University of La Laguna

Study of pseudocircular orbits in binary systems

Author:
Alejandro López Morales

Supervisor:
Ignacio González Martínez-Pais

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Abstract

A large number of stars are found in binary systems, which makes them of wide interest, in addition, the study of the external orbits of these systems has not been a highly developed field, which opens the door to obtaining results about planetary orbits in binary systems that have not been analyzed before. The study focuses on the development and study of the restricted problem of the three bodies, as well as the study of the problem of the two bodies in the corotating system, based on Newtonian physics and focusing especially on pseudo-circular, simple and periodic orbits.

Through the equations of motion, and the theoretical development of terms such as Poincaré's theorem, Jacobi's integral, or stability, a large number of orbits have been calculated and studied. A numerical method of equation solving is developed based on the Runge - Kutta method, which is particularized for the restricted problem of the three bodies, and through it, the equations of system motion are integrated. From these equations results have been obtained as the minimum radius for which it is possible to generate external orbits in binary systems, families of orbits that can be generated by modifying parameters such as the mass or angular velocity.

Contents

1 Objectives

In the universe there is a high percentage of stars that belong to binary systems and therefore the study of them is a very important subject. It should also be noted that these systems are little studied, and therefore it is possible to obtain very interesting results in their study. Our study will focus on external object orbits with particular characteristics, including pseudo-circular orbits, simple (not cutting the orbit itself at any point) and external orbits in the orbital plane (plane in which the orbit is located).

2 Restricted three-body problem

2.1 Introduction

The theoretical development has been based on the book of Ignacio González ("Introducción a la Mecánica Celeste" [1])Be a two-particle system with masses m_1 and $m_2 \leq m_1$ moving around their common center of mass under its gravitational pull with circular orbits. It is known as the restricted three- body problem or Roche's problem, the study of the motion of a third particle of negligible mass with respect to m_2 (therefore its mass does not affect the orbit of the two other bodies).

In this chapter we will make a theoretical introduction to the problem of the three bodies, from a Newtonian physics and studying the particular cases of the movement of two bodies. In addition, important concepts such as Jacobi's interal or Poincaré's theorem will be developed, which will be important for the development of the work.

2.2 Equations of motion

To describe the motion, we will choose an inertial reference system (IRS), S' centered at the mass center of the system, with the plane $X'Y'$ coinciding with the orbit plane and Z' in the direction of the orbital angular momentum of the system. Another rotating system (RS) will also be considered, co-rotating with the binary system also centered at the center of mass with $Z \equiv Z'$ and with the X axis in the line that joins the two finite mass components and pointing towards the less massive component.

We start from the analysis of the position vector of each of the bodies in the orbital plane:

Being a the separation between two masses

$$\left(-\frac{m_2}{m_1+m_2}a, 0, 0\right), \qquad \left(\frac{m_1}{m_1+m_2}a, 0, 0\right) \tag{2.1}$$

Be **R** the position vector of the infinitesimal body in the system S and **R'** the same vector but in system S'. Knowing that the rotating system S rotates relative to S', Kepler's third law can be written as $\Omega^2 = G(m_1+m_2)/a^3$ we can obtain the equation of motion:

$$\ddot{\mathbf{R}}' = -\frac{Gm_1}{{R_1'}^3}\mathbf{R_1'} - \frac{Gm_2}{{R_2'}^3}\mathbf{R_2'} = \mathbf{f} \tag{2.2}$$

The force acting on the particle measured in S' is conservative:

$$\mathbf{f} = \nabla\psi' \tag{2.3}$$

with:

$$\psi' = -\frac{Gm_1}{R_1'} - \frac{Gm_2}{R_2'} \tag{2.4}$$

The centrifugal potential ψ_c is defined below and it has been added to the ψ' potential to give the total potential function in the system S:

$$\psi = \psi' + \psi_c = -\frac{Gm_1}{R_1} - \frac{Gm_2}{R_2} - \frac{1}{2}\Omega^2 P^2 \tag{2.5}$$

