mathcal{P}(B_X) \to {}_{\mathbf{Sets}}\mathcal{S}(B_X)$$

is a reflector, i.e. ${}_{\mathbf{Sets}}\mathcal{S}(B_X)$ is a reflective subcategory of ${}_{\mathbf{Sets}}\mathcal{P}(B_X)$.

Proof. First of all it may be checked (cf. [34, 4.4.2]) that the collection of morphisms $\{\Phi(P) : P \to TSP\}_{P \in {}_{\mathbf{Sets}}\mathcal{P}(B_X)}$ give rise to the natural transformation $\Phi : \mathrm{id}_{{}_{\mathbf{Sets}}\mathcal{P}(B_X)} \to TS$, since the diagram

$$\begin{tikzcd} P \arrow[r, "\Phi(P)"] \arrow[d, "\nu"'] & TSP \arrow[d, "TS\nu"] \\ Q \arrow[r, "\Phi(Q)"'] & TSQ \end{tikzcd}$$

is commutative for any morphism $\nu : P \to Q$ in ${}_{\mathbf{Sets}}\mathcal{P}(B_X)$.

Secondly, let us prove that there exists an isomorphism of functors

$$\Psi : ST \to \mathrm{id}_{CSh(X)}.$$

For every $\mathbb{E} = \ulcorner E, \pi, X \urcorner \in CSh(X)$, let us define an isomorphism of concrete sheaves $\Psi(\mathbb{E}) : ST\mathbb{E} \to \mathbb{E}$. We recall from 1.4.8 and 1.4.10 that $ST\mathbb{E} = \ulcorner F, \pi', X \urcorner$, where the total space is the disjoint union of stalks $F = \bigsqcup_{x \in X} \Gamma(\,\cdot\,, E)_x$, whose elements are of the form $s_x = \eta_U(s)$, for some $s \in \Gamma(U, E)$, $U \in V_{B_X}(x)$, and $x \in X$; the projection is given by $\pi'(s_x) = x$, and the topology defined on F has a basis

$$\{\tilde{s}(U) \mid U \in B_X, s \in \Gamma(U, E)\},$$

where $\tilde{s} : U \to F;\, x \mapsto s_x \in \Gamma(\cdot, E)_x$. Thus, the morphism $\Psi(\mathbb{E})$ is well-defined on every $s_x \in F$ by

$$\Psi(\mathbb{E})(s_x) = s(x).$$

Indeed, if $s_x = s'_x$ with $s_x = \eta_U(s)$ and $s'_x = \eta_V(s')$ then, by 1.4.2 ii), there exists $W \in V_{B_X}(x)$ with $W \subseteq U \cap V$ and $\Gamma(\,\cdot\,, E)_{UW}(s) = \Gamma(\,\cdot\,, E)_{VW}(s')$, that is, with $s|_W = s'|_W$, and therefore $s(x) = s'(x)$.
Conversely, if $s(x) = s'(x)$, by 1.4.7 iii), there exists an open $W \in V_{B_X}(x)$ such that $s|_W = s'|_W$. Then, by 1.4.2 ii), $s_x = s'_x$. Thus $\Psi(\mathbb{E})$ is injective.
Moreover, $\Psi(\mathbb{E})$ is open since $\Psi(\mathbb{E})(\tilde{s}(U)) = s(U)$ and s is open (1.4.7 iv)). From the previous equality it follows that $\Psi(\mathbb{E})^{-1}(s(U)) = \tilde{s}(U)$ since $\Psi(\mathbb{E})$ is injective, and consequently $\Psi(\mathbb{E})$ is continuous.
It remains to check $\pi \circ \Psi(\mathbb{E}) = \pi'$ in order to have $\Psi(\mathbb{E})$ in the category of concrete sheaves, and this is straightforward since

$$\pi \circ \Psi(\mathbb{E})(s_x) = \pi(s(x)) = x = \pi'(s_x).$$

On the other hand, let $e \in E$. By 1.4.7 v), there exists a section s on an open subset U such that $e \in s(U)$. Therefore, there exists $x \in U$ with $e = s(x)$. Let $V = U \cap U' \in V_{B_X}(x)$ with fixed arbitrary $U' \in V_{B_X}(x)$. Then $e = s|_V(x) = \Psi(\mathbb{E})((s|_V)_x)$ and therefore, $\Psi(\mathbb{E})$ is surjective.

To derive the isomorphism of functors it remains to prove that for every $f \in \mathrm{Hom}_{CSh(X)}(\mathbb{E}_1, \mathbb{E}_2)$, where $\mathbb{E}_1 = \ulcorner E_1, \pi_1, X \urcorner$ and $\mathbb{E}_2 = \ulcorner E_2, \pi_2, X \urcorner$ are two arbitrary concrete sheaves, the following diagram is commutative

$$\begin{tikzcd}
ST\mathbb{E}_1 \arrow[r, "\Psi(\mathbb{E}_1)"] \arrow[d, "ST(f)"'] & \mathbb{E}_1 \arrow[d, "f"] \\
ST\mathbb{E}_2 \arrow[r, "\Psi(\mathbb{E}_2)"'] & \mathbb{E}_2
\end{tikzcd}$$

Let $s_x \in ST(\mathbb{E}_1)$ with $s \in \Gamma(U, E_1)$. Taking into account the definitions of the functors S and T on the morphisms, it follows in a straightforward way that

$$\begin{aligned} f(\Psi(\mathbb{E}_1)(s_x)) &= f(s(x)) = (f \circ s)(x) \\ &= (T(f)(U)(s))(x) = \Psi(\mathbb{E}_2)((T(f)(U)(s))_x) \\ &= \Psi(\mathbb{E}_2)((T(f))_x(s_x)) = \Psi(\mathbb{E}_2)(ST(f)(s_x)). \end{aligned}$$

Finally, it may be checked that

$$(SP \overset{S\Phi(P)}{\longrightarrow} STSP \overset{\Psi(SP)}{\longrightarrow} SP) = (SP \overset{\mathrm{id}_{SP}}{\longrightarrow} SP),$$

for every $P \in {}_{\mathbf{Sets}}\mathcal{P}(B_X)$, and

$$(T\mathbb{E} \overset{\Phi(T\mathbb{E})}{\longrightarrow} TST\mathbb{E} \overset{T(\Psi(\mathbb{E}))}{\longrightarrow} T\mathbb{E}) = (T\mathbb{E} \overset{\mathrm{id}_{T\mathbb{E}}}{\longrightarrow} T\mathbb{E}),$$

for every $\mathbb{E} \in CSh(X)$, (cf. [34, Prop.4.4.4]). Hence, by 1.1.18 we have the adjoint situation

$${}_{\mathbf{Sets}}\mathcal{P}(B_X) \overset{S}{\longrightarrow} CSh(X) \overset{T}{\longrightarrow} {}_{\mathbf{Sets}}\mathcal{P}(B_X),$$

and since Ψ is an isomorphism, by 1.1.21 it follows that the functor T is fully faithful.

On the other hand, we have the commutative diagram

$$\begin{tikzcd}
{}_{\mathbf{Sets}}\mathcal{P}(B_X) \arrow[r, "S"] \arrow[dr, "S'"'] & CSh(X) \arrow[r, "T"] \arrow[d, "T'"] & {}_{\mathbf{Sets}}\mathcal{P}(B_X) \\
& {}_{\mathbf{Sets}}\mathcal{S}(B_X) \arrow[ur, hook, "i"'] &
\end{tikzcd}$$

By [34, thm.1.7.11] it follows that T' is an equivalence of categories, since T is fully faithful and for every $P \in {}_{\mathbf{Sets}}\mathcal{S}(B_X)$ there exists $\mathbb{E} \in CSh(X)$ such that $P \cong T\mathbb{E}$. Therefore

$$ {}_{\mathbf{Sets}}\mathcal{P}(B_X) \xrightarrow{S'} {}_{\mathbf{Sets}}\mathcal{S}(B_X) \overset{i}{\hookrightarrow} {}_{\mathbf{Sets}}\mathcal{P}(B_X) $$

is an adjoint situation and ${}_{\mathbf{Sets}}\mathcal{S}(B_X)$ is a reflective subcategory of ${}_{\mathbf{Sets}}\mathcal{P}(B_X)$. □

1.4.15 Proposition. *${}_{R\text{-}\mathbf{mod}}\mathcal{S}(B_X)$ is a Giraud subcategory of ${}_{R\text{-}\mathbf{mod}}\mathcal{P}(B_X)$.*

Proof. The previous result for presheaves on the category of **Sets** may be extended to presheaves on the category of left R-modules. Thus, it only remains to prove that the functor S' preserves kernels:
Let $\ker f : \operatorname{Ker} f \to P$ be the kernel of a morphism $f : P \to P'$ in ${}_{R\text{-}\mathbf{mod}}\mathcal{P}(B_X)$. Then the sequence

$$ 0 \to \operatorname{Ker} f \xrightarrow{\ker f} P \xrightarrow{f} P' $$

is exact in ${}_{R\text{-}\mathbf{mod}}\mathcal{P}(B_X)$, which means that for every $U \in B_X$

$$ 0 \to (\operatorname{Ker} f)(U) \xrightarrow{(\ker f)(U)} P(U) \xrightarrow{f(U)} P'(U) $$

is an exact sequence in the Grothendieck category R-**mod**. Consequently,

$$ 0 \to (\operatorname{Ker} f)_x \xrightarrow{(\ker f)_x} P_x \xrightarrow{f_x} P'_x $$

is an exact sequence of left R-modules (by $Ab5$), which is equivalent to the exact sequence

$$ 0 \to (S(\operatorname{Ker} f))_x \to (SP)_x \to (SP')_x, $$

by (1.2). Taking into account 1.4.9, for every $x \in X$ we obtain the exact sequence

$$ (T(S(\operatorname{Ker} f)))_x \to (TSP)_x \to (TSP')_x. $$

Therefore,

$$ TS(\operatorname{Ker} f) \to TSP \to TSP' $$

is an exact sequence of sheaves, i.e. $TS(f) \circ TS(\ker f) = 0$, and consequently $S'(f) \circ S'(\ker f) = 0$, since $TS = iT'S = iS'$.
Now let $g : P'' \to S'P$ be a morphism in ${}_{R\text{-}\mathbf{mod}}\mathcal{S}(B_X)$ such that $S'f \circ g = 0$.

Then the following diagram is commutative

$$\begin{tikzcd}
S'\mathrm{Ker}\, f \arrow[dr, "S'\mathrm{ker}\, f"'] \arrow[drr, bend left, "0"] & & \\
 & S'P \arrow[r, "S'f"] & S'P' \\
P'' \arrow[ur, "g"] \arrow[urr, bend right, "0"'] & &
\end{tikzcd}$$

To obtain that $S'(\mathrm{ker}\, f)$ is indeed the kernel of $S'f$ it remains to find a unique morphism of presheaves $\xi : P'' \to S'\mathrm{Ker}\, f$ such that $S'\mathrm{ker}\, f \circ \xi = g$.
From $S'f \circ g = 0$ it follows for every $x \in X$ that $(S'f)_x \circ g_x = 0$. On the other hand, for all $Q \in {}_{R\text{-mod}}\mathcal{P}(B_X)$,

$$(S'Q)_x = (TSQ)_x \cong (SQ)_x = Q_x.$$

Therefore, the following diagram is commutative

$$\begin{tikzcd}
P''_x \arrow[dr, "g_x"'] \arrow[drr, bend left, "0"] & & \\
 & (S'P)_x \arrow[r, "(S'f)_x"'] \arrow[d, "\cong"'] & (S'P')_x \arrow[d, "\cong"] \\
 & P_x \arrow[r, "f_x"'] & P'_x
\end{tikzcd}$$

and consequently, for every $x \in X$, there exists a unique

$$\xi_x : P''_x \to \mathrm{Ker}\, f_x = (\mathrm{Ker}\, f)_x = (S'\mathrm{Ker}\, f)_x$$

such that $\mathrm{ker}\, f_x \circ \xi_x = g_x$ (since the composition $P''_x \to P_x \to P'_x$ is equal to 0). Finally, from the maps $\{\xi_x\}_{x \in X}$ we derive the unique morphism ξ of sheaves such that $S'\mathrm{ker}\, f \circ \xi = g$. Therefore, $S'\mathrm{ker}\, f$ is the kernel of $S'f$.

□

1.4.16 Corollary. *The category ${}_{R\text{-mod}}\mathcal{S}(B_X)$ of sheaves of left R-modules is a Grothendieck category.*

Proof. This follows from the previous result and 1.3.3. □

1.5 Localization

In this section we consider *localization functors*, which allows to prove that every quotient category of a Grothendieck category $\mathcal{C}$ is a Giraud subcategory of $\mathcal{C}$, and consequently, a Grothendieck category itself, and to obtain a bijective correspondence between Giraud subcategories and torsion theories.

Let $\mathcal{C}$ be a Grothendieck category and $\sigma \in K(\mathcal{C})$.

1.5.1 Definition. A *σ-injective hull* of an object C of $\mathcal{C}$ is an essential extension $C \to E$ such that E is σ-injective and $E/C \in \mathcal{T}_\sigma$.

1.5.2 Any $C \in \mathcal{F}_\sigma$ possesses an essentially unique σ-injective hull, denoted by $E_\sigma(C)$ (cf. [23, (3.6)]). It may be constructed as the inverse image of the σ-torsion of $E(C)/C$ via the projection map π, as in the diagram

$$\begin{tikzcd}
0 \arrow[r] & C \arrow[r, hook] \arrow[dr, hook] & E(C) \arrow[r, two heads, "\pi"] & E(C)/C \arrow[r] & 0 \\
& & E_\sigma(C) \arrow[u, hook] & \sigma(E(C)/C) \arrow[u, hook] \arrow[l, dashed, "\pi^{-1}"'] &
\end{tikzcd}$$

i.e. $E_\sigma(C) = \pi^{-1}(\sigma(E(C)/C))$. Then it obviously contains C and the extension $C \to E_\sigma(C)$ is essential since $C \hookrightarrow E(C)$ is essential (cf. 1.1.31).

The object $E_\sigma(C)$ is not only σ-injective but also σ-torsion-free since it is contained in the σ-torsion-free injective hull of a σ-torsion-free object (recall that the class of σ-torsion-free objects is closed under injective hulls). Thus, the σ-injective hull of C may be redefined as the essentially unique σ-closed object E such that $E/C \in \mathcal{T}_\sigma$.

1.5.3 To every $\sigma \in K(\mathcal{C})$ we may associate a functor $a_\sigma : \mathcal{C} \to \mathcal{C}(\sigma)$ from the category $\mathcal{C}$ to the quotient category $\mathcal{C}(\sigma)$, which is defined on every object $C \in \mathcal{C}$ as the σ-injective hull of the quotient $C/\sigma C$, i.e. for all $C \in \mathcal{C}$

$$a_\sigma C = E_\sigma(C/\sigma C)$$

(this is well-defined in view of 1.5.2 since the quotient is torsion free), and on every morphism $f \in \operatorname{Hom}_{\mathcal{C}}(C, C')$ as the unique extension of the morphism $\bar{f} : C/\sigma C \to C'/\sigma C'$ induced by f (which is well-defined since σ is a subfunctor of the identity and therefore $f(\sigma C) \subseteq \sigma C'$) making the following diagram commutative

$$\begin{tikzcd}
0 \arrow[r] & \overline{C} \arrow[r, hook] \arrow[d, "\bar{f}"'] & a_\sigma C \arrow[r, two heads, "\pi"] \arrow[d, dashed, "a_\sigma(f)"] & a_\sigma C/\overline{C} \arrow[r] & 0 \\
& \overline{C'} \arrow[r, hook, "i"'] & a_\sigma C' & &
\end{tikzcd}$$

where $\overline{C}$ and $\overline{C'}$ denote the quotient objects $C/\sigma C$ and $C'/\sigma C'$ respectively. Indeed the quotient $a_\sigma C/\overline{C} = E_\sigma(\overline{C})/\overline{C}$ is obviously σ-torsion by the definition of σ-injective hulls, and $a_\sigma C'$ is σ-closed, therefore there exists a unique extension $a_\sigma(f)$ of $i \circ \bar{f}$ making the diagram commutative.

1.5.4 Theorem. *The functor a_σ is a left adjoint of the canonical inclusion $i_\sigma : \mathcal{C}(\sigma) \to \mathcal{C}$.*

Proof. Cf. [23, (3.7)]. □

1.5.5 Remark. As an immediate consequence of this theorem the quotient category $\mathcal{C}(\sigma)$ is a reflective subcategory of $\mathcal{C}$.

1.5.6 Definition. The *localization functor* at σ in $\mathcal{C}$, denoted by Q_σ, is defined as the functor composition $\mathcal{C} \xrightarrow{a_\sigma} \mathcal{C}(\sigma) \xrightarrow{i_\sigma} \mathcal{C}$. Thus, Q_σ assigns to every C in $\mathcal{C}$ the object

$$Q_\sigma(C) = i_\sigma(E_\sigma(C/\sigma C)) \in \mathcal{C},$$

called the *localization of C at σ*, which may be constructed by the pullback diagram

$$\begin{tikzcd}
Q_\sigma(C) \arrow[r] \arrow[d] & E_\sigma(\overline{C}) \arrow[d, "\pi"] \\
\sigma(E_\sigma(\overline{C})/\overline{C}) \arrow[r] & E_\sigma(\overline{C})/\overline{C}
\end{tikzcd}$$

where $\overline{C}$ denotes the quotient object $C/\sigma C$ (cf. [46, (1.11.)]), and for every morphism $f : C \to C'$ in $\mathcal{C}$, the morphism $Q_\sigma(f) = i_\sigma(a_\sigma(f))$ is the extension of the morphism $\bar{f} : C/\sigma C \to C'/\sigma C'$ induced by f, viewed within $\mathcal{C}$

$$\begin{tikzcd}
0 \arrow[r] & \overline{C} \arrow[r, hook] \arrow[d, "\bar{f}"'] & Q_\sigma C \arrow[r, two heads, "\pi"] \arrow[d, dashed, "Q_\sigma(f)"] & Q_\sigma C/\overline{C} \arrow[r] & 0 \\
 & \overline{C'} \arrow[r, hook, "i"'] & Q_\sigma C' & &
\end{tikzcd}$$

The *localization morphism* is the canonical morphism $j_{\sigma,C} : C \to Q_\sigma(C)$ defined as the composition

$$C \twoheadrightarrow C/\sigma C \hookrightarrow Q_\sigma(C)$$

where the first arrow is the canonical projection and the second is an essential extension.

1.5.7 Properties of the localization functor.

1. $Q_\sigma(C) = Q_\sigma(\overline{C})$, for all C in $\mathcal{C}$, since $\overline{\overline{C}} = \overline{C}$.
2. If $C \in \mathcal{T}_\sigma$ then $Q_\sigma(C) = 0$.
3. $Q_\sigma(C)$ is σ-injective and σ-torsion-free, i.e. it is σ-closed, for all $C \in \mathcal{C}$.

4. The quotient $Q_\sigma(C)/\overline{C}$ is σ-torsion, i.e. the cokernel of the localization morphism $j_{\sigma,C}$ is σ-torsion, for all C in $\mathcal{C}$.

5. For every object C in $\mathcal{C}$, the kernel of the localization morphism $j_{\sigma,C}$ is σC. Therefore,

   i) $C \in \mathcal{F}_\sigma \;\Leftrightarrow\; j_{\sigma,C}$ injective;
   ii) $C \in \mathcal{T}_\sigma \;\Leftrightarrow\; j_{\sigma,C} = 0$.

6. C belongs to the quotient category $\mathcal{C}(\sigma)$ if and only if the localization morphism is an isomorphism. Thus, the class of σ-closed objects is

$$\mathcal{C}(\sigma) = \{C \in \mathcal{C} \mid C \cong Q_\sigma(C) = i_\sigma a_\sigma C\}.$$

7. The functor Q_σ is idempotent (this follows easily from the fact that $E_\sigma(E_\sigma(\overline{C})) = E_\sigma(\overline{C})$ and $\overline{E_\sigma(\overline{C})} = E_\sigma(\overline{C})$, for all C in $\mathcal{C}$).

8. The functor Q_σ is left exact.

1.5.8 Remark. From the fact that the localization functor is a left exact endofunctor in $\mathcal{C}$, one obtains that the functor a_σ is exact. Therefore the quotient category $\mathcal{C}(\sigma)$ is not only a reflective subcategory but even a Giraud subcategory of $\mathcal{C}$. Consequently, in view of 1.3.3, the quotient category $\mathcal{C}(\sigma)$ of a Grothendieck category $\mathcal{C}$ is also a Grothendieck category. For instance **mod**-(R, σ) is a Giraud subcategory of **mod**-R and hence a Grothendieck category itself (cf. [33, X,thm.1.6]).

1.5.9 Bijection between torsion theories and Giraud subcategories. As we have already mentioned in 1.2.8, to every torsion theory of $\mathcal{C}$ corresponds a unique $\sigma \in K(\mathcal{C})$ and from σ we obtain the unique strict Giraud subcategory $\mathcal{C}(\sigma)$ which corresponds to the initial torsion theory.

Conversely, to any strict Giraud subcategory of $\mathcal{C}$ with reflector $a : \mathcal{C} \to \mathcal{D}$ corresponds the torsion theory

$$\begin{aligned} \mathcal{T} &= \{C \in \mathcal{C} \mid aC = 0\}; \\ \mathcal{F} &= \{C \in \mathcal{C} \mid \text{ there exists } D \in \mathcal{D} \text{ s.t. } C \text{ is a subobject of } iD\} \\ &= \{C \in \mathcal{C} \mid \text{ the reflection } C \to iaC \text{ is a monomorphism}\}. \end{aligned}$$

In particular, the reflector a corresponds to the radical $\sigma \in K(\mathcal{C})$ defined on any $C \in \mathcal{C}$ by

$$\sigma C = \sum_{C' \subseteq C,\ aC' = 0} C',$$

and it may be proved that $\mathcal{D} = \mathcal{C}(\sigma)$. Thus, every strict Giraud subcategory of $\mathcal{C}$ is a quotient category $\mathcal{C}(\sigma)$, for some σ. Indeed, this correspondence between torsion theories for $\mathcal{C}$ and strict Giraud subcategories of $\mathcal{C}$ is a bijection, as a consequence of Gabriel's theorem (cf. [33, X,prop.1.5.] or [23, (3.31)]).

1.6 Localization in R-mod

Let R be an (associative) ring with unity and $\sigma \in K(R)$, where $K(R)$ denotes the class of all left exact radicals in R-**mod**.

All the results from 1.5 are applicable to the category R-**mod**, as a well-known example of a Grothendieck category. In this section we recall these results in particular for left R-modules, but exactly the same may be done for the category **mod**-R.

Let us give a complete description of the functor Q_σ in R-**mod**.

1.6.1 Recalling the definitions from 1.2, a left R-module E is *σ-injective* if for any left submodule N of a left R-module M with $M/N \in \mathcal{T}_\sigma$, every morphism $f : N \to E$ extends to a morphism $\bar{f} : M \to E$, i.e. such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, hook, "i"] \arrow[dr, "f"'] & M \arrow[r, two heads, "\pi"] \arrow[d, dashed, "\bar{f}"] & M/N \arrow[r] & 0 \\
 & & E & &
\end{tikzcd}$$

We say that E is *σ-closed* if $\bar{f}$ is unique as such. The quotient category of R-**mod** with respect to σ is the Giraud subcategory given by the class of all σ-closed left R-modules, and is denoted by (R, σ)-**mod**.

1.6.2 Proposition. *The category* (R, σ)-**mod** *is complete.*

Proof. This follows from the fact that reflective subcategories of a complete Grothendieck category are also complete ([33, X,prop.1.2]). ☐

1.6.3 Proposition. *The category* (R, σ)-**mod** *has enough injectives.*

Proof. Let $M \in (R, \sigma)$-**mod**. The injective hull $E(M)$ is an injective left R-module and then it is also σ-injective. Moreover, it is σ-torsion-free, therefore it is σ-closed, i.e. $E(M) \in (R, \sigma)$-**mod**. By 1.3.4, $E(M)$ is also injective in (R, σ)-**mod**. Therefore, M is a subobject of the injective object $E(M) \in (R, \sigma)$-**mod** and thus the category has enough injectives. ☐

1.6.4 Recalling 1.5.1 and 1.5.2, a σ-injective hull of a left R-module M is an essential extension $M \to E$ of left R-modules such that E is σ-injective and $E/M \in \mathcal{T}_\sigma$, and if M is σ-torsion-free, its σ-injective hull is essentially unique and is denoted by $E_\sigma(M)$.

1.6.5 Definition. The *localization functor* Q_σ *of* R-**mod** *at* σ, i.e. the left exact functor defined as the composition $R\text{-}\mathbf{mod} \overset{a_\sigma}{\to} (R,\sigma)\text{-}\mathbf{mod} \overset{i_\sigma}{\to} R\text{-}\mathbf{mod}$, assigns to every left R-module M the left R-module

$$Q_\sigma(M) = E_\sigma(M/\sigma M)$$

which, together with the localization morphism $j_{\sigma,M} : M \to Q_\sigma M$, is called the *module of quotients of* M *at* σ.
For every homomorphism of left R-modules $f : M \to N$, the localization $Q_\sigma(f) = i_\sigma(a_\sigma(f))$ is the extension of the morphism $\bar{f} : M/\sigma M \to N/\sigma N$ induced by f

$$\begin{tikzcd}
0 \arrow[r] & \overline{M} \arrow[r, hook] \arrow[d, "\bar{f}"'] & Q_\sigma(M) \arrow[r, two heads, "\pi"] \arrow[d, dashed, "Q_\sigma(f)"] & Q_\sigma(M)/\overline{M} \arrow[r] & 0 \\
 & \overline{N} \arrow[r, hook, "i"'] & Q_\sigma(N) & &
\end{tikzcd}$$

where $\overline{M}$ resp. $\overline{N}$ denotes the quotient $M/\sigma M$ resp. $N/\sigma N$.

In particular, $Q_\sigma(R)$ together with the localization morphism $j_{\sigma,R}$ is called the *ring of quotients of* R *at* σ, since it has a ring structure in virtue of the following result which will play a fundamental role in section 2.3.

1.6.6 Proposition.

i) *$Q_\sigma(R)$ is a ring containing $R/\sigma R$ as a subring; its ring structure is uniquely determined by its left R-module structure.*

ii) *If M is a faithfully σ-injective left R-module, then $Q_\sigma(M)$ has a left $Q_\sigma(R)$-module structure uniquely determined by its left R-module structure.*

Proof. Cf. [37, I] □

1.6.7 Bijection between torsion theories and Giraud subcategories in R-mod.
To every torsion theory $(\mathcal{T}, \mathcal{F})$ in the category of left R-modules corresponds

a unique $\sigma \in K(R)$, as in 1.5.9, which on every left R-module M takes the form

$$\sigma M = \sum_{N \subseteq M, N \in \mathcal{T}} N = \sum_{m \in M, Rm \in \mathcal{T}} Rm. \tag{1.3}$$

Thus, to $(\mathcal{T}, \mathcal{F})$ corresponds a unique strict Giraud subcategory (R, σ)-**mod**. Conversely, to every strict Giraud subcategory $\mathcal{C}$ of R-**mod** with reflector $a : R\text{-}\mathbf{mod} \to \mathcal{C}$ corresponds a unique torsion theory

$$\begin{aligned} \mathcal{T} &= \{M \in R\text{-}\mathbf{mod} \mid aM = 0\}; \\ \mathcal{F} &= \{M \in R\text{-}\mathbf{mod} \mid M \hookrightarrow iaM\}, \end{aligned}$$

and there exists a unique $\sigma \in K(R)$ such that $\mathcal{C} = (R, \sigma)$-**mod**.

1.6.8 To every ring homomorphism $f : R \to S$ we associate the functor

$$f_* : S\text{-}\mathbf{mod} \to R\text{-}\mathbf{mod}$$

which assigns to any S-module M the same M with the R-module structure obtained by scalar restriction via f.
There is also a lattice homomorphism $\bar{f} : K(R) \to K(S)$ induced by f on the classes of radicals which assigns to any radical $\sigma \in K(R)$ the radical $\bar{f}(\sigma)$ on left S-modules defined by the torsion class

$$\mathcal{T}_{\bar{f}(\sigma)} = \{M \in S\text{-}\mathbf{mod} \mid f_*(M) \in \mathcal{T}_\sigma\}.$$

Note that this is indeed a torsion class since f_* is exact and commutes with direct sums.

1.6.9 Proposition. *Let $f : R \to S$ be a ring homomorphism and σ a radical in R-**mod** then*

*i) for all $M \in S$-**mod** we have $f_*(\bar{f}(\sigma)(M)) \subseteq \sigma(f_*(M))$;*

ii) if f is surjective, then we have equality, i.e. $f_* \circ \bar{f}(\sigma) = \sigma \circ f_*$.

Proof.

i) Let M be a left S-module. By (1.3),

$$\bar{f}(\sigma)(M) = \sum_{N \in \mathcal{T}_{\bar{f}(\sigma)}, N \subseteq M} N = \sum_{f_*(N) \in \mathcal{T}_\sigma, N \subseteq M} N.$$

On the other hand,

$$\sigma(f_*(M)) = \sum_{N' \in \mathcal{T}_\sigma, N' \subseteq f_*(M)} N' \supseteq \sum_{f_*(N) \in \mathcal{T}_\sigma, N \subseteq M} f_*(N).$$

Therefore, $f_*(\bar{f}(\sigma)(M)) \subseteq \sigma(f_*(M))$.

ii) Since

$$f_*(\bar{f}(\sigma)(M)) = \sum_{f_*(N)\in\mathcal{T}_\sigma, N\subseteq M} f_*(N)$$

and $\sigma(f_*(M))$ is obviously in $\mathcal{T}_\sigma$, if $\sigma(f_*(M))$ is also a left S-submodu of M then it is contained in $f_*(\bar{f}(\sigma)(M))$. Indeed, let $s \in S$ ar $x \in \sigma(f_*(M))$. As f is surjective, $s = f(r)$ for some $r \in R$, and

$$sx = f(r)x = rx,$$

by the definition of the left R-module structure on M, and sx belon to $\sigma(f_*(M))$ because it contains rx as a left R-module.

We end this chapter by introducing so-called *Gabriel filters* and their c rresponding version of the Gabriel-Popescu theorem. In the category of le R-modules, $K(R)$ may be determined not only by torsion theories but al by *Gabriel filters*, as we explain in what follows.

1.6.10 Definition. A *Gabriel filter* on R is a nonempty family $\mathcal{L}$ of le R-ideals satisfying the properties:

i) if I and J are left R-ideals with $I \subseteq J$ and $I \in \mathcal{L}$, then $J \in \mathcal{L}$;

ii) if $I, J \in \mathcal{L}$, then $I \cap J \in \mathcal{L}$;

iii) if $I \in \mathcal{L}$ and $s \in R$, then

$$(I : s) = \{r \in R \mid rs \in I\} \in \mathcal{L};$$

iv) if I is a left R-ideal and there exists $J \in \mathcal{L}$ with the property th $(I : r) \in \mathcal{L}$ for all $r \in J$, then $I \in \mathcal{L}$.

1.6.11 Lemma. *Let $\mathcal{L}$ be a Gabriel filter. If I, J belong to $\mathcal{L}$, then so do IJ.*

Proof. Cf. [33, VI,lemma 5.3].

1.6.12 Definition. Let $\sigma \in K(R)$. The family of left R-ideals

$$\mathcal{L} = \mathcal{L}(\sigma) = \{L \text{ left ideal of } R \mid \sigma(R/L) = R/L\}$$

satisfies the previous properties i)-iv); it is called the *Gabriel filter associat to σ.*

1.6.13 Bijection between Giraud subcategories of left R-modules and Gabriel filters on R.

By 1.6.7, to every strict Giraud subcategory of R-**mod** corresponds a unique torsion theory, and then a unique $\sigma \in K(R)$. Hence, to every strict Giraud subcategory we may associate the corresponding Gabriel filter $\mathcal{L}(\sigma)$.

Conversely, to every Gabriel filter $\mathcal{L}$, we may associate the strict Giraud subcategory (R, σ)-**mod** where σ is the radical determined by assigning to any left R-module M the submodule

$$\sigma M = \{m \in M \mid \exists L \in \mathcal{L}, Lm = 0\} = \{m \in M \mid \operatorname{Ann}_R(m) \in \mathcal{L}\}, \qquad (1.4)$$

and it may be verified that $\mathcal{L}(\sigma) = \mathcal{L}$ (cf. [23, (4.4)]).
This actually yields a bijection between strict Giraud subcategories of R-**mod** and Gabriel filters on R.

1.6.14 Note. In an arbitrary Grothendieck category $\mathcal{C}$ with generator U, we may still associate to any $\sigma \in K(\mathcal{C})$, some kind of Gabriel filter $\mathcal{L}(U, \sigma)$ consisting of all subobjects D of U such that $U/D \in \mathcal{T}_\sigma$. However, in general $\mathcal{L}(U, \sigma)$ does not uniquely determine σ anymore ([37, II.6], [46, 1.9]).

As an immediate consequence of 1.6.13, the category **mod**-(R, σ) may be uniquely denoted by **mod**-$(R, \mathcal{L}(\sigma))$, and conversely we may define the quotient category associated to an arbitrary Gabriel filter $\mathcal{L}$ as the quotient category **mod**-(R, σ), where σ is the radical corresponding to $\mathcal{L}$, and denote it by **mod**-$(R, \mathcal{L})$. Thus, the Gabriel-Popescu theorem stated in 1.2.13 may be reformulated as follows:

1.6.15 Theorem. *Let $\mathcal{C}$ be a Grothendieck category with generator U. Put $R = \operatorname{Hom}_{\mathcal{C}}(U, U)$ and let $T : \mathcal{C} \to$ **mod**-R be given by $T(C) = \operatorname{Hom}_{\mathcal{C}}(U, C)$. Then,*

- *i) T is full and faithful;*
- *ii) T induces an equivalence between $\mathcal{C}$ and the category* **mod**-$(R, \mathcal{L})$, *where $\mathcal{L}$ is the largest Gabriel filter on R for which all modules $T(C)$ are $\mathcal{L}$-closed.*

1.6.16 Lattice stucture in $K(R)$.

It is obvious that $\sigma_1 \leq \sigma_2$ if and only if $\mathcal{L}(\sigma_1) \subseteq \mathcal{L}(\sigma_2)$, which characterizes the partial ordering in $K(R)$ in terms of Gabriel filters. The meet and the join of any family of radicals can also be described using the Gabriel filter. Indeed, from the fact that $\mathcal{T}_{\bigwedge_{i\in I}\sigma_i} = \bigcap_{i\in I} \mathcal{T}_{\sigma_i}$, it follows immediately that

$$\mathcal{L}(\bigwedge_{i\in I}\sigma_i) = \bigcap_{i\in I}\mathcal{L}(\sigma_i),$$

and it can be proved that the join $\sigma = \bigvee_{i\in I}\sigma_i$ is the radical which assigns to any R-module M the submodule

$$\sigma M = \{m \in M \mid \text{ there exists } L \in \mathcal{L} \text{ s.t. } Lm = 0\},$$

where $\mathcal{L}$ is the filter of all ideals of R containing some finite product $\prod_j I_j$, with $I_j \in \bigcup_{i\in I}\mathcal{L}(\sigma_i)$.

1.6.17 The Gabriel filter associated to a radical $\sigma \in K(R)$ may be used to determine the module of quotients of any left R-module M at σ by the formulas:

i) $Q_\sigma(M) = \{e \in E(M/\sigma M) \mid \text{ there exists } L \in \mathcal{L}(\sigma) \text{ s.t. } Le \subseteq M/\sigma M\}$;

ii) $Q_\sigma(M) = \varinjlim_{L\in\mathcal{L}(\sigma)} \mathrm{Hom}_R(L, M/\sigma M)$.

(Cf. [44, 1] and [46, (1.14)] resp.)

Finally let us give some examples of Gabriel filters for particular radicals in R-**mod**:

1.6.18 Example. Let R be a commutative ring and S a multiplicatively closed subset of R. The class of all $M \in R$-**mod** such that for all $m \in M$ there exists $s \in S$ with $sm = 0$, is a torsion class denoted by $\mathcal{T}_S$. Its associated radical, denoted by σ_S, assigns to every R-module M the classical S-torsion submodule

$$\sigma_S M = \sum_{m\in M, Rm\in\mathcal{T}_S} Rm = \{m \in M \mid \text{ there exists } s \in S \text{ s.t. } sm = 0\},$$

and the corresponding Gabriel filter, $\mathcal{L}(\sigma_S)$, consists of all the ideals with empty intersection with S. In this case the localization functor Q_{σ_S} is precisely the functor $S^{-1} : R\text{-}\mathbf{mod} \to S^{-1}R\text{-}\mathbf{mod}$ described in 1.3.5.

In particular, let S be the multiplicatively closed subset $R - \mathfrak{p}$ with $\mathfrak{p}$ a prime ideal of R. Then the associated radical, usually denoted by $\sigma_\mathfrak{p}$, describes the classical torsion at $\mathfrak{p}$ and its corresponding Gabriel filter is the set of all ideals not contained in $\mathfrak{p}$. The localization functor at the radical $\sigma_\mathfrak{p}$ is just the usual localization at $\mathfrak{p}$, i.e. $Q_{\sigma_\mathfrak{p}}(M)$ coincides with the $R_\mathfrak{p}$-module $M_\mathfrak{p}$, for every R-module M.

If R is a noncommutative ring and S a multiplicatively closed subset of R, then for any left R-module M the set $\sigma_S M$ is no longer a left R-module, in general. However, we may obtain a left R-module structure on $\sigma_S M$ if we choose S to be a *left Ore set*, i.e. a multiplicatively closed subset of R satisfying

i) for all $r \in R$ and $s \in S$ there exists $r' \in R$ and $s' \in S$ such that $s'r = r's$;

ii) for all $r \in R$ and $s \in S$ such that $rs = 0$, there exists $s' \in S$ such that $s'r = 0$.

The localization of a left R-module M at the radical σ_S is the left module of fractions

$$S^{-1}M = S \times M \ / \ \sim \ ,$$

where $(s_1, m_1) \sim (s_2, m_2)$ if and only if there exists some $s \in S$ such that $s(s_1 m_2 - s_2 m_1) = 0$. In this case, the corresponding Gabriel filter associated to S consists of all left ideals L satisfying $(L : r) \cap S \neq \emptyset$, for all $r \in R$.

1.6.19 Example. Let M be a left R-module. The radical $\chi(M)$ defined on every left R-module N by

$$\chi(M)N = \bigcap_{f \in \mathrm{Hom}_R(N, E(M))} \mathrm{Ker}\, f,$$

(where $E(M)$ denotes the injective hull of M) is the largest radical such that M is torsion-free. Indeed, let σ be a radical such that M is σ-torsion-free and let us check that $\sigma \leq \chi(M)$, i.e. that $\mathcal{T}_\sigma \subseteq \mathcal{T}_{\chi(M)}$ (by 1.2.8).
Let $N \in \mathcal{T}_\sigma$. Since M is σ-torsion-free, $E(M)$ is also σ-torsion-free and $f(N)$ is σ-torsion for every $f \in \mathrm{Hom}_R(N, E(M))$ since it is the image of a σ-torsion R-module. On the other hand, $f(N)$ is contained in the σ-torsion-free module $E(M)$, so $f(N) = 0$ and $\mathrm{Hom}_R(N, E(M)) = \{0\}$. Hence, $\chi(M)N = N$.
In a comparable way, to any prime ideal $\mathfrak{p}$ one may associate the radical $\chi(R/\mathfrak{p})$, obviously equal to $\sigma_\mathfrak{p}$.
In a similar way one may prove that the radical defined as

$$\xi(M) = \bigwedge_{\mathcal{T}_\sigma \ni M} \sigma$$

is the smallest with respect to which M is torsion.

1.6.20 Example. Let I be a finitely generated ideal of a commutative ring R. The radical σ_I assigns to every R-module M the set

$$\sigma_I M = \{m \in M \mid \text{ there exists } n \geq 0 \text{ s.t. } I^n m = 0\}.$$

In particular,

$$\sigma_I(R/L) = \{r + L \mid r \in R \text{ s.t. there exists } n \geq 0 \text{ with } I^n r \in L\},$$

for every ideal L, so R/L is σ_I-torsion if and only if $I^n \subseteq L$ for some positive integer n. Therefore, the associated Gabriel filter $\mathcal{L}_I$ is the set of ideals which contains some positive power of I.
Note that if I is a principal ideal generated by an element $r \in R$ and S is the multiplicative subset generated by r, then $\sigma_I = \sigma_S$.

Chapter 2

Presheaves

In this chapter we study the category of *presheaves on an arbitrary poset E*, which, as a functor category, inherits a Grothendieck category structure when it takes values in a Grothendieck category. On the other hand, when R is a *presheaf of not necessarily commutative rings on E*, we study the category of *presheaves of left R-modules.* Although this is not a functor category, we prove that it also has a Grothendieck category structure when it takes values in a Grothendieck category. Finally, we study how the *localization functor* acts *on presheaves of left R-modules.*

2.1 What is a presheaf?

Let X be a topological space and $\mathcal{C}$ an arbitrary category. It is well-know that a presheaf P on X with values in $\mathcal{C}$ consists of the data:

i) an object $P(U)$ of $\mathcal{C}$, for every open subset U of X;

ii) for every inclusion $V \subseteq U$ of open subsets of X, a morphism in $\mathcal{C}$

$$P_{UV} : P(U) \to P(V);\ s \mapsto P_{UV}(s) := s|_V$$

called the *restriction morphism*,

subject to the conditions:

1) $P_{UU} = id_{P(U)}$ for every open subset U of X;

2) if $W \subseteq V \subseteq U$ are open subsets of X, then $P_{UW} = P_{VW} \circ P_{UV}$.

Indeed, P is a contravariant functor from $\mathcal{O}(X)$ to $\mathcal{C}$, where $\mathcal{O}(X)$ is the category whose objects are the open subsets of X and the set of morphisms $\operatorname{Hom}_{\mathcal{O}(X)}(V, U)$ is a singleton when $V \subseteq U$ and is empty otherwise. In particular, the set of open subsets of X may be regarded as a poset ordered by the inclusion.
More generally, one may take any arbitrary poset $(E, \leq)$ viewed as a category $\mathcal{E}$ (recall the notation from 1.1.8), and define presheaves on $\mathcal{E}$ with values in $\mathcal{C}$. In fact, the category of presheaves is the *functor category* from $\mathcal{E}^{opp}$ to $\mathcal{C}$. In this section we explain this concept emphasizing the fact that functor categories on a Grothendieck category are also Grothendieck categories.

2.1.1 Definition. Let $\mathcal{E}$ be a small category and $\mathcal{C}$ an arbitrary category. We denote by $\operatorname{Fun}(\mathcal{E}, \mathcal{C})$ the category whose objects are the covariant functors from $\mathcal{E}$ to $\mathcal{C}$ and whose morphisms are the natural transformations between functors, that is, if F, G are two functors in $\operatorname{Fun}(\mathcal{E}, \mathcal{C})$, a morphism $\nu : F \to G$ gives rise to the following commutative diagram of morphisms in $\mathcal{C}$

$$\begin{tikzcd}
F(a) \arrow[r, "\nu_a"] \arrow[d, "F(f)"'] & G(a) \arrow[d, "G(f)"] \\
F(b) \arrow[r, "\nu_b"'] & G(b)
\end{tikzcd}$$

for every a, b in $\mathcal{E}$ and for every morphism $f : a \to b$ in $\mathcal{E}$.

Note that the class of all natural transformations from F to G can be regarded as a subclass of the cartesian product $\prod_{a\in\mathcal{E}} \operatorname{Hom}_{\mathcal{C}}(F(a), G(a))$, which is a set since $\mathcal{E}$ is small, hence so is $\operatorname{Hom}_{\operatorname{Fun}(\mathcal{E},\mathcal{C})}(F, G)$, and therefore $\operatorname{Fun}(\mathcal{E}, \mathcal{C})$ is indeed a category, which is called a *functor category*. This category is small as long as $\mathcal{C}$ is small.

2.1.2 Remark. If $\mathcal{E}$ is a small category, the category of functors $\operatorname{Fun}(\mathcal{E}, \mathcal{C})$ tends to inherit properties of the target category $\mathcal{C}$ (cf. [34, 2.9] or [33, IV,§7.]), for instance:

i) if every family of objects in $\mathcal{C}$ has a product (coproduct), then every family of objects in $\operatorname{Fun}(\mathcal{E}, \mathcal{C})$ has a product (coproduct);

ii) if every morphism in $\mathcal{C}$ has a kernel (cokernel), then every morphism in $\operatorname{Fun}(\mathcal{E}, \mathcal{C})$ has a kernel (cokernel);

iii) if $\mathcal{C}$ is complete (cocomplete), then so is $\operatorname{Fun}(\mathcal{E}, \mathcal{C})$;

iv) if $\mathcal{C}$ is abelian, then so is $\operatorname{Fun}(\mathcal{E}, \mathcal{C})$;

v) if $\mathcal{C}$ satisfies *Ab5*, then so does $\mathrm{Fun}(\mathcal{E}, \mathcal{C})$.

Moreover, if $\mathcal{C}$ has generators then we may also obtain a generator for the functor category, according to a result found in [27, VI,thm.4.3]:

2.1.3 Theorem. *Let $\mathcal{E}$ be a small category and $\mathcal{C}$ a category with coproducts and a null object. If U_a is a generator in $\mathcal{C}$ for each $a \in E$, then the covariant functor*

$$G = \bigoplus_{a \in E} X_a$$

is a generator for $\mathrm{Fun}(\mathcal{E}, \mathcal{C})$*; where $X_a : \mathcal{E} \to \mathcal{C}$ is given on every $b \in \mathcal{E}$ by*

$$X_a(b) = \begin{cases} U_a, & \textit{if } \mathrm{Hom}_{\mathcal{E}}(a, b) \neq \emptyset; \\ 0, & \textit{otherwise,} \end{cases}$$

and on every morphism $f : b \to c$ in $\mathcal{E}$ by

$$X_a(f) = \begin{cases} \mathrm{id}_{U_a}, & \textit{if } \mathrm{Hom}_{\mathcal{E}}(a, b) \neq \emptyset; \\ 0, & \textit{otherwise,} \end{cases}$$

2.1.4 Corollary. *If $\mathcal{C}$ is a Grothendieck category then so is* $\mathrm{Fun}(\mathcal{E}, \mathcal{C})$*, for every small category $\mathcal{E}$.*

Proof. This follows immediately from 2.1.2 and the previous theorem. □

Let E be a poset viewed as a category $\mathcal{E}$ and $\mathcal{C}$ an arbitrary category.

2.1.5 Definition. The *category of presheaves on E with values in $\mathcal{C}$* is the functor category $\mathrm{Fun}(\mathcal{E}^{opp}, \mathcal{C})$, and it is denoted by ${}_{\mathcal{C}}\mathcal{P}(E)$. Thus, a *presheaf* P on E with values in $\mathcal{C}$ consists of the data:

i) for all $a \in E$ an object $P(a)$ in $\mathcal{C}$;

ii) for every $b \leq a$ in E, a morphism in $\mathcal{C}$

$$P_{ab} : P(a) \to P(b);\ s \mapsto P_{ab}(s) := s|_b$$

(we call P_{ab} the *restriction morphism* and $s|_b$ the *restriction of s to b*),

subject to the conditions:

1) $P_{aa} = id_{P(a)},\ \forall a \in E$;

2) if $c \leq b \leq a$ in E then $P_{ac} = P_{bc} \circ P_{ab}$.

Moreover, a *morphism* $f : P \to P'$ *of presheaves* on E with values in $\mathcal{C}$ consists of a family $\{f(a) : P(a) \to P'(a)\}_{a \in E}$ of morphisms in $\mathcal{C}$ such that whenever $b \leq a$ in E, the following diagram is commutative

$$\begin{tikzcd} P(a) \arrow[r, "f(a)"] \arrow[d, "P_{ab}"'] & P'(a) \arrow[d, "P'_{ab}"] \\ P(b) \arrow[r, "f(b)"'] & P'(b) \end{tikzcd}$$

where P_{ab} and P'_{ab} are the restriction morphisms of P and P' respectively. The *identity presheaf* $\mathrm{id}_{{}_{\mathcal{C}}\mathcal{P}(E)}$ on E is given by

$$\mathrm{id}_{{}_{\mathcal{C}}\mathcal{P}(E)}(a) = a; \quad \forall a \in E,$$

and an *isomorphism of presheaves* is a morphism which has a two-sided inverse with respect to this identity.

2.1.6 Proposition. *If $\mathcal{C}$ is a complete additive category then so is the category ${}_{\mathcal{C}}\mathcal{P}(E)$ of presheaves on E with values in $\mathcal{C}$.*

Although this result is an immediate corollary of 2.1.2 iv) (since the category of presheaves is a functor category), we are interested in specifying how limits are defined in order to simplify other proofs. This is why we give the following sketch of proof, based on [33, IV,§8], which is also a good example of showing how a functor category inherits the properties of $\mathcal{C}$.

Proof. Let I be an small category and $F : I \to {}_{\mathcal{C}}\mathcal{P}(E)$ a functor. We have to check the existence of the limit of F in ${}_{\mathcal{C}}\mathcal{P}(E)$.
For every morphism $\lambda : i \to j$ in I we obtain a morphism $F_\lambda : F(i) \to F(j)$ in ${}_{\mathcal{C}}\mathcal{P}(E)$ given by a set $\{F_\lambda(a) : F(i)(a) \to F(j)(a)\}_{a \in E}$ of morphisms in $\mathcal{C}$. Consequently, for every $a \in E$, we may define a functor $F_a : I \to \mathcal{C}$ given on every $i \in I$ by $F(i)(a)$, and on every morphism $\lambda : i \to j$ by $F_\lambda(a)$.
Since $\mathcal{C}$ is a complete category, we may construct the limit of F_a in $\mathcal{C}$. Thus, the presheaf $\varprojlim F$ is defined on every $a \in E$ by $(\varprojlim F)(a) = \varprojlim F_a$, which, according to [33, IV,8.2], coincides with the kernel of the morphism

$$\prod_{i \in I} F(i) \longrightarrow \prod_{\lambda : i \to j} F(j),$$

with canonical morphism for all $i \in I$

$$\pi_i^a : \varprojlim F_a \hookrightarrow \prod_{j \in I} F(j) \xrightarrow{p_i} F(i); \quad s = (s_i)_{i \in I} \mapsto s_i.$$

On the other hand, whenever $b \leq a$ in E, the restriction morphism $(\varprojlim F)_{ab}$ is obtained by the universal property of the limit, as the unique morphism making the following diagram commutative:

$$\begin{tikzcd}
\varprojlim F_a \arrow[r, "\pi_i^a"] \arrow[dr, dashed, "(\varprojlim F)_{ab}"'] & F(i)(a) \arrow[r, "F(i)_{ab}"] & F(i)(b) \\
& \varprojlim F_b \arrow[ur, "\pi_i^b"'] &
\end{tikzcd}$$

That is, for all $i \in I$ and $s = (s_i)_{i \in I} \in \varprojlim F_a$, the projections of the restriction of s to b are given by

$$\pi_i^b((\varprojlim F)_{ab})(s)) = F(i)_{ab}(s_i). \tag{2.1}$$

□

Moreover:

2.1.7 Proposition. *Let E be a poset. If $\mathcal{C}$ is a Grothendieck category, then the category of presheaves ${}_{\mathcal{C}}\mathcal{P}(E)$ is a Grothendieck category.*

Proof. This is clear by 2.1.4, since $\mathcal{E}^{opp}$ is small and ${}_{\mathcal{C}}\mathcal{P}(E)$ is the functor category $\text{Fun}(\mathcal{E}^{opp}, \mathcal{C})$. In particular, as in 2.1.3, if U is a generator for $\mathcal{C}$, then the presheaf $G = G_{{}_{\mathcal{C}}\mathcal{P}(E)}$ given for all $b \in E$ by

$$G(b) = \bigoplus_{a \in E} X_a(b) \cong U^{(E_b)}, \tag{2.2}$$

is a generator for ${}_{\mathcal{C}}\mathcal{P}(E)$, where $E_b = \{a \in E \mid b \leq a\}$, and for every $c \leq b$ the restriction morphism is given by the canonical injection

$$G_{bc} : U^{(E_b)} \to U^{(E_c)}; \quad s = (s_a)_{a \geq b} \mapsto s|_c = (s'_a)_{a \geq c}, \tag{2.3}$$

where

$$s'_a = \begin{cases} s_a, & \text{if } a \geq b; \\ 0, & \text{otherwise.} \end{cases}$$

□

2.1.8 Examples.

1. Let E be a poset, $\mathcal{C}$ an arbitrary category, and take an arbitrary fixed object C in $\mathcal{C}$. The *constant presheaf* $P_E^C \in {}_{\mathcal{C}}\mathcal{P}(E)$ *with fiber* C is defined by putting $P_E^C(a) = C$, for every $a \in E$, and $(P_E^C)_{ab} = \text{id}_C$, for every $b \leq a$ in E.

2. Let X be a topological space and B_X a basis for the topology on X. For every concrete sheaf $\mathbb{F}$ on X we may define the *presheaf of sections* $T\mathbb{F}$ in the poset B_X as in 1.4.8.

3. Let R be a commutative ring and take the poset $E = \{X(s)\}_{s\in R-\{0\}}$, where $X(s) = \{\mathfrak{p} \in \operatorname{Spec} R \mid s \in \mathfrak{p}\}$, which constitutes a basis for the Zarisky topology on $\operatorname{Spec} R$. The *structure presheaf* $Q_R \in {}_{\mathbf{Rings}}\mathcal{P}(E)$ of the ring R is defined by putting $Q_R(X(s)) = R_s$, the ring of fractions of R at the multiplicative subset of R generated by s, and by defining the restriction morphism $Q_R(X(s)) \to Q_R((X(t))$, for any $X(t) \subseteq X(s)$ (hence $t^n = rs$ for some $r \in R - \{0\}$) to be the map
$$(Q_R)_{X(s)X(t)} : R_s \longrightarrow R_t;\ u/s^m \mapsto ur^m/t^{nm}.$$

4. Let X, Y be topological spaces and B a basis for the topology on X. The *presheaf P of Y-valued continuous functions* on the poset B is defined by $P(U) = \{f : U \to Y \mid f \text{ continuous}\}$, for every $U \in B$, and for every $V \subseteq U \in B$ the restriction morphism is given by
$$P_{UV} : P(U) \longrightarrow P(V);\ f \mapsto f|_V.$$

2.2 The category R-pre-Mod

In order to study the localization in categories of presheaves of left R-modules on arbitrary posets, we begin by endowing this category with a Grothendieck structure, which is not straightforward since we are no longer dealing with functor categories, as we did in the previous section.

From hereon, let E be an arbitrary poset and R a presheaf of not necessarily commutative rings on E.

2.2.1 Definition. A *presheaf of left R-modules* on E, or shortly, an *R-pre-Module* on E, is a presheaf M on E such that $M(a)$ is an $R(a)$-module, for all $a \in E$, and for all $b \leq a$ in E the restriction morphism $M_{ab} : M(a) \to M(b)$ is R_{ab}-semilinear, i.e. it is compatible with the module structure via the restriction ring homomorphism R_{ab}, i.e. for all $r \in R(a)$ and $x \in M(a)$,
$$M_{ab}(rx) = R_{ab}(r)M_{ab}(x).$$

A *morphism $f : M \to N$ of R-pre-Modules* is a morphism of presheaves such that $f(a) : M(a) \to N(a)$ is a morphism of left $R(a)$-modules, for all $a \in E$. We denote this category by R-**pre-Mod**.

2.2.2 Proposition. *The category R-**pre-Mod** of presheaves of left R-modules on E is abelian.*

Proof. Let M, N be R-pre-Modules on E. The set $\mathrm{Hom}_{R\text{-}\mathbf{pre\text{-}Mod}}(M, N)$ is an abelian group with the 0-morphism and the sum defined in the obvious way, and it is very easy to check that the composition of morphisms is distributive with respect to the sum, therefore the category R-**pre-Mod** is preadditive.

Let $\{M_i\}_{i \in I}$ be a finite family of R-pre-Modules. Since $R(a)$-**mod** is an abelian category for every $a \in E$, the family of left $R(a)$-modules $\{M_i(a)\}_{i \in I}$ has the product

$$(\prod_{i \in I} M_i(a)\ ,\ \{\pi_i(a) : \prod_{j \in I} M_j(a) \to M_i(a)\}_{i \in I}).$$

We define a presheaf of left R-modules by

$$(\prod_{i \in I} M_i)(a) = \prod_{i \in I} M_i(a);\ \forall a \in E,$$

with the restriction morphisms given by

$$(\prod_{i \in I} M_i)_{ab} : \prod_{i \in I} M_i(a) \to \prod_{i \in I} M_i(b);\ (x_i)_{i \in I} \mapsto ((M_i)_{ab}(x_i))_{i \in I},$$

for every $b \leq a$ in E, which is clearly R_{ab}-semilinear since $(M_i)_{ab}$ is R_{ab}-semilinear for all $i \in I$. If for every $i \in I$ we define a morphism of R-pre-Modules $\pi_i : \prod_{j \in I} M_j \to M_i$ by the family

$$\{\pi_i(a) : \prod_{j \in I} M_j(a) \to M_i(a)\}_{a \in E},$$

then it is easy to check that the pair $(\prod_{i \in I} M_i, \{\pi_i\}_{i \in I})$ is a product in R-**pre-Mod**.
Similarly, making use of the coproduct of the family of left $R(a)$-modules $\{M_i(a)\}_{i \in I}$, for every $a \in E$, we may check that the family $\{M_i\}_{i \in I}$ has a coproduct.

It is also easy to see that every morphism of R-pre-Modules $f : M \to N$ has a kernel and a cokernel in R-**pre-Mod**. Indeed, for every $a \in E$ there exists an $R(a)$-homomorphism $\ker f(a) : \mathrm{Ker}\, f(a) \to M(a)$ which is the kernel of $f(a)$. We define the R-pre-Module $\mathrm{Ker}\, f$ by $(\mathrm{Ker}\, f)(a) = \mathrm{Ker}\, f(a)$, for every

$a \in E$, and the restriction morphism $(\mathrm{Ker}\, f)_{ab} : \mathrm{Ker}\, f(a) \to \mathrm{Ker}\, f(b)$ is defined for every $b \leq a$ in E, as the restriction $M_{ab}|_{\mathrm{Ker}\, f(a)}$, which is well-defined since $M_{ab}(\mathrm{Ker}\, f(a))$ is clearly contained in $\mathrm{Ker}\, f(b)$.
It is now a straightforward exercise to see that the morphism of R-pre-Modules $\ker f : \mathrm{Ker}\, f \to M$ defined by the class of left $R(a)$-homomorphisms $\{(\ker f)(a) = \ker f(a)\}_{a \in E}$ is the kernel of f in R-**pre-Mod**.
Similarly we may check that $\mathrm{coker}\, f : N \to \mathrm{Coker}\, f$ is given by the class of cokernels of left $R(a)$-modules $\{\mathrm{coker}\, f(a)\}_{a \in E}$, where $\mathrm{Coker}\, f$ is the R-pre-module defined by $(\mathrm{Coker}\, f)(a) = \mathrm{Coker}\, f(a)$, for all $a \in E$, with restriction morphism $(\mathrm{Coker}\, f)_{ab} = N_{ab}|_{\mathrm{Coker}\, f(a)}$, for every $b \leq a$ in E.

Finally it remains to check that the *Ab2* condition holds in R-**pre-Mod**, but this is straightforward since for every morphism $f : M \to N$ of R-pre-Modules, the canonical morphism of left $R(a)$-modules

$$\overline{f(a)} : \mathrm{Coker}\, (\ker f(a)) \to \mathrm{Ker}\, (\mathrm{coker}\, f(a))$$

is an isomorphism, for every $a \in E$, and hence $\bar{f}$, which is given by the class $\{\bar{f}(a) = \overline{f(a)}\}_{a \in E}$, is an isomorphism of R-pre-Modules. □

2.2.3 Proposition. *The category R-**pre-Mod** of presheaves of left R-modules on E is cocomplete.*

Proof. Let I be a small category and $F : I \to R$-**pre-Mod** a functor. We have to construct the colimit of F in R-**pre-Mod**.
For every $a \in E$ we define a functor $F_a : I \to R(a)$-**mod** which assigns to every $i \in I$ the left $R(a)$-module $F(i)(a)$.
It is then easy to see that we may define the R-pre-Module $\varinjlim F$ on every $a \in E$ by

$$(\varinjlim F)(a) = \varinjlim F_a,$$

which comes equipped with maps $\eta_i(a) : F(i)(a) \to \varinjlim F_a$, for all $i \in I$. The restriction morphism is given, for every $b \leq a$ in E, by

$$(\varinjlim F)_{ab} : \varinjlim F_a \to \varinjlim F_b; \quad \eta_i(a)(x) \mapsto \eta_i(b)(F(i)_{ab}(x)).$$

A straightforward verification then shows that

$$(\varinjlim F, \{\eta_i : F(i) \to \varinjlim F\}_{i \in I})$$

is the inductive limit of F, indeed, where η_i is the morphism of R-pre-Modules given by $\{\eta_i(a)\}_{a \in E}$. □

One now easily verifies:

2.2.4 Proposition. *The category R-***pre-Mod** *of presheaves of left R-modules on E satisfies the Grothendieck condition Ab5.*

We now finally have:

2.2.5 Theorem. *The category R-***pre-Mod** *of presheaves of left R-modules on E is a Grothendieck category.*

Proof. In view of the previous results, it only remains to prove that R-**pre-Mod** has a generator. Let R_a be the R-pre-Module defined by

$$R_a(b) = \begin{cases} R(b), & \text{if } b \leq a; \\ 0, & \text{otherwise,} \end{cases}$$

with restriction morphism R_{bc} when $c \leq b \leq a$ and 0 otherwise, for every $c \leq b$ in E. We assert that $\{R_a\}_{a \in E}$ is a family of generators. Indeed, let $\alpha : M \to M'$ be a nonzero morphism in R-**pre-Mod**, so there exists $a \in E$ such that $\alpha(a) \neq 0$. Since $R(a)$ is a generator in $R(a)$-**mod**, there exists an $R(a)$-homomorphism $\beta_a : R(a) \to M(a)$ such that $\alpha(a) \circ \beta_a \neq 0$. We define a morphism $\beta : R_a \to M$ by the family

$$\beta(b) = \begin{cases} \beta_b, & \text{if } b \leq a; \\ 0, & \text{otherwise,} \end{cases}$$

where β_b is the $R(b)$-homomorphism defined through the following commutative diagram

$$\begin{tikzcd} R(a) \arrow[r, "R_{ab}"] \arrow[d, "\beta_a"'] & R(b) \arrow[d, "\beta_b"] \\ M(a) \arrow[r, "M_{ab}"'] & M(b) \end{tikzcd}$$

It is very easy to check that β is a morphism of presheaves and it is obvious that $\alpha \circ \beta \neq 0$ since $\alpha(a) \circ \beta_a \neq 0$. Therefore, $\bigoplus_{a \in E} R_a$ is a generator in R-**pre-Mod**. □

2.3 Localization in R-pre-Mod

The aim of this section is to describe the localization functor in the category of *presheaves of left R-modules on posets*. After having proved in 2.2.5 that this category is a Grothendieck category, the main result of this chapter is that, under certain assumptions, the localization functor "locally" acts as the localization functor in the category of modules. For this reason, proposition

1.6.6 which explains the module structure of the localization of a left R-module, plays a fundamental role. This study was first done in [37] for the category of presheaves of left R-modules on ordinary topological spaces, here it is generalized to posets. Moreover, it appears that some results in [37] have redundant or missing hypotheses; this section thus also aims to place these in the correct framework.

Let E be a poset viewed as a category $\mathcal{E}$.

2.3.1 We recall from 2.1.7 that ${}_{\mathcal{C}}\mathcal{P}(E)$ is a Grothendieck category for every Grothendieck category $\mathcal{C}$. Consequently, all the results from section 1.5 apply to it. In particular,

i) an extension Q of a presheaf P in ${}_{\mathcal{C}}\mathcal{P}(E)$ is essential if for any nonzero subpresheaf Q' of Q the intersection (pullback) $P \cap Q'$ is nonzero, i.e. if there exists some $a \in E$ such that $P(a) \cap Q'(a) \neq 0$.

ii) If σ is a radical in ${}_{\mathcal{C}}\mathcal{P}(E)$, an object Q in ${}_{\mathcal{C}}\mathcal{P}(E)$ is σ-injective if for any subfunctor P' of P in ${}_{\mathcal{C}}\mathcal{P}(E)$ with $P/P' \in \mathcal{T}_\sigma$, every morphism $f : P' \to Q$ extends to a morphism $\bar{f} : P \to Q$; i.e. if for any subfunctor P' of P such that for all $a \in E$,

$$\sigma(P/P')(a) = P(a)/P'(a),$$

every morphism f extends to a morphism $\bar{f}$ such that for all $a \in E$ the following diagram commutes.

$$\begin{tikzcd}
0 \arrow[r] & P'(a) \arrow[r, hook, "i"] \arrow[dr, "f(a)"'] & P(a) \arrow[r, two heads, "\pi"] \arrow[d, dashed, "\bar{f}(a)"] & P(a)/P'(a) \arrow[r] & 0 \\
 & & Q(a) & &
\end{tikzcd}$$

iii) The localization of P at $\sigma \in K({}_{\mathcal{C}}\mathcal{P}(E))$ is the unique σ-injective presheaf $Q_\sigma P \in {}_{\mathcal{C}}\mathcal{P}(E)$ such that $\overline{P} \hookrightarrow Q_\sigma P$ is an essential extension and $Q_\sigma P/\overline{P}$ is σ-torsion, where $\overline{P}$ is the quotient presheaf $P/\sigma P$; i.e. $\overline{P} \hookrightarrow Q_\sigma P$ is an essential extension and for all $a \in E$ we have

$$\sigma(Q_\sigma P/\overline{P})(a) = Q_\sigma P(a)/\overline{P}(a).$$

2.3.2 Definition. Let $\mathcal{C}$ be an arbitrary category and $P \in {}_{\mathcal{C}}\mathcal{P}(E)$. We call P *flabby* (or *flasque*) if all the restriction morphisms $P_{ab} : P(a) \to P(b)$ are surjective.

2.3.3 Lemma. *Let $\mathcal{C}$ be an arbitrary category. If $P \in {}_{\mathcal{C}}\mathcal{P}(E)$ is flabby and $f : P \twoheadrightarrow P'$ is an epimorphism in ${}_{\mathcal{C}}\mathcal{P}(E)$, then P' is flabby.*

Proof. For every $b \leq a$ in E we have the commutative diagram

$$\begin{tikzcd}
P(a) \arrow[r, two heads, "f(a)"] \arrow[d, two heads, "P_{ab}"'] & P'(a) \arrow[d, "P'_{ab}"] \\
P(b) \arrow[r, two heads, "f(b)"'] & P'(b)
\end{tikzcd}$$

and therefore, P'_{ab} is surjective. □

From hereon, let R be a presheaf of rings on E.

2.3.4 Suppose that for every $a \in E$ we are given a radical $\sigma(a)$ in $R(a)$-**mod** and that for all $a, b \in E$ such that $b \leq a$ the corresponding radicals satisfy

L1) for all $M \in R(a)$-**mod** and for all $N \in R(b)$-**mod**, if $f : M \to N$ is an R_{ab}-semilinear map, then

$$f(\sigma(a)M) \subseteq \sigma(b)N.$$

Obviously, if the pair of radicals $\sigma(a)$ and $\sigma(b)$ satisfy L1), then the following diagram is commutative

$$\begin{tikzcd}
\sigma(a)M \arrow[r, "f|_{\sigma(a)M}"] \arrow[d, hook] & \sigma(b)N \arrow[d, hook] \\
M \arrow[r, "f"'] & N
\end{tikzcd}$$

The following proposition characterizes this condition for the case of R being flabby.

2.3.5 Proposition. *Let R be flabby. Condition L1) is satisfied if and only if*

$$\overline{R_{ab}}(\sigma(a)) \leq \sigma(b), \tag{2.4}$$

where $\overline{R_{ab}}$ is the lattice morphism induced by the ring homomorphism R_{ab}.

Proof. Let M be an arbitrary left $R(b)$-module. The identity map on M, $id : (R_{ab})_*(M) \to M$, satisfies for all $r \in R(a)$ and all $m \in M$ that

$$id(rm) = id(R_{ab}(r)m) = R_{ab}(r)m = R_{ab}(r)id(m),$$

i.e. it is R_{ab}-semilinear. Thus by L1), $\sigma(a)((R_{ab})_*(M)) \subseteq \sigma(b)M$ as sets and consequently

$$\sigma(a)((R_{ab})_*(M)) \subseteq (R_{ab})_*(\sigma(b)M)$$

as left $R(a)$-modules. On the other hand, by 1.6.9 ii),

$$(R_{ab})_*(\overline{R_{ab}}(\sigma(a))M) = \sigma(a)((R_{ab})_*(M)).$$

Therefore, $(R_{ab})_*(\overline{R_{ab}}(\sigma(a))M) \subseteq (R_{ab})_*(\sigma(b)M)$ as left $R(a)$-modules and then $\overline{R_{ab}}(\sigma(a))M \subseteq \sigma(b)M$ as left $R(b)$-modules.

Conversely, suppose that the pair of radicals $\sigma(a)$ and $\sigma(b)$ satisfy (2.4) and let $f : M \to N$ be an R_{ab}-semilinear map.
By assumption, to prove that $f(\sigma(a)M) \subseteq \sigma(b)N$ it is sufficient to check that $f(\sigma(a)M) \subseteq \overline{R_{ab}}(\sigma(a))N$. Since

$$\overline{R_{ab}}(\sigma(a))N = \sum_{N' \subseteq N \text{ s.t. } (R_{ab})_*(N') \in \mathcal{T}_{\sigma(a)}} N',$$

$f(\sigma(a)M)$ is a subset of $\overline{R_{ab}}(\sigma(a))N$ if it is a left $R(b)$-submodule of N and is $\sigma(a)$-torsion as an $R(a)$-module via R_{ab}.
Let $s \in R(b)$. Since R is flabby, there exists $r \in R(a)$ such that $s = R_{ab}(r)$. Thus, for all $x \in \sigma(a)M$, we have

$$sf(x) = R_{ab}(r)f(x) = f(rx).$$

This product obviously belongs to $f(\sigma(a)M)$ since $\sigma(a)M$ is a left $R(a)$-module. Therefore, $f(\sigma(a)M)$ is a left $R(b)$-submodule of N. Finally, let us see that every element $f(x) \in f(\sigma(a)M)$ belongs to $\sigma(a)(R_{ab})_*(f(\sigma(a)M))$. Indeed, since $x \in \sigma(a)M$, there exists $L \in \mathcal{L}(\sigma(a))$ such that $Lx = 0$, hence

$$Lf(x) = R_{ab}(L)f(x) = f(Lx) = f(0) = 0,$$

by the definition of the $R(a)$-module structure of $(R_{ab})_*(f(\sigma(a)M))$ and by the R_{ab}-semilinearity of f. Therefore, $f(x)$ belongs to $\sigma(a)(R_{ab})_*(f(\sigma(a)M))$.
□

2.3.6 Remark. In [37, prop.III.2.2] there is a similar result for the case of a topological space X instead of an arbitrary poset E. However, according to the previous proof of the converse, the condition of R being flabby should be added to the hypotheses of [37, prop.III.2.2] since it has been necessary to check that $f(\sigma(a)M)$ is a left $R(b)$-submodule of N, for every R_{ab}-semilinear map f.

2.3.7 Lemma. *If condition L1) is satisfied by the pair of radicals $\sigma(a)$ and $\sigma(b)$, then they also satisfy*

*L2) for all $M, M' \in R(a)$-**mod** and for all $N, N' \in R(b)$-**mod** such that the following diagram is commutative*

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, "g_a"'] & N \arrow[d, "g_b"] \\
M' \arrow[r, "f'"'] & N'
\end{tikzcd}$$

where f, f' are R_{ab}-semilinear and g_a, g_b are resp. $R(a)$,$R(b)$-linear, the induced diagram

$$\begin{tikzcd}
\sigma(a)M \arrow[r, "f|_{\sigma(a)M}"] \arrow[d, "\sigma(a)g_a"'] & \sigma(b)N \arrow[d, "\sigma(b)g_b"] \\
\sigma(a)M' \arrow[r, "f'|_{\sigma(a)M'}"'] & \sigma(b)N'
\end{tikzcd}$$

is also commutative.

2.3.8 Remark. As an immediate consequence of this lemma, let us point out that in the statements of [37, III] only condition $L1)$ is necessary, instead of imposing both $L1)$ and $L2)$.

Proof. Since $\sigma(a)$ resp. $\sigma(b)$ are subfunctors of the identity functor in $R(a)$-**mod** resp. $R(b)$-**mod**, it follows that

$$\begin{aligned}
\sigma(a)g_a &= g_a|_{\sigma(a)M} : \sigma(a)M \to \sigma(a)M'; \\
\sigma(b)g_b &= g_b|_{\sigma(b)N} : \sigma(b)N \to \sigma(b)N'.
\end{aligned}$$

On the other hand, by L1)

$$\begin{aligned}
f|_{\sigma(a)M} &: \sigma(a)M \to \sigma(b)N; \\
f'|_{\sigma(a)M'} &: \sigma(a)M' \to \sigma(b)N'.
\end{aligned}$$

So,

$$\begin{aligned}
\sigma(b)g_b \circ f|_{\sigma(a)M} &= g_b|_{\sigma(b)N} \circ f|_{\sigma(a)M} = g_b|_{f(\sigma(a)M)} \circ f|_{\sigma(a)M} \\
&= (g_b \circ f)|_{\sigma(a)M} = (f' \circ g_a)|_{\sigma(a)M} \\
&= f'|_{g_a(\sigma(a)M)} \circ g_a|_{\sigma(a)M} = f'|_{\sigma(a)M'} \circ \sigma(a)g_a.
\end{aligned}$$

□

2.3.9 Proposition. *Let* $\{\sigma(a) \in K(R(a))\}_{a\in E}$ *be a family of radicals. If for all* $a, b \in E$ *such that* $b \leq a$*, the corresponding radicals satisfy L1), then there exists a unique radical* $\sigma \in K(R\text{-}\mathbf{pre}\text{-}\mathbf{Mod})$ *such that, for all* $M \in R\text{-}\mathbf{pre}\text{-}\mathbf{Mod}$ *and* $a \in E$,

$$(\sigma M)(a) = \sigma(a)M(a). \tag{2.5}$$

Proof. We have to check that there exists a left exact subfunctor of the identity functor in R-**pre-Mod** such that $\sigma(M/\sigma M) = 0$, for every $M \in R$-**pre-Mod**. First of all, let us define σ. For every $M \in R$-**pre-Mod**, let σM be given on every $a \in E$ by

$$(\sigma M)(a) = \sigma(a)M(a) \subseteq M(a).$$

For every $b \leq a$ in E we get the restriction morphism

$$(\sigma M)_{ab} = (M_{ab})|_{\sigma M(a)} : (\sigma M)(a) = \sigma(a)M(a) \longrightarrow (\sigma M)(b) = \sigma(b)M(b)$$

by applying L1) to the restriction map M_{ab} which is R_{ab}-semilinear. Then it is clear that σM is a subpresheaf of M which satisfies (2.5), and it is a presheaf of left R-modules since $(\sigma M)(a)$ is a left $R(a)$-module and $(\sigma M)_{ab}$ is R_{ab}-semilinear because it is the restriction of an R_{ab}-semilinear map. Therefore, σ is well-defined on the objects.