P being the projection on the orbital plane of the position vector of the particle

The equation of motion in S:

$$\ddot{\mathbf{R}} = -\frac{Gm_1}{R_1^3}\mathbf{R_1} - \frac{Gm_2}{R_2^3}\mathbf{R_2} - \Omega \times (\Omega \times \mathbf{R}) - 2\Omega \times \dot{\mathbf{R}} \tag{2.6}$$

can be written as:

$$\ddot{\mathbf{R}} + 2\Omega \times \dot{\mathbf{R}} = -\nabla\psi \tag{2.7}$$

Once we have this equation we are interested in making it adimensional; for this reason we multiply (2.5) by $a/(G(m_1 + m_2))$, define the normalized position vectors as:

$$\mathbf{r} = \frac{\mathbf{R}}{a}, \qquad \rho = \frac{\mathbf{P}}{a}, \qquad \mathbf{r_i} = \frac{\mathbf{R_i}}{a}, \qquad \tau = \Omega t \tag{2.8}$$

and define the parameter μ:

$$\mu = \frac{m_2}{m_1 + m_2} \tag{2.9}$$

we can finally define the Roche potential as:

$$\psi_R \equiv -\frac{1-\mu}{r_1} - \frac{\mu}{r_2} - \frac{1}{2}\rho^2 \tag{2.10}$$

obtaining from (2.7) the equation of motion in the co-rotating system:

$$\ddot{\mathbf{r}} + 2\mathbf{k} \times \dot{\mathbf{r}} = -\nabla\psi_R \tag{2.11}$$

where the derivatives are defined by

$$\ddot{\mathbf{r}} = \frac{d^2 r}{d\tau^2}, \qquad \dot{\mathbf{r}} = \frac{dr}{d\tau} \tag{2.12}$$

2.3 The Jacobi integral

It is possible to obtain one integral of motion.Let's consider an infinitesimal displacement of the particle $d\mathbf{r} = \dot{\mathbf{r}}dt$. If (2.11) is multiplied by the displacement vector:

$$\ddot{\mathbf{r}} \cdot d\mathbf{r} + 2(\mathbf{k} \times \dot{\mathbf{r}}) \cdot d\mathbf{r} = \frac{d\dot{\mathbf{r}}}{dt}\dot{\mathbf{r}} \cdot dt = \frac{1}{2}d(\dot{\mathbf{r}}^2) \tag{2.13}$$

$$\nabla\psi_R \cdot d\mathbf{r} = d\psi_R \tag{2.14}$$

that is:

$$\frac{1}{2}d(v^2) = -d\psi_R \tag{2.15}$$

integrating:

$$v^2 = -2\psi_R - C \tag{2.16}$$

with C the Jacobi constant.

This constant is similar to the energy, but it is defined in a no inertial reference system. The energy is conserved in a inertial reference system, but not the one we're working on. On the other hand, this energy that represents the Jacobi constant is conserved in the co-rotating system.

2.4 Orbital motion in polar coordinates

The study of the orbits will be performed in the orbital plane of the system, so, we will use the polar coordinates (r, θ) (polar coordinates are studied in the orbital plane of the binary system, the centre of mass being the origin of the polar coordinate system) obtaining from the expression (2.11) the components of the equation of motion:

$$\ddot{r} - r\dot{\theta}^2 - 2r\dot{\theta} = -\frac{\partial\psi_R}{\partial r} \tag{2.17}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} + 2\dot{r} = -\frac{1}{r}\frac{\partial\psi_R}{\partial\theta} \tag{2.18}$$

which will be very useful for later sections.