Let $f : M \to N$ be a morphism in R-**pre-Mod**, i.e. a family

$$\{f(a) : M(a) \longrightarrow N(a)\}_{a\in E}$$

of homomorphisms of left $R(a)$-modules such that for all $b \leq a$ in E the following diagram is commutative

$$\begin{tikzcd} M(a) \arrow[r, "M_{ab}"] \arrow[d, "f(a)"'] & M(b) \arrow[d, "f(b)"] \\ N(a) \arrow[r, "N_{ab}"'] & N(b) \end{tikzcd}$$

By L2) we obtain a commutative diagram

$$\begin{tikzcd} \sigma(a)M(a) \arrow[r, "(M_{ab})|_{\sigma(a)M(a)}"] \arrow[d, "\sigma(a)f(a)"'] & \sigma(b)M(b) \arrow[d, "\sigma(b)f(b)"] \\ \sigma(a)N(a) \arrow[r, "(N_{ab})|_{\sigma(a)N(a)}"'] & \sigma(b)N(b) \end{tikzcd}$$

and by the definition of the R-pre-Module σM,

$$(M_{ab})|_{\sigma(a)M(a)} = (M_{ab})|_{(\sigma M)(a)} = (\sigma M)_{ab}.$$

That is, if for all $a \in E$ we define $(\sigma f)(a) = \sigma(a)f(a)$, we obtain the set

$$\{(\sigma f)(a) : (\sigma M)(a) \to (\sigma N)(a)\}_{a \in E}$$

of left $R(a)$-module homomorphisms which yields a morphism of R-pre-Modules $\sigma f : \sigma M \longrightarrow \sigma N$. Finally $\sigma f = f|_{\sigma M}$ if and only if for all $a \in E$

$$(\sigma f)(a) = f(a)|_{(\sigma M)(a)},$$

and this is obvious since $\sigma(a) \in K(R(a))$. Therefore, σ is a subfunctor of the identity functor in R-**pre-Mod** satisfying (2.5) and its uniqueness follows from its construction.

Next, to check that $M/\sigma M$ is σ-torsion-free we have to prove that the presheaf $\sigma(M/\sigma M)$ is the zero presheaf. Let us mention first that the quotient presheaf $M/\sigma M \in R$-**pre-Mod** is defined on every $a \in E$ as the quotient module

$$(M/\sigma M)(a) = M(a)/(\sigma M)(a) = M(a)/\sigma(a)M(a), \tag{2.6}$$

which is a left $R(a)$-module. For every $b \leq a$ in E, the restriction morphism

$$(M/\sigma M)_{ab} : M(a)/\sigma(a)M(a) \longrightarrow M(b)/\sigma(b)M(b)$$

is the R_{ab}-semilinear map which assigns to every element $x + \sigma(a)M(a)$ in $M(a)/\sigma(a)M(a)$, the element

$$M_{ab}(x) + \sigma(b)M(b) \in M(b)/\sigma(b)M(b).$$

Now, by the definition of σ on the objects, for all $a \in E$

$$(\sigma(M/\sigma M))(a) = \sigma(a)((M/\sigma M)(a)).$$

By (2.6) this may be written as

$$\sigma(a)(M(a)/(\sigma M)(a)) = \sigma(a)(M(a)/\sigma(a)M(a)),$$

which is obviously equal to 0 since $\sigma(a) \in K(R(a))$. Thus, $M/\sigma M \in \mathcal{F}_\sigma$.

Finally, the left exactness of the functor $\sigma : R$-**pre-Mod** $\longrightarrow R$-**pre-Mod** is trivially obtained from the left exactness of each $\sigma(a) \in K(R(a))$. $\square$

2.3.10 Definition. A radical σ in R-**pre-Mod** given, as in prop. 2.3.9, by a family of radicals $\Gamma = \{\sigma(a) \in K(R(a))\}_{a \in E}$ such that for every pair of elements $b \leq a$ in E the corresponding radicals in Γ satisfy L1) is said to be a *local radical.*

2.3.11 Definition. Let σ be a local radical in R-**pre-Mod** defined by the family of radicals $\{\sigma(a) \in K(R(a))\}_{a \in E}$. Considering $R(a)$ as a left $R(a)$-module itself, we obtain the left $R(a)$-submodule $\sigma(a)R(a)$, which is also a two-sided ideal of the ring. The quotient $R/\sigma R \in {}_{\mathbf{Ring}}\mathcal{P}(E)$ is given on every $a \in E$ by the quotient ring

$$(R/\sigma R)(a) = R(a)/\sigma(a)R(a).$$

For every $b \leq a$ in E, the restriction morphism

$$(R/\sigma R)_{ab} : R(a)/\sigma(a)R(a) \to R(b)/\sigma(b)R(b)$$

is the ring homomorphism which assigns to every $x + \sigma(a)R(a)$ in $(R/\sigma R)(a)$ the element

$$R_{ab}(x) + \overline{R_{ab}}(\sigma(a))R(b) \in R(b)/\sigma(b)R(b)$$

(this is well-defined since $R_{ab}(\sigma(a)R(a)) \subseteq \sigma(b)R(b)$ by L1) applied to R_{ab}).

2.3.12 Remark. In view of 2.3.3, if $R \in {}_{\mathbf{Ring}}\mathcal{P}(E)$ is flabby then the quotient $R/\sigma R \in {}_{\mathbf{Ring}}\mathcal{P}(E)$ is also flabby.

2.3.13 Theorem. *If R is a flabby presheaf of rings on E and σ is a local radical in R-**pre-Mod**, then $Q_\sigma R$ is a presheaf of rings whose structure is uniquely determined by R.*

Proof. Let Q be the presheaf of rings which assigns to any $a \in E$ the ring of quotients of $R(a)$ at $\sigma(a)$, i.e. which is defined on the objects $a \in E$ by

$$Q(a) = Q_{\sigma(a)}R(a).$$

For every $b \leq a$ in E, the restriction morphism $Q_{ab} : Q(a) \to Q(b)$ is obtained (by the $\sigma(b)$-injectivity of $Q(b)$),

$$\begin{tikzcd}
0 \arrow[r] & (R/\sigma R)(a) \arrow[r, hook, "i_a"] \arrow[d, "(R/\sigma R)_{ab}"'] & Q(a) \arrow[r, two heads, "\pi"] \arrow[d, dashed, "Q_{ab}"] & Q(a)/(R/\sigma R)(a) \arrow[r] & 0 \\
& (R/\sigma R)(b) \arrow[r, hook, "i_b"'] & Q(b) & &
\end{tikzcd}$$

as the extension of the ring homomorphism $i_b \circ (R/\sigma R)_{ab}$, since the quotient ring $Q(a)/(R/\sigma R)(a)$ is not only $\sigma(a)$-torsion but also $\sigma(b)$-torsion. Indeed, since $R_{ab} : R(a) \to R(b)$ is surjective, by 1.6.9 ii) it follows for all $N \in R(b)$-**mod** that

$$(R_{ab})_*(\overline{R_{ab}}(\sigma(a))N) = \sigma(a)((R_{ab})_*N); \tag{2.7}$$

and consequently we get

$$\mathcal{T}_{\sigma(a)} = (R_{ab})_*(\mathcal{T}_{\overline{R_{ab}}\sigma(a)}).$$

On the other hand, from (2.4) we obtain $\mathcal{T}_{\overline{R_{ab}}\sigma(a)} \subseteq \mathcal{T}_{\sigma(b)}$. Therefore,

$$\mathcal{T}_{\sigma(a)} \subseteq (R_{ab})_*(\mathcal{T}_{\sigma(b)})$$

so $Q(a)/(R/\sigma R)(a)$ is $\sigma(b)$-torsion.

The constructed presheaf is exactly the ring of quotients of R at σ, i.e. Q is the localization functor of R at σ, since it verifies the conditions explained in 2.3.1 which uniquely determine $Q_\sigma R$ in the category ${}_{\mathbf{Ring}}\mathcal{P}(E)$.
Indeed, let P' be a subpresheaf of P in ${}_{\mathbf{Ring}}\mathcal{P}(E)$ such that P/P' is σ-torsion, i.e. since σ is local, such that $P(a)/P'(a)$ is $\sigma(a)$-torsion, for all $a \in E$, and let $f : P' \to Q$ be a morphism in ${}_{\mathbf{Ring}}\mathcal{P}(E)$. By the $\sigma(a)$-injectivity of $Q(a)$, there exists $\bar{f}(a) : P(a) \to Q(a)$ such that

$$\begin{tikzcd}
0 \arrow[r] & P'(a) \arrow[r, hook, "i"] \arrow[dr, "f(a)"'] & P(a) \arrow[r, two heads, "\pi"] \arrow[d, dashed, "\bar{f}(a)"] & P(a)/P'(a) \arrow[r] & 0 \\
 & & Q(a) & &
\end{tikzcd}$$

for all $a \in E$. Therefore, in view of definition 2.3.1 ii), Q is a σ-injective presheaf of rings since it is locally $\sigma(a)$-injective, for all $a \in E$.
Moreover, the extension $R/\sigma R \hookrightarrow Q$ in ${}_{\mathbf{Ring}}\mathcal{P}(E)$ is essential since for all $a \in E$ we have the essential extension $R(a)/\sigma(a)R(a) \hookrightarrow Q(a)$.

Finally the quotient presheaf $Q/(R/\sigma R)$ is σ-torsion since for all $a \in E$ it is locally $\sigma(a)$-torsion. Indeed, for all $a \in E$,

$$\begin{aligned}\sigma(Q/(R/\sigma R))(a) &= \sigma(a)(Q/(R/\sigma R))(a) = \sigma(a)(Q(a)/(R(a)/\sigma(a)R(a))) \\ &= Q(a)/(R(a)/\sigma(a)R(a)) = (Q/(R/\sigma R))(a).\end{aligned}$$

Therefore, $Q = Q_\sigma R$. □

2.3.14 Definition. Let $\sigma \in K(R\text{-}\mathbf{pre\text{-}Mod})$. We say that σ is *reducing* R or, equivalently, that R is *reduced by* σ, if $\operatorname{Ker} R_{ab} \subseteq (\sigma R)(a)$, for every $b \leq a$ in E, that is,

$$\operatorname{Ker} R_{ab} \hookrightarrow (\sigma R)(a) \hookrightarrow R(a) \xrightarrow{R_{ab}} R(b).$$

for every $b \leq a$ in E. More generally, an R-pre-Module M is said to be *reduced by σ* if

$$\operatorname{Ker} M_{ab} \hookrightarrow (\sigma M)(a) \hookrightarrow M(a) \xrightarrow{M_{ab}} M(b),$$

for every $b \leq a$ in E.

2.3.15 Note. Let $\sigma \in K(R\text{-}\mathbf{pre\text{-}Mod})$ be local. If σ reduces R, then for every $b \leq a \in E$,

$$\operatorname{Ker} R_{ab} \subseteq \sigma(a)R(a). \tag{2.8}$$

More generally, if $M \in R$-pre-Module is reduced by σ, then for every $b \leq a$ in E we have $\operatorname{Ker} M_{ab} \subseteq \sigma(a)M(a)$.

2.3.16 Remark. In theorem 2.3.13, which generalizes [37, thm.III.4.2], we obtained the localization presheaf $Q_\sigma R$ without having to assume that σ *reduces* R. Therefore, in [37, thm.III.4.2] it is a redundant hypothesis.

2.3.17 Theorem. *Let R be a flabby presheaf of rings, $\sigma \in K(R\text{-}\mathbf{pre\text{-}Mod})$ a local radical reducing R, and M an R-pre-Module, then $Q_\sigma M$ is in a natural way a left $Q_\sigma R$-pre-Module given on every $a \in E$ by*

$$Q_\sigma M(a) = Q_{\sigma(a)} M(a),$$

and whose structure is uniquely determined by its R-pre-Module structure.

Proof. Let Q be the presheaf of left R-modules which assigns to any $a \in E$ the module of quotients of $M(a)$ at $\sigma(a)$, i.e. defined on the objects by

$$Q(a) = Q_{\sigma(a)} M(a).$$

For every $b \leq a$ in E, the restriction morphism $Q_{ab} : Q(a) \longrightarrow Q(b)$ is obtained in three steps.

First of all, considering $M(b)$ as a left $R(a)$-module by scalar restriction via R_{ab}, it is obvious that M_{ab} is a homomorphism of $R(a)$-modules. Therefore, there exists

$$Q_{\sigma(a)}(M_{ab}) : Q(a) = Q_{\sigma(a)} M(a) \longrightarrow Q_{\sigma(a)} M(b)$$

as the image of M_{ab} by the functor $Q_{\sigma(a)}$ in $R(a)$-**mod** (recall this definition from 1.6.5).

On the other hand, let $\sigma'(a)$ denote the radical $\overline{R_{ab}}(\sigma(a)) \in K(R(b))$. Since σ is local, by (2.4) we have $\sigma'(a)M(b) \subseteq \sigma(b)M(b)$, so we may define the surjection of left $R(b)$-modules $\psi : M(b)/\sigma'(a)M(b) \longrightarrow M(b)/\sigma(b)M(b)$. We obtain the homomorphism of left $R(b)$-modules

$$\overline{\psi} : Q_{\sigma'(a)}M(b) \longrightarrow Q(b) = Q_{\sigma(b)}M(b)$$

by the $\sigma(b)$-injectivity of $Q(b)$, as the extension of $i \circ \psi$.

$$\begin{tikzcd}
M(b)/\sigma'(a)M(b) \arrow[r, hook, "i'"] \arrow[d, "\psi"'] & Q_{\sigma'(a)}M(b) \arrow[d, dashed, "\overline{\psi}"] \\
M(b)/\sigma(b)M(b) \arrow[r, hook, "i"'] & Q(b)
\end{tikzcd}$$

Indeed, all the objects in the diagram above are left $R(b)$-modules and the quotient

$$Q_{\sigma'(a)}M(b)/(M(b)/\sigma'(a)M(b))$$

is not only $\sigma'(a)$-torsion but also $\sigma(b)$-torsion since $\sigma'(a) \leq \sigma(b)$.

We finally prove the equality

$$Q_{\sigma(a)}M(b) = Q_{\sigma'(a)}M(b) \tag{2.9}$$

which allows us to obtain Q_{ab} as the composition

$$Q(a) \xrightarrow{Q_{\sigma(a)}(M_{ab})} Q_{\sigma(a)}M(b) = Q_{\sigma'(a)}M(b) \xrightarrow{\overline{\psi}} Q(b),$$

which is R_{ab}-semilinear in view of the compatibility of the module structures via R_{ab}.

To prove (2.9) it is sufficient to check that $Q_{\sigma(a)}M(b)$ is a $\sigma'(a)$-injective object in $R(b)$-**mod** and its quotient by $M(b)/\sigma'(a)M(b)$ is $\sigma'(a)$-torsion:

First of all, since R is flabby, for all $s \in R(b)$ there exists some $r \in R(a)$ such that $s = R_{ab}(r)$. Thus, we may define on the left $R(a)$-module $Q_{\sigma(a)}M(b)$ the left $R(b)$-module structure via R_{ab}, given for all $x \in Q_{\sigma(a)}M(b)$ and $s \in R(b)$ by

$$sx = rx, \text{ with } s = R_{ab}(r) \text{ for some } r \in R(a).$$

This structure is well-defined, i.e. for all $r \in \operatorname{Ker} R_{ab}$ we have $rx = 0$, since R is reduced by σ. Indeed, in view of by (2.8), if $r \in \operatorname{Ker} R_{ab}$ then $r \in \sigma(a)R(a)$, so there exists $L \in \mathcal{L}(\sigma(a))$ such that $Lr = 0$ by (1.4). Consequently, $Lrx = 0$, so rx is in the $\sigma(a)$-torsion of the $\sigma(a)$-torsion-free object $Q_{\sigma(a)}M(b)$. Therefore, $rx = 0$.

On the other hand, $Q_{\sigma(a)}M(b)$ is $\sigma'(a)$-injective since for all $N' \subseteq N$ in $R(b)$-**mod** with $N/N' \in \mathcal{T}_{\sigma'(a)}$, we have $N/N' \in \mathcal{T}_{\sigma(a)}$ in view of (2.9). Moreover, every homomorphism of left $R(b)$-modules $f : N' \to Q_{\sigma(a)}M(b)$ is also a left $R(a)$-homomorphism. Indeed, for all $r \in R(a)$ and $n \in N'$, we have

$$f(rn) = f(R_{ab}(r)n) = R_{ab}(r)f(n) = rf(n),$$

considering the left $R(a)$-module structure of N' by scalar restriction via R_{ab} and the left $R(a)$-module structure defined on $Q_{\sigma(a)}M(b)$. Therefore, by the $\sigma(a)$-injectivity of $Q_{\sigma(a)}M(b)$ as an $R(a)$-module, there exists a left $R(a)$-homomorphism $\bar{f} : N \to Q_{\sigma(a)}M(b)$ which is the extension of f. This is also a left $R(b)$-homomorphism, since for all $s \in R(b)$ with $s = R_{ab}(r)$, and for all $n \in N$, we have

$$\bar{f}(sn) = \bar{f}(rn) = r\bar{f}(n) = R_{ab}(r)\bar{f}(n) = s\bar{f}(n),$$

again considering the left $R(a)$-module structure of N by scalar restriction via R_{ab} and the left $R(a)$-module structure defined on $Q_{\sigma(a)}M(b)$.
Finally, by (2.7) it follows that $\sigma'(a)M(b)$ as a left $R(a)$-module is equivalent to $\sigma(a)(R_{ab})_*M(b)$. Therefore, considering $M(b)$ as a left $R(a)$-module via R_{ab}, the quotient

$$Q_{\sigma(a)}M(b)/(M(b)/\sigma'(a)M(b))$$

viewed within $R(a)$-**mod** is equivalent to

$$Q_{\sigma(a)}M(b)/(M(b)/\sigma(a)M(b)),$$

which is obviously $\sigma(a)$-torsion, and again taking into account (2.7) it is $\sigma'(a)$-torsion too. Therefore, (2.9) holds.

The presheaf Q we just constructed is exactly the localization functor of M at σ, since it verifies the conditions which uniquely determine $Q_\sigma M$ in R-**pre-Mod**. Indeed, let M' be a subpresheaf of M in R-**pre-Mod** such that M/M' is σ-torsion, i.e. (since σ is local) such that $M(a)/M'(a)$ is $\sigma(a)$-torsion, for all $a \in E$, and let $f : M' \to Q$ be a morphism in R-**pre-Mod**. By the $\sigma(a)$-injectivity of $Q(a)$, there exists a morphism of left $R(a)$-modules $\bar{f}(a) : M(a) \to Q(a)$ such that

$$\begin{tikzcd}
0 \arrow[r] & M'(a) \arrow[r, hook, "i"] \arrow[dr, "f(a)"'] & M(a) \arrow[r, two heads, "\pi"] \arrow[d, dashed, "\bar{f}(a)"] & M(a)/M'(a) \arrow[r] & 0 \\
 & & Q(a) & &
\end{tikzcd}$$

for all $a \in E$, which determines the extension of f in $\pi_\Lambda(R)$. Therefore, Q is a σ-injective presheaf of R-modules since for all $a \in E$ it is locally $\sigma(a)$-injective.

On the other hand, the essential extension $M/\sigma M \hookrightarrow Q$ in ${}_{\mathbf{Ring}}\mathcal{P}(E)$ is given by the family of essential extensions

$$\{M(a)/\sigma(a)M(a) \hookrightarrow Q(a)\}_{a \in E}.$$

Finally the quotient presheaf $Q/(M/\sigma M)$ is σ-torsion since it is locally $\sigma(a)$-torsion, for all $a \in E$. Therefore, $Q = Q_\sigma M$. $\square$

Chapter 3

Noncommutative Topologies

In the previous chapter we could have been talking about presheaves strictly as algebraic objects, forgetting everything related to topologies. However, to be able to deal with sheaves one chapter ahead, the topology turns out to be indispensable. In fact, very roughly speaking, a sheaf is a presheaf that *glues well*, and the gluing condition involves topological concepts such as covering or intersection of open subsets. Of course, our noncommutative sheaves earn this extra qualification from the noncommutativity of the topology on which they are defined, which is basically found in the fact that intersections are no longer commutative. Hence, it is time to present some of the examples of noncommutative topologies that may be found in the literature.

We begin by recalling the concept of *site*, as the best categorical approach to topological spaces, since the examples of noncommutative topologies may also be studied from this point of view. After having summarized the main characteristics and examples of some different noncommutative sites, we will finally present our own type of (noncommutative) site, which we call *Q-site*, whose construction is based on *quantales*.

3.1 Grothendieck topologies

For a better approach to this matter, a good starting point is to look for a more categorical definition of the notion of topology, that is, the *Grothendieck topology of a site*, since many noncommutative topologies follow this *site* philosophy too.

Let X be an arbitrary topological space. Then, it is well known that every open subset is covered by itself, a covering of coverings is again a covering,

and coverings may be induced on subsets.
Consider the small category $\mathcal{O}(X)$ whose underlying set of objects is the set of open subsets $O(X)$, and where for every $U, V \in O(X)$ the set $Hom_{\mathcal{O}(X)}(U, V)$ is the singleton $\{U \hookrightarrow V\}$ if and only if $U \subseteq V$, and the empty set otherwise. If we define for every $U \in O(X)$ a set $\mathrm{Cov}(U)$ given by

$$\{U_i \hookrightarrow U\}_{i \in I} \in \mathrm{Cov}(U) \Longleftrightarrow \bigcup_{i \in I} U_i = U,$$

then the three basic properties above translate to:

G1) for all $U \in O(X)$, $\{U \hookrightarrow U\} \in \mathrm{Cov}(U)$;

G2) if $\{U_i \hookrightarrow U\}_{i \in I} \in \mathrm{Cov}(U)$ and $\{U_{ij} \hookrightarrow U_i\}_{j \in J_i} \in \mathrm{Cov}(U_i)$, for all $i \in I$, then $\{U_{ij} \hookrightarrow U\}_{\substack{j \in J_i \\ i \in I}} \in \mathrm{Cov}(U)$;

G3) if $\{U_i \hookrightarrow U\}_{i \in I} \in \mathrm{Cov}(U)$ and $U' \subseteq U$ then $\{U' \cap U_i \hookrightarrow U\}_{i \in I} \in \mathrm{Cov}(U')$.

Taking into account that the intersection is (categorically) just a pullback, we just established the motivation for the following concept:

3.1.1 Definition. A *Grothendieck topology* on a small category $\mathcal{E}$ consists of a family $\{\mathrm{Cov}(A)\}_{A \in \mathcal{E}}$ where for every object $A \in \mathcal{E}$ the elements $\mathrm{Cov}(A)$ are families of morphisms in $\mathcal{E}$ with common target A, satisfying

G1) $\{A \to A\} \in \mathrm{Cov}(A)$, for every $A \in \mathcal{E}$;

G2) If $\{A_i \to A\}_{i \in I} \in \mathrm{Cov}(A)$ and $\{A_{ij} \to A_i\}_{j \in J_i} \in \mathrm{Cov}(A_i)$, for all $i \in I$, then $\{A_{ij} \to A_i \to A\}_{j \in J_i, i \in I} \in \mathrm{Cov}(A)$;

G3) If $A' \to A$ is a morphism in $\mathcal{E}$ and $\{A_i \to A\}_{i \in I} \in \mathrm{Cov}(A)$, then for every $i \in I$ the pullback $A' \times_A A_i$ exists and $\{A' \times_A A_i \to A'\}_{i \in I} \in \mathrm{Cov}(A')$.

The pair ($\mathcal{E}$, $\{\mathrm{Cov}(A)\}_{A \in \mathcal{E}}$) is called a *site*, and the elements of $\mathrm{Cov}(A)$ are called the *coverings of* A.

In view of all the former arguments and the previous definition, we obviously obtain the following:

3.1.2 Example. For every topological space X, the pair

$$(\mathcal{O}(X), \{\mathrm{Cov}(U)\}_{U \in O(X)})$$

is a site.

3.2 *À la* García Román

This noncommutative topology, which may be found in [15] and [17], is based on Gabriel filters and their (noncommutative) composition, which plays the role of the intersection of open subsets in the commutative case. It has to be stressed that it generalizes the Zariski topology on the prime spectrum of a commutative ring.

Let R be an (associative) ring with unity.

Before giving the construction of the *noncommutative site à la García Román*, we first study the lattice of uniform filters a little more in depth. Let us recall the definition of Gabriel filter given in 1.6.10, but this time specifying the following concepts:

3.2.1 Definition. A nonempty family $\mathcal{L}$ of left R-ideals that satisfies

1. if L and H are left R-ideals with $L \subseteq H$ and $L \in \mathcal{L}$, then $H \in \mathcal{L}$;
2. if $L, H \in \mathcal{L}$, then $L \cap H \in \mathcal{L}$

or, equivalently,

i) if $L, L' \in \mathcal{L}$ then $H \in \mathcal{L}$ for all left R-ideal H such that $L \cap L' \subseteq H$,

is called a *filter*. If $\mathcal{L}$ also satisfies

ii) if $L \in \mathcal{L}$ and $s \in R$, then $(L : s) = \{r \in R \mid rs \in L\} \in \mathcal{L}$,

is called a *uniform filter*. We say that $\mathcal{L}$ is a *Gabriel filter* if it is a uniform filter that also verifies

iii) if L is a left R-ideal and there exists $H \in \mathcal{L}$ with the property that $(L : r) \in \mathcal{L}$ for all $r \in H$, then $L \in \mathcal{L}$.

3.2.2 Definition. Let $\mathcal{L}$ and $\mathcal{H}$ be two filters. The *composition* $\mathcal{L} \circ \mathcal{H}$ *of* $\mathcal{L}$ *and* $\mathcal{H}$ is the set of left R-ideals L with the property that there exists $H \in \mathcal{H}$ such that $(L : r) \in \mathcal{L}$, for all $r \in H$. It may easily be checked that $\mathcal{L} \circ \mathcal{H}$ is a filter as well (cf. [15, 3.1]).

3.2.3 Note. Using this notation, properties ii) and iii) of the former definition for uniform filters and Gabriel filters are reduced to $\mathcal{L} \subseteq \mathcal{L} \circ \{\mathcal{R}\}$ and $\mathcal{L} = \mathcal{L} \circ \mathcal{L}$ respectively.

Note that in general, from its very definition it follows that the composition of filters is not necessarily commutative. Besides, amongst all the nice properties of this binary operation, let us point out the following ones which will be useful for our aims (see [15, 3] and [16, 1] for more details):

3.2.4 Properties of the composition of filters.

i) If $\mathcal{L} \subseteq \mathcal{L}'$ and $\mathcal{H}$ are filters then $\mathcal{L} \circ \mathcal{H} \subseteq \mathcal{L}' \circ \mathcal{H}$ and $\mathcal{H} \circ \mathcal{L} \subseteq \mathcal{H} \circ \mathcal{L}'$;

ii) $\mathcal{H} \subseteq \mathcal{L} \circ \mathcal{H}$, for all filters $\mathcal{L}, \mathcal{H}$;

iii) $\{R\} \circ \mathcal{L} = \mathcal{L}$, for all filters $\mathcal{L}$;

iv) the composition of filters is associative;

v) if $L \in \mathcal{L}$ and $H \in \mathcal{H}$ then $LH \in \mathcal{L} \circ \mathcal{H}$;

vi) let $\mathcal{L}$ be a filter, then $\mathcal{L} \subseteq \mathcal{L} \circ \{\mathcal{R}\} \Leftrightarrow \mathcal{L} = \mathcal{L} \circ \{\mathcal{R}\} \Leftrightarrow \mathcal{L} \subseteq \mathcal{L} \circ \mathcal{H}$, for all filters $\mathcal{H}$;

vii) the composition of uniform filters is again a uniform filter;

viii) the intersection of uniform (resp. Gabriel) filters is again a uniform (resp. Gabriel) filter;

ix) let $\mathcal{H}$ be a uniform filter and $\{\mathcal{L}_a\}_{a \in A}$ a family of uniform filters, then

$$\bigcap_{a \in A} (\mathcal{H} \circ \mathcal{L}_a) = \mathcal{H} \circ (\bigcap_{a \in A} \mathcal{L}_a)$$

and

$$(\bigcap_{a \in A} \mathcal{L}_a) \circ \mathcal{H} \subseteq \bigcap_{a \in A} (\mathcal{L}_a \circ \mathcal{H}).$$

3.2.5 The lattice of uniform filters.

The set of uniform filters ordered by inclusion is a complete lattice where any family $\{\mathcal{L}_a\}_{a \in A}$ of uniform filters has meet

$$\bigwedge_{a \in A} \mathcal{L}_a = \bigcap_{a \in A} \mathcal{L}_a,$$

and where the join $\bigvee_{a \in A} \mathcal{L}_a$ consists of the left R-ideals I with the property that there exist $a_1, \ldots, a_n \in A$ with corresponding $I_i \in \mathcal{L}_{a_i}$ such that $\bigcap_{i=1}^{n} I_i \subseteq I$ (cf. [16, 1]).

As we already mentioned, Gabriel filters play a fundamental role in this noncommutative site, let us see how.

Let R be a left noetherian ring and $\mathcal{G}$ a set of Gabriel filters of R that contains $\{R\}$. Let us first fix some notation:

3.2.6 Notation. By $< \mathcal{G} >$ we denote the free monoid generated by $\mathcal{G}$ with the Gabriel filter $\mathcal{L}_R = \{R\}$ as the unit element. For any arbitrary element $\mathbf{L} = \mathcal{L}_1 \cdots \mathcal{L}_n$ of $< \mathcal{G} >$, the uniform filter $\mathcal{L}_1 \circ \cdots \circ \mathcal{L}_n$ is denoted by $\varepsilon(\mathbf{L})$, and the composition of the localization functors $Q_{\sigma_n} \circ \cdots \circ Q_{\sigma_1}$ is denoted by $Q_{\mathbf{L}}$, where σ_i is the unique radical that corresponds to the Gabriel filter $\mathcal{L}_i$ (recall the bijection described in 1.6.13).

3.2.7 Definition. $\mathbb{T}(\mathcal{G})$ is the small category with underlying set of objects $< \mathcal{G} > / \sim$, where two elements $\mathbf{L}, \mathbf{L}'$ of $< \mathcal{G} >$ are defined to be equivalent (we write $\mathbf{L} \sim \mathbf{L}'$) if and only if for every left R-module M there exists an isomorphism of modules $\eta_M : Q_{\mathbf{L}'}M \to Q_{\mathbf{L}}M$ such that the following diagram is commutative:

$$\begin{tikzcd}
Q_{\mathbf{L}'}M \arrow[rr, "\eta_M"] & & Q_{\mathbf{L}}M \\
& M \arrow[ul, "j_{\mathbf{L}',M}"] \arrow[ur, "j_{\mathbf{L},M}"'] &
\end{tikzcd}$$

For every $\mathbf{L} \in < \mathcal{G} >$, we denote by $[\mathbf{L}]$ its corresponding class in $\mathbb{T}(\mathcal{G})$. For two arbitrary $[\mathbf{L}]$, $[\mathbf{H}]$ in $\mathbb{T}(\mathcal{G})$, a morphism in $\operatorname{Hom}_{\mathbb{T}(\mathcal{G})}([\mathbf{L}], [\mathbf{H}])$ is a natural transformation $\eta : Q_{\mathbf{H}} \to Q_{\mathbf{L}}$ over the identity, i.e. such that for every left R-module M the following diagram is commutative:

$$\begin{tikzcd}
Q_{\mathbf{H}}M \arrow[rr, "\eta_M"] & & Q_{\mathbf{L}}M \\
& M \arrow[ul, "j_{\mathbf{H},M}"] \arrow[ur, "j_{\mathbf{L},M}"'] &
\end{tikzcd}$$

If $\mathbf{L}$ and $\mathbf{H}$ are just Gabriel filters $\mathcal{L}$ and $\mathcal{H}$ in $\mathcal{G}$, then the set of morphisms $\operatorname{Hom}_{\mathbb{T}(\mathcal{G})}([\mathbf{L}], [\mathbf{H}])$ is a singleton when $\mathcal{H} \subseteq \mathcal{L}$, and the empty set otherwise (cf. [15, prop.4.2.4] or [17, (3.7)]).

3.2.8 Remark. The idea is to consider the objects of $\mathbb{T}(\mathcal{G})$ as the open sets of a noncommutative topology. This is inspired by the fact that when R is a commutative noetherian ring, if we take the set of Gabriel filters $\mathcal{G}_{Zar} = \{\mathcal{L}_I\}_{I \leq R}$, where $\mathcal{L}_I$ is the Gabriel filter defined in 1.6.20, then the objects of $\mathbb{T}_{Zar} = \mathbb{T}(\mathcal{G}_{Zar})$ are in bijection with the open sets of the Zariski topology of $\operatorname{Spec} R$ (cf. [15, 4.2.6] or [17, (3.4)]).

3.2.9 Definition. The *noncommutative site* associated to the family of Gabriel filters $\mathcal{G}$ is defined to be the pair

$$(\ \mathbb{T}(\mathcal{G}),\ \{\mathrm{Cov}([\mathbf{H}])\}_{[\mathbf{H}]\in\mathbb{T}(\mathcal{G})}\ )$$

where a *covering of* $[\mathbf{H}]$ is defined as a finite family of classes $\{[\mathbf{L}_\alpha]\}_{\alpha=1}^{n}$ satisfying

i) $\mathcal{L}=\bigcap_{\alpha=1}^{n}\varepsilon(\mathbf{L}_\alpha)$ is a Gabriel filter;

ii) $[\mathcal{L}][\mathbf{H}]=[\mathbf{H}]$.

3.2.10 Remarks.

i) This noncommutative site verifies the axiom G1) of a Grothendieck topology since $\mathcal{L}_R$ is a covering of $[\mathbf{H}]$ for every $[\mathbf{H}]\in\ \mathbb{T}(\mathcal{G})$, and also verifies G2) since if $\{[\mathbf{L}_\alpha]\}_{\alpha=1}^{n}$ is a covering of $[\mathbf{H}]$, and $\{[\mathbf{L}_\beta^\alpha]\}_{\beta=1}^{m_\alpha}$ is a covering of $[\mathbf{L}_\alpha]$ for every $\alpha\in\{1,\ldots,n\}$, then

$$\{[\mathbf{L}_\beta^\alpha][\mathbf{L}_\alpha]\}_{1\leq\alpha\leq n,\,1\leq\beta\leq m_\alpha}$$

is a covering of $[\mathbf{H}]$, in view of [15, (4.2.16)].

ii) According to the previous definition, a *global covering in* $\mathbb{T}(\mathcal{G})$ is a covering of $[\mathcal{L}_R]$, i.e. a family of classes $\{[\mathbf{L}_\alpha]\}_{\alpha=1}^{n}$ satisfying

i) $\mathcal{L}=\bigcap_{\alpha=1}^{n}\varepsilon(\mathbf{L}_\alpha)$ is a Gabriel filter;

ii) $[\mathcal{L}][\mathcal{L}_R]=[\mathcal{L}_R]$.

The radical associated to $\mathcal{L}_R$ is zero on every $M\in R$-**mod**, and hence the localization functor $Q_{\mathcal{L}_R}$ is the identity functor. Therefore, the word $\mathcal{L}\mathcal{L}_R\in<\mathcal{G}>$ is equivalent to $\mathcal{L}_R$ since $Q_{\mathcal{L}_R}\circ Q_{\mathcal{L}}$ is naturally equivalent to $Q_{\mathcal{L}}$. Hence, $[\mathcal{L}][\mathcal{L}_R]=[\mathcal{L}\mathcal{L}_R]=[\mathcal{L}]$ and condition ii) reduces to $[\mathcal{L}]=[\mathcal{L}_R]$, which is equivalent to assert that $\mathcal{L}=\mathcal{L}_R$. Indeed, let $L\in\mathcal{L}$. By definition 1.6.12, the quotient R/L is an $\mathcal{L}$-torsion ideal and by hypothesis $\mathcal{L}_R$-torsion too since $Q_{\mathcal{L}}=Q_{\mathcal{L}_R}$. This means that $Rm\in L$ for every $\overline{m}\in R/L$, in particular for $\overline{1}$. Therefore $L=R$ and consequently $\mathcal{L}=\mathcal{L}_R$.

Thus, a global covering is a family of classes $\{[\mathbf{L}_\alpha]\}_{\alpha=1}^{n}$ such that $\bigcap_{\alpha=1}^{n}\varepsilon(\mathbf{L}_\alpha)=\mathcal{L}_R$.

iii) A global covering $\{[\mathbf{L}_\alpha]\}_{\alpha=1}^{n}$ in $\mathbb{T}(\mathcal{G})$ is a covering of $[\mathbf{H}]$, for every $[\mathbf{H}]\in\ \mathbb{T}(\mathcal{G})$, since $[\mathbf{H}]=[\mathcal{L}_R][\mathbf{H}]$ and in general, according to [15, 4.2.17], a covering of an arbitrary $[\mathbf{L}]\in\ \mathbb{T}(\mathcal{G})$ is a covering of each $[\mathbf{H}]$ which admits a factorization of the form $[\mathbf{L}][\mathbf{K}]$, for some $[\mathbf{K}]\in\ \mathbb{T}(\mathcal{G})$.

3.2.11 Example. The previous concept of covering, for the case of $\mathcal{G}_{Zar}$ defined in 3.2.8 when R is a commutative noetherian ring, coincides with the classical notion in the Zariski topology, since in this case $\{[\mathcal{L}_{I_\alpha}]\}_{\alpha=1}^{n}$ is a covering of $[\mathcal{L}_I]$ if and only if the family of open subsets $\{D(I_\alpha)\}_{\alpha=1}^{n}$ in $\operatorname{Spec} R$ is a covering of $D(I)$ (cf. [15, 4.2.13]). It fact, to the intersection $D(I) \cap D(J) = D(IJ)$ corresponds the Gabriel filter $\mathcal{L}_{IJ}$ which is equal to

$$\mathcal{L}_I \circ \mathcal{L}_J = \{L \leq R \mid \exists n, m \in \mathbb{N}, I^n J^m \subseteq L\}.$$

This is the reason why we announced at the beginning of this section that the composition of uniform filters plays the role of the intersection of open subsets in the commutative case.

3.3 *À la* Van Oystaeyen

In [36] the author presents a notion of noncommutative topology based on an axiomatic system. The axioms are established on a *poset* Λ with two operations $\vee$, $\wedge$ having as the basic noncommutative aspect that not all the elements $\lambda \in \Lambda$ satisfy $\lambda \wedge \lambda = \lambda$ (apart from the necessary noncommutativity of $\wedge$). On this noncommutative topology one defines a concept of *noncommutative Grothendieck topology of a noncommutative site* in a similar way as in 3.1.1 for a commutative space (X, T).

3.3.1 Definition. A *noncommutative topology* consists of a poset Λ with elements 0 and 1, together with two operations $\vee$ and $\wedge$ satisfying the following axioms:

A1) $x \wedge y \leq y$, $\forall x, y \in \Lambda$;

A2) $x \wedge 1 = 1 \wedge x = x$, $x \wedge 0 = 0 \wedge x = 0$, $x \wedge \cdots \wedge x = 0 \Leftrightarrow x = 0$, $\forall x \in \Lambda$;

A3) $x \wedge y \wedge z = (x \wedge y) \wedge z = x \wedge (y \wedge z)$, $\forall x, y, z \in \Lambda$;

A4) if $a \leq b$ then $x \wedge a \leq x \wedge b$ and $a \wedge x \leq b \wedge x$, $\forall x \in \Lambda$;

A5) $y \leq x \vee y$, $\forall x, y \in \Lambda$;

A6) $x \vee 1 = 1 \vee x = 1$, $x \vee 0 = 0 \vee x = x$, $x \vee \cdots \vee x = 1 \Leftrightarrow x = 1$, $\forall x \in \Lambda$;

A7) $x \vee y \vee z = (x \vee y) \vee z = x \vee (y \vee z)$, $\forall x, y, z \in \Lambda$;

A8) if $a \leq b$ then $x \vee a \leq x \vee b$ and $a \vee x \leq b \vee x$, $\forall x \in \Lambda$;

A9) let $i_\wedge(\Lambda)$ be the subset $\{x \in \Lambda \mid x \wedge x = x\}$, called the set of *idempotent elements* of Λ; for every idempotent x and every $y \in \Lambda$, if $x \leq y$ then

  i) $x \vee (x \wedge y) \leq (x \vee x) \wedge y$;
  ii) $x \vee (y \wedge x) \leq (x \vee y) \wedge x$.

A10) if $x_1 \vee \cdots \vee x_n = 1$ then $(x \wedge x_1) \vee \cdots \vee (x \wedge x_n) = x$, $\forall x \in \Lambda$.

3.3.2 Note. In this context the elements of the poset are considered to be the *open subsets* and their *intersection* is given by the operation $\wedge$, which is not necessary commutative (whence the terminology *noncommutative topology*). Even more, we point out that in a noncommutative topology *à la* Van Oystaeyen the set $i_\wedge(\Lambda)$ may be a strict subset of Λ, i.e. not all the elements are necessarily idempotent. A noncommutative topology in which there exists no element A such that $A \cap A \neq A$ would have to be called *commutative* according to this philosophy.

3.3.3 Properties.

1) From A1), ..., A4) it follows that $x \wedge y, y \wedge x \leq x, y$, $\forall x, y \in \Lambda$.

2) If $x \in i_\wedge(\Lambda)$ and $x \leq y$ then $x \wedge y = y \wedge x = x$, $\forall y \in \Lambda$.

3) $x \wedge y = y \wedge x \Leftrightarrow x \wedge y$ and $y \wedge x$ are in $i_\wedge(\Lambda)$, for every $x, y \in i_\wedge(\Lambda)$.

4) If Λ satisfies A1), ..., A9) then $i_\wedge(\Lambda) \subseteq i_\vee(\Lambda)$, where $i_\vee(\Lambda)$ is defined to be the subset $\{x \in \Lambda \mid x \vee x = x\}$.

5) $x \vee (y \wedge z) \leq ((x \vee y) \wedge z)^{\vee 2}$ and $(x \vee (y \wedge z))^{\wedge 2} \leq (x \vee y) \wedge z$, for every $x, y, z \in \Lambda$ with $x \in i_\wedge(\Lambda)$ and $x \leq z$, where $\wedge 2$ and $\vee 2$ are exponent notation with respect to $\wedge$ and $\vee$ respectively.

6) $x \wedge (y \vee z) \geq ((x \wedge y) \vee (x \wedge z))^{\wedge 2}$ and $((x \wedge y) \vee (x \wedge z))^{\wedge 2} \geq x \vee (y \wedge z)$, for every $x, y, z \in \Lambda$ with $x \in i_\wedge(\Lambda)$.

3.3.4 Definition. A *global covering* in Λ is a family of elements $\{\lambda_1, \ldots, \lambda_n\}$ in Λ such that $\lambda_1 \vee \cdots \vee \lambda_n = 1$. A *covering* of an element $x \in \Lambda$ is a family $\{x \wedge \lambda_1, \ldots, x \wedge \lambda_n\}$ where $\{\lambda_1, \ldots, \lambda_n\}$ is a global covering, and hence $x = (x \wedge \lambda_1) \vee \cdots \vee (x \wedge \lambda_n)$ according to A10).

3.3.5 Definition. A relation $x \leq y$ in Λ is said to be *generic* if it is a consequence of the axioms of the noncommutative topology. For instance, $x \leq x \vee y$ is always generic. On the contrary, if x, y are idempotent elements, the relation $x \leq y$ is not viewed as generic; however it is if x is for example equal to $a \wedge y$, or if $y = a \vee x$, for some $a \in \Lambda$.

3.3.6 Example. In a noncommutative topology one may consider the subset $T(\Lambda) \subseteq \Lambda$ consisting of all finite length bracketed expressions involving $\wedge$, $\vee$ and elements of $i_{\wedge}(\Lambda)$. Then $T(\Lambda)$ also satisfies the axioms A1), ..., A10), and it is called the *noncommutative topology generated by* $i_{\wedge}(\Lambda)$.

3.3.7 Example. Let Λ be a poset. We denote by $D(\Lambda)$ the set of all directed subsets of Λ, and we say that two directed subsets $X, Y \in D(\Lambda)$ are equivalent, written $X \sim Y$, if

i) for every $x \in X$ there exists $x' \in X$ such that $x' \leq x$ and there exist $y, y' \in Y$ such that $y \leq x' \leq y'$;

ii) for every $y \in Y$ there exists $y' \in Y$ such that $y' \leq y$ and there exist $x, x' \in X$ such that $x \leq y' \leq x'$.

We denote by $[X]$ the $\sim$-equivalence class of $X \in D(\Lambda)$, and by $C(\Lambda)$ the quotient set $D(\Lambda)/\sim$.
A subset $X \in D(\Lambda)$ is said to be a *filter* if $x \leq y$ with $x \in X$ entails $y \in X$. One may associate to any subset $X \in D(\Lambda)$ the filter

$$\overline{X} = \{\lambda \in \Lambda \mid \exists x \in X \text{with } x \leq \lambda\}.$$

For every $X, Y \in D(\Lambda)$, we say that $X \leq Y$ if

i) for every $x \in X$ there exists $x' \in X$ such that $x' \leq x$ and there exists $y \in Y$ such that $x' \leq y$;

ii) for every $y \in Y$ there exists $x \in X$ such that $x \leq y$.

It may be proved that $X \leq Y$ if and only if $\overline{X} \subseteq \overline{Y}$, and $X \sim Y$ if and only if $X \leq Y$ and $Y \leq X$. In view of these properties the induced ordering on $C(\Lambda)$ is well-defined, therefore $C(\Lambda)$ is a poset. Moreover, if Λ is a noncommutative topology, so is $C(\Lambda)$ with respect to the operations given by

$$[X] \wedge [Y] = [X \dot{\wedge} Y]; \quad [X] \vee [Y] = [X \dot{\vee} Y],$$

for all $[X], [Y] \in C(\Lambda)$, where

$$\begin{aligned} X \dot{\wedge} Y &= \{x \wedge y \mid x \in X, y \in Y\}; \\ X \dot{\vee} Y &= \{x \vee y \mid x \in X, y \in Y\}. \end{aligned}$$

3.3.8 Proposition. *Let R be a domain. The poset of uniform filters ordered by the inverse inclusion, with the zero element $\mathcal{L}_R = \{R\}$, the unit element $\mathcal{L}_0 = \{L \leq_l R\}$, the join given by the intersection, and the meet given by the composition of filters, is a noncommutative topology where the idempotent elements are the Gabriel filters.*

Proof. Taking into account that $\{R\} \subseteq \mathcal{L}$ and $\mathcal{L} \subseteq \{L \leq_l R\}$, for every filter $\mathcal{L}$, thus $\mathcal{L}_R$ and $\mathcal{L}_0$ are indeed the zero element and the unit element respectively in the poset of uniform filters with the inverse inclusion. We have to check the axioms A1), ..., A10). In view of 3.2.4, we see that from properties ii) and vi) follows directly A1), i.e. $\mathcal{L} \circ \mathcal{H} \supseteq \mathcal{L}, \mathcal{H}$. The fact that $\mathcal{L}_R$ is the unit for the composition of uniform filters follows from properties iii) and vi). To obtain A2) it remains to prove that

$$\mathcal{L} = \mathcal{L}_0 \Leftrightarrow \mathcal{L} \circ \cdots \circ \mathcal{L} = \mathcal{L}_0.$$

If $\mathcal{L} = \mathcal{L}_0$ then it obviously follows that $\mathcal{L} \circ \cdots \circ \mathcal{L} = \mathcal{L}_0$. For the converse, it is sufficient to check that if $\mathcal{L} \circ \mathcal{H} = \mathcal{L}_0$ then $\mathcal{L} = \mathcal{L}_0$ or $\mathcal{H} = \mathcal{L}_0$ or, equivalently, if $0 \in \mathcal{L} \circ \mathcal{H}$ then $0 \in \mathcal{L}$ or $0 \in \mathcal{H}$. Indeed, if $0 \in \mathcal{L} \circ \mathcal{H}$ then there exists $H \in \mathcal{H}$ such that $(0 : r) \in \mathcal{L}$, for all $r \in H$. If we suppose $0 \notin \mathcal{H}$ then $H \neq 0$, thus there exists a nonzero element $r \in H$ such that $(0 : r) \in \mathcal{L}$, therefore $0 \in \mathcal{L}$ since $(0 : r) = 0$ in the domain R.

Axioms A3) and A4) are exactly properties iv) and i) respectively. The axioms A5),...,A8) follow immediately since the operation meet is in this example the intersection of sets.

In view of 3.2.3, the set $\{\mathcal{L} \text{ uniform filter} \mid \mathcal{L} \circ \mathcal{L} = \mathcal{L}\}$ of idempotent elements (having the composition of filters as the meet) is the set of Gabriel filters. In order to check A9), let $\mathcal{L}$ be an arbitrary Gabriel filter and $\mathcal{H}$ a uniform filter contained in $\mathcal{L}$. Then, by applying property i) we get $\mathcal{L} \circ \mathcal{H} \subseteq \mathcal{L} \circ \mathcal{L} = \mathcal{L}$, therefore

$$\mathcal{L} \circ \mathcal{H} \subseteq \mathcal{L} \cap (\mathcal{L} \circ \mathcal{H}).$$

On the other hand, $(\mathcal{L} \cap \mathcal{H}) \circ \mathcal{L} \subseteq \mathcal{H} \circ \mathcal{L}$ and $(\mathcal{L} \cap \mathcal{H}) \circ \mathcal{L} \subseteq \mathcal{L} \circ \mathcal{L} = \mathcal{L}$, both again by 3.2.4 i). Therefore $(\mathcal{L} \cap \mathcal{H}) \circ \mathcal{L} \subseteq \mathcal{L} \cap (\mathcal{H} \circ \mathcal{L})$, hence we obtain A9).

Finally suppose $\mathcal{L}_1 \cap \cdots \cap \mathcal{L}_n = \mathcal{L}_R$ and let $\mathcal{L}$ be a uniform filter. Then by 3.2.4 ix),

$$(\mathcal{L} \circ \mathcal{L}_1) \cap \cdots \cap (\mathcal{L} \circ \mathcal{L}_n) = \mathcal{L} \circ (\bigcap_{i=1}^{n} \mathcal{L}_i) = \mathcal{L} \circ \mathcal{L}_R = \mathcal{L},$$

therefore axiom A10) is proved too. □

Similar to the commutative case, this noncommutative topology may be categorically generalized by a *noncommutative site à la Van Oystaeyen*, which is a site where instead of $G3$ we have a skew axiom $NCG3$ subject to the concept of *noncommutative pullback*.

3.3.9 Noncommutative Grothendieck topologies.

Let Λ be a noncommutative topological space. One may associate to Λ a small category $\mathcal{O}(\Lambda)$, whose objects are the elements in Λ and where for every $x, y \in \Lambda$, the set $\mathrm{Hom}_\Lambda(x, y)$ is defined to be the singleton $\{x \hookrightarrow y\}$ if and only if the relation $x \leq y$ is generic in Λ and the empty set otherwise.

3.3.10 Definition. A *noncommutative Grothendieck topology* on $\mathcal{O}(\Lambda)$ consists of a family $\{\mathrm{Cov}(x)\}_{x\in\Lambda}$, where $\mathrm{Cov}(x)$ is a set of subsets of morphisms in $\mathcal{O}(\Lambda)$ with common target x satisfying G1), G2) (as in 3.1.1) and the following noncommutative axiom:

NCG3) if $x' \to x$ is a morphism in $\mathcal{O}(\Lambda)$ and $\{x_i \to x\}_{i\in I} \in \mathrm{Cov}(x)$, then

i) for every $i \in I$ there exists $x' \times_x x_i \in \Lambda$ satisfying the following *noncommutative pullback* property:
for all morphisms $s \to x_i$, $s \to x'$ and $t \to x_i$, $t \to x'$ in $\mathcal{O}(\Lambda)$, there exist morphisms $s \wedge t \to x' \times_x x_i$ and $t \wedge s \to x' \times_x x_i$ fitting in the commutative diagram

$$\begin{tikzcd}
 & & x & & \\
 & x_i \arrow[ur] & & x' \arrow[ul] & \\
t \arrow[ur] \arrow[urrr] & & x' \times_x x_i \arrow[ul] \arrow[ur] & & s \arrow[ul] \arrow[ulll] \\
 & t \wedge s \arrow[ul] \arrow[ur, dashed] & & s \wedge t \arrow[ur] \arrow[ul, dashed] &
\end{tikzcd}$$

(Note that when $s = t$ is an idempotent element then the diagram reduces to the pullback as in G3.)

ii) $\{x' \times_x x_i \to x'\}_{i\in I} \in \mathrm{Cov}(x')$.

The pair ($\mathcal{O}(\Lambda)$, $\{\mathrm{Cov}(x)\}_{x\in\Lambda}$) is called a *noncommutative site.*

3.3.11 Theorem. *Let Λ be a noncommutative topology. If for every $x \in \Lambda$ we define the coverings of x by*

$$\{x \wedge \lambda_i \to x\}_{i=1}^{n} \in \mathrm{Cov}(x) \Leftrightarrow \{\lambda_1, \ldots, \lambda_n\} \text{ is a global covering,}$$

then ($\mathcal{O}(\Lambda)$, $\{\mathrm{Cov}(x)\}_{x\in\Lambda}$) is a noncommutative site.

Proof. Cf. [36, Thm.7.2.2]. ☐

3.4 *À la* Borceux-Van den Bossche

Inspired by the philosophy of *quantales* (explained in the following section), the authors of [12] propose *quantum spaces* for being considered as noncommutative topological spaces. Roughly speaking, these spaces are defined to satisfy all the properties of a classical topological space except for the finite intersection of open subsets, which is no longer necessarily open but contained in a specific open subset given by a *multiplication* (actually playing the role of the intersection). Thus, the interest is fixed on the cases when this *multiplication* is noncommutative, otherwise we would be dealing with classical commutative topologies.

3.4.1 Definition. A *quantum space* consists of a set X provided with a family $O(X)$ of subsets and a binary operation

$$\& : O(X) \times O(X) \to O(X),$$

called *multiplication*, satisfying the following axioms:

S1) $\emptyset$ and X are in $O(X)$;

S2) for any family $\{U_i\}_{i\in I}$ in $O(X)$ we have $\bigcup_{i\in I} U_i \in O(X)$;

S3) for every $U, V \in O(X)$ we have $U \cap V \subseteq U \,\&\, V$;

S4) for every $U, V, W \in O(X)$ we have $U \,\&\, (V \,\&\, W) = (U \,\&\, V) \,\&\, W$;

S5) for all $U \in O(X)$ we have $U \,\&\, X = U$;

S6) for any family $\{U_i\}_{i\in I}$ in $O(X)$, and for all $U \in O(X)$ we have

$$U \,\&\, (\bigcup_{i\in I} U_i) = \bigcup_{i\in I} (U \,\&\, U_i);$$

S7) for any family $\{U_i\}_{i\in I}$ in $O(X)$, and for all $U \in O(X)$ we have

$$(\bigcup_{i\in I} U_i) \,\&\, U = \bigcup_{i\in I} (U_i \,\&\, U).$$

The elements in $O(X)$ are called the *open subsets.*

3.4.2 Remark. This definition generalizes the concept of commutative topology in the sense that every commutative topology satisfies the axioms S1), …, S7) just by defining $U \,\&\, V = U \cap V$.

3.4.3 Properties.

For every open subsets $T, U, V, W \in O(X)$ the following relations hold:

1) if $T \subseteq U$ and $V \subseteq W$ then $T \,\&\, V \subseteq U \,\&\, W$;

2) $U \,\&\, \emptyset = \emptyset = \emptyset \,\&\, U$;

3) $U \cap V \subseteq U \,\&\, V \subseteq U \cap (X \,\&\, V)$; in particular, $U \,\&\, V \subseteq U$;

4) $U \,\&\, U = U$ (*idempotent property*);

5) if $U \subseteq V$ then $U \,\&\, V = U$;

6) $U \,\&\, V \,\&\, W = U \,\&\, W \,\&\, V$.

3.4.4 Example. Let X be a topological space and let us denote by $\tilde{}$ the closure operation on the set $\mathcal{P}(X)$ of subsets of X. We may endow X with a quantum space structure by taking $\mathcal{P}(X)$ as the set of open subsets and the multiplication given by

$$\& : \mathcal{P}(X) \times \mathcal{P}(X) \longrightarrow \mathcal{P}(X); \quad (Y, Z) \mapsto Y \,\&\, Z = Y \cap \widehat{Z},$$

where $\widehat{}$ is an operation in $\mathcal{P}(X)$ defined for every $Z \in \mathcal{P}(X)$ by

$$\widehat{Z} = \{x \in X \mid \exists z \in Z, \tilde{x} = \tilde{z}\}.$$

This quantum structure is denoted by (X, Q_1) (cf. [12, 1,ex.1]).

3.4.5 Example. Another example is to consider $\{Y \subseteq X \mid \widehat{Y} \in O(X)\}$ as the set of open subsets, with the same multiplication as in the previous example. In this case we obtain another quantum structure on X denoted by (X, Q_2), (cf. [12, 1,ex.2]).

For the following fundamental example of what the authors call a quantum space, we need to recall some basic machinery from functional analysis, which we summarize in the following definitions.

Let K be the field $\mathbb{R}$ or $\mathbb{C}$, of real or complex numbers respectively.

3.4.6 Definitions. A *Banach space* V is a complete normed K-vector space, that is, a K-vector space with a norm $\|\cdot\|$ such that every Cauchy sequence in V has a limit in V (with respect to the metric $d(x, y) = \|x - y\|$).
A *Banach algebra* A is an associative K-algebra which at the same time is a Banach space such that for all $x, y \in A$ the algebra multiplication and the

Banach space norm satisfy $\|xy\| \leq \|x\|\|y\|$, which ensures that the multiplication is continuous.
Let A be a Banach algebra. A map $*: A \to A$ is called an *involution* if it satisfies the following properties:

i) for all $x, y \in A$, $(x+y)^* = x^* + y^*$;

ii) for every $\lambda \in \mathbb{C}$ and every $x \in A$, $(\lambda x)^* = \lambda^* x^*$ (where λ^* stands for the complex conjugation of λ);

iii) for all $x, y \in A$, $(xy)^* = y^* x^*$;

iv) for all $x \in A$, $(x^*)^* = x$.

A Banach algebra A over $\mathbb{C}$ is said to be a $\mathbb{C}^*$-*algebra* if it has an involution * which satisfies for all $x \in A$ that $\|x^* x\| = \|x\|^2$.

Let A be a $\mathbb{C}^*$-algebra with unit e, where we choose to work with right ideals. A $\mathbb{C}$-form $f : A \to \mathbb{C}$ on A is said to be *positive* if for all $x \in A$ we have $f(xx^*) \geq 0$. If f also satisfies $f(e) = 1$, then it is said to be a *state*. An extreme point of the convex set of states of A is called a *pure state*, that is, a state f such that for any two states g_1 and g_2 and every real number t, $0 \leq t \leq 1$, the condition $f = tg_1 + (1-t)g_2$ implies $g_1 = g_2 = f$.

3.4.7 Example. Let A be a $\mathbb{C}^*$-algebra. The set of all $\mathbb{C}$-forms which are a pure state of A, denoted by Spec A, is called the *spectrum* of A.
On Spec A, we define a commutative topology whose open subsets are of the form

$$\mathcal{O}_I = \{f \in \operatorname{Spec} A \mid \exists x \in I, f(xx^*) \neq 0\},$$

for every closed two-sided ideal I of A. In [12, 6,thm.2] the authors prove that on Spec A one may also define a quantum space structure which generalizes this classical topology. It consists of considering not only two-sided but all right ideals, i.e. of taking $\{\mathcal{O}_I \mid I \text{ closed right ideal of } A\}$ as the set of open subsets. This set is provided with a noncommutative intersection, that is, the multiplication given by $\mathcal{O}_I \,\&\, \mathcal{O}_I = \mathcal{O}_{\overline{IJ}}$ (where $\overline{IJ}$ stands for the closure of IJ in A). Thus, Spec A has a quantum space structure.

3.5 Q-sites

It is well-known that the poset of open subsets of a topological space X is a locale with the partial ordering given by the inclusion, and the meet and join given by the union and intersection of open subsets respectively (cf. [6, ex.1.3.4.a]). In fact, the notion of locale is the best algebraic approach to topological spaces. Locales find their noncommutative analog during the 80's with the appearing of *quantales*, a kind of lattices with an interesting noncommutative logic.

In this structure we aim to define the proper coverings to obtain a noncommutative site, which we will call Q-*site*. This will be our candidate for noncommutative topological space, on which we will develop the results of the last chapter.

3.5.1 Definition. A *quantale* is a complete lattice $(\mathcal{Q}, \leq)$ (cf. 1.1.4) provided with an additional binary operation

$$\& : \mathcal{Q} \times \mathcal{Q} \to \mathcal{Q},$$

called *multiplication*, satisfying the following axioms

Q1) $U \,\&\, (V \,\&\, W) = (U \,\&\, V) \,\&\, W$;

Q2) $U \,\&\, 1 = U$;

Q3) $U \,\&\, U = U$;

Q4) $U \,\&\, (\bigvee_{i \in I} V_i) = \bigvee_{i \in I}(U \,\&\, V_i)$;

Q5) $(\bigvee_{i \in I} U_i) \,\&\, V = \bigvee_{i \in I}(U_i \,\&\, V)$,

where I is a set, U, V, W, U_i, V_i are elements of $\mathcal{Q}$ and $1 = \vee \mathcal{Q}$ is the greatest element of $\mathcal{Q}$.

An element $U \in \mathcal{Q}$ that verifies the condition Q2) also on the left, i.e. such that $1 \,\&\, U = U$, is called a *two-sided element*.

3.5.2 Properties.

Let $(\mathcal{Q}, \leq, \&)$ be a quantale. For all $U, V, W \in \mathcal{Q}$ the following relations hold:

1) if $V \leq W$ then $U \,\&\, V \leq U \,\&\, W$;

2) if $U \leq V$ then $U \,\&\, W \leq V \,\&\, W$;

3) $U \,\&\, 0 = 0 = 0 \,\&\, U$;

4) $U \,\&\, V \leq U$;

5) if $U \leq W$ and $V \leq W$ then $U \,\&\, V \leq W$;

6) if $U \leq V$ then $U = U \,\&\, V$;

7) $U \mathbin{\&} V \mathbin{\&} W = U \mathbin{\&} W \mathbin{\&} V$.

(Cf. [11, prop.1]).

3.5.3 Example. A locale $(L, \leq)$ is a quantale just by taking $\& = \wedge$. Conversely, in [11, corollaries 1,2] it is proved that the set of two-sided elements of an arbitrary quantale constitutes a locale, and moreover, a quantale is a locale as soon as every element is two-sided.

3.5.4 Example. Let R be an arbitrary ring with a unit. A right ideal I of R satisfying that for all $a \in I$ there exists $e \in I$ such that $a \cdot e = a$, is called a *neat ideal*. The set of neat ideals of R with the multiplication given by the product of ideals constitute a quantale (cf. [11, ex.3]).

3.5.5 Definition. Let $(\mathcal{Q}_1, \leq, \&)$ and $(\mathcal{Q}_2, \leq, \&)$ be two arbitrary quantales. A *morphism of quantales* is a map $f : \mathcal{Q}_1 \to \mathcal{Q}_2$ satisfying the following axioms

M1) $f(1) = 1$;

M2) $f(\bigvee_{i\in I} U_i) = \bigvee_{i\in I} f(U_i)$;

M3) $f(U \mathbin{\&} V) \geq f(U) \mathbin{\&} f(V)$,

for all elements U, V, U_i in $\mathcal{Q}_1$ and every set I. The morphism is called *strict* if axiom M3) is satisfied with equality, i.e.

M3') if for all $U, V \in \mathcal{Q}_1$, then $f(U \mathbin{\&} V) = f(U) \mathbin{\&} f(V)$.

3.5.6 Example. Quantum spaces (cf. 3.4) are algebraically approached by quantales in a similar way as locales are an algebraic approach for topological spaces. Indeed, in view of the definition 3.4.1 and the idempotent property, a quantum space $(X, O(X), \&)$ may be redefined as a set X provided with a quantale $O(X)$ of *open subsets* such that the inclusion

$$(O(X), \subseteq, \&) \hookrightarrow (2^X, \subseteq, \cap)$$

is a morphism of quantales (cf. [12]).

3.5.7 Example. The complete lattice of uniform filters ordered by the inverse inclusion, described in 3.2.5, with the multiplication given by the composition of filters, is a *quasi-quantale* in the sense that instead of satisfying the axiom Q3) it satisfies that every element is left sided (i.e. all the elements are two sided instead of idempotent), and instead of Q5) it satisfies the weaker condition Q5)' which states that

$$(\bigvee_{i\in I} U_i) \mathbin{\&} V \geq \bigvee_{i\in I} (U_i \mathbin{\&} V),$$

with equality only when I is finite. This follows from the properties of the composition listed in 3.2.4. Indeed, property iv) asserts that the composition of filters is associative, by properties iii) and vi) all the uniform filters are two-sided, and finally the property ix) translates into the axioms Q4) and Q5)' since in this opposite lattice the join is the intersection of filters.

Let $(\mathcal{Q}, \leq, \&)$ be a quantale.

Since $\mathcal{Q}$ is a poset, it can be considered as a small category (recall 1.1.8). Thus, with the goal of giving a Grothendieck topology on $\mathcal{Q}$ (cf. 3.1.1), the following step is to assign to every element in $\mathcal{Q}$ a set of *Q-coverings*.

3.5.8 Definition. Let $U \in \mathcal{Q}$. We say that the family $\{U_i\}_{i\in I}$ of elements of $\mathcal{Q}$ is a *Q-covering* of U if

C1) $U = \bigvee_{i\in I} U_i$;

C2) $U_i = U \,\&\, U_i$, for all $i \in I$.

We denote by $\mathrm{Cov}(U)$ the set of all Q-coverings of U.

Taking into account the axioms of a quantale (cf. 3.5.1) and the properties listed in 3.5.2 we obtain the following properties for the coverings we have just defined, which will be very useful in the sequel.

3.5.9 Lemma. *Let $U \in \mathcal{Q}$ and let $\{U_i\}_{i\in I}$, $\{U'_j\}_{j\in J}$ be two Q-coverings of U. For all $i \in I$ and $j \in J$ we have:*

1) $U_i \leq U$;

2) $U_i \,\&\, U = U_i$;

3) $U_i \,\&\, U'_j = U'_j \,\&\, U_i \leq U_i, U'_j$; *in particular, elements of the same Q-covering are $\&$-commutative;*

4) *if* $V \leq U$ *then* $V \,\&\, U_i \leq V, U_i$.

Proof.

1) This fact follows directly from 3.5.2.4 since $U_i = U \,\&\, U_i$. (Note that for an arbitrary $V \leq U$ in $\mathcal{Q}$ we do not obtain $V = U \,\&\, V$);

2) this follows just by applying 3.5.2.7 since

$$U_i = U \,\&\, U_i = U \,\&\, U \,\&\, U_i = U \,\&\, U_i \,\&\, U = U_i \,\&\, U;$$

3) from Q1) and 3.5.2.7 we obtain

$$U_i \,\&\, U'_j = (U \,\&\, U_i \,\&\, U) \,\&\, U'_j = (U \,\&\, U \,\&\, U_i) \,\&\, U'_j,$$

which is equal to $U \,\&\, U_i \,\&\, U'_j$ by Q3), and again by 3.5.2.7 equal to $(U \,\&\, U'_j) \,\&\, U_i = U'_j \,\&\, U_i$. The inequality follows by 3.5.2.4 since U_i and U'_j are $\&$-commutative;

4) $V \,\&\, U_i \leq V$ by 3.5.2.4, and by applying 3.5.2.2 we also obtain that $V \,\&\, U_i \leq U \,\&\, U_i = U_i$.

□

Let $U \in \mathcal{Q}$. We continue by endowing $\mathrm{Cov}(U)$ with an ordering.

3.5.10 Definition. Let $\mathcal{U} = \{U_i\}_{i \in I}$ and $\mathcal{U}' = \{U'_j\}_{j \in J}$ be two Q-coverings of U. In $\mathrm{Cov}(U)$ we define an ordering "$\preceq$" given by $\mathcal{U}' \preceq \mathcal{U}$ if and only if there exists a map $\delta : J \to I$ such that for all $j \in J$ we have $U'_j \leq U_{\delta(j)}$. In this case we say that $\mathcal{U}'$ is a *sub-Q-covering* of $\mathcal{U}$.

With this ordering one may easily check that $(\mathrm{Cov}(U), \preceq)$ is a poset. Hence $\mathrm{Cov}(U)$ has the structure of a small category. We write $\mathcal{C}ov(U)$ whenever we are referring to this category structure. Moreover, $\mathrm{Cov}(U)$ has the following property which is going to be fundamental for our construction in chapter 5:

3.5.11 Lemma. *$\mathrm{Cov}(U)$ is a directed set with the inverse ordering of $\preceq$.*

Proof. We will check that for all Q-coverings $\mathcal{U} = \{U_i\}_{i \in I}$ and $\mathcal{U}' = \{U_j\}_{j \in J}$ of U, there exists $\mathcal{U}'' \in \mathrm{Cov}(U)$ such that $\mathcal{U}'' \preceq \mathcal{U}, \mathcal{U}'$.
Let $K = I \times J$ and $\mathcal{U}'' = \{U_i \,\&\, U'_j\}_{(i,j) \in I \times J}$. First of all, $\mathcal{U}''$ is indeed a Q-covering of U since for all $(i,j) \in I \times J$ clearly

$$U_i \,\&\, U'_j = (U \,\&\, U_i) \,\&\, U'_j = U \,\&\, (U_i \,\&\, U'_j)$$

and, by Q4),

$$\bigvee_{(i,j) \in I \times J} U_i \,\&\, U'_j = \bigvee_{i \in I} (U_i \,\&\, (\bigvee_{j \in J} U'_j)) = \bigvee_{i \in I} (U_i \,\&\, U),$$

which, by 3.5.9.2, coincides with $\bigvee_{i \in I} U_i = U$.
On the other hand, in virtue of 3.5.9.3, to check that $\mathcal{U}'' \preceq \mathcal{U}$ (resp. $\mathcal{U}'' \preceq \mathcal{U}'$) it is sufficient to take $\delta_1 : K \to I$ (resp. $\delta_2 : K \to J$) as the projection map onto the first (resp. second) component. □

3.5.12 Definition. The pair $(\mathcal{Q}, \{\mathrm{Cov}(U)\}_{U \in \mathcal{Q}})$ is called a *Q-site*.

A Q-site is indeed a site in the sense of definition 3.1.1, as we prove in the following:

3.5.13 Proposition. *The family* $\{\mathrm{Cov}(U)\}_{U\in\mathcal{Q}}$ *constitutes a Grothendieck topology on* $\mathcal{Q}$.

Proof. Let $U \in \mathcal{Q}$. We obviously have $\{U\} \in \mathrm{Cov}(U)$ by Q3), i.e. G1) is obviously satisfied. Let $\{U_i\}_{i\in I} \in \mathrm{Cov}(U)$ and suppose that for all $i \in I$ we have $\{U_{j_i}\}_{j_i\in J_i} \in \mathrm{Cov}(U_i)$. Then, for all $i \in I$ and $j_i \in J_i$,

$$U_{j_i} = U_i \,\&\, U_{j_i} = (U \,\&\, U_i) \,\&\, U_{j_i} = U \,\&\, (U_i \,\&\, U_{j_i}) = U \,\&\, U_{j_i},$$

by C2) and Q1); and

$$\bigvee_{j_i\in J_i, i\in I} U_{j_i} = \bigvee_{i\in I}(\bigvee_{j_i\in J_i} U_{j_i}) = \bigvee_{i\in I} U_i = U.$$

Therefore $\{U_{j_i}\}_{j_i\in J_i, i\in I} \in \mathrm{Cov}(U)$, whence G2) holds.
Finally, suppose $V \leq U$. We claim that $\{V \,\&\, U_i\}_{i\in I}$ is a covering of V. Indeed,

1) for all $i \in I$, by Q1) and Q3), we have $V \,\&\, U_i = V \,\&\, (V \,\&\, U_i)$; and
2) applying Q4) it follows $\bigvee_{i\in I}(V \,\&\, U_i) = V \,\&\, (\bigvee_{i\in I} U_i) = V \,\&\, U$ which coincides with V by 3.5.2.6.