2.5 Coordinate system centered on the center of mass of the system

In this case we have to define both components as:

$$r_1^2 = r^2 + \mu^2 + 2\mu r\cos\theta, \qquad r_2^2 = r^2 + (1-\mu)^2 - 2(1-\mu)r\cos\theta, \qquad \rho = r \tag{2.19}$$

obtaining the equations of motion:

$$\ddot{r} - r\ddot{\theta}^2 - 2r\dot{\theta} - r = -\frac{1-\mu}{r_1^3}(r+\mu\cos\theta) - \frac{\mu}{r_2^3}[r-(1-\mu)\cos\theta] \tag{2.20}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} + 2\dot{r} = \mu(1-\mu)(\frac{1}{r_1^3} - \frac{1}{r_2^3})\sin\theta \tag{2.21}$$

being the Jacobi integral:

$$v^2 = 2\frac{1-\mu}{r_1} + 2\frac{\mu}{r_2} + r^2 - C \tag{2.22}$$

2.6 The problem of the two bodies in a rotating system

We are interested in obtaining pseudocircular orbits in the restricted three body problem, i.e simple (and therefore periodic) closed orbits (i.e., such that when θ increases 2π, the movement repeats itself) . And among them we are interested in those that in the limit when $\mu \to 0$ become the circular orbits of the problem of the two bodies (Szebehely, Victor "Theory of orbits: The restricted problem of three bodies" [2]). We're interested in what, in addition, the orbits are external, i.e. around the center of mass $1-\mu$. In this section we will see that these orbits exist, and that they can be obtained as an analytical continuation of the orbits of the problem of the two bodies. In the following chapters we will see methods to obtain these orbits based on the results of this one.

2.6.1 The problem of the two bodies in a rotating system

In this subsection we are going to study the case $\mu = 0$, that is the problem of the two bodies in a reference system that rotates with angular velocity $\Omega = 1$

In particular, we want to study the orbits that are closed in the co-rotating reference system (CRS). This problem is solved and is well known in an inertial reference system (IRS).

We will define as w the average movement of a periodical orbit, $w = \frac{2\pi}{T}$ being T the orbital period. The sub-indice i will be used to indicate that this is the measured motion in the inertial system.

For an orbit to be closed in the CRS, it must be closed in the IRS, i.e. elliptical or circular. However, this condition is not enough, as there may be orbits that are closed in the IRS but not closed in the CRS. Specifically, for an elliptical orbit in the IRS to be closed (and therefore periodic) in the CRS, it must be fulfilled that when giving the system an integer number n of full cycles, the particle gives an integer number m of cycles in its orbit in the IRS, i.e. if T_i is the period of the orbit in the IRS, it must be verified:

$$mT_i = n2\pi \tag{2.23}$$

with n and m integers and n positive. The last expression can be writen as:

$$w_i = \frac{m}{n} \tag{2.24}$$

The period of the corresponding orbit in the CRS is mT_i , however, although in the IRS the particle has travelled in that time an angle of $2\pi m$, in the CRS it will have travelled $2\pi m - 2\pi n$, and therefore the mean measured mean motion in that system will be:

$$w = \frac{2\pi m - 2\pi n}{mT_i} = w_i \frac{m-n}{m} \tag{2.25}$$

On the other hand, any orbit that is internal ($a < 1$, with a the semi-major axis of the ellipse described in the IRS) in one of the two reference systems, is also the same in the other, and the same happens if it is external ($a > 1$). So, from Kepler's third law:

$$w_i^2 a^3 = 1 \rightarrow w_i = a^{-3/2} \tag{2.26}$$

so we arrive to the conclusion that for internal orbits: $|w_i| > 1$, and for external orbits: $|w_i| < 1$

in this way it is possible to construct a table in which the different orbits are represented as much as possible in the co-rotating and inertial system:

IRS		CRS	
$w_i > 1$	Direct Internal	w >0	Direct Internal
$w_i < -1$	Internal Retrograde	w <-2	Internal Retrograde
$0 < w_i < 1$	Direct External	-1 <w <0	Direct External
$-1 < w_i < 0$	External Retrograde	-2 <w <-1	External Retrograde

the Jacobi integral (2.22), which has the form:

$$v^2 = \frac{2}{r} + r^2 - C \tag{2.27}$$

with v the speed of the particle in the CRS. The zero velocity surfaces, $v^2 = 0$, which limit the regions where the particle can move ($C \leq 2/r + r^2$), from those forbidden because in them it would be $v^2 < 0$ ($C > 2/r + r^2$) has the equation:

$$r^3 - Cr + 2 = 0. \tag{2.28}$$

There are two possible cases to solve this equation:

1. For $C < 3$, the equation has not real positive solutions, so there are no sign changes of v^2 at any point. So the particle can move around the entire space, so we're not interested in this case (we are interested in closed orbits).