Moreover, we claim that for all $i \in I$, the element $V \,\&\, U_i$ is a pullback in $\mathcal{Q}$ (seen as a small category). Indeed, $V \,\&\, U_i \leq V$ and $V \,\&\, U_i \leq U_i$ by 3.5.9.4, and if $W \in \mathcal{Q}$ is another element such that $W \leq V$ and $W \leq U_i$ then, by 3.5.2.1, 3.5.2.2 and Q3), it follows that

$$W = W \,\&\, W \leq V \,\&\, U_i.$$

Therefore, G3) is also satisfied since we have the following commutative diagram of morphisms in $\mathcal{Q}$:

$$\begin{tikzcd}
W \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& V \,\&\, U_i \arrow[r] \arrow[d] & U_i \arrow[d] \\
& V \arrow[r] & U
\end{tikzcd}$$

(Note that all the sets of morphisms in $\mathcal{Q}$ are singletons, thus in particular the arrow $W \dashrightarrow V \,\&\, U_i$ is obviously unique). □

3.5.14 Note. We must draw the attention to the fact that the noncommutativity of a Q-site is based on the noncommutativity of the multiplication &, thus we focus our interest on the cases when & is not commutative.

3.5.15 Example. Let $(O(X), \subseteq, \&)$ be the quantale of open subsets of a quantum space X (cf. 3.5.6). For every $U \in O(X)$, the subset $\{U_i\}_{i \in I}$ of $O(X)$ is a *quantum covering* of U if

i) $U = \bigcup_{i \in I} U_i$;

ii) for all $i \in I$ we have $U_i = U \,\&\, U_i$.

Then then pair $(O(X), \{\mathrm{Cov}(U)\}_{U \in O(X)})$ is a Q-site.
In this case, we should remark that $U \,\&\, U_i = U \cap U_i$, since $U_i \subseteq U$ by 3.5.9.1. For this reason, this example becomes much more commutative than expected, as we will see in 4.2.3.4.

3.5.16 Example. Let R be an arbitrary ring with a unit and let us consider the quantale of neat ideals of R. (cf. 3.5.4). For every neat ideal I of R, according to definition 3.5.8, the family $\{I_a\}_{a \in A}$ in the quantale of neat ideals is a covering of I if

i) $I = \sum_{a \in A} I_a$;

ii) for all $a \in A$ we have $I_a = I \cdot I_a$.

Provided with such coverings, the quantale of neat ideals is another example of Q-site.

Chapter 4

Sheaves and sheafification

In this chapter we will see how one may define the category of sheaves in a way which applies to both commutative and noncommutative sheaves, i.e. for arbitrary topologies, which can be an ordinary topology but also a not necessarily commutative topology. In order to avoid the classical use of stalks (as in section 1.4), we make use of the theory of localization in Grothendieck categories. In this way we introduce the category of sheaves and at the same time a sheafification functor for this category, i.e. we obtain the sheafification functor even in noncommutative situations.

4.1 Separated presheaves

This section is devoted to the study of the category of separated presheaves on a poset with the minimum topological structure in order to obtain a general definition, valid at the same time for commutative and noncommutative contexts, in the majority of examples of topologies we know. In this general situation, we find a generator and prove that this category is complete when it takes values in a complete category; moreover, we prove that the class of objects of this category is a torsion-free class for some torsion theory in the category of preshaves (cf. 4.1.9).

Let E be a poset and $\mathcal{C}$ an arbitrary category.

In 2.1.5 we define the category ${}_{\mathcal{C}}\mathcal{P}(E)$ of presheaves on E with values in $\mathcal{C}$, let us now give some particular examples for different posets:

4.1.1 Definitions.

1) Taking into account that in a noncommutative topology *à la* Van Oystaeyen the space Λ is a poset, we may define presheaves with values in $\mathcal{C}$ *à la* Van Oystaeyen just through this definition. In this way we obtain the category ${}_{\mathcal{C}}\mathcal{P}(\Lambda)$ of *noncommutative presheaves with values in* $\mathcal{C}$. In the same way we obtain the category ${}_{\mathcal{C}}\mathcal{P}(\mathcal{O}(\Lambda))$, considering the poset Λ with the ordering given by the generic relations (cf. 3.3.5).

2) Similarly, we define *quantum presheaves à la Borceux-Van den Bossche* through definition 2.1.5, using the poset $O(X)$ of a quantum space X. Thus we obtain the category ${}_{\mathcal{C}}\mathcal{P}(O(X))$.

3) More generally, for every quantale $\mathcal{Q}$ we may also define the category ${}_{\mathcal{C}}\mathcal{P}(\mathcal{Q})$ of presheaves on $\mathcal{Q}$ with values in $\mathcal{C}$, since $\mathcal{Q}$ is also a poset. When $\mathcal{Q}$ is the quantale of a Q-site, we shortly call ${}_{\mathcal{C}}\mathcal{P}(\mathcal{Q})$ the *category of Q-presheaves.*

We remark that we do not consider presheaves *à la* García Román mainly because $\mathbb{T}(\mathcal{G})$ is a small category whose underlying set of objects is not a poset, so a nonempty set of morphisms is not necessarily a singleton.

On the other hand, the concept of presheaf on a Grothendieck topology is well known (cf. [3, 6.7] or [8, 3.2.3] for instance). However, to prove that the noncommutative site *à la* García Román is a classical site still remains an open problem, so presheaves here are out of consideration from this point of view. Moreover, not only this case is excluded but also the presheaves on a noncommutative topology *à la* Van Oystaeyen. Thus, although our point of view is less general than considering a site instead of a poset, we choose to study it for the case of posets which at least does include the latter example.

Now let $\mathcal{C}$ be an abelian category and let us recall what is classically understood by a separated presheaf:

4.1.2 If X is a topological space and B_X a basis for the topology on X, it is well known that a presheaf P on B_X with values in $\mathcal{C}$ is said to be a *separated presheaf* if it also satisfies the axiom Sh1) established in 1.4.1 or, equivalently, if for every $U \in B_X$ and every covering $\{U_i\}_{i\in I}$ of U in B_X, the map

$$\xi : P(U) \longrightarrow \prod_{i\in I} P(U_i); \quad s \mapsto (s|_{U_i})_{i\in I}$$

is injective.

This definition involves coverings, thus if we want to define separated presheaves on a poset, we would need at least to provide the poset with some kind of coverings. With the aim of imposing minimal conditions, in this section we will just consider:

4.1.3 Let T be a poset such that to every $a \in T$ is assigned a set, denoted by $C(a)$, which consists of families $\{a_i\}_{i\in I}$ in T such that for all $i \in I$ we have $a_i \leq a$. The elements of $C(a)$ are called the *quasi-coverings* of a.

4.1.4 Examples. This is obviously the case for the poset of open subsets in an ordinary (commutative) topological space, if for every open subset U we take $C(U)$ as the set $\mathrm{Cov}(U)$ of strict coverings, but also:

1) for the poset Λ of a noncommutative topology *à la* Van Oystaeyen, since if $\{x \wedge \lambda_1, \dots, x \wedge \lambda_n\}$ is a covering of an element $x \in \Lambda$ (cf. 3.3.4) then for all $i \in \{1, \dots, n\}$ by 3.3.3.1 we have $x \wedge \lambda_i \leq x$; and

2) for the poset of a Q-site $(\mathcal{Q}, \{\mathrm{Cov}(U)\}_{U\in\mathcal{Q}})$, since for all $U \in \mathcal{Q}$, if $\{U_i\}_{i\in I}$ is a Q-covering of U (cf. 3.5.8), then for all $i \in I$ we have by 3.5.9.1 that $U_i \leq U$.
   (Recall that this case includes quantum spaces, –example 3.5.15–, since the set of open subsets in a quantum space has a quantale structure).

Under the assumptions of 4.1.3 we are able to give the following definition:

4.1.5 Definition. The *category of separated presheaves on T with values in $\mathcal{C}$*, denoted by ${}_{\mathcal{C}}\mathcal{F}(T)$, is the full subcategory of ${}_{\mathcal{C}}\mathcal{P}(T)$ whose objects are the presheaves P which verify, for every $a \in T$ and for every quasi-covering $\{a_i\}_{i\in I}$ of a, that the map

$$\xi : P(a) \to \prod_{i\in I} P(a_i); \quad s \mapsto (s|_{a_i})_{i\in I},$$

is injective or, equivalently, which satisfy axiom Sh1) whose statement in the present context is the following:

Sh1) if $a \in T$ and $\{a_i\}_{i\in I} \in C(a)$, then for every $s \in P(a)$ we have $s = 0$ whenever $s|_{a_i} = 0$ for all $i \in I$.

(Note that it makes sense to consider the restriction $s|_{a_i} = P_{aa_i}(s)$ since $a_i \leq a$.)

This definition (which obviously generalizes the classical case) applied to the examples of 4.1.4 translates to the following ones:

4.1.6 Examples.

1) Let Λ be a noncommutative topology *à la* Van Oystaeyen. The category ${}_{\mathcal{C}}\mathcal{F}(\Lambda)$ of *noncommutative separated presheaves* is the full subcategory of ${}_{\mathcal{C}}\mathcal{P}(\Lambda)$ whose objects are the presheaves P such that if $x \in \Lambda$ and $\{\lambda_1, \ldots, \lambda_n\}$ is a global covering, then for every $s \in P(x)$ we have $s = 0$ whenever $s|_{x\wedge\lambda_i} = 0$ for all $i \in \{1, \ldots, n\}$.

2) Let $(\mathcal{Q}, \{\mathrm{Cov}(U)\}_{U\in\mathcal{Q}})$ be a Q-site. The category ${}_{\mathcal{C}}\mathcal{F}(\mathcal{Q})$ of *separated Q-presheaves* is the full subcategory of ${}_{\mathcal{C}}\mathcal{P}(\mathcal{Q})$ whose objects are the Q-presheaves P such that for every $U \in \mathcal{Q}$ and every $\{U_i\}_{i\in I} \in \mathrm{Cov}(U)$, if $s \in P(U_i)$ and if for all $i \in I$ we have $s|_{U_i} = 0$, then $s = 0$.

3) In particular, if we consider the Q-site $(O(X), \{\mathrm{Cov}(U)\}_{U\in O(X)})$ associated to the quantale of open subsets of a quantum space X (cf. 3.5.15), then we obtain the category ${}_{\mathcal{C}}\mathcal{F}(O(X))$ of *separated quantum presheaves*.

Additionally, these minimum specifications of 4.1.3 are sufficient to obtain the following results, which hold in the different already mentioned topologies, in both commutative and noncommutative contexts:

4.1.7 Lemma. *If $\mathcal{C}$ is complete then so is the category ${}_{\mathcal{C}}\mathcal{F}(T)$ of separated presheaves on T with values in $\mathcal{C}$.*

Proof. Let $\{P_k\}_{k\in K}$ be an arbitrary family of separated presheaves and let us verify that $\prod_{k\in K} P_k$ is separated too. Let $a \in T$ and $s \in (\prod_{k\in K} P_k)(a)$. Every $s \in (\prod_{k\in K} P_k)(a)$ is of the form $(s_k)_{k\in K}$ with s_k in $P_k(a)$. If $\{a_i\}_{i\in I}$ is an arbitrary quasi-covering of a such that for all $i \in I$ we have $s|_{a_i} = 0$, then by assumption $0 = s|_{a_i} = (s_k|_{a_i})_{k\in K}$. Hence, for every $k \in K$ the separated presheaf P_k verifies for all $i \in I$ that $s_k|_{a_i} = 0$. Therefore for all $k \in K$, $s_k = 0$, and consequently $s = 0$, i.e. $\prod_{k\in K} P_k$ is separated.
(Note that we need $\mathcal{C}$ complete by assumption just to obtain a well-defined $\prod_{k\in K} P_k$, for an arbitrary family.) □

4.1.8 Remark. Similarly, we may prove that if $\mathcal{C}$ is cocomplete then so is the category ${}_{\mathcal{C}}\mathcal{F}(T)$.

4.1.9 Proposition. *Let $\mathcal{C}$ be a complete category. The class of objects of the category ${}_{\mathcal{C}}\mathcal{F}(T)$ is a torsion-free class for some torsion theory in ${}_{\mathcal{C}}\mathcal{P}(T)$.*

Proof. We have just proved that it is closed under products. On the other hand it is straightforward to check that if P is a separated presheaf, then so is every subpresheaf P' of P.

Finally, let $0 \to P' \xrightarrow{f} P \xrightarrow{g} P'' \to 0$ be an exact sequence in ${}_{\mathcal{C}}\mathcal{P}(T)$ with $P', P'' \in {}_{\mathcal{C}}\mathcal{F}(T)$. We have to check that P also belongs to ${}_{\mathcal{C}}\mathcal{F}(T)$. Let $a \in T$, $s \in P(a)$, and let $\{a_i\}_{i \in I}$ be an arbitrary quasi-covering of a in T such that for all $i \in I$, $s|_{a_i} = 0$. Let us verify that $s = 0$. In view of the exact sequence we have the following commutative diagram in $\mathcal{C}$

$$\begin{tikzcd}
0 \arrow[r] & P'(a) \arrow[r, "f(a)"] \arrow[d, "P'_{aa_i}"'] & P(a) \arrow[r, "g(a)"] \arrow[d, "P_{aa_i}"'] & P''(a) \arrow[r] \arrow[d, "P''_{aa_i}"'] & 0 \\
0 \arrow[r] & P'(a_i) \arrow[r] & P(a_i) \arrow[r, "f(a_i)"] & P''(a_i) \arrow[r, "g(a_i)"] & 0
\end{tikzcd}$$

Let s'' denote $g(a)(s) \in P''(a)$. Then for all $i \in I$

$$s''|_{a_i} = (P''_{aa_i} \circ g(a))(s) = g(a_i)(s|_{a_i}) = 0.$$

Hence, $s'' = g(a)(s) = 0$ since P'' is separated. Therefore, we may find some $s' \in P'(a)$ such that $s = f(a)(s')$, since $\operatorname{Ker} g(a) = \operatorname{Im} f(a)$. Moreover, by assumption for all $i \in I$ we have

$$0 = s|_{a_i} = (P_{aa_i} \circ f(a))(s') = f(a_i)(s'|_{a_i}).$$

Hence, by the injectivity of $f(a_i)$ for all $i \in I$ it follows that $s'|_{a_i} = 0$. Therefore $s' = 0$ since P' is separated, and consequently $s = 0$.
In view of proposition 1.2.4 ii), we may then conclude that the class of objects of the category ${}_{\mathcal{C}}\mathcal{F}(T)$ is a torsion-free class for some torsion theory. □

Now let us recall that if $\mathcal{C}$ is an abelian category and U is a generator for $\mathcal{C}$, then the presheaf $G = G_{{}_{\mathcal{C}}\mathcal{P}(T)}$, given for all $b \in T$ by $U^{(T_b)}$, where

$$T_b = \{a \in T \mid b \leq a\},$$

is a generator for ${}_{\mathcal{C}}\mathcal{P}(T)$ (cf. 2.1.7). Moreover:

4.1.10 Proposition. *The presheaf G is separated and therefore is a generator for the category ${}_{\mathcal{C}}\mathcal{F}(T)$ of separated presheaves on T.*

Proof. Let $b \in T$ and $\{b_i\}_{i \in I} \in C(b)$, and let us suppose the existence of $s \in G(b)$ such that $s|_{b_i} = 0$ for all $i \in I$. We have to verify that $s = 0$.
Since $G(b) = U^{(T_b)}$, clearly s is of the form $(s_a)_{a \geq b}$, with $s_a \in U$ for all $a \geq b$. Thus, it is sufficient to check that $s_a = 0$ for all $a \geq b$. On the other hand, we recall from (2.3) that for all $i \in I$ we have $s|_{b_i} = (s'_a)_{a \geq b_i}$, where

$$s'_a = \begin{cases} s_a, & \text{if } a \geq b; \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, for all $a \geq b_i$ we have by assumption that $s'_a = 0$, whence $s_a = 0$ if $a \geq b$. □

4.1.11 Note. The presheaf G satisfies even a stronger condition than Sh1), which is: if $c \leq b$ in T and $s \in G(b)$ such that $s|_c = 0$, then $s = 0$.

4.2 The sheafification functor S

In this section we present the central goals of this chapter: we obtain a category of sheaves on a general not necessarily commutative topology and at the same time an associated sheafification functor. The main tool we use is the theory of localization, since we define sheaves as the objects of a quotient category of the category of presheaves.

Again we consider a poset T as in 4.1.3 but this time we need to impose a rather natural condition to be satisfied by the quasi-coverings in order to obtain the sheafification functor. This is what we call the

4.2.1 *C-condition*:

(C) for all $b \leq a$ in T, if $\{a_i\}_{i \in I}$ is a quasi-covering of a then there exists a quasi-covering $\{b_i\}_{i \in I}$ of b such that for all $i \in I$ we have $b_i \leq a_i$.

4.2.2 Remark. If the family $\{C(a)\}_{a \in T}$ defines a Grothendieck topology on T, then from axiom G3) we obviously derive that C-condition holds. Exactly the same occurs if that family defines a noncommutative Grothendieck topology (in the sense of 3.3.10), from axiom NCG3. For this reason we may understand the C-condition as a weaker axiom we want to impose instead of G3) or NCG3).

4.2.3 Examples. Although from the previous remark it clearly follows that all the already mentioned examples satisfy (C), let us specify how the C-condition is verified in each case. We will see that, as in the classical case, b_i is just the *intersection* of a_i with b (understanding by *intersection* the not necessarily commutative binary operation which plays the role of $\cap$ in each context):

1) let X be a topological space and let $O(X)$ denote its set of open subsets. For all $V \subseteq U$ in $O(X)$, if $\{U_i\}_{i \in I}$ is a covering of U then the set $\{V \cap U_i\}_{i \in I}$ is obviously a covering of V satisfying (C);

2) let Λ be a noncommutative topology *à la* Van Oystaeyen. For all $y \leq x$ in Λ, if $\{x \wedge \lambda_1, \ldots, x \wedge \lambda_n\}$ is a covering of x then, according to definition 3.3.4, $\{y \wedge \lambda_1, \ldots, y \wedge \lambda_n\}$ is a covering of y and, in virtue of axiom A4), for all $i \in \{1, \ldots, n\}$ we have $y \wedge \lambda_i \leq x \wedge \lambda_i$. Therefore, the coverings

in Λ satisfy (C). In a similar way, one checks that the coverings in $\mathcal{O}(\Lambda)$ satisfy (C);

3) let $(\mathcal{Q}, \{\mathrm{Cov}(U)\}_{U\in\mathcal{Q}})$ be a Q-site. It is very easy to check that with our very definition of Q-covering we have guaranteed the C-condition. Indeed, this is already done in the proof of 3.5.13 where we check that if $U \in \mathcal{Q}$ and $\{U_i\}_{i\in I}$ is a Q-covering of U then for every $V \leq U$, the set $\{V \,\&\, U_i\}_{i\in I}$ is a Q-covering of V such that for all $i \in I$ we have $V \,\&\, U_i \leq U_i$. Therefore, Q-coverings also satisfy (C);

4) in particular, if $(O(X), \subseteq, \&)$ is the quantale of open subsets of a given quantum space X, then its quantum coverings (defined as in 3.5.15) also satisfy (C). However, in this case $V \,\&\, U_i = V \cap U_i$. Indeed, we obtain $V \,\&\, U_i \subseteq V \cap U_i$ from 3.5.9.4, and the inverse inclusion holds by S3). Thus, although this Q-site has a noncommutative structure given by &, this multiplication appears to be the ordinary intersection of sets, not only among elements of quantum coverings of the same $U \in O(X)$, but also between any open subset of U and any element of a quantum covering of U.

From hereon let T be a poset with quasi-coverings satisfying (C).

A fundamental fact needed in order to obtain the sheafification functor is the following property of the category ${}_{\mathcal{C}}\mathcal{F}(T)$, on which depend the main results of this chapter:

4.2.4 Theorem. *Let $\mathcal{C}$ be an arbitrary abelian category. The category ${}_{\mathcal{C}}\mathcal{F}(T)$ is closed under essential extensions.*

Proof. Let $F \hookrightarrow P$ be an essential extension in ${}_{\mathcal{C}}\mathcal{P}(T)$ with $F \in {}_{\mathcal{C}}\mathcal{F}(T)$. We have to check that P is a separated presheaf, i.e. for every $a \in T$ and every $\{a_i\}_{i\in I} \in C(a)$, we have to check that the map

$$\xi : P(a) \longrightarrow \prod_{i\in I} P(a_i); \quad s \mapsto (s|_{a_i})_{i\in I},$$

is injective. Indeed, if we suppose that $\mathrm{Ker}\,\xi \neq 0$ then we may define a nonzero subpresheaf $P' \subseteq P$ given by

$$P'(b) = \begin{cases} P_{ab}(\mathrm{Ker}\,\xi), & \text{if } b \leq a; \\ 0, & \text{otherwise,} \end{cases}$$

with restriction morphisms

$$P'_{bc} = P_{bc}|_{P'(b)} = \begin{cases} P_{bc}|_{P_{ab}(\mathrm{Ker}\,\xi)}, & \text{if } c \leq b \leq a; \\ 0, & \text{otherwise,} \end{cases}$$

for every $c \leq b$ in T. It is easy to check that P' is in fact a subpresheaf of P, and it is nonzero since $P'(a) = \operatorname{Ker} \xi \neq 0$. Then, as the extension $F \hookrightarrow P$ is essential, it follows that $P' \cap F \neq 0$. Hence, we may find $b \in T$ such that $P'(b) \cap F(b) \neq 0$. In view of the definition of P' this means that there exists a nonzero element $t \in F(b)$ with $t = P_{ab}(s) = s|_b$, for some $s \in \operatorname{Ker} \xi$ (i.e. for some $s \in P(a)$ such that for all $i \in I$, $s|_{a_i} = 0$).

The C-condition on the quasi-covering $\{a_i\}_{i \in I}$ of a with $b \leq a$, yields a quasi-covering $\{b_i\}_{i \in I}$ of b such that for all $i \in I$ we have $b_i \leq a_i$. This guarantees that the restriction morphisms $\{F_{bb_i}\}_{i \in I}$ on t are zero. Indeed, $F_{bb_i}(t) = P_{bb_i}(t) = t|_{b_i}$ since $F \subseteq P$, and for all $i \in I$

$$t|_{b_i} = (s|_b)|_{b_i} = s|_{b_i} = (P_{a_i b_i} \circ P_{aa_i})(s) = P_{a_i b_i}(s|_{a_i}) = 0.$$

Therefore, since F is separated it follows that $t = 0$, which is a contradiction that comes from having supposed that $\operatorname{Ker} \xi \neq 0$. ☐

We henceforth assume that $\mathcal{C}$ is a Grothendieck category.

Taking into account that injective hulls are in particular essential extensions (cf. definition 1.1.32) and having proved that the category ${}_{\mathcal{C}}\mathcal{F}(T)$ is a torsion-free class for some torsion theory in ${}_{\mathcal{C}}\mathcal{P}(T)$ (cf. 4.1.9), as a consequence of this theorem now we may also assert that:

4.2.5 Corollary. *The class of objects of the category ${}_{\mathcal{C}}\mathcal{F}(T)$ is a torsion-free class for some hereditary torsion theory in ${}_{\mathcal{C}}\mathcal{P}(T)$.*

4.2.6 To this torsion theory of ${}_{\mathcal{C}}\mathcal{P}(T)$ corresponds (in view of the bijection 1.2.8) a unique radical $\tau_S \in K({}_{\mathcal{C}}\mathcal{P}(T))$ defined for all $P \in {}_{\mathcal{C}}\mathcal{P}(T)$ by

$$\tau_S P = \sum_{P' \subseteq P,\ P' \in \mathcal{T}_S} P',$$

where $\mathcal{T}_S$ is the torsion class $\{P \in {}_{\mathcal{C}}\mathcal{P}(T) \mid \forall F \in {}_{\mathcal{C}}\mathcal{F}(T), \operatorname{Hom}(P, F) = 0\}$ and $\mathcal{F}_S = {}_{\mathcal{C}}\mathcal{F}(T)$.

4.2.7 Definition. The *category of sheaves on T with values in $\mathcal{C}$* is defined to be the quotient category

$${}_{\mathcal{C}}\mathcal{P}(T)(\tau_S) = \{P \in {}_{\mathcal{C}}\mathcal{P}(T) \mid P \text{ is } \tau_S\text{-closed}\} \subseteq {}_{\mathcal{C}}\mathcal{F}(T)$$

of ${}_{\mathcal{C}}\mathcal{P}(T)$ with respect to τ_S; we will denote it by ${}_{\mathcal{C}}\mathcal{S}(T)$.

4.2.8 Examples. We may obtain this quotient category for all the examples of topologies listed in 4.2.3. In this way we define: the category ${}_{\mathcal{C}}\mathcal{S}(\Lambda)$ of *noncommutative sheaves à la Van Oystaeyen*, the category ${}_{\mathcal{C}}\mathcal{S}(\mathcal{Q})$ of *Q-sheaves* and the category ${}_{\mathcal{C}}\mathcal{S}(O(X))$ of *quantum sheaves.*

4.2.9 Corollary. *For every Grothendieck category $\mathcal{C}$ there exists a functor $S : {}_{\mathcal{C}}\mathcal{P}(T) \longrightarrow {}_{\mathcal{C}}\mathcal{S}(T)$, left adjoint of the inclusion functor i_{τ_S}, and such that*

$$ {}_{\mathcal{C}}\mathcal{S}(T) = \{P \in {}_{\mathcal{C}}\mathcal{P}(T) \mid P = i_{\tau_S}SP = Q_{\tau_S}P\}. \tag{4.1}$$

Proof. Taking into account that in 2.1.7 we prove that ${}_{\mathcal{C}}\mathcal{P}(T)$ is a Grothendieck category, we obtain this corollary by applying general localization theory in Grothendieck categories. In fact, S is the functor associated to the radical $\tau_S \in K({}_{\mathcal{C}}\mathcal{P}(T))$ as in 1.5.3 (where it is denoted by a_σ). The equality follows directly from 1.5.7 (property 6 of the localization functor). ☐

4.2.10 Definition. The functor $S : {}_{\mathcal{C}}\mathcal{P}(T) \longrightarrow {}_{\mathcal{C}}\mathcal{S}(T)$ is called the *sheafification functor.*

4.2.11 Corollary. *The category ${}_{\mathcal{C}}\mathcal{S}(T)$ of sheaves on T with values in $\mathcal{C}$ is a Grothendieck category.*

Proof. Recalling 1.5.8 we obtain that ${}_{\mathcal{C}}\mathcal{S}(T) = {}_{\mathcal{C}}\mathcal{P}(T)(\tau_S)$ is a Giraud subcategory of the Grothendieck category ${}_{\mathcal{C}}\mathcal{P}(T)$, and therefore a Grothendieck category itself. ☐

4.2.12 In particular we may conclude that:

1) there exists a *noncommutative sheafification functor* in every noncommutative topology Λ *à la* Van Oystaeyen

$$S_\Lambda : {}_{\mathcal{C}}\mathcal{P}(\Lambda) \longrightarrow {}_{\mathcal{C}}\mathcal{S}(\Lambda),$$

   and a *Q-sheafification functor* in every Q-site $\mathcal{Q}$,

$$S_{\mathcal{Q}} : {}_{\mathcal{C}}\mathcal{P}(\mathcal{Q}) \longrightarrow {}_{\mathcal{C}}\mathcal{S}(\mathcal{Q}),$$

   (thus, in particular, in every quantum space with coverings defined as in 3.5.15);

2) if $\mathcal{C}$ is a Grothendieck category then so are the categories of sheaves ${}_{\mathcal{C}}\mathcal{S}(\Lambda)$, ${}_{\mathcal{C}}\mathcal{S}(\mathcal{Q})$ and ${}_{\mathcal{C}}\mathcal{S}(O(X))$.

4.3 Sheafification in R-pre-Mod

Let T be a poset as in 4.1.3 and R a presheaf of not necessarily commutative rings on T.

In the Grothendieck category of R-pre-Modules on T (cf. section 2.2) we may also obtain a sheafification functor in a similar way as in section 4.2 for the category ${}_{\mathcal{C}}\mathcal{P}(T)$. This will be the aim of this section.

4.3.1 Definition. The category of *separated presheaves of left R-modules* on T, or shortly, of *separated R-pre-Modules* on T, is the full subcategory of R-**pre-Mod** whose objects are the R-pre-Modules M which verify, for every $a \in T$ and for every quasi-covering $\{a_i\}_{i\in I}$ of a, that the map

$$\xi : M(a) \longrightarrow \prod_{i\in I} M(a_i); \quad s \mapsto (s|_{a_i})_{i\in I},$$

is injective or, equivalently, which satisfy

Sh1) if $a \in T$ and $\{a_i\}_{i\in I} \in C(a)$, then for every $s \in M(a)$ we have $s = 0$ whenever $s|_{a_i} = 0$ for all $i \in I$.

We remark that $\prod_{i\in I} M(a_i)$ has an $R(a)$-module structure since each $M(a_i)$ is an $R(a)$-module by scalar restriction via R_{aa_i}. Moreover, ξ is a homomorphism of left $R(a)$-modules, due to the fact that each M_{aa_i} is R_{aa_i}-semilinear. By this reason, we may consider $\operatorname{Ker}\xi$ as an $R(a)$-submodule of $M(a)$.

We may verify that the category of separated R-pre-Modules on T is complete, with a similar proof as in 4.1.7, taking into account that in this case the product of an arbitrary family of R-pre-Modules is well defined since we are dealing with a Grothendieck category. Moreover, by using a proof similar to the one in 4.1.9, we also obtain:

4.3.2 Proposition. *The class of separated R-pre-Modules on T is a torsion-free class for some torsion theory in R-**pre-Mod**.*

Now let T be a poset with quasi-coverings satisfying (C) and R a flabby sheaf of not necessarily commutative rings on T.

Under this assumptions, for every R-pre-Module M we obtain a well defined subpresheaf $M' \subseteq M$ given by

$$M'(b) = \begin{cases} M_{ab}(\operatorname{Ker}\xi), & \text{if } b \leq a; \\ 0, & \text{otherwise.} \end{cases} \tag{4.2}$$

Indeed, if $M'(b) \neq 0$, for every $m' \in M'(b)$ and $r' \in R(b)$ there exists $m \in \operatorname{Ker}\xi$ and $r \in R(a)$ (in virtue of the flabbiness of R) such that

$$r' \cdot m' = R_{ab}(r) \cdot M_{ab}(m) = M_{ab}(r \cdot m) \in M_{ab}(\operatorname{Ker}\xi) = M'(b).$$

Therefore $M'(b)$ is a left $R(b)$-submodule of $M(b)$, for all $b \in T$. Moreover, for every $c \leq b$ in T, the restriction morphisms $M'_{bc} = M_{bc}|_{M'(b)}$ are clearly R_{bc}-semilinear.

Thus, making use of this subpresheaf, we may prove as in proposition 4.2.4 that the class of separated R-pre-Modules on T is closed under essential extensions and consequently:

4.3.3 Theorem. *The class of separated R-pre-Modules on T is a torsion-free class for some hereditary torsion theory in R-**pre-Mod**.*

To this torsion theory corresponds (in view of the bijection 1.2.8) a unique radical $\tau_R \in K(R\text{-}\mathbf{pre\text{-}Mod})$ defined for all $M \in R$-**pre-Mod** by

$$\tau_R M = \sum_{M' \subseteq M,\ M' \in \mathcal{T}_R} M',$$

where $\mathcal{T}_R$ is the torsion class

$$\{M \in R\text{-}\mathbf{pre\text{-}Mod} \mid \ \forall N \in \mathcal{F}_R, \operatorname{Hom}(M,N) = 0\}$$

and $\mathcal{F}_R$ is the class of separated R-pre-Modules. Thus,

4.3.4 Definition. We define the category of *sheaves of left R-modules*, or shortly, *R-Modules*, as the quotient category

$$R\text{-}\mathbf{pre\text{-}Mod}(\tau_R) = \{M \in R\text{-}\mathbf{pre\text{-}Mod} \mid M \text{ is } \tau_R\text{-closed}\}$$

of R-**pre-Mod** with respect to τ_R; we will denote it by R-**Mod**.

4.3.5 Corollary. *There exists a functor,*

$$S_R : R\text{-}\mathbf{pre\text{-}Mod} \longrightarrow R\text{-}\mathbf{Mod},$$

*left adjoint of the inclusion functor $i_{\tau_R} : R$-**Mod** $\hookrightarrow R$-**pre-Mod**, such that*

$$R\text{-}\mathbf{Mod} = \{M \in R\text{-}\mathbf{pre\text{-}Mod} \mid M = i_{\tau_R} S_R M = Q_{\tau_R} M\}.$$

4.3.6 Definition. We call S_R the *sheafification functor* in R-**pre-Mod**.

4.3.7 Corollary. *The category R-**Mod** of sheaves of left R-modules on T is a Grothendieck category.*

Chapter 5

Sheaves and sheafification on Q-sites

In this chapter we will concentrate on one type of noncommutative topology: Q-sites (cf. 3.5). We will consider noncommutative separated presheaves and sheaves over them, which we appropriately will refer to as *separated Q-presheaves* and *Q-sheaves*. We will show that there is a left adjoint of the inclusion functor from the category of Q-sheaves into that of Q-presheaves.

At this point the reader might object: "but, didn't we already do that in the previous chapter?". Actually, in the previous chapter we defined the category of Q-sheaves as a localization of the category of Q-presheaves, a procedure which naturally generalizes the Giraud and quotient category point of view in the ordinary *commutative* case. What we will do here is to introduce, in the Q-site setting, a more natural, intuitive definition of Q-sheaf and construct an associated Q-sheafification functor.

Rather surprisingly, however, it appears that in this particular case both constructions coincide: the category of Q-sheaves obtained through localization in the previous chapter and our new more intuitive, alternative category of Q-sheaves are essentially the same, as well as their associated Q-sheafification functors.

Throughout this chapter, let $(\mathcal{Q}, \leq, \&)$ be a quantale and $(\mathcal{Q}, \{\mathrm{Cov}(U)\}_{U\in\mathcal{Q}})$ a Q-site defined on $\mathcal{Q}$.

5.1 Sheaves on Q-sites

Let $\mathcal{C}$ be an arbitrary abelian category.

First of all, by way of introduction, let us recall what is classically understood by a sheaf on a (commutative) topological space, from two different point of views.

5.1.1 What is a sheaf? The classical definitions.

Let X be a topological space and B_X a basis for the topology on X.

1. **Algebraically:** We have already defined (in 1.4.1) the categories of sheaves of sets and left R-modules on B_X. In a similar way, the category ${}_{\mathcal{C}}\mathcal{S}(B_X)$ of sheaves on B_X with values in $\mathcal{C}$ is defined to be the full subcategory of ${}_{\mathcal{C}}\mathcal{P}(B_X)$ whose objects are the presheaves P which satisfies Sh1) and Sh2), or equivalently, such that for every open subset $U \in B_X$ and every open covering $\{U_i\}_{i\in I}$ of U in B_X, the sequence
$$0 \longrightarrow P(U) \xrightarrow{\xi} \prod_{i\in I} P(U_i) \xrightarrow{\theta} \prod_{(i,j)\in I\times I} P(U_i \cap U_j)$$
is exact. Here θ is given for all $(s_i)_{i\in I} \in \prod_{i\in I} P(U_i)$ by
$$(s_i|_{U_i\cap U_j} - s_j|_{U_i\cap U_j})_{(i,j)\in I\times I}.$$
Note that for an arbitrary presheaf what we have is only $\theta \circ \xi = 0$. Indeed, for all $s \in P(U)$,
$$(\theta\circ\xi)(s) = \theta((s|_{U_i})_{i\in I}) = ((s|_{U_i})|_{U_i\cap U_j} - (s|_{U_j})|_{U_i\cap U_j})_{(i,j)\in I\times I} = 0.$$
2. **Geometrically:** Let $\mathbb{E} = \ulcorner E, \pi, X \urcorner$ be a concrete sheaf on X. As a well known example of sheaf we have the presheaf of sections $\Gamma(\,\cdot\,, E)$ of $\mathbb{E}$ on B_X described in 1.4.8, for which axioms Sh1) and Sh2) are easily checked ([45, (1.29.)]). In fact, in proposition 1.4.11 we stated that all the sheaves of sets on B_X are essentially of this type, and the same happens in the category R-**mod** ([45, (2.3.)]).

Inspired by this definition, which involves coverings but in which the role of the intersection of open subsets also is fundamental, taking into account that this role in Q-sites is played by the multiplication &, we define:

5.1.2 Definition. The category ${}_{\mathcal{C}}Sh(\mathcal{Q})$ of *Q-sheaves with values in $\mathcal{C}$* is the full subcategory of ${}_{\mathcal{C}}\mathcal{P}(\mathcal{Q})$ whose objects are the Q-presheaves P which satisfy Sh1) and the following *gluing condition*:

Sh2) if $U \in \mathcal{Q}$, if $\{U_i\}_{i\in I} \in \mathrm{Cov}(U)$, and if for all $i \in I$ there is given $s_i \in P(U_i)$ verifying for all $i, j \in I$ that $s_i|_{U_i \,\&\, U_j} = s_j|_{U_i \,\&\, U_j}$, then there exists some $s \in P(U)$ such that $s|_{U_i} = s_i$, for all $i \in I$.

(Note that, in virtue of 3.5.9.3, it makes perfect sense to consider the restrictions of $s_i \in P(U_i)$ and $s_j \in P(U_j)$ to the open subset $U_i \,\&\, U_j$ since we have not only $U_i \,\&\, U_j \leq U_i$ but also $U_i \,\&\, U_j \leq U_j$.)
It may be checked (as in the classical case) that P is a Q-sheaf if and only if, for every open subset U and every Q-covering $\{U_i\}_{i\in I}$ of U, the sequence

$$0 \longrightarrow P(U) \xrightarrow{\xi} \prod_{i\in I} P(U_i) \xrightarrow{\theta} \prod_{(i,j)\in I\times I} P(U_i \,\&\, U_j)$$

is exact, where $\xi(s) = (s|_{U_i})_{i\in I}$, for every $s \in P(U)$, and θ is given for all $(s_i)_{i\in I} \in \prod_{i\in I} P(U_i)$ by $(s_i|_{U_i \,\&\, U_j} - s_j|_{U_i \,\&\, U_j})_{(i,j)\in I\times I}$.
Again, as in the classical case, what we have for an arbitrary Q-presheaf is only $\theta \circ \xi = 0$.

5.2 The Q-presheaf LP

For a given Q-presheaf P, we construct in detail the Q-presheaf LP on which is based the whole notion of *Q-sheafification*. Instead of using stalks (as done in 1.4 for the classical case), we make use of inverse and direct limits (on sets of indices of Q-coverings and on directed sets of Q-coverings respectively), and their well known universal properties. Generalizing similar properties in the classical case, we finally prove two fundamental results in order to define the *Q-sheafification functor*, stating that if P is a Q-presheaf resp. a separated Q-presheaf, then LP is a separated Q-presheaf resp. a Q-sheaf.

Let $\mathcal{C}$ be an arbitrary Grothendieck category, $U \in \mathcal{Q}$ and $P \in {}_{\mathcal{C}}\mathcal{P}(\mathcal{Q})$.

5.2.1 For every $\mathcal{U} = \{U_i\}_{i\in I} \in \operatorname{Cov}(U)$, we denote by $P\mathcal{U}$ the system on $\mathcal{C}$ which consists of all the diagrams

$$\begin{tikzcd}
P(U_i) \arrow[dr, "p_{U_i,U_i \,\&\, U_j}"] & \\
& P(U_i \,\&\, U_j), \\
P(U_j) \arrow[ur, "p_{U_j,U_i \,\&\, U_j}"'] &
\end{tikzcd}$$

for all $i, j \in I$. It may be checked that $P\mathcal{U}$ is an inverse system on the quasi-ordered set $(\{U_i, U_i \,\&\, U_j\}_{i,j\in I}, \leq)$. Therefore, its *inverse limit* exists (i.e. the limit on the small category defined by I); it is of the form

$$\varprojlim P\mathcal{U} = \left\{ (s_i)_{i\in I} \in \prod_{i\in I} P(U_i) \mid \forall i, j \in I, s_i|_{U_i \,\&\, U_j} = s_j|_{U_i \,\&\, U_j} \right\},$$

and comes equipped with projection maps $\pi_i : \varprojlim P\mathcal{U} \longrightarrow P(U_i)$, for every $i \in I$.

5.2.2 For all Q-coverings $\mathcal{U} = \{U_i\}_{i\in I}$ and $\mathcal{U}' = \{U'_j\}_{j\in J}$ of U such that $\mathcal{U}' \preceq \mathcal{U}$, we define a map

$$P^U_{\mathcal{U}\mathcal{U}'} : \varprojlim P\mathcal{U} \longrightarrow \varprojlim P\mathcal{U}'; \quad (s_i)_{i\in I} \mapsto (s_{\delta(j)}|_{U'_j})_{j\in J}, \tag{5.1}$$

where $\delta : J \to I$ is the map satisfying for all $j \in J$ that $U'_j \leq U_{\delta(j)}$. Indeed, for every $j \in J$, there exists

$$\varprojlim P\mathcal{U} \xrightarrow{\pi_{\delta(j)}} P(U_{\delta(j)}) \xrightarrow{P_{U_{\delta(j)}U'_j}} P(U'_j).$$

Thus, the universal property of the inverse limit guarantees the existence of a unique morphism $\varprojlim P\mathcal{U} \to \varprojlim P\mathcal{U}'$ making the following diagram commutative

$$\begin{tikzcd}
\varprojlim P\mathcal{U} \arrow[rr, "P^U_{\mathcal{U}\mathcal{U}'}"] \arrow[dr, "P_{U_{\delta(j)}U'_j}\circ\pi_{\delta(j)}"'] & & \varprojlim P\mathcal{U}' \arrow[dl, "\pi_j"] \\
& P(U'_j) &
\end{tikzcd}$$

for all $j \in J$; therefore the j^{th} component of $P^U_{\mathcal{U}\mathcal{U}'}((s_i)_{i\in I})$ is

$$(P_{U_{\delta(j)}U'_j} \circ \pi_{\delta(j)})(s_i)_{i\in I} = P_{U_{\delta(j)}U'_j}(s_{\delta(j)}) = s_{\delta(j)}|_{U'_j}.$$

5.2.3 For every morphism of Q-presheaves $f : P \to P'$ and every Q-covering $\mathcal{U} = \{U_i\}_{i\in I}$ of U, we denote by $f^U(\mathcal{U})$ the unique morphism in $\mathcal{C}$ (given by the universal property of the inverse limit) which makes the following diagram commutative, for all $i \in I$:

$$\begin{tikzcd}
\varprojlim P\mathcal{U} \arrow[rr, "f^U(\mathcal{U})"] \arrow[dr, "f(U_i)\circ\pi_i"'] & & \varprojlim P'\mathcal{U} \arrow[dl, "\pi'_i"] \\
& P'(U_i) &
\end{tikzcd}$$

Therefore, for every $(s_i)_{i\in I} \in \varprojlim P\mathcal{U}$,

$$f^U(\mathcal{U})((s_i)_{i\in I}) = (f(U_i)(s_i))_{i\in I}. \tag{5.2}$$

5.2.4 The construction of LP.

For every $P \in {}_{\mathcal{C}}\mathcal{P}(\mathcal{Q})$ we define another Q-presheaf, denoted by LP, as follows: for every $U \in \mathcal{Q}$, there exists a direct system

$$(\{\varprojlim P\mathcal{U}\}_{\mathcal{U}\in\mathrm{Cov}(U)}, \{P^U_{\mathcal{U}\mathcal{U}'}\}_{\mathcal{U}'\preceq\mathcal{U}\in\mathrm{Cov}(U)})$$

defined on the directed set $\mathrm{Cov}(U)$ (cf. 3.5.11), whose direct limit (colimit) belongs to $\mathcal{C}$. Thus, it makes sense to define LP on the open subsets by

$$LP(U) = \varinjlim_{\mathcal{U}\in\mathrm{Cov}(U)}(\varprojlim P\mathcal{U}), \tag{5.3}$$

which comes equipped with maps

$$\eta_{\mathcal{U}} : \varprojlim P\mathcal{U} \to LP(U),$$

for all $\mathcal{U} \in \mathrm{Cov}(U)$. Hence, for every $s \in LP(U)$ there exists a Q-covering $\mathcal{U} = \{U_i\}_{i\in I}$ of U such that $s = \eta_{\mathcal{U}}(x)$ for some $x = (x_i)_{i\in I} \in \varprojlim P\mathcal{U}$ (where $x_i \in P(U_i)$ and for all $i, j \in I$ we have $x_i|_{U_i \,\&\, U_j} = x_j|_{U_i \,\&\, U_j}$).
Moreover, if $\mathcal{U}' = \{U'_j\}_{j\in J}$ is another Q-covering and $y = (y_j)_{j\in J} \in \varprojlim P\mathcal{U}'$, then $\eta_{\mathcal{U}}(x) = \eta_{\mathcal{U}'}(y)$ if and only if there exists a sub-Q-covering $\mathcal{U}'' = \{U''_k\}_{k\in K}$ of $\mathcal{U}$ and $\mathcal{U}'$ such that

$$P^U_{\mathcal{U}\mathcal{U}''}((x_i)_{i\in I}) = P^U_{\mathcal{U}'\mathcal{U}''}((y_j)_{j\in J}),$$

i.e. such that, for all $k \in K$,

$$x_{\delta_1(k)}|_{U''_k} = y_{\delta_2(k)}|_{U''_k}, \tag{5.4}$$

where $\delta_1 : K \to I$ and $\delta_2 : K \to J$ are the maps satisfying for all $k \in K$ that $U''_k \leq U_{\delta_1(k)}, U'_{\delta_2(k)}$. This defines an equivalence relation such that

$$LP(U) = (\bigsqcup_{\mathcal{U}\in\mathrm{Cov}(U)} \varprojlim P\mathcal{U}) \;/\sim\;;$$

the map $\eta_{\mathcal{U}}$ is the composition

$$\varprojlim P\mathcal{U} \hookrightarrow \bigsqcup_{\mathcal{U}\in\mathrm{Cov}(U)} \varprojlim P\mathcal{U} \twoheadrightarrow LP(U).$$

We note that $\eta_{\mathcal{U}}((x_i)_{i\in I}) = 0$ if and only if there exists a sub-Q-covering $\mathcal{U}' = \{U'_j\}_{j\in J}$ of $\mathcal{U}$ such that $P^U_{\mathcal{U}\mathcal{U}'}((x_i)_{i\in I}) = 0$, i.e. if and only if for all $j \in J$ we have $x_{\delta(j)}|_{U'_j} = 0$, where $\delta : J \to I$ is the map satisfying for all $j \in J$ that $U'_j \leq U_{\delta(j)}$.

On the other hand, let $V \leq U$ in $\mathcal{Q}$ and let us define the restriction morphism $(LP)_{UV} : LP(U) \to LP(V)$ in the following steps:

1) From each $\mathcal{U} = \{U_i\}_{i\in I} \in \mathrm{Cov}(U)$ we obtain a Q-covering

$$\mathcal{V} = \{V \,\&\, U_i\}_{i\in I} \in \mathrm{Cov}(V)$$

such that $V \mathbin{\&} U_i \leq U_i$, in virtue of the C-condition (cf. 4.2.1). From this fact, for all $i, j \in I$ by 3.5.2.1 and 3.5.2.2 we also obtain

$$(V \mathbin{\&} U_i) \& (V \mathbin{\&} U_j) \leq U_i \mathbin{\&} U_j.$$

Thus, for all $i, j \in I$ we have the following commutative diagram

$$\begin{tikzcd}
 & P(U_i) \arrow[r] \arrow[dd] & P(V \mathbin{\&} U_i) \arrow[dd] \\
\varprojlim P\mathcal{U} \arrow[ur, "\pi_i"] \arrow[dr, "\pi_j"'] & & \\
 & P(U_i \mathbin{\&} U_j) \arrow[r] & P((V \mathbin{\&} U_i) \& (V \mathbin{\&} U_j))
\end{tikzcd}$$

(where the triangle commutes by the definition of limit and the square by the definition of Q-presheaf).
Therefore, by the universal property of the inverse limit, there exists a unique morphism $f_I : \varprojlim P\mathcal{U} \to \varprojlim P\mathcal{V}$ that fits in the following commutative diagram (where the triangle $\bullet$ is commutative since it coincides with the previous pentagonal diagram)

$$\begin{tikzcd}
\varprojlim P\mathcal{V} \arrow[dr] \arrow[ddr, bend right] & & \varprojlim P\mathcal{U} \arrow[ll, dashed, "f_I"'] \arrow[dl] \arrow[ddl, bend left] \\
 & P(V \mathbin{\&} U_i) \arrow[d] \arrow[r, phantom, "\bullet"] & \\
 & P((V \mathbin{\&} U_i) \& (V \mathbin{\&} U_j)) &
\end{tikzcd}$$

Hence, f_I is given for all $(s_i)_{i \in I} \in \varprojlim P\mathcal{U}$ by

$$f_I((s_i)_{i \in I}) = (s_i|_{V \mathbin{\&} U_i})_{i \in I}. \tag{5.5}$$

2) For all Q-coverings $\mathcal{U} = \{U_i\}_{i \in I}$ and $\mathcal{U}' = \{U'_j\}_{j \in J}$ of U, if there exists a map $\delta : J \to I$ such that $\mathcal{U}' \leq \mathcal{U}$, then we get the same relation on the coverings $\mathcal{V}' \leq \mathcal{V}$ induced by the same δ. Indeed, for all $j \in J$ by 3.5.2.1 it follows that $V \mathbin{\&} U'_j \leq V \mathbin{\&} U_{\delta(j)}$. Hence, it makes sense to

consider the following diagram:

$$\begin{tikzcd}
\varprojlim P\mathcal{U} \arrow[r, "f_I"] \arrow[dd, "P^U_{\mathcal{U}\mathcal{U}'}"'] & \varprojlim P\mathcal{V} \arrow[dd, "P^V_{\mathcal{V}\mathcal{V}'}"'] \arrow[dr, "\eta_{\mathcal{V}}"] & \\
 & & LP(V) \\
\varprojlim P\mathcal{U}' \arrow[r, "f_J"'] & \varprojlim P\mathcal{V}' \arrow[ur, "\eta_{\mathcal{V}'}"'] &
\end{tikzcd}$$

where the triangle is commutative by the definition of direct limit. Let $(s_i)_{i\in I} \in \varprojlim P\mathcal{U}$ and let us check that the square is also commutative.

$$\begin{aligned}
(P^V_{\mathcal{V}\mathcal{V}'} \circ f_I)((s_i)_{i\in I}) &= P^V_{\mathcal{V}\mathcal{V}'}((s_i|_{V \,\&\, U_i})_{i\in I}) \\
&= P_{V \,\&\, U_{\delta(j)}, V \,\&\, U'_j}((s_{\delta(j)}|_{V \,\&\, U_{\delta(j)}})_{j\in J}) \\
&= (s_{\delta(j)}|_{V \,\&\, U'_j})_{j\in J} \\
&= f_J((s_{\delta(j)}|_{U'_j})_{j\in J}) \\
&= (f_J \circ P^U_{\mathcal{U}\mathcal{U}'})((s_i)_{i\in I}).
\end{aligned}$$

Therefore, the triangle $\bullet$ in the diagram below is commutative.

3) Hence, by the universal property of the direct limit, there exists a unique $(LP)_{UV} : LP(U) \to LP(V)$ that fits in the following commutative diagram.

$$\begin{tikzcd}
LP(U) \arrow[rr, dashed, "(LP)_{UV}"] & & LP(V) \\
 & \varprojlim P\mathcal{U} \arrow[ul, "\eta_{\mathcal{U}}"'] \arrow[ur, "\eta_{\mathcal{V}}\circ f_I"] \arrow[d, "P^U_{\mathcal{U}\mathcal{U}'}"] & \\
 & \varprojlim P\mathcal{U}' \arrow[uul, bend left, "\eta_{\mathcal{U}'}"] \arrow[uur, bend right, "\eta_{\mathcal{V}'}\circ f_J"'] \arrow[ur, phantom, "\bullet"] &
\end{tikzcd}$$

Consequently, for every $s \in LP(U)$ there exists $\mathcal{U} = \{U_i\}_{i\in I} \in \mathrm{Cov}(U)$ such that $s = \eta_{\mathcal{U}}(x)$, for some $x \in \varprojlim P\mathcal{U}$ and

$$(LP)_{UV}(s) = \eta_{\mathcal{V}}(f_I((x_i)_{i\in I})) = \eta_{\mathcal{V}}((x_i|_{V \,\&\, U_i})_{i\in I}), \tag{5.6}$$

where $\mathcal{V} = \{V \,\&\, U_i\}_{i\in I} \in \mathrm{Cov}(V)$ and $x = (x_i)_{i\in I}$ with $x_i \in P(U_i)$, for all $i \in I$, such that $x_i|_{U_i \,\&\, U_j} = x_j|_{U_i \,\&\, U_j}$, for all $i, j \in I$.

5.2.5 Theorem. *If $P \in {}_{\mathcal{C}}\mathcal{P}(\mathcal{Q})$ then $LP \in {}_{\mathcal{C}}\mathcal{F}(\mathcal{Q})$.*

Proof. Let $U \in \mathcal{Q}$ and $\mathcal{U} = \{U_i\}_{i\in I} \in \mathrm{Cov}(U)$, and let us check that

$$\xi : LP(U) \to \prod_{i\in I} LP(U_i); \quad s \mapsto ((LP)_{UU_i}(s))_{i\in I}$$

is injective. If $s \in \operatorname{Ker}\xi$ then there exists a Q-covering $\mathcal{U}' = \{U'_a\}_{a\in A}$ of U such that $s = \eta_{\mathcal{U}'}(x)$, for some $x = (x_a)_{a\in A} \in \varprojlim P\mathcal{U}'$, and for all $i \in I$,

$$0 = LP_{UU_i}(s) = \eta_{\mathcal{U}'_i}((x_a|_{U_i \,\&\, U'_a})_{a\in A}),$$

where $\mathcal{U}'_i$ denotes the Q-covering $\{U_i \,\&\, U'_a\}_{a\in A}$ of U_i. Therefore, for all $i \in I$ there exists a sub-Q-covering $\mathcal{U}_i$ of $\mathcal{U}'_i$ such that

$$P^{U_i}_{\mathcal{U}'_i\mathcal{U}_i}((x_a|_{U_i \,\&\, U'_a})_{a\in A}) = 0$$

Suppose $\mathcal{U}_i = \{U_{b_i}\}_{b_i\in B_i}$ and let $\delta_i : B_i \to A$ be the map such that for all $b_i \in B_i$ we have $U_{b_i} \le U_i \,\&\, U'_{\delta_i(b_i)}$.
Then, for all $b_i \in B_i$, we obtain by assumption that $(x_{\delta_i(b_i)}|_{U_i \,\&\, U'_{\delta_i(b_i)}})|_{U_{b_i}} = 0$, i.e. that

$$x_{\delta_i(b_i)}|_{U_{b_i}} = 0. \tag{5.7}$$

To be able to assert that $s = 0$ it is sufficient to find a sub-Q-covering $\mathcal{U}''$ of $\mathcal{U}'$ such that $P^{U}_{\mathcal{U}'\mathcal{U}''}((x_a)_{a\in A}) = 0$.
We claim that

$$\mathcal{U}'' = \bigcup_{i\in I} \mathcal{U}_i = \{U_{b_i}\}_{b_i\in B_i, i\in I}$$

is a sub-Q-covering satisfying this condition.
Indeed, $\mathcal{U}''$ is a Q-covering of U just by axiom G2) of a Grothendieck topology (cf. 3.5.13). Moreover, $\mathcal{U}'' \preceq \mathcal{U}'$ by using $\delta : \bigsqcup_{i\in I} B_i \to A$ given, for all $b_i \in B_i$ and $i \in I$, by

$$\delta(b_i) = \delta_i(b_i).$$

This is easily checked since we have by assumption that $U_{b_i} \le U_i \,\&\, U'_{\delta(b_i)}$, so $U_{b_i} \le U'_{\delta(b_i)}$ by 3.5.9.3.
Finally,

$$P^{U}_{\mathcal{U}'\mathcal{U}''}((x_a)_{a\in A}) = (x_{\delta(b_i)}|_{U_{b_i}})_{b_i\in B_i, i\in I},$$

which is equal to 0 just by (5.7). □

5.2.6 Theorem. *If $P \in {}_{\mathcal{C}}\mathcal{F}(\mathcal{Q})$ then $LP \in {}_{\mathcal{C}}\mathcal{S}h(\mathcal{Q})$.*

Proof. Let $U \in \mathcal{Q}$ and $\{U_i\}_{i\in I} \in \mathrm{Cov}(U)$. We have to verify the exactness of the sequence

$$0 \longrightarrow LP(U) \xrightarrow{\ \xi\ } \textstyle\prod_{i\in I} LP(U_i) \xrightarrow{\ \theta\ } \prod_{(i,j)\in I\times I} LP(U_i \,\&\, U_j).$$

Since P is a Q-presheaf, LP is separated, i.e. ξ is injective (cf. 5.2.5). Besides $\theta \circ \xi = 0$ for every Q-presheaf (in particular for LP). Thus, it remains to check that $\mathrm{Ker}\,\theta \subseteq \mathrm{Im}\,\xi$.
Let $(s_i)_{i\in I} \in \mathrm{Ker}\,\theta$. Then for all $i \in I$ there exists $\mathcal{U}_i = \{U_{a_i}\}_{a_i\in A_i} \in \mathrm{Cov}(U_i)$ such that $s_i = \eta_{\mathcal{U}_i}(x_i)$, for some $x_i \in \varprojlim P\mathcal{U}_i$, where

$$\eta_{\mathcal{U}_i} : \varprojlim P\mathcal{U}_i \longrightarrow LP(U_i),$$

and $x_i = (m_{a_i})_{a_i\in A_i}$ with $m_{a_i} \in P(U_{a_i})$ such that for all $a_i, b_i \in A_i$,

$$m_{a_i}|_{U_{a_i}\,\&\,U_{b_i}} = m_{b_i}|_{U_{a_i}\,\&\,U_{b_i}}. \tag{5.8}$$

On the other hand, in view of G2) we obtain that

$$\mathcal{U} = \bigcup_{i\in I} \mathcal{U}_i = \{U_{a_i}\}_{a_i\in A_i, i\in I} \in \mathrm{Cov}(U).$$

Now let us consider the element

$$x = (m_{a_i})_{a_i\in A_i, i\in I} \in \prod_{a_i\in A_i, i\in I} P(U_{a_i})$$

and let us verify that $x \in \varprojlim P\mathcal{U}$ (whence $\eta_{\mathcal{U}}(x) \in LP(U)$). It is sufficient to check for all $i, j \in I$, $a_i \in A_i$ and $b_j \in A_j$ that

$$m_{a_i}|_{U_{a_i}\,\&\,U_{b_j}} = m_{b_j}|_{U_{a_i}\,\&\,U_{b_j}}. \tag{5.9}$$

Let us fix arbitrary $i, j \in I$. When $i = j$ this equality is given by (5.8). Otherwise, let V denote the element $U_i \,\&\, U_j$. Then, by (5.6), the restriction of $s_i \in LP(U_i)$ to V coincides with

$$\eta_{\mathcal{V}_1}((m_{a_i}|_{V\,\&\,U_{a_i}})_{a_i\in A_i}) \in LP(V),$$

where $\mathcal{V}_1$ denotes the Q-covering $\{V \,\&\, U_{a_i}\}_{a_i\in A_i}$ of V. On the other hand, denoting $\mathcal{U}_j = \{U_{b_j}\}_{b_j\in A_j} \in \mathrm{Cov}(U_j)$, the restriction of $s_j \in LP(U_j)$ to V is equal to

$$\eta_{\mathcal{V}_2}((m_{b_j}|_{V\,\&\,U_{b_j}})_{b_j\in A_j}) \in LP(V),$$

with $\mathcal{V}_2 = \{V \,\&\, U_{b_j}\}_{b_j\in A_j} \in \mathrm{Cov}(V)$.
Both restrictions coincide by assumption (since $(s_i)_{i\in I} \in \mathrm{Ker}\,\theta$), therefore

there exists a sub-Q-covering $\mathcal{V}_3 = \{V_k\}_{k\in K}$ of $\mathcal{V}_1$ and $\mathcal{V}_2$ such that for all $k \in K$

$$(m_{\delta_i(k)}|_{V \,\&\, U_{\delta_i(k)}})|_{V_k} = (m_{\delta_j(k)}|_{V \,\&\, U_{\delta_j(k)}})|_{V_k},$$

where $\delta_i : K \to A_i$ and $\delta_j : K \to A_j$ are the maps respectively satisfying for all $k \in K$ that $V_k \leq V \,\&\, U_{\delta_i(k)}$ and $V_k \leq V \,\&\, U_{\delta_j(k)}$, (cf. 5.4). Consequently, for all $k \in K$:

$$m_{\delta_i(k)}|_{V_k} = m_{\delta_j(k)}|_{V_k}. \tag{5.10}$$

On the other hand, by Q3) we obtain for all $k \in K$ that

$$U_{a_i} \,\&\, U_{b_j} \,\&\, V_k = (U_{a_i} \,\&\, U_{b_j}) \,\&\, (U_{a_i} \,\&\, U_{b_j} \,\&\, V_k),$$

and from Q4) it follows that

$$\bigvee_{k\in K} (U_{a_i} \,\&\, U_{b_j} \,\&\, V_k) = U_{a_i} \,\&\, U_{b_j} \,\&\, (\bigvee_{k\in K} V_k) = (U_{a_i} \,\&\, U_{b_j}) \,\&\, V,$$

which coincides with $U_{a_i} \,\&\, U_{b_j}$ by 3.5.2.6. Therefore $\{U_{a_i} \,\&\, U_{b_j} \,\&\, V_k\}_{k\in K}$ is a Q-covering of $U_{a_i} \,\&\, U_{b_j}$, so the map

$$\xi' : P(U_{a_i} \,\&\, U_{b_j}) \longrightarrow \prod_{k\in K} P(U_{a_i} \,\&\, U_{b_j} \,\&\, V_k); \quad t \mapsto (t|_{U_{a_i} \,\&\, U_{b_j} \,\&\, V_k})_{k\in K}$$

is injective (since P is separated). Hence, in order to obtain (5.9) it is sufficient to prove that

$$\xi'(m_{a_i}|_{U_{a_i} \,\&\, U_{b_j}}) = \xi'(m_{b_j}|_{U_{a_i} \,\&\, U_{b_j}}).$$

This is equivalent to checking, for all $k \in K$, that

$$m_{a_i}|_{U_{a_i} \,\&\, U_{b_j} \,\&\, V_k} = m_{b_j}|_{U_{a_i} \,\&\, U_{b_j} \,\&\, V_k}. \tag{5.11}$$

First of all, for all $k \in K$, taking into account that $V_k \leq V \,\&\, U_{\delta_i(k)}$ and the properties listed in 3.5.2 and 3.5.9, we get that

$$\begin{aligned}(U_{a_i} \,\&\, U_{b_j}) \,\&\, V_k &\leq U_{a_i} \,\&\, V_k \leq (U_{a_i} \,\&\, U_i) \,\&\, U_j \,\&\, U_{\delta_i(k)}\\ &= U_{a_i} \,\&\, U_j \,\&\, U_{\delta_i(k)} = (U_{a_i} \,\&\, U_{\delta_i(k)}) \,\&\, U_j \leq U_{a_i} \,\&\, U_{\delta_i(k)}.\end{aligned}$$

Therefore,

$$m_{a_i}|_{U_{a_i} \,\&\, U_{b_j} \,\&\, V_k} = (m_{a_i}|_{U_{a_i} \,\&\, U_{\delta_i(k)}})|_{U_{a_i} \,\&\, U_{b_j} \,\&\, V_k},$$

which, by (5.8), coincides with

$$(m_{\delta_i(k)}|_{U_{a_i} \,\&\, U_{\delta_i(k)}})|_{U_{a_i} \,\&\, U_{b_j} \,\&\, V_k} = m_{\delta_i(k)}|_{U_{a_i} \,\&\, U_{b_j} \,\&\, V_k}.$$

On the other hand,

$$U_{a_i} \mathbin{\&} U_{b_j} \mathbin{\&} V_k \leq U_i \mathbin{\&} U_j \mathbin{\&} V_k = V \mathbin{\&} V_k = V_k,$$

so

$$m_{a_i}|_{U_{a_i} \& U_{b_j} \& V_k} = (m_{\delta_i(k)}|_{V_k})|_{U_{a_i} \& U_{b_j} \& V_k}.$$

With a similar procedure, taking into account that

$$\begin{aligned} U_{a_i} \mathbin{\&} V_k &\leq U_i \mathbin{\&} V \mathbin{\&} U_{\delta_j(k)} = V \mathbin{\&} U_{\delta_j(k)} \\ &= U_j \mathbin{\&} U_i \mathbin{\&} U_{\delta_j(k)} = U_j \mathbin{\&} U_{\delta_j(k)} \mathbin{\&} U_i = U_{\delta_j(k)} \mathbin{\&} U_i \leq U_{\delta_j(k)}, \end{aligned}$$

whence

$$U_{a_i} \mathbin{\&} U_{b_j} \mathbin{\&} V_k = (U_{a_i} \mathbin{\&} V_k) \mathbin{\&} U_{b_j} \leq U_{\delta_j(k)} \mathbin{\&} U_{b_j},$$

we may obtain that

$$m_{b_j}|_{U_{a_i} \& U_{b_j} \& V_k} = (m_{\delta_j(k)}|_{V_k})|_{U_{a_i} \& U_{b_j} \& V_k}.$$

Therefore, the equality (5.11) holds as a consequence of (5.10). Consequently $s = \eta_{\mathcal{U}}(x) \in LP(U)$, so it makes sense to consider $\xi(s)$. Finally, let us verify that $(s_i)_{i \in I} \in \operatorname{Im} \xi$ by checking that it precisely coincides with $\xi(s)$.

By (5.6), clearly $\xi(s)$ coincides with $(\eta_{\mathcal{U}'_i}((m_{c_t}|_{U_i \& U_{c_t}})_{c_t \in A_t, t \in I}))_{i \in I}$, where $\mathcal{U}'_i = \{U_i \mathbin{\&} U_{c_t}\}_{c_t \in A_t, t \in I} \in \operatorname{Cov}(U_i)$. On the other hand, $(s_i)_{i \in I}$ is equal to $(\eta_{\mathcal{U}_i}((m_{a_i})_{a_i \in A_i}))_{i \in I}$. Thus, it is sufficient to verify for all $i \in I$ that

$$\eta_{\mathcal{U}'_i}((m_{c_t}|_{U_i \& U_{c_t}})_{c_t \in A_t, t \in I}) = \eta_{\mathcal{U}_i}((m_{a_i})_{a_i \in A_i}).$$

Taking into account that $\mathcal{U}_i \preceq \mathcal{U}'_i$ (just by choosing δ_1 as the inclusion map $A_i \hookrightarrow \bigsqcup_{t \in I} A_t$) and that $\mathcal{U}_i \preceq \mathcal{U}_i$, it is then sufficient to check for all $a_i \in A_i$ that

$$(m_{\delta_1(a_i)}|_{U_i \& U_{\delta_1(a_i)}})|_{U_{a_i}} = m_{a_i}.$$

Indeed,

$$(m_{\delta_1(a_i)}|_{U_i \& U_{\delta_1(a_i)}})|_{U_{a_i}} = (m_{a_i}|_{U_i \& U_{a_i}})|_{U_{a_i}} = (m_{a_i}|_{U_{a_i}})|_{U_{a_i}} = m_{a_i}.$$

Therefore, $(s_i)_{i \in I} = \xi(s)$. □

As an immediate consequence of theorems 5.2.5 and 5.2.6 it follows:

5.2.7 Corollary. *If* $P \in {}_{\mathcal{C}}\mathcal{P}(\mathcal{Q})$ *then* $L^2 P = L(LP) \in {}_{\mathcal{C}}\mathcal{S}h(\mathcal{Q})$.

5.3 The Q-sheafification functor a

The aim of this section is, once we have previously established all the necessary conditions, to describe the announced *Q-sheafification functor* and give some of its properties.

Let $\mathcal{C}$ be an arbitrary Grothendieck category.

5.3.1 The functor L.

In virtue of 5.2.5 we may define a functor

$$L : {}_{\mathcal{C}}\mathcal{P}(\mathcal{Q}) \longrightarrow {}_{\mathcal{C}}\mathcal{F}(\mathcal{Q})$$

given on every $P \in {}_{\mathcal{C}}\mathcal{P}(\mathcal{Q})$ by $L(P) = LP$ (recall the description of the Q-presheaf LP from 5.2.4).
For every morphism $f : P \to P'$ in ${}_{\mathcal{C}}\mathcal{P}(\mathcal{Q})$, it remains to define the morphism of separated Q-presheaves $L(f) : L(P) \to L(P')$, which is given by a collection $\{L(f)(U) : LP(U) \longrightarrow LP'(U)\}_{U\in\mathcal{Q}}$ of morphisms in $\mathcal{C}$ such that the diagram

$$\begin{tikzcd}
LP(U) \arrow[r, "(LP)_{UV}"] \arrow[d, "L(f)(U)"'] & LP(V) \arrow[d, "L(f)(V)"] \\
LP'(U) \arrow[r, "(LP')_{UV}"'] & LP'(V)
\end{tikzcd} \tag{5.12}$$

is commutative for every $V \leq U$ in $\mathcal{Q}$.
For every $U \in \mathcal{Q}$ we define $L(f)(U)$ as the direct limit of the family

$$\{f^U(\mathcal{U}) : \varprojlim P\mathcal{U} \longrightarrow \varprojlim P'\mathcal{U}\}_{\mathcal{U}\in\mathrm{Cov}(U)}$$

of morphisms in $\mathcal{C}$, which takes the form

$$L(f)(U) : LP(U) \longrightarrow LP'(U); \quad s \mapsto \eta'_{\mathcal{U}}((f(U_i)(x_i))_{i\in I}), \tag{5.13}$$

where $s = \eta_{\mathcal{U}}(x)$ with $x = (x_i)_{i\in I} \in \varprojlim P\mathcal{U}$, and $\eta'_{\mathcal{U}}$ denotes the map $\varprojlim P'\mathcal{U} \longrightarrow LP'(U)$.

Finally, let us verify the commutativity of diagram (5.12). Let $\mathcal{U}$ be the Q-covering $\{U_i\}_{i\in I}$ and $s = \eta_{\mathcal{U}}((x_i)_{i\in I}) \in LP(U)$. On one hand, we obtain

$$\begin{aligned} L(f)(V)((LP)_{UV}(s)) &= L(f)(V)(\eta_{\mathcal{U}}(P_{U_i,V\,\&\,U_i}((x_i)_{i\in I}))) \\ &= \eta'_{\mathcal{V}}((f(V\,\&\,U_i)(P_{U_i,V\,\&\,U_i}(x_i)))_{i\in I}), \end{aligned}$$

where $\mathcal{V}$ denotes $\{V \,\&\, U_i\}_{i\in I} \in \mathrm{Cov}(V)$. On the other hand, we obtain

$$\begin{aligned}(LP')_{UV}(L(f)(U)(s)) &= (LP')_{UV}(\eta'_{\mathcal{U}}((f(U_i)(x_i))_{i\in I}))\\ &= \eta'_{\mathcal{V}}((P'_{U_i,V \,\&\, U_i}((f(U_i)(x_i)))_{i\in I}).\end{aligned}$$

Both results coincide by definition of morphism, since for all $i \in I$ we have $V \,\&\, U_i \leq U_i$ (by 3.5.9.4), so

$$f(V \,\&\, U_i) \circ P_{U_i,V \,\&\, U_i} = P'_{U_i,V \,\&\, U_i} \circ f(U_i).$$

The functor we have just described has the following property:

5.3.2 Proposition. *The functor $L : {}_{\mathcal{C}}\mathcal{P}(\mathcal{Q}) \to {}_{\mathcal{C}}\mathcal{F}(\mathcal{Q})$ is left exact.*

Proof. Let $0 \to P \xrightarrow{f} P' \xrightarrow{g} P'' \to 0$ be a short exact sequence in ${}_{\mathcal{C}}\mathcal{P}(\mathcal{Q})$ and let us check that $0 \longrightarrow LP \xrightarrow{L(f)} LP' \xrightarrow{L(g)} LP''$ is exact in ${}_{\mathcal{C}}\mathcal{F}(\mathcal{Q})$, i.e. that for all $U \in \mathcal{Q}$ the following sequence in $\mathcal{C}$ is exact:

$$0 \longrightarrow LP(U) \xrightarrow{L(f)(U)} LP'(U) \xrightarrow{L(g)(U)} LP''(U).$$

Let us fix an arbitrary $U \in \mathcal{Q}$. From the fact that direct limits in Grothendieck categories over directed families of indices are exact, it is sufficient to prove that for all $\mathcal{U} \in \mathrm{Cov}(U)$ the following sequence in $\mathcal{C}$ is exact:

$$0 \longrightarrow \varprojlim P'\mathcal{U} \xrightarrow{\quad f^U(\mathcal{U}) \quad} \varprojlim P\mathcal{U} \xrightarrow{\quad g^U(\mathcal{U}) \quad} \varprojlim P''\mathcal{U}.$$

Indeed, by assumption, for all $i \in I$ the morphism $f(U_i)$ is injective. Thus, in view of (5.2), it directly follows that the morphism $f^U(\mathcal{U})$ is injective.
On the other hand, $\mathrm{Im}\, f^U(\mathcal{U}) \subseteq \mathrm{Ker}\, g^U(\mathcal{U})$ since for all $i \in I$ we have $g(U_i) \circ f(U_i) = 0$, and therefore $g^U(\mathcal{U}) \circ f^U(\mathcal{U}) = 0$. Hence, only the inverse inclusion remains to be verified.
Let $(s_i)_{i\in I} \in \mathrm{Ker}\, g^U(\mathcal{U})$. Then, for all $i, j \in I$ we have $s_i \in \mathrm{Ker}\, g(U_i)$ (i.e. $s_i \in \mathrm{Im}\, f(U_i)$), and $P_{U_i,U_i \,\&\, U_j}(s_i) = P_{U_j,U_i \,\&\, U_j}(s_j)$. Therefore, for all $i \in I$ there exists $s'_i \in P'(U_i)$ such that $s_i = f(U_i)(s'_i)$ with $(s'_i)_{i\in I} \in \varprojlim P'\mathcal{U}$, since

$$\begin{aligned}f(U_i \,\&\, U_j)(P'_{U_i,U_i \,\&\, U_j}(s'_i)) &= P_{U_i,U_i \,\&\, U_j}(f(U_i)(s'_i)) = P_{U_i,U_i \,\&\, U_j}(s_i)\\ &= P_{U_j,U_i \,\&\, U_j}(s_j) = P_{U_j,U_i \,\&\, U_j}(f(U_j)(s'_j))\\ &= f(U_i \,\&\, U_j)(P'_{U_j,U_i \,\&\, U_j}(s'_j)),\end{aligned}$$

and so, $P'_{U_i,U_i \,\&\, U_j}(s'_i) = P'_{U_j,U_i \,\&\, U_j}(s'_j)$ by injectivity of $f(U_i \,\&\, U_j)$. Consequently, $(s_i)_{i\in I} = f^U(U_I)((s'_i)_{i\in I}) \in \mathrm{Im}\, f^U(\mathcal{U})$. □

In particular, by (cf. 1.1.15), L preserves kernels, pullbacks and finite limits.