2. For $C \geq 3$, the equation has two real positive roots r_m and r_M, such that $0 < r_m < 1 < r_M < C$ (in the extreme case $C = 3$, $r_m = r_M = 1$). In this case the motion is possible inside the radius $r = r_m$ and outside $r = r_M$. The first case corresponds to internal orbits and the second to orbits around the system as a whole, that is, external.

Therefore, we will restrict ourselves to the case where $C > 3$.

The relationship between the velocity $\vec{V}$ of a particle in the IRS and the velocity $\vec{v}$ in the SRC is:

$$\vec{v} = \vec{V} - \vec{\Omega} \times \vec{r} \tag{2.29}$$

with $\vec{\Omega} = \vec{k}$

Lets consider a particle in the IRS that follows a elliptical orbit. Its velocity in the pericenter of the orbit measured at the IRS is strictly azimuthal ($\dot{r} = 0$) and is:

$$V_Q = \pm[\frac{1+e}{a(1-e)}]^{1/2} \tag{2.30}$$

with the top sign corresponding to direct orbits and the bottom to retrograde orbits.

We will now study the orbit in the rotating system, to do this we will calculate the velocity of the partıcle in this system by means of (2.29) the speed of the system as it passes through Q (the pericenter):

$$v_Q = V_Q - r_Q = \pm[\frac{1+e}{a(1-e)}]^{1/2} - a(1-e) \tag{2.31}$$

combining this expression with the Jacobi's integral expression evaluated in $r = r_Q$:

$$v_Q^2 = \frac{2}{a(1-e)} + a^2(1-e)^2 - C \tag{2.32}$$

it results:

$$\pm a(1-e^2)^{1/2} = \frac{C}{2}a^{1/2} - \frac{1}{2a^{1/2}} \tag{2.33}$$

in term of the minor semiaxis of the ellipse: $b = a(1-e^2)^{1/2}$:

$$\pm b = \frac{C}{2}a^{1/2} - \frac{1}{2a^{1/2}} \tag{2.34}$$

b will always be positive and lower than a, so $0 \leq b \leq a$. Two expression can be obtained from (2.34), one with positive sign, that corresponds with direct orbits, and the other one with negative

sign, that corresponds with retrograde orbits. Each of these cases gives us a relationship between a and b, for a given value of Jacobi's constant C. The cut of the curve is:

$$f(a) = \frac{C}{2}a^{1/2} - \frac{1}{2a^{1/2}} \tag{2.35}$$

the straights $g_+(a) = a$ and $g_-(a) = -a$ give us the possible circular orbits for a given value of C, and elliptical orbits can exist if $|f(a)| \pm a$, that is to say, in the parts of the curve below of $g_+(a)$ and above of $g_-(a)$. In the figure (1) the function $f(a)$ is represented for $C > 3$ values. $f(a)$ is always increasing and convex and is characterized by:

$$\lim_{a\to 0} f(a) = -\inf, \quad \lim_{a\to \inf} = \inf, \quad f(\frac{1}{C}) = 0, \quad f(1) = \frac{C-1}{2} > 1 \tag{2.36}$$

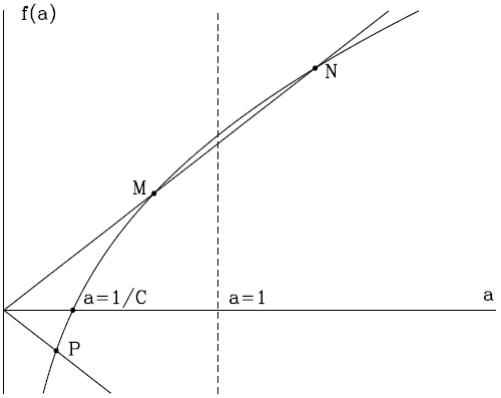


Figure 1: Relation between the major and minor semiaxis of a elliptical orbit in a IRS, for a given value of Jacobi's constant. N and M correspond to direct circular orbits in the IRS. P corresponds with a an internal retrograde orbit in the IRS.