5.3.3 The Q-sheafification functor a.

At this point, in virtue of 5.2.7 and 5.3.1, it becomes evident that we define what we call the *Q-sheafification functor*,

$$a : {}_{\mathcal{C}}\mathcal{P}(\mathcal{Q}) \longrightarrow {}_{\mathcal{C}}\mathcal{S}h(\mathcal{Q}),$$

as the composition $L \circ L$. However, in order to prove that it is a reflector, let us include the following detailed description:
Let $P \in {}_{\mathcal{C}}\mathcal{P}(\mathcal{Q})$. The Q-sheaf aP is given on every $U \in \mathcal{Q}$ by

$$aP(U) = \varinjlim_{\mathcal{U}\in\mathrm{Cov}(U)}(\varprojlim (LP)\mathcal{U}),$$

which coincides with the quotient set $(\bigsqcup_{\mathcal{U}\in\mathrm{Cov}(U)} \varprojlim (LP)\mathcal{U}) \ / \sim$, where for every Q-covering $\mathcal{U}$, the limit $\varprojlim (LP)\mathcal{U}$ is equivalent to

$$\left\{(s_i)_{i\in I} \in \textstyle\prod_{i\in I}(LP)(U_i) \mid \forall i,j \in I, (LP)_{U_i,U_i \,\&\, U_j}(s_i) = (LP)_{U_j,U_i \,\&\, U_j}(s_j)\right\};$$

it has the associated map

$$\eta^2_{\mathcal{U}} : \varprojlim (LP)\mathcal{U} \longrightarrow aP(U).$$

To aP corresponds, whenever $V \leq U$ in $\mathcal{Q}$, a restriction morphism

$$(aP)_{UV} : aP(U) \longrightarrow aP(V);\ \ s \mapsto \eta^2_{\mathcal{V}}(((LP)_{U_i,V \,\&\, U_i}(x_i))_{i\in I}),$$

where $s = \eta^2_{\mathcal{U}}(x)$ with $x = (x_i)_{i\in I} \in \varprojlim (LP)\mathcal{U}$, and where $\mathcal{U}$, $\mathcal{V}$ denote respectively the Q-coverings $\{U_i\}_{i\in I}$ and $\{V \,\&\, U_i\}_{i\in I}$. Note that every x_i belongs to $LP(U_i)$, so for all $i \in I$ there exists $\mathcal{U}_i \in \mathrm{Cov}(U_i)$ such that $x_i = \eta_{\mathcal{U}_i}(r_i)$, for some $r_i \in \varprojlim P\mathcal{U}_i$.
On every morphism $f : P \to P'$ of Q-presheaves, the functor a is defined by $a(f) = L(L(f)) : aP \to aP'$. Thus, by (5.13), it consists of a collection $\{a(f)(U)\}_{U\in\mathcal{Q}}$ of morphisms in $\mathcal{C}$ which are given for all $U \in \mathcal{Q}$ by

$$a(f)(U) : aP(U) \longrightarrow aP'(U);\ \ s \mapsto (\eta')^2_{\mathcal{U}}((L(f)(U_i)(x_i))_{i\in I}),$$

where $s = \eta^2_{\mathcal{U}}(x)$ with $x = (x_i)_{i\in I} \in \varprojlim (LP)\mathcal{U}$, and where $(\eta')^2_{\mathcal{U}}$ denotes the map $\varprojlim (LP')\mathcal{U} \longrightarrow aP'(U)$.

Since $a = L \circ L$, it follows from 5.3.2:

5.3.4 Proposition. *The functor $a : {}_{\mathcal{C}}\mathcal{P}(\mathcal{Q}) \to {}_{\mathcal{C}}\mathcal{S}h(\mathcal{Q})$ is left exact, hence it preserves kernels, pullbacks and finite limits.*

5.4 Useful machinery

In order to simplify the proof of the main result of the next section (theorem 5.5.1) we collect here some preliminary results.

5.4.1 Lemma. *To every $P \in {}_{\mathcal{C}}\mathcal{P}(\mathcal{Q})$ corresponds a morphism of Q-presheaves $P \overset{\zeta_P}{\to} LP$. Besides, if P is separated then ζ_P is injective.*

Proof. For every $U \in \mathcal{Q}$ let $\mathcal{U}_0$ denote the trivial Q-covering $\{U\}$. Then $\varprojlim P\mathcal{U}_0$ coincides with $P(U)$, so its corresponding map $\eta_{\mathcal{U}_0}$ is the canonical morphism $P(U) \to LP(U)$. We define ζ_P to be given by the family

$$\{\zeta_P(U) : P(U) \to LP(U)\}_{U \in \mathcal{Q}}$$

of morphisms in $\mathcal{C}$, where $\zeta_P(U) = \eta_{\mathcal{U}_0}$. Let us verify that this is indeed a morphism of presheaves, i.e. that whenever $V \leq U$ in $\mathcal{Q}$, the following diagram is commutative:

$$\begin{tikzcd} P(U) \arrow[r, "P_{UV}"] \arrow[d, "\eta_{\mathcal{U}_0}"'] & P(V) \arrow[d, "\eta_{\mathcal{V}_0}"] \\ LP(U) \arrow[r, "(LP)_{UV}"'] & LP(V) \end{tikzcd}$$

This follows in a straightforward way from (5.6) since for every $s \in P(U)$ we have $(LP)_{UV}(\eta_{\mathcal{U}_0}(s)) = \eta_{\mathcal{V}_0}(s|_{V \,\&\, U})$, which is equal to $\eta_{\mathcal{V}_0}(s|_V)$ by 3.5.2.6.

Moreover, if P is separated then for every $U \in \mathcal{Q}$ we may easily check that $\zeta_P(U)$ is injective as follows. If $s \in \operatorname{Ker} \zeta_P(U)$ then, since $\eta_{\mathcal{U}_0}(s) = 0$, there exists a sub-Q-covering of $\mathcal{U}_0$, i.e. another Q-covering $\mathcal{U} = \{U_i\}_{i \in I}$ of U, such that all the restrictions $s|_{U_i}$ are 0. Hence $s = 0$, indeed, as P is separated.

□

5.4.2 Corollary. *There exists a natural transformation*

$$\zeta : \mathrm{id}_{{}_{\mathcal{C}}\mathcal{P}(\mathcal{Q})} \longrightarrow i \circ L,$$

where i is the inclusion functor ${}_{\mathcal{C}}\mathcal{F}(\mathcal{Q}) \hookrightarrow {}_{\mathcal{C}}\mathcal{P}(\mathcal{Q})$.

Proof. We define ζ to be given by the family of morphisms of Q-presheaves $\zeta_P : P \to iLP$, for all $P \in {}_{\mathcal{C}}\mathcal{P}(\mathcal{Q})$. By (5.13), it is easy to verify that, for every morphism $f : P \to P'$ of Q-presheaves, we have $L(f) \circ \zeta_P = \zeta_{P'} \circ f$, i.e. that for every $U \in \mathcal{Q}$ we obtain the following commutative square in $\mathcal{C}$,

$$\begin{tikzcd}
P(U) \arrow[r, "\eta_{\mathcal{U}_0}"] \arrow[d, "f(U)"'] & LP(U) \arrow[d, "L(f)(U)"] \\
P'(U) \arrow[r, "\eta'_{\mathcal{U}_0}"'] & LP'(U)
\end{tikzcd}$$

Indeed, for all $s \in P(U)$ we have

$$L(f)(U)(\eta_{\mathcal{U}_0}(s)) = \eta'_{\mathcal{U}_0}(f(U)(s)).$$

□

Let P be a Q-presheaf. Then the image $L(\zeta_P)$ of the corresponding morphism ζ_P via the functor L is a morphism in $\operatorname{Hom}(LP, aP)$. On the other hand, we have another morphism $\zeta_{LP} \in \operatorname{Hom}(LP, aP)$ which is the one that corresponds to the Q-presheaf LP, as in 5.4.1.
Let us prove that they coincide:

5.4.3 Proposition. *For every $P \in {}_{\mathcal{C}}\mathcal{P}(\mathcal{Q})$ we have $L(\zeta_P) = \zeta_{LP}$.*

Proof. Let $U \in \mathcal{Q}$. We have to verify that $L(\zeta_P)(U) = \zeta_{LP}(U)$. Let $s = \eta_{\mathcal{U}}(x) \in LP(U)$, where $\mathcal{U} = \{U_i\}_{i \in I} \in \operatorname{Cov}(U)$ and $x = (x_i)_{i \in I} \in \varprojlim P\mathcal{U}$. By (5.13), we obtain

$$L(\zeta_P)(U)(s) = \eta^2_{\mathcal{U}}((\zeta_P(U_i)(x_i))_{i \in I}) = \eta^2_{\mathcal{U}}((\eta_{(\mathcal{U}_i)_0}(x_i))_{i \in I}),$$

where $\eta^2_{\mathcal{U}} : \varprojlim (LP)\mathcal{U} \longrightarrow aP(U)$ and $\eta_{(\mathcal{U}_i)_0} : P(U_i) \longrightarrow LP(U_i)$.
On the other hand,

$$\zeta_{LP}(U)(s) = \eta^2_{\mathcal{U}_0}(s),$$

where $\eta^2_{\mathcal{U}_0} : LP(U) \longrightarrow aP(U)$.
Taking into account that $\mathcal{U} \preceq \mathcal{U}, \mathcal{U}_0$, in order to assert that both images coincide it is sufficient, by (5.4), to verify that $\eta_{(\mathcal{U}_i)_0}(x_i) = LP_{UU_i}(s)$, for all $i \in I$.
Indeed, by (5.6) we obtain $LP_{UU_i}(s) = \eta_{\mathcal{U}_i}((x_l|_{U_i \,\&\, U_l})_{l \in I})$, where $\mathcal{U}_i$ denotes the covering $\{U_i \,\&\, U_l\}_{l \in I}$. Thus, to prove that the latter element coincides with $\eta_{(\mathcal{U}_i)_0}(x_i)$ it is sufficient, again by (5.4), to check for all $l \in I$ that

$$x_l|_{U_i \,\&\, U_l} = x_i|_{U_i \,\&\, U_l},$$

(since $\mathcal{U}_i \preceq \mathcal{U}_i, (\mathcal{U}_i)_0$). In fact, this equality holds for all $i, l \in I$ just because $(x_i)_{i \in I} \in \varprojlim P\mathcal{U}$. □

5.4.4 Lemma. *To every $P \in {}_{\mathcal{C}}\mathcal{S}h(\mathcal{Q})$ corresponds an isomorphism of Q-sheaves $\varphi_P : LP \xrightarrow{\sim} P$.*

Proof. Let $U \in \mathcal{Q}$. In general, for every $s \in LP(U)$ there exists a Q-covering $\mathcal{U} = \{U_i\}_{i\in I}$ of U and $x = (x_i)_{i\in I} \in \varprojlim P\mathcal{U}$ such that $s = \eta_{\mathcal{U}}(x)$, with $x_i|_{U_i \,\&\, U_j} = x_j|_{U_i \,\&\, U_j}$ for all $i, j \in I$. In this case, since P is a Q-sheaf, we can go further and assert that there exists a unique $t^x \in P(U)$ such that $t^x|_{U_i} = x_i$, for all $i \in I$. Thus, we may define a morphism

$$\varphi_P(U) : LP(U) \longrightarrow P(U); \quad s \mapsto t^x.$$

If there exists another $\mathcal{U}' = \{U'_j\}_{j\in J} \in \mathrm{Cov}(U)$ and $y = (y_j)_{j\in J} \in \varprojlim P\mathcal{U}'$ such that $s = \eta_{\mathcal{U}'}(y)$ then let us prove that $t^y = t^x$. By (5.4), there exists a sub-Q-covering $\mathcal{U}'' = \{U''_k\}_{k\in K} \in \mathrm{Cov}(U)$ of $\mathcal{U}$ and $\mathcal{U}'$ such that for all $k \in K$ the restrictions of $x_{\delta_1(k)}$ and $y_{\delta_2(k)}$ to U''_k coincide (where $\delta_1 : K \to I$ and $\delta_2 : K \to J$ are the maps satisfying $U''_k \le U_{\delta_1(k)}, U'_{\delta_2(k)}$). Thus, for all $k \in K$ we obtain the following sequence of equalities from which we derive that $t^x = t^y$, as P is separated:

$$t^x|_{U''_k} = (t^x|_{U_{\delta_1(k)}})|_{U''_k} = x_{U_{\delta_1(k)}}|_{U''_k} = y_{U'_{\delta_2(k)}}|_{U''_k} = (t^y|_{U'_{\delta_2(k)}})|_{U''_k} = t^y|_{U''_k}.$$

Therefore, $\varphi_P(U)$ is well defined. Besides if $s = \eta_{\mathcal{U}}(x) \in \mathrm{Ker}\, \varphi_P(U)$ then for all $i \in I$ we obtain from the very definition that $x_i = t^x|_{U_i} = 0$, and consequently $s = 0$. Hence, $\varphi_P(U)$ is injective. Finally it is also surjective since for every $t \in P(U)$ we may choose $s = \eta_{\mathcal{U}_0}(t)$ as the element in $LP(U)$ such that $t = \varphi_P(U)(s)$, since $t|_U = t$.
Thus, we assume φ_P to be given by the family of isomorphisms $\varphi_P(U)$ in $\mathcal{C}$. Let us verify that this is indeed an isomorphism of Q-sheaves, i.e. that whenever $V \le U$ in $\mathcal{Q}$, the following diagram is commutative:

$$\begin{tikzcd}
LP(U) \arrow[r, "(LP)_{UV}"] \arrow[d, "\varphi_P(U)"'] & LP(V) \arrow[d, "\varphi_P(V)"] \\
P(U) \arrow[r, "P_{UV}"'] & P(V)
\end{tikzcd}$$

Let $s = \eta_{\mathcal{U}}(x) \in LP(U)$ as before. Then,

$$P_{UV}(\varphi_P(U)(s)) = t^x|_V.$$

On the other hand, $\varphi_P(V)(\eta_{\mathcal{V}}((x_i|_{V \,\&\, U_i})_{i\in I}))$ is by definition the unique q such that $q|_{V \,\&\, U_i} = x_i|_{V \,\&\, U_i}$. Hence, to derive that $q = t^x|_V$ it is sufficient to verify that $(t^x|_V)|_{V \,\&\, U_i} = x_i|_{V \,\&\, U_i}$. Indeed,

$$(t^x|_V)|_{V \,\&\, U_i} = t^x|_{V \,\&\, U_i} = (t^x|_{U_i})|_{V \,\&\, U_i} = x_i|_{V \,\&\, U_i},$$

(recall that $V \,\&\, U_i \le U_i$ by 3.5.9.4). ☐

5.4.5 Corollary. *There exists a natural equivalence*

$$\varphi : L \circ j \longrightarrow \mathrm{id}_{{}_{\mathcal{C}}\mathcal{S}(\mathcal{Q})},$$

where j is the inclusion functor ${}_{\mathcal{C}}\mathcal{S}h(\mathcal{Q}) \hookrightarrow {}_{\mathcal{C}}\mathcal{F}(\mathcal{Q})$, and L is considered to act from ${}_{\mathcal{C}}\mathcal{F}(\mathcal{Q})$ to ${}_{\mathcal{C}}\mathcal{S}h(\mathcal{Q})$ (in virtue of 5.2.6).

Proof. The family $\{\varphi_P : LjP \longrightarrow P\}_{P \in {}_{\mathcal{C}}\mathcal{S}(\mathcal{Q})}$ of isomorphisms of Q-sheaves defines such a φ. Indeed, if $f : P \to P'$ is a morphism of Q-sheaves then $f \circ \varphi_P = \varphi_{P'} \circ L(f)$, i.e. for every $U \in \mathcal{Q}$ the square

$$\begin{tikzcd}
LP(U) \arrow[r, "\varphi_P(U)"] \arrow[d, "L(f)(U)"'] & P(U) \arrow[d, "f(U)"] \\
LP'(U) \arrow[r, "\varphi_{P'}(U)"'] & P'(U)
\end{tikzcd}$$

in $\mathcal{C}$ is commutative, as we verify in what follows: let $s \in LP(U)$. Then, there exists $\mathcal{U} = \{U_i\}_{i \in I} \in \mathrm{Cov}(U)$ and $x = (x_i)_{i \in I} \in \varprojlim P\mathcal{U}$ such that $s = \eta_{\mathcal{U}}(x)$. If t^x denotes the unique element in $P(U)$ such that for all $i \in I$ we have $P_{UU_i}(t^x) = x_i$, then

$$f(U)(\varphi_P(U)(s)) = f(U)(t^x).$$

On the other hand,

$$\varphi_{P'}(U)(L(f)(U)(s)) = \varphi_{P'}(U)(\eta'_{\mathcal{U}}((f(U_i)(x_i))_{i \in I}))$$

is by definition the unique q such that $P'_{UU_i}(q) = f(U_i)(x_i)$, for all $i \in I$. Hence, to derive $q = f(U)(t^x)$, it is sufficient to verify

$$P'_{UU_i}(f(U)(t^x)) = f(U_i)(x_i).$$

Since $P'_{UU_i} \circ f(U) = f(U_i) \circ P_{UU_i}$, this follows directly from the assumption $P_{UU_i}(t^x) = x_i$. □

Making use of 5.4.1 and 5.4.4 we obtain:

5.4.6 Lemma. *For every $P \in {}_{\mathcal{C}}\mathcal{S}h(\mathcal{Q})$ we have the following commutative diagram*

$$\begin{tikzcd}
P \arrow[r, hook, "\zeta_P"] \arrow[rr, bend right, "\mathrm{id}_P"'] & LP \arrow[r, "\varphi_P"] \arrow[rr, bend left, "\mathrm{id}_{LP}"] & P \arrow[r, hook, "\zeta_P"] & LP.
\end{tikzcd}$$

Proof. Let $U \in \mathcal{Q}$. For all $x \in P(U)$,

$$(\varphi_P(U) \circ \zeta_P(U))(x) = \varphi_P(U)(\eta_{\mathcal{U}_0}(x))$$

is by definition the unique $t \in P(U)$ such that $t|_U = x$, thus $t = x$. Therefore, for all $U \in \mathcal{Q}$ we obtain $\varphi_P(U) \circ \zeta_P(U) = \mathrm{id}_{P(U)}$, i.e. $\varphi_P \circ \zeta_P = \mathrm{id}_P$. Since φ_P is an isomorphism of Q-sheaves, it follows that ζ_P is the inverse of φ_P. Hence it also follows that $\zeta_P \circ \varphi_P = \mathrm{id}_{LP}$. □

5.5 An adjoint pair

What really permits to call a the *Q-sheafification functor* is the fact that it is a left adjoint of the inclusion functor ${}_{\mathcal{C}}\mathcal{S}h(\mathcal{Q}) \hookrightarrow {}_{\mathcal{C}}\mathcal{P}(\mathcal{Q})$, i.e. that it is a reflector. In this section we prove that this is indeed the case, and we also state some of its consequences.

5.5.1 Theorem. *The functor $a : {}_{\mathcal{C}}\mathcal{P}(\mathcal{Q}) \longrightarrow {}_{\mathcal{C}}\mathcal{S}h(\mathcal{Q})$ is a left adjoint of the inclusion functor $i : {}_{\mathcal{C}}\mathcal{S}h(\mathcal{Q}) \longrightarrow {}_{\mathcal{C}}\mathcal{P}(\mathcal{Q})$.*

Proof. In view of proposition 1.1.18 it is sufficient to check the following points:

i) the existence of a natural transformation $\phi : \mathrm{id}_{{}_{\mathcal{C}}\mathcal{P}(\mathcal{Q})} \longrightarrow i \circ a$;

ii) the existence of a natural equivalence $\psi : a \circ i \longrightarrow \mathrm{id}_{{}_{\mathcal{C}}\mathcal{S}h(\mathcal{Q})}$;

iii) the commutativity of the following diagram, for all $P \in {}_{\mathcal{C}}\mathcal{S}h(\mathcal{Q})$,

$$\begin{tikzcd} iP \arrow[r, "\phi_{iP}"] \arrow[rr, bend right, "\mathrm{id}_{iP}"'] & iaiP \arrow[r, "i\psi_P"] & iP; \end{tikzcd} \tag{5.14}$$

iv) the commutativity of the following diagram, for all $P \in {}_{\mathcal{C}}\mathcal{P}(\mathcal{Q})$,

$$\begin{tikzcd} aP \arrow[r, "a(\phi_P)"] \arrow[rr, bend right, "\mathrm{id}_{aP}"'] & aiaP \arrow[r, "\psi_{aP}"] & aP. \end{tikzcd} \tag{5.15}$$

First of all, in view of 5.4.2, we may define the natural transformation ϕ to be given by the family of morphisms of Q-presheaves $\phi_P = \zeta_{LP} \circ \zeta_P$, for all $P \in {}_{\mathcal{C}}\mathcal{P}(\mathcal{Q})$,

$$P \xrightarrow{\zeta_P} LP \xrightarrow{\zeta_{LP}} aP = i(aP).$$

Thus, for every morphism $f : P \to P'$ of Q-presheaves, we easily derive $a(f) \circ (\zeta_{LP} \circ \zeta_P) = (\zeta_{LP'} \circ \zeta_{P'}) \circ f$ since the following diagram is commutative, being a composition of two commutative squares.

$$\begin{tikzcd}
P \arrow[r, "\zeta_P"] \arrow[d, "f"'] & LP \arrow[r, "\zeta_{LP}"] \arrow[d, "L(f)"] & L(LP) \arrow[d, "L(L(f))"] \\
P' \arrow[r, "\zeta_{P'}"'] & LP' \arrow[r, "\zeta_{LP'}"'] & L(LP')
\end{tikzcd}$$

Secondly, by 5.4.5, we may define the natural equivalence ψ by the family of isomorphisms of Q-sheaves $\varphi_P \circ \varphi_{LP}$, for all $P \in {}_{\mathcal{C}}\mathcal{S}h(\mathcal{Q})$,

$$\begin{tikzcd}
aiP = L(LP) \arrow[r, "\varphi_{LP}"] \arrow[rr, bend right, "\psi_P"'] & LP \arrow[r, "\varphi_P"] & P.
\end{tikzcd}$$

Moreover, for every morphism $f : P \to P'$ of Q-sheaves the following diagram is also commutative (being the composition of two commutative squares):

$$\begin{tikzcd}
L(LP) \arrow[r, "\varphi_{LP}"] \arrow[d, "L(L(f))"'] & LP \arrow[r, "\varphi_P"] \arrow[d, "L(f)"] & P \arrow[d, "f"] \\
L(LP') \arrow[r, "\varphi_{LP'}"'] & LP' \arrow[r, "\varphi_{P'}"'] & P'
\end{tikzcd}$$

Therefore, $f \circ (\varphi_P \circ \varphi_{LP}) = (\varphi_{P'} \circ \varphi_{LP'}) \circ a(f)$.
Thirdly, let $P \in {}_{\mathcal{C}}\mathcal{S}h(\mathcal{Q})$ and let us check the commutativity of diagram (5.14). Making use of lemma 5.4.6 it easily follows that

$$\begin{aligned} i\psi_P \circ \phi_{iP} &= (\varphi_P \circ \varphi_{LP}) \circ (\zeta_{LP} \circ \zeta_P) = \varphi_P \circ (\varphi_{LP} \circ \zeta_{LP}) \circ \zeta_P \\ &= \varphi_P \circ \mathrm{id}_{LP} \circ \zeta_P = \varphi_P \circ \zeta_P = \mathrm{id}_P = \mathrm{id}_{iP}. \end{aligned}$$

Finally, it only remains to verify the commutativity of diagram (5.15). Let $P \in {}_{\mathcal{C}}\mathcal{P}(\mathcal{Q})$. Since a is a functor, we obtain

$$a(\phi_P) = a(\zeta_{LP} \circ \zeta_P) = a(\zeta_{LP}) \circ a(\zeta_P).$$

But, in proposition 5.4.3 we proved that $L(\zeta_P) = \zeta_{LP}$, so

$$a(\zeta_P) = L(L(\zeta_P)) = L(\zeta_{LP}) = \zeta_{aP}.$$

In a similar way, $a(\zeta_{LP}) = \zeta_{a(LP)} = \zeta_{L(aP)}$ which is indeed equal to ζ_{aP} since $\varphi_{aP} : L(aP) \longrightarrow aP$ is an isomorphism (recall 5.4.4). Hence, we conclude that $a(\phi_P) = (\zeta_{aP})^2$. On the other hand,

$$\psi_{aP} = \varphi_{aP} \circ \varphi_{L(aP)} = (\varphi_{aP})^2.$$

Consequently, $\psi_{aP} \circ a(\phi_P) = \mathrm{id}_{aP}$, since in 5.4.6 we checked for every Q-sheaf P' that $\zeta_{P'}$ is the inverse of $\varphi_{P'}$, and this holds in particular for the Q-sheaf aP. □

From the very definition of an adjoint pair (cf. 1.1.16) it now follows:

5.5.2 Corollary. *For all $P_1 \in {}_{\mathcal{C}}\mathcal{P}(\mathcal{Q})$ and all $P_2 \in {}_{\mathcal{C}}\mathcal{S}h(\mathcal{Q})$ there exists a canonical bijection*

$$\mathrm{Hom}_{\,{}_{\mathcal{C}}\mathcal{S}h(\mathcal{Q})}(aP_1, P_2) \longleftrightarrow \mathrm{Hom}_{\,{}_{\mathcal{C}}\mathcal{P}(\mathcal{Q})}(P_1, iP_2),$$

which assigns to every $f : aP_1 \to P_2$ the morphism $i(f) \circ \phi_{P_1}$, and whose inverse assigns $\psi_{P_2} \circ a(g)$ to every $g : P_1 \to iP_2$.

5.5.3 Remark. The functor a is exact for being a reflector (see 1.3.2).

5.5.4 Corollary. *Let $\mathcal{C}$ be a Grothendieck category. The category ${}_{\mathcal{C}}\mathcal{S}h(\mathcal{Q})$ of Q-sheaves on $\mathcal{C}$ is a Grothendieck category.*

Proof. The functor a is left exact and a left adjoint of the inclusion functor ${}_{\mathcal{C}}\mathcal{S}h(\mathcal{Q}) \hookrightarrow {}_{\mathcal{C}}\mathcal{P}(\mathcal{Q})$. Therefore, ${}_{\mathcal{C}}\mathcal{S}h(\mathcal{Q})$ is a Giraud subcategory of the Grothendieck category ${}_{\mathcal{C}}\mathcal{P}(\mathcal{Q})$, whence a Grothendieck category itself. □

5.5.5 Remark. Let P be a Q-presheaf. In view of the proof of theorem 5.5.1 we may also assert:

i) if P is separated then the morphism of Q-presheaves $\phi_P : P \longrightarrow iaP$ is a monomorphism (being the composition of the monomorphisms ζ_P and ζ_{LP}, cf. 5.4.1);

ii) if P is a Q-sheaf then $\psi_P : aiP \xrightarrow{\sim} P$ is an isomorphism;

iii) if P is a Q-sheaf then ϕ_{iP} and $i\psi_P$ are isomorphisms of Q-presheaves;

iv) $a(\phi_P)$ and ψ_{aP} are isomorphisms of Q-sheaves.

5.6 Relation between S_Q and a

Let $\mathcal{C}$ be an arbitrary Grothendieck category.

In section 4.2 we defined the category of Q-sheaves as the quotient category ${}_{\mathcal{C}}\mathcal{S}(\mathcal{Q}) = {}_{\mathcal{C}}\mathcal{P}(\mathcal{Q})(\tau_{S_Q})$, where S_Q denotes the Q-sheafification functor

$$S_Q : {}_{\mathcal{C}}\mathcal{P}(\mathcal{Q}) \longrightarrow {}_{\mathcal{C}}\mathcal{S}(\mathcal{Q}).$$

On the other hand, in this chapter we have defined the category ${}_{\mathcal{C}}\mathcal{S}h(\mathcal{Q})$ of Q-sheaves and a Q-sheafification functor denoted by

$$a : {}_{\mathcal{C}}\mathcal{P}(\mathcal{Q}) \longrightarrow {}_{\mathcal{C}}\mathcal{S}h(\mathcal{Q}).$$

Thus, the obvious question we propose to solve in this section is whether ${}_{\mathcal{C}}\mathcal{S}h(\mathcal{Q})$ and a respectively coincide with ${}_{\mathcal{C}}\mathcal{S}(\mathcal{Q})$ and $S_{\mathcal{Q}}$. It appears, as we will see (cf. 5.6.6), that the answer is affirmative.
First of all, let us point out that

$${}_{\mathcal{C}}\mathcal{S}h(\mathcal{Q}) = \{aP \mid P \in {}_{\mathcal{C}}\mathcal{P}(\mathcal{Q})\}.$$

Indeed, for every $P \in {}_{\mathcal{C}}\mathcal{P}(\mathcal{Q})$ we obtain $aP \in {}_{\mathcal{C}}\mathcal{S}h(\mathcal{Q})$ (cf. 5.2.7), and conversely, for every Q-sheaf P we proved in 5.5.1 that $\psi_P : aiP \xrightarrow{\sim} P$.
On the other hand, taking into account (4.1),

$${}_{\mathcal{C}}\mathcal{S}(\mathcal{Q}) = \{SP \mid P \in {}_{\mathcal{C}}\mathcal{P}(\mathcal{Q})\}.$$

Hence, to show that ${}_{\mathcal{C}}\mathcal{S}(\mathcal{Q})$ and ${}_{\mathcal{C}}\mathcal{S}h(\mathcal{Q})$ are isomorphic, it is sufficient to prove for every $P \in {}_{\mathcal{C}}\mathcal{P}(\mathcal{Q})$ that $aP = S_{\mathcal{Q}}P$, up to canonical isomorphism.

Once again, we make use of the theory of localization in Grothendieck categories as the main tool in order to achieve our aim:

5.6.1 The torsion theory associated to a.

We proved in theorem 5.5.1 that the functor a is a (right exact) reflector. It is also left exact (cf. proposition 5.3.4), hence to a corresponds the hereditary torsion theory

$$\begin{aligned} \mathcal{T}_a &= \{P \in {}_{\mathcal{C}}\mathcal{P}(\mathcal{Q}) \mid aP = 0\}; \\ \mathcal{F}_a &= \{P \in {}_{\mathcal{C}}\mathcal{P}(\mathcal{Q}) \mid \phi_P : P \hookrightarrow iaP \text{ is a monomorphism}\}, \end{aligned}$$

whose associated radical $\tau_a \in K({}_{\mathcal{C}}\mathcal{P}(\mathcal{Q}))$ is defined on any Q-presheaf P by

$$\tau_a P = \sum_{P' \subseteq P, aP' = 0} P'.$$

5.6.2 Proposition. $\mathcal{F}_a = {}_{\mathcal{C}}\mathcal{F}(\mathcal{Q})$.

Proof. In view of 5.5.5 i), from the very definition of $\mathcal{F}_a$ it follows that it contains all the separated Q-presheaves. Conversely, if $P \in \mathcal{F}_a$, then let us verify that P is separated, i.e. that for every $U \in \mathcal{Q}$ and every Q-covering $\{U_j\}_{j \in J}$ of U, the map

$$\xi : P(U) \longrightarrow \prod_{j \in J} P(U_j); \quad s \mapsto (P_{UU_j}(s))_{j \in J}$$

is injective.
First of all we consider (as in the proof of 4.2.4) the sub-Q-presheaf $P' \subseteq P$ given by

$$P'(V) = \begin{cases} P_{UV}(\operatorname{Ker}\xi), & \text{if } V \leq U; \\ 0, & \text{otherwise,} \end{cases}$$

with restriction morphisms

$$P'_{VW} = P_{VW}|_{P'(V)} = \begin{cases} P_{VW}|_{P_{UV}(\operatorname{Ker}\xi)}, & \text{if } W \leq V \leq U; \\ 0, & \text{otherwise.} \end{cases}$$

Thus, to be able to assert that $\operatorname{Ker}\xi = 0$, it is sufficient to check that $P' = 0$ (since $\operatorname{Ker}\xi = P'(U)$). Moreover, since $\mathcal{F}_a$ is closed under taking subobjects, we may assert that $P' \in \mathcal{F}_a$, i.e. that $\phi_{P'} : P' \hookrightarrow iaP'$ is a monomorphism. Therefore, it is sufficient to prove that $LP' = 0$.
So, let $V \in \mathcal{Q}$ and $t \in LP'(V)$, and let us check that $t = 0$: there exists $\mathcal{V} = \{V_i\}_{i \in I} \in \operatorname{Cov}(V)$ such that $t = \eta_{\mathcal{V}}(x)$ for some $x = (x_i)_{i \in I} \in \varprojlim P\mathcal{V}$. Let K be the set of indices $I_1 \sqcup (I_2 \times J)$, where $I_1 = \{i \in I \mid V_i \not\leq U\}$ and $I_2 = I - I_1$, and let us consider the set

$$\mathcal{V}' = \{V'_k\}_{k \in K} = \{V_{i_1}\}_{i_1 \in I_1} \;\cup\; \{V_{i_2} \,\&\, U_j\}_{i_2 \in I_2, j \in J}.$$

We claim that $\mathcal{V}' \preceq \mathcal{V}$. Indeed, for all $i_1 \in I_1$ (resp. $i_2 \in I_2, j \in J$) we obviously have $V_{i_1} = V \,\&\, V_{i_1}$ (resp. $V_{i_2} \,\&\, U_j = V \,\&\, (V_{i_2} \,\&\, U_j)$), and by Q4),

$$\begin{aligned}(\bigvee_{i_1 \in I_1} V_{i_1}) \vee (\bigvee_{i_2 \in I_2, j \in J} (V_{i_2} \,\&\, U_j)) &= (\bigvee_{i_1 \in I_1} V_{i_1}) \vee (\bigvee_{i_2 \in I_2} (V_{i_2} \,\&\, (\bigvee_{j \in J} U_a))) \\ &= (\bigvee_{i_1 \in I_1} V_{i_1}) \vee (\bigvee_{i_2 \in I_2} (V_{i_2} \,\&\, U)),\end{aligned}$$

which is equal to

$$(\bigvee_{i_1 \in I_1} V_{i_1}) \vee (\bigvee_{i_2 \in I_2} V_{i_2}) = \bigvee_{i \in I} V_i = V,$$

since for all $i_2 \in I_2$ we have $V_{i_2} \,\&\, U = V_{i_2}$ by 3.5.2.6. Therefore $\mathcal{V}'$ is a Q-covering of V. Moreover, the map

$$\delta : K \longrightarrow I; \quad i_1 \mapsto i_1, \; (i_2, j) \mapsto i_2,$$

satisfies $V_{i_1} \leq V_{\delta(i_1)}$ and $V_{i_2} \,\&\, U_j \leq V_{\delta(i_2,j)}$. Hence, $\mathcal{V}' \leq \mathcal{V}$. Thus, to prove that $t = 0$, it is sufficient to check for all $k \in K$ that $x_{\delta(k)}|_{V'_k} = 0$ (cf. 5.2.4). Indeed, for all $i_1 \in I_1$ this is clear from the very definition of P', since $P'(V_{i_1}) = 0$ and $x_{i_1}|_{V_{i_1}} = x_{i_1} \in P'(V_{i_1})$. Finally, for all $i_2 \in I_2$ and $j \in J$

we have $x_{i_2} \in P'(V_{i_2}) = P_{UV_{i_2}}(\mathrm{Ker}\,\xi)$ (so there exists $s_{i_2} \in \mathrm{Ker}\,\xi$ such that $x_i = P_{UV_{i_2}}(s_{i_2})$) and $P'_{V_{i_2},V_{i_2}\,\&\,U_j} = P_{V_{i_2},V_{i_2}\,\&\,U_j}|_{P'(V_{i_2})}$. Hence,

$$P'_{V_{i_2},V_{i_2}\,\&\,U_j}(x_i) = P_{V_{i_2},V_{i_2}\,\&\,U_j}(P_{UV_{i_2}}(s_{i_2})) = P_{U,V_{i_2}\,\&\,U_j}(s_{i_2}),$$

which coincides with

$$P_{U_j,V_{i_2}\,\&\,U_j}(P_{UU_j}(s_{i_2})) = P_{U,V_{i_2}\,\&\,U_j}(0) = 0,$$

since $V_{i_2} \,\&\, U_j \leq U \,\&\, U_j \leq U$ (by 3.5.2.2 and 3.5.2.4) and $s_{i_2} \in \mathrm{Ker}\,\xi$. $\square$

5.6.3 Corollary. *Let $P \in {}_{\mathcal{C}}\mathcal{P}(\mathcal{Q})$, then $P \in \mathcal{T}_{S_{\mathcal{Q}}}$ if and only if $aP = 0$.*

Proof. We already know that ${}_{\mathcal{C}}\mathcal{F}(\mathcal{Q})$ is also the torsion-free class $\mathcal{F}_{S_{\mathcal{Q}}}$ associated to the radical $\tau_{S_{\mathcal{Q}}}$ (recall 4.2.6 for the particular case $T = \mathcal{Q}$). Thus, the previous theorem is equivalent to asserting that $\mathcal{T}_{S_{\mathcal{Q}}} = \mathcal{T}_a$, which is precisely the set $\{P \in {}_{\mathcal{C}}\mathcal{P}(\mathcal{Q}) \mid aP = 0\}$ (cf. 5.6.1). $\square$

5.6.4 Lemma. *LF is $\tau_{S_{\mathcal{Q}}}$-injective, for all $F \in {}_{\mathcal{C}}\mathcal{F}(\mathcal{Q})$.*

Proof. Let $P' \subseteq P$ in ${}_{\mathcal{C}}\mathcal{P}(\mathcal{Q})$ with $P/P' \in \mathcal{T}_{S_{\mathcal{Q}}}$ and $f : P' \to LF$, and let us prove that f extends to a morphism $\bar{f} : P \to LF$.
Since a is an exact functor (cf. 5.5.3) and $a(P/P') = 0$ by 5.6.3, from the exactness of the sequence

$$0 \to P' \overset{i}{\hookrightarrow} P \twoheadrightarrow P/P' \to 0$$

we obtain that $a(i)$ is an isomorphism. On the other hand, the following diagram is obviously commutative:

$$\begin{tikzcd}
LF \arrow[d, "\phi_{LF}"'] & P' \arrow[l, "f"'] \arrow[r, hook, "i"] \arrow[d, "\phi_{P'}"'] & P \arrow[d, "\phi_P"] \\
a(LF) & aP' \arrow[l, "a(f)"] \arrow[r, "\sim", "a(i)"'] & aP
\end{tikzcd}$$

Therefore, $\phi_{LF} \circ f = a(f) \circ (a(i))^{-1} \circ \phi_P \circ i$. Besides $LF \in {}_{\mathcal{C}}\mathcal{S}h(\mathcal{Q})$ by 5.2.6, whence ϕ_{LF} is also invertible (recall 5.5.5 iii)). Consequently, it is sufficient to take $\bar{f}$ as the composition $(\phi_{LF})^{-1} \circ a(f) \circ (a(i))^{-1} \circ \phi_P \circ i$. $\square$

5.6.5 Theorem. *For all $P \in {}_{\mathcal{C}}\mathcal{P}(\mathcal{Q})$ we have $aP = S_{\mathcal{Q}}P$, up to canonical isomorphism.*

Proof. Since $S_{\mathcal{Q}}P = E_{\tau_{S_{\mathcal{Q}}}}(P/\tau_{S_{\mathcal{Q}}}P)$, it is then sufficient to check the following points:

i) $P/\tau_{S_Q}P \hookrightarrow aP$ is an essential extension of Q-presheaves;

ii) aP is τ_{S_Q}-injective;

iii) $aP/(P/\tau_{S_Q}P) \in \mathcal{T}_{S_Q}$.

First, let P' be a nonzero sub-Q-presheaf of aP and let us check that the pullback $P' \times_{aP} (P/\tau_{S_Q}P)$ is nonzero. In view of the exactness of a (recall 5.5.3) we obtain

$$a(P/\tau_{S_Q}P) = aP/a(\tau_{S_Q}P),$$

which coincides with aP, since $a(\tau_{S_Q}P) = 0$ by 5.6.3 (as $\tau_{S_Q}P \in \mathcal{T}_{\tau_{S_Q}}$). Consequently, taking into account that a preserves pullbacks (cf. 5.3.4), if we suppose that $P' \times_{aP} (P/\tau_{S_Q}P) = 0$ then

$$aP' \times_{a(aP)} aP = 0.$$

From this we derive that $aP' = 0$, since $a(aP) \cong aP$ (by 5.5.5 ii)) and $aP' \times_{aP} aP = aP'$. Moreover, since P' is a sub-Q-presheaf of a (separated) sheaf, it is separated itself, so the morphism

$$\phi_{P'} : P' \longrightarrow iaP' = 0$$

is an injection (recall 5.5.5 i)). This yields that $P' = 0$, which is a contradiction that comes from having supposed that the above pullback was zero.

Secondly, aP is indeed τ_{S_Q}-injective by 5.6.4 since LP is separated (cf. 5.2.5). Finally, by 5.6.3, it only remains to verify that $a(aP/(P/\tau_{S_Q}P)) = 0$. Indeed,

$$a(aP/(P/\tau_{S_Q}P)) = a(aP)/a(P/\tau_{S_Q}P) = aP/aP = 0.$$

□

5.6.6 Corollary. *The category of Q-sheaves ${}_{\mathcal{C}}\mathcal{S}h(\mathcal{Q})$ is isomorphic to the quotient category ${}_{\mathcal{C}}\mathcal{S}(\mathcal{Q})$, and the Q-sheafification functor a is naturally equivalent to $S_{\mathcal{Q}} : {}_{\mathcal{C}}\mathcal{P}(\mathcal{Q}) \longrightarrow {}_{\mathcal{C}}\mathcal{S}(\mathcal{Q})$.*

5.7 Functoriality

In this section we study how the Q-sheafification behaves with respect to Q-sites, having given a strict morphism between their corresponding quantales. First of all, for each morphism f, we define a functor f_* which relates Q-presheaves among the corresponding Q-sites, called *direct image functor*. We prove that when f is strict then f_* restricts well to the full subcategories of separated Q-presheaves and Q-sheaves. Moreover, we see in 5.7.5 that the direct image functor commutes with the Q-sheafification functor when f is also a bijection.

Let $(\mathcal{Q}_1, \{\mathrm{Cov}(U)\}_{U \in \mathcal{Q}_1})$ and $(\mathcal{Q}_2, \{\mathrm{Cov}(A)\}_{A \in \mathcal{Q}_2})$ be two Q-sites and let us consider an arbitrary morphism of quantales $f : \mathcal{Q}_1 \to \mathcal{Q}_2$.

5.7.1 Definition. We define a functor, called the *direct image functor*,

$$f_* : {}_{\mathcal{C}}\mathcal{P}(\mathcal{Q}_2) \longrightarrow {}_{\mathcal{C}}\mathcal{P}(\mathcal{Q}_1),$$

by assigning to every $P \in {}_{\mathcal{C}}\mathcal{P}(\mathcal{Q}_2)$ the *direct image* $f_*P \in {}_{\mathcal{C}}\mathcal{P}(\mathcal{Q}_1)$, which is defined on every $U \in \mathcal{Q}_1$ by

$$(f_*P)(U) = P(f(U)),$$

with restriction morphisms when $V \leq U$ in $\mathcal{Q}_1$ given by

$$(f_*P)_{UV} = P_{f(U)f(V)}.$$

Note that $f(V) \leq f(U)$ by M2), since

$$f(U) = f(U \vee V) = f(U) \vee f(V) \geq f(V).$$

The *direct image* of a morphism of Q-presheaves $\alpha : P \to P'$ in $\mathcal{Q}_2$ is the Q-morphism $f_*(\alpha) : f_*P \to f_*P'$ in $\mathcal{Q}_1$ given by the family

$$\{f_*(\alpha)(U) = \alpha(f(U))\}_{U \in \mathcal{Q}_1}.$$

5.7.2 Remark. When $f : \mathcal{Q}_1 \to \mathcal{Q}_2$ is a strict, then for all $U \in \mathcal{Q}_1$ and $\{U_i\}_{i \in I} \in \mathrm{Cov}(U)$ we obtain $\{f(U_i)\}_{i \in I} \in \mathrm{Cov}(f(U))$. Indeed,

$$f(U) = f(\bigvee_{i \in I} U_i) = \bigvee_{i \in I} f(U_i)$$

by M2), and $f(U_i) = f(U \,\&\, U_i) = f(U) \,\&\, f(U_i)$ by M3').

5.7.3 Lemma. *If $f : \mathcal{Q}_1 \to \mathcal{Q}_2$ is strict then:*

1) *for all $P \in {}_{\mathcal{C}}\mathcal{F}(\mathcal{Q}_2)$, we have $f_*P \in {}_{\mathcal{C}}\mathcal{F}(\mathcal{Q}_1)$;*

2) *for all $P \in {}_{\mathcal{C}}\mathcal{S}(\mathcal{Q}_2)$, we have $f_*P \in {}_{\mathcal{C}}\mathcal{S}(\mathcal{Q}_1)$.*

Proof. Let $P \in {}_{\mathcal{C}}\mathcal{F}(\mathcal{Q}_2)$, $U \in \mathcal{Q}_1$, and $\{U_i\}_{i\in I} \in \mathrm{Cov}(U)$. If $s \in (f_*P)(U)$ satisfies for all $i \in I$ that $s|_{U_i} = 0$, then we directly derive that $s = 0$ taking into account that

$$s|_{U_i} = P_{f(U)f(U_i)}(s)$$

and that $\{f(U_i)\}_{i\in I} \in \mathrm{Cov}(f(U))$. Thus, f_*P is separated.
Now let us suppose that P also satisfies the gluing condition. If we have a family $\{s_i \in (f_*P)(U_i)\}_{i\in I}$ satisfying for all $i, j \in I$ that $s_i|_{U_i \,\&\, U_j} = s_j|_{U_i \,\&\, U_j}$, then the existence of $s \in (f_*P)(U_i)$ such that $s|_{U_i} = s_i$ follows directly from the very assumption, since

$$s_i|_{U_i \,\&\, U_j} = P_{f(U)f(U_i \,\&\, U_j)}(s_i) = P_{f(U),f(U_i)\,\&\,f(U_j)}(s_i)$$

and $\{f(U_i)\}_{i\in I} \in \mathrm{Cov}(f(U))$. Therefore, $f_*P \in {}_{\mathcal{C}}\mathcal{S}(\mathcal{Q}_1)$. □

5.7.4 Proposition. *Let* $(\mathcal{Q}_1, \{\mathrm{Cov}(U)\}_{U\in\mathcal{Q}_1})$ *and* $(\mathcal{Q}_2, \{\mathrm{Cov}(A)\}_{A\in\mathcal{Q}_2})$ *be two Q-sites and* $f : \mathcal{Q}_1 \to \mathcal{Q}_2$ *a strict morphism of quantales. Then, there exists a natural transformation*

$$\varphi : L_1 \circ f_* \longrightarrow f_* \circ L_2,$$

where L_1 *(resp.* L_2*) denotes the functor described in 5.3.1 considered on the Q-site* $\mathcal{Q}_1$ *(resp.* $\mathcal{Q}_2$*). Moreover, when* f *is bijective then we obtain a natural equivalence.*

Proof. First of all, let us prove for each $P \in {}_{\mathcal{C}}\mathcal{P}(\mathcal{Q}_2)$ the existence of a morphism $\varphi_P : L_1(f_*P) \longrightarrow f_*(L_2P)$ in ${}_{\mathcal{C}}\mathcal{F}(\mathcal{Q}_1)$:
let $U \in \mathcal{Q}_1$ and $\mathcal{U} = \{U_i\}_{i\in I} \in \mathrm{Cov}(U)$. The inverse system $(f_*P)\mathcal{U}$ consists of all the diagrams

$$\begin{tikzcd}
P(f(U_i)) \arrow[dr, "P_{f(U_i),f(U_i)\,\&\,f(U_j)}"] & \\
& P(f(U_i)\,\&\,f(U_j)), \\
P(f(U_j)) \arrow[ur, "P_{f(U_j),f(U_i)\,\&\,f(U_j)}"'] &
\end{tikzcd}$$

for all $i, j \in I$, i.e. it coincides with the inverse system $P\mathcal{A}$, where $\mathcal{A}$ denotes the Q-covering $\{f(U_i)\}_{i\in I} \in \mathrm{Cov}(f(U))$. Therefore, for all $\mathcal{U}' \preceq \mathcal{U}$ we obtain a commutative diagram

$$\begin{tikzcd}
L_1(f_*P)(U) \arrow[rr, dashed, "\varphi_P(U)"] & & L_2P(f(U)) \\
& \varprojlim (f_*P)\mathcal{U} = \varprojlim P\mathcal{A} \arrow[ul, "\eta_{\mathcal{U}}"] \arrow[ur, "\eta_{\mathcal{A}}"'] \arrow[d] & \\
& \varprojlim (f_*P)\mathcal{U}' = \varprojlim P\mathcal{A}' \arrow[uul, bend left, "\eta_{\mathcal{U}'}"] \arrow[uur, bend right, "\eta_{\mathcal{A}'}"'] &
\end{tikzcd}$$

where $\varphi_P(U)$ is given by the universal property of the direct limit as the morphism in $\mathcal{C}$ which assigns to any $t \in L_1(f_*P)(U)$ the element

$$\varphi_P(U)(t) = \eta_{\mathcal{A}}(x) \in L_2P(f(U)),$$

where $t = \eta_{\mathcal{U}}(x)$ for some $\mathcal{U} = \{U_i\}_{i\in I} \in \operatorname{Cov}(U)$, and $x \in \varprojlim (f_*P)\mathcal{A}$ with $\mathcal{A} = \{f(U_i)\}_{i\in I} \in \operatorname{Cov}(f(U))$.
Thus, the family

$$\{\varphi_P(U) : L_1(f_*P)(U) \longrightarrow f_*(L_2P)(U)\}_{U\in\mathcal{Q}_1}$$

of morphisms in $\mathcal{C}$ defines the morphism of separated Q-presheaves φ_P. Indeed, it is an exercise to verify for all $V \leq U$ in $\mathcal{Q}_1$ that the following diagram is commutative:

$$\begin{tikzcd}
L_1(f_*P)(U) \arrow[r, "\varphi_P(U)"] \arrow[d, "(L_1(f_*P))_{UV}"'] & L_2P(f(U)) \arrow[d, "(L_2P)_{f(U)f(V)}"] \\
L_1(f_*P)(V) \arrow[r, "\varphi_P(V)"'] & L_2P(f(V))
\end{tikzcd}$$

Moreover, the family

$$\{\varphi_P : L_1(f_*P) \longrightarrow f_*(L_2P)\}_{P\in {}_{\mathcal{C}}\mathcal{P}(\mathcal{Q}_2)}$$

defines a natural transformation φ. Indeed, if $\alpha : P \to P'$ is a morphism in ${}_{\mathcal{C}}\mathcal{P}(\mathcal{Q}_2)$ then $f_*(L_2(\alpha)) \circ \varphi_P = \varphi_{P'} \circ L_1(f_*(\alpha))$, i.e. for every $U \in \mathcal{Q}_1$ the square

$$\begin{tikzcd}
L_1(f_*P)(U) \arrow[r, "\varphi_P(U)"] \arrow[d, "L_1(f_*(\alpha))(U)"'] & f_*(L_2P)(U) \arrow[d, "(L_2(\alpha))(f(U))"] \\
L_1(f_*P')(U) \arrow[r, "\varphi_{P'}(U)"'] & f_*(L_2P')(U)
\end{tikzcd}$$

in $\mathcal{C}$ is commutative, as we verify in what follows:
let $t \in L_1(f_*P)(U)$. Then, there exists some $\mathcal{U} = \{U_i\}_{i\in I} \in \mathrm{Cov}(U)$ and $x = (x_i)_{i\in I} \in \varprojlim (f_*P)\mathcal{U}$ such that $t = \eta_{\mathcal{U}}(x)$. Let $\mathcal{A}$ denote the Q-covering $\{f(U_i)\}_{i\in I}$ of $f(U)$, then

$$\begin{aligned}(L_2(\alpha)(f(U)) \circ \varphi_P(U))(t) &= L_2(\alpha)(f(U))(\eta_{\mathcal{A}}(x)) \\ &= \eta'_{\mathcal{A}}((\alpha(f(U_i))(x_i))_{i\in I}).\end{aligned}$$

On the other hand,

$$\begin{aligned}(\varphi_{P'}(U) \circ L_1(f_*(\alpha))(U))(t) &= \varphi_{P'}(U)(\eta'_{\mathcal{U}}((\alpha(f(U_i))(x_i))_{i\in I})) \\ &= \eta'_{\mathcal{A}}((\alpha(f(U_i))(x_i))_{i\in I}).\end{aligned}$$

Finally, let us suppose that f is bijective. To be able to assert that φ is a natural equivalence it is sufficient to check for all $U \in \mathcal{Q}_1$ that $\varphi_P(U)$ is an isomorphism. Let $U \in \mathcal{Q}_1$:
if there exists $t = \eta_{\mathcal{U}}(x)$ and $t' = \eta_{\mathcal{U}'}(y)$ in $L_1(f_*P)(U)$, for some Q-coverings $\mathcal{U} = \{U_i\}_{i\in I}$ and $\mathcal{U}' = \{U'_j\}_{j\in J}$ of U, such that

$$\varphi_P(U)(t) = \varphi_P(U)(t'),$$

then there exists some $x = (x_i)_{i\in I} \in \varprojlim (f_*P)\mathcal{A}$ and $y = (y_j)_{j\in J} \in \varprojlim (f_*P)\mathcal{A}'$ such that

$$\eta_{\mathcal{A}}(x) = \eta_{\mathcal{A}'}(y),$$

where $\mathcal{A}$ and $\mathcal{A}'$ respectively denote the Q-coverings $\{f(U_i)\}_{i\in I}$ and $\{f(U'_j)\}_{j\in J}$ of $f(U)$. Consequently, there exists a sub-Q-covering $\mathcal{A}'' = \{A''_k\}_{k\in K}$ of $\mathcal{A}$ and $\mathcal{A}'$ given by maps $\delta_1 : I \to K$ and $\delta_2 : J \to K$ such that for all $k \in K$ we have

$$P_{f(U_{\delta_1(k)})A''_k}(x_{\delta_1(k)}) = P_{f(U'_{\delta_2(k)})A''_k}(y_{\delta_2(k)}).$$

Since f is bijective one may easily check that $\{f^{-1}(A''_k)\}_{k\in K}$ is a sub-Q-covering of $\mathcal{U}$ and $\mathcal{U}'$, so the previous equality is equivalent to

$$(f_*P)_{U_{\delta_1(k)}f^{-1}(A''_k)}(x_{\delta_1(k)}) = (f_*P)_{U'_{\delta_2(k)}f^{-1}(A''_k)}(y_{\delta_2(k)}).$$

This is sufficient to assert that $\eta_{\mathcal{U}}(x) = \eta_{\mathcal{U}'}(y)$, i.e. that $t = t'$. Therefore $\varphi_P(U)$ is injective.

On the other hand, if $s \in L_2P(f(U))$, then there exists some Q-covering $\mathcal{A} = \{A_i\}_{i\in I}$ of $f(U)$ and $x = (x_i)_{i\in I} \in \varprojlim P\mathcal{A}$ such that $s = \eta_{\mathcal{A}}(x)$. In this case, $s = \varphi_P(U)(t)$ where $t = \eta_{\mathcal{U}}(x)$ with $\mathcal{U} = \{f^{-1}(A_i)\}_{i\in I} \in \mathrm{Cov}(U)$. Therefore, $\varphi_P(U)$ is also surjective. $\square$

5.7.5 Corollary. *Let* $(\mathcal{Q}_1, \{\mathrm{Cov}(U)\}_{U \in \mathcal{Q}_1})$ *and* $(\mathcal{Q}_2, \{\mathrm{Cov}(A)\}_{A \in \mathcal{Q}_2})$ *be two Q-sites and* $f : \mathcal{Q}_1 \to \mathcal{Q}_2$ *a strict isomorphism of quantales. Then there exists a natural equivalence between the functor compositions*

$$\begin{tikzcd} {}_{\mathcal{C}}\mathcal{P}(\mathcal{Q}_2) \arrow[r, "f_*"] & {}_{\mathcal{C}}\mathcal{P}(\mathcal{Q}_1) \arrow[r, "a_1"] & {}_{\mathcal{C}}\mathcal{S}(\mathcal{Q}_1) \end{tikzcd}$$

and

$$\begin{tikzcd} {}_{\mathcal{C}}\mathcal{P}(\mathcal{Q}_2) \arrow[r, "a_2"] & {}_{\mathcal{C}}\mathcal{S}(\mathcal{Q}_1) \arrow[r, "f_*"] & {}_{\mathcal{C}}\mathcal{S}(\mathcal{Q}_1) \end{tikzcd}.$$

In particular, for all $P \in {}_{\mathcal{C}}\mathcal{P}(\mathcal{Q}_2)$, *we obtain the following isomorphism of Q-sheaves:*

$$a_1(f_*P) \cong f_*(a_2P).$$

Proof. Let $P \in {}_{\mathcal{C}}\mathcal{P}(\mathcal{Q}_2)$, then

$$a_1(f_*P) = L_1(L_1(f_*P)) \cong L_1(f_*(L_2P)) \cong f_*(L_2(L_2P)) = f_*(a_2P).$$

□

5.8 The category R-${}_{\mathcal{Q}}$Mod

Let R be a Q-sheaf of not necessarily commutative rings.

In section 2.2 we defined the category of R-pre-Modules on an arbitrary poset and proved that it is a Grothendieck category. In this section we will consider the Grothendieck category of R-pre-Modules on $\mathcal{Q}$; for this particular poset we will denote it by R-**pre**-${}_{\mathcal{Q}}$**Mod**.

We will see that the same aims already reached in the category ${}_{\mathcal{C}}\mathcal{P}(\mathcal{Q})$ may similarly be obtained in R-**pre**-${}_{\mathcal{Q}}$**Mod**, i.e. we may explicitly define a category of sheaves of left R-modules, construct a sheafification functor, and also prove that this is naturally equivalent to the one we categorically obtained in section 4.3 (when we R is flabby).

Although R-**pre**-${}_{\mathcal{Q}}$**Mod** is not a functor category like ${}_{\mathcal{C}}\mathcal{P}(\mathcal{Q})$, the *semilinearity* of the restriction morphisms of R-pre-${}_{\mathcal{Q}}$Modules allows that different left $R(U)$-module structures, for different $U \in \mathcal{Q}$, glue well in the cases we require it. This property together with the fact that categories of modules are Grothendieck categories, permit to mimic all the constructions and proofs of the previous sections. Thus, what we will do in this section is basically to summarize these constructions and results, reformulated in terms of the category of presheaves of left R-modules on $\mathcal{Q}$.

5.8.1 Definition. A *sheaf of left R-modules on* $\mathcal{Q}$, or shortly, an R-${}_{\mathcal{Q}}$*Module*, is a separated R-pre-Module M on $\mathcal{Q}$ which satisfies the *gluing condition* Sh2), or equivalently, such that for every open subset U and every Q-covering $\{U_i\}_{i\in I}$ of U, the sequence

$$0 \longrightarrow M(U) \xrightarrow{\xi} \prod_{i\in I} M(U_i) \xrightarrow{\theta} \prod_{(i,j)\in I\times I} M(U_i \,\&\, U_j)$$

is exact, where $\xi(s) = (s|_{U_i})_{i\in I}$, for every $s \in M(U)$, and θ is given for all $(s_i)_{i\in I} \in \prod_{i\in I} M(U_i)$ by $(s_i|_{U_i \,\&\, U_j} - s_j|_{U_i \,\&\, U_j})_{(i,j)\in I\times I}$.
We denote this full subcategory of R-**pre**-${}_{\mathcal{Q}}$**Mod** by R-${}_{\mathcal{Q}}$**Mod**.

For a given R-pre-${}_{\mathcal{Q}}$Module M we may construct another R-pre-${}_{\mathcal{Q}}$Module LM similarly in section 5.2:

5.8.2 The construction of LM**.**
Let $U \in \mathcal{Q}$. For every $\mathcal{U} = \{U_i\}_{i\in I} \in \mathrm{Cov}(U)$, the system $M\mathcal{U}$ which consists of all the diagrams

$$\begin{tikzcd}
M(U_i) \arrow[dr, "M_{U_i,U_i \,\&\, U_j}"] & \\
& M(U_i \,\&\, U_j), \\
M(U_j) \arrow[ur, "M_{U_j,U_i \,\&\, U_j}"'] &
\end{tikzcd}$$

for all $i, j \in I$, may be considered as an inverse system on $R(U)$-**mod**, just by taking in each $M(U_i)$ (resp. $M(U_i \,\&\, U_j)$) the left $R(U)$-module structure given by scalar restriction via R_{UU_i} (resp. $R_{U,U_i \,\&\, U_j}$).
Its inverse limit, $\varprojlim M\mathcal{U}$, is then a left $R(U)$-module and comes equipped with projection maps $\pi_i : \varprojlim M\mathcal{U} \longrightarrow M(U_i)$, for every $i \in I$, which are homomorphisms of left $R(U)$-modules.
In this way, we may also obtain a presheaf $M^U \in {}_{R(U)\text{-}\mathbf{mod}}\mathcal{P}(\mathrm{Cov}(U))$ following the construction done in 5.2.2, and even a left exact functor

$$F^U : R\text{-}\mathbf{pre}\text{-}{}_{\mathcal{Q}}\mathbf{Mod} \longrightarrow {}_{R(U)-\mathbf{mod}}\mathcal{P}(\mathrm{Cov}(U)),$$

as in 5.2.3.
Thus, we may define the R-pre-${}_{\mathcal{Q}}$Module LM given on every $U \in \mathcal{Q}$ by the left $R(U)$-module

$$LM(U) = \varinjlim_{\mathcal{U}\in \mathrm{Cov}(U)} (\varprojlim M\mathcal{U}),$$

which comes equipped with $R(U)$-homomorphisms $\eta_{\mathcal{U}} : \varprojlim M\mathcal{U} \to LM(U)$, for all $\mathcal{U} \in \mathrm{Cov}(U)$.
For every $V \leq U$ in $\mathcal{Q}$, in order to obtain the restriction morphisms

$$(LM)_{UV} : LM(U) \longrightarrow LM(V); \quad \eta_{\mathcal{U}}((x_i)_{i\in I}) \mapsto \eta_{\mathcal{V}}(f_I((x_i)_{i\in I})),$$

as in 5.2.4, for every $\mathcal{U} = \{U_i\}_{i\in I} \in \mathrm{Cov}(U)$ we must first consider in $\varprojlim M\mathcal{V}$ the left $R(U)$-module structure given by scalar restriction via R_{UV} in order to obtain f_I as a homomorphism of left $R(U)$-modules given by the universal property of $\varprojlim M\mathcal{U}$. Similarly $(LM)_{UV}$ is also a homomorphism of left $R(U)$-modules, considering in $LM(V)$ the structure given by scalar restriction via R_{UV}, i.e. $(LM)_{UV}$ is R_{UV}-semilinear.

We may also obtain similar results to 5.2.5 and 5.2.6 in this case, just taking into account that the structures *glue* well thanks to the semilinearity of each restriction morphism of an R-pre-${}_{\mathcal{Q}}$Module, in this way:

5.8.3 Theorem. *If $M \in R$-**pre**-${}_{\mathcal{Q}}$**Mod** then $L^2M = L(LM) \in R$-${}_{\mathcal{Q}}$**Mod**.*

5.8.4 The sheafification functor a_R in R-pre-${}_{\mathcal{Q}}$Mod.
Similarly as it is done in sections 5.3.1 and 5.3.3, we may obtain a reflector

$$a_R : R\text{-}\mathbf{pre}\text{-}{}_{\mathcal{Q}}\mathbf{Mod} \longrightarrow R\text{-}{}_{\mathcal{Q}}\mathbf{Mod},$$

given on every $M \in R$-**pre**-${}_{\mathcal{Q}}$**Mod** by the R-${}_{\mathcal{Q}}$Module L^2M; on every morphism of R-pre-${}_{\mathcal{Q}}$Modules $f : M \to N$, the morphism of R-${}_{\mathcal{Q}}$Modules $a_R(f)$ is given by a family of homomorphism of $R(U)$-modules,

$$\{a_R(f)(U) : a_RM(U) \longrightarrow a_RN(U)\}_{U\in\mathcal{Q}}.$$

Once we have this well defined functor we may check that it is indeed a left adjoint of the inclusion functor

$$i : R\text{-}{}_{\mathcal{Q}}\mathbf{Mod} \hookrightarrow R\text{-}\mathbf{pre}\text{-}{}_{\mathcal{Q}}\mathbf{Mod},$$

as it was the case for the Q-sheafification functor in ${}_{\mathcal{C}}\mathcal{P}(\mathcal{Q})$ (cf. section 5.5).

Therefore, the category $R_{\mathcal{Q}}$-**Mod** is a Giraud subcategory of the Grothendieck category R-**pre**-${}_{\mathcal{Q}}$**Mod**, whence a Grothendieck category itself.

Finally we may also prove that, when R is flabby, this functor is naturally equivalent to the sheafification functor S_R obtained in section 4.3 for an arbitrary poset with quasi-coverings satisfying (C). In order to do this we just need to translate to the present context what is done in section 5.6. We remark that to be able to check that the torsion-free classes associated to both reflectors coincide, we will need a flabby Q-presheaf of rings because in this proof we make use of a subpresheaf defined as in (4.2), which needs flabbiness to be well defined.

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Index

List of Symbols

$(E, \leq)$	poset, 1
$(L, \leq)$	lattice, 1
$\mathbf{Lat}$	category of lattices, 2
$\mathbf{ab}$	category of abelian groups, 3
$\mathbf{gr}$	category of graded abelian groups, 3
R-$\mathbf{mod}$	category of left R-modules, 3
$\mathbf{mod}$-R	category of right R-modules, 3
R-$\mathbf{gr}$	category of graded left R-modules, 4
$\mathbf{gr}$-R	category of graded right R-modules, 4
X	topological space, 4
P_{UV}	restriction morphism, 4
${}_{\mathbf{ab}}\mathcal{P}(X)$	category of presheaves of abelian groups on X, 4
${}_{\mathbf{ab}}\mathcal{P}(B_X)$	category of presheaves of abelian groups on a basis, 5
${}_{R\text{-}\mathbf{mod}}\mathcal{P}(X)$	category of presheaves of left R-modules on X, 5
${}_{R\text{-}\mathbf{mod}}\mathcal{P}(B_X)$	category of presheaves of left R-modules on a basis, 5
${}_{\mathbf{Sets}}\mathcal{P}(X)$	category of presheaves of sets on X, 5
${}_{\mathbf{Sets}}\mathcal{P}(B_X)$	category of presheaves of sets on a basis, 5
$O(X)$	set of open subsets, $8, 68$
$\mathcal{T}$	torsion class, 10
$\mathcal{F}$	torsion-free class, 10
σ	left exact radical, 11
$K(C)$	class of all left exact radicals in $\mathcal{C}$, 11
$\mathcal{T}_\sigma$	σ-torsion class, 12
$\mathcal{F}_\sigma$	σ-torsion-free class, 12
$\mathcal{C}(\sigma)$	quotient category of $\mathcal{C}$ at σ, 13
$\mathbf{mod}$-(R, σ)	quotient category of $\mathbf{mod}$-R at σ, 13

${}_{\mathbf{Sets}}\mathcal{S}(X)$, ${}_{\mathbf{Sets}}\mathcal{S}(B_X)$	category of sheaves of sets, 15
${}_{R\text{-}\mathbf{mod}}\mathcal{S}(X)$, ${}_{R\text{-}\mathbf{mod}}\mathcal{S}(B_X)$	category of sheaves of left R-modules, 16
P_x	stalk of P in x, 16
$\mathbb{E} = \ulcorner E, \pi, X \urcorner$	concrete sheaf or sheaf space, 17
$\Gamma(U,E)$	collection of sections over U, 17
$\Gamma(U,X)$	collection of global sections, 17
$CSh(X)$	category of concrete sheaves, 17
S'	sheafification functor on X, 19
${}_{R}CSh(X)$	category of concrete sheaves in R-$\mathbf{mod}$, 19
$E_\sigma(C)$	σ-injective hull of C, 24
$a_\sigma : \mathcal{C} \to \mathcal{C}(\sigma)$	letf adjoint of $\mathcal{C}(\sigma) \hookrightarrow \mathcal{C}$, 24
$a_\sigma C$	σ-injective hull of $C/\sigma C$, 24
Q_σ	localization functor at σ, 25
$Q_\sigma(C)$	localization of C at σ, 25
$j_{\sigma,C}$	localization morphism, 25
$K(R)$	class of all left exact radicals in R-$\mathbf{mod}$, 27
(R,σ)-$\mathbf{mod}$	quotient category of R-$\mathbf{mod}$ at σ, 27
$Q_\sigma(M)$	module of quotients of M at σ, 28
$Q_\sigma(R)$	ring of quotients of R at σ, 28
$\mathcal{L}(\sigma)$	Gabriel filter associated to σ, 31
$\mathcal{O}(X)$	category of open subsets of X, 36
$\mathcal{E}$	small category, 36
$\mathrm{Fun}(\mathcal{E},\mathcal{C})$	functor category, 36
${}_{\mathcal{C}}\mathcal{P}(E)$	category of presheaves on a poset, 37
P_{ab}	restriction morphism, 37
$s\vert_b$	restriction of s to b, 37
R-pre-$\mathbf{Mod}$	category of R-pre-Modules on a poset, 40
$\mathrm{Cov}(A)$	set of coverings of A, 58
$(\mathcal{E}, \{\mathrm{Cov}(A)\}_{A\in\mathcal{E}})$	site, 58
$\mathcal{G}$	set of Gabriel filters of R, 61
$(\mathbb{T}(\mathcal{G}), \{\mathrm{Cov}([\mathbf{H}])\}_{[\mathbf{H}]\in\mathbb{T}(\mathcal{G})})$	noncommutative site associated to $\mathcal{G}$, 62
Λ	noncummtutative topology, 63
$i_\wedge(\Lambda)$	set of idempotent elements, 64
$\mathcal{O}(\Lambda)$	category defined on Λ, 67
$(\mathcal{O}(\Lambda), \{\mathrm{Cov}(x)\}_{x\in\Lambda})$	noncommutative site, 67
$\&$	multiplication, 68
$(X, O(X), \&)$	quantum space, 68

$(\mathcal{Q}, \leq, \&)$	quantale, 71
$\mathrm{Cov}(U)$	set of Q-coverings of U, 73
$\mathcal{U}$	Q-covering of U, 74
$\preceq$	ordering in the set of Q-coverings, 74
$\mathcal{C}ov(U)$	category of Q-coverings of U, 74
$(\mathcal{Q}, \{\mathrm{Cov}(U)\}_{U \in \mathcal{Q}})$	Q-site, 74
${}_{\mathcal{C}}\mathcal{P}(\Lambda)$	category of noncommutative presheaves, 78
${}_{\mathcal{C}}\mathcal{P}(O(X))$	category of quantum presheaves, 78
${}_{\mathcal{C}}\mathcal{P}(\mathcal{Q})$	category of Q-presheaves, 78
T	poset with quasi-coverings, 79
$C(a)$	set of quasi-coverings of a, 79
${}_{\mathcal{C}}\mathcal{F}(T)$	category of separated presheaves, 79
${}_{\mathcal{C}}\mathcal{F}(\Lambda)$	category of noncommutative separated presheaves, 80
${}_{\mathcal{C}}\mathcal{F}(\mathcal{Q})$	category of separated Q-presheaves, 80
${}_{\mathcal{C}}\mathcal{F}(O(X))$	category of separated quantum presheaves, 80
τ_S	radical in $K({}_{\mathcal{C}}\mathcal{P}(T))$, 84
$\mathcal{T}_S$	torsion class associated to τ_S, 84
$\mathcal{F}_S$	torsion-free class associated to τ_S, 84
${}_{\mathcal{C}}\mathcal{S}(T)$	category of sheaves, 84
${}_{\mathcal{C}}\mathcal{S}(\Lambda)$	category of noncommutative sheaves, 85
${}_{\mathcal{C}}\mathcal{S}(\mathcal{Q})$	category of Q-sheaves, 85
${}_{\mathcal{C}}\mathcal{S}(O(X))$	category of quantum sheaves, 85
S	sheafification functor on T, 85
S_Λ	noncommutative sheafification functor , 85
$S_\mathcal{Q}$	Q-sheafification functor, 85
τ_R	radical in $K(R\text{-pre-Mod})$, 87
$\mathcal{T}_R$	torsion class associated to τ_R, 87
$\mathcal{F}_R$	torsion-free class associated to τ_R, 87
R-Mod	category of R-Modules on T, 87
S_R	sheafification functor in R-pre-Mod, 87
${}_{\mathcal{C}}\mathcal{S}h(\mathcal{Q})$	category of Q-sheaves, 90
$P\mathcal{U}$	inverse system associated to $\mathcal{U}$, 91
LP	Q-presheaf associated to P, 92
L	functor from ${}_{\mathcal{C}}\mathcal{P}(\mathcal{Q})$ to ${}_{\mathcal{C}}\mathcal{F}(\mathcal{Q})$, 100
a	Q-sheafification functor, 102
$\mathcal{T}_a$	torsion class associated to a, 110
$\mathcal{F}_a$	torsion-free class associated to a, 110

τ_a	radical associated to a, 110
R-pre-${}_Q$Mod	category of R-pre-Modules on a Q-site, 118
R-${}_Q$Mod	category of R-Modules on a Q-site, 119
LM	R-pre-${}_Q$Module associated to M, 119
a_R	sheafification functor in R-pre-${}_Q$Mod, 120