For a given value $C > 3$, there are always three circular orbits, which can be represented according to their radii r_1 and r_2:

$$C = \frac{1}{r_2} + 2r_2^{1/2} \tag{2.37}$$

$$C = \frac{1}{r_1} - 2r_1^{1/2} \tag{2.38}$$

being (2.37) for direct orbits in the IRS, and (2.38) for retrograde orbits in the IRS. This expressions has been obtained through making the change $b = a = r_2$ and $-b = a = r_1$

2.7 Analytical continuation for $\mu \neq 0$

The equations (2.20) and (2.21) can be writen as:

$$\ddot{r} - r\ddot{\theta}^2 - 2r\dot{\theta} - r = -\frac{1}{r^2} + f(\mu, r, \theta) \tag{2.39}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} + 2\dot{r} = g(\mu, r, \theta) \tag{2.40}$$

being f(μ, r, θ) and g(μ, r, θ) such that:

$$f(0, r\theta) = g(0, r, \theta) = 0 \tag{2.41}$$

also the expression of the integral of Jacobi can be written as:

$$v^2 = r^2 + \frac{2}{r} - C + h(\mu, r, \theta) \tag{2.42}$$

It is also important to note that for $\mu = 0$ that is to say $h(0, r, \theta) = 0$, we have the equations of the problem of the two bodies in the rotating system, of which we know that for any value of C such that $C > 3$, there are three circular orbits: one internal, another internal, and a third, also retrograde, but external. The initial conditions for circular orbits are:

$$\begin{array}{c} r(0) = r_0 \\ \theta(0) = 0 \\ \dot{r}(0) = 0 \\ \dot{\theta}(0) = \frac{1}{r_0^{3}/2} - 1 \end{array} \tag{2.43}$$

being r_0 related to the constant of Jacobi by means of the expression:

$$C = \frac{1}{r_0} \pm 2r_0^{1/2} \tag{2.44}$$

depending on the sign of whether the orbit is direct (+) or retrograde (-). if we write this equation in the CRS of the circular orbits:

$$r = R_0(t) = r_0, \qquad \theta = \Theta_0 = (w_i - 1)t = wt \tag{2.45}$$

with the subindex 0 indicating that they are orbits for the case $\mu = 0$.

We had previously defined the average angular velocity as $w_i = 2\pi/T_i$ in the IRS, which can be written in this case as $w_i = r_0^{-3/2}$.

If we consider the case for $\mu \neq 0$ the general solution can be written :

$$r = R(\mu, r(0), \theta(0), \dot{r}(0), \dot{\theta}(0), t); \qquad \theta = \Theta(\mu, r(0), \theta(0), \dot{r}(0), \dot{\theta}(0), t) \tag{2.46}$$

Our interest now lies in asking ourselves how the initial conditions that creates circular orbits for two-body problems behave for values of $\mu \neq 0$. Remembering, the orbits we are interested in are closed, simple and symmetrical with respect to the polar axis, because Roche's potential goes from having spherical symmetry for $\mu = 0$ to being only axis symmetrical for the polar axis for $\mu \neq 0$. Therefore, for the last case, the orbits have to satisfy the initial conditions:

$$\theta(0) = 0, \qquad \dot{r}(0) = 0 \tag{2.47}$$

so the equation of the orbit (2.46) can now be written as:

$$r = R(\mu, r(0), \dot{\theta}(0), t); \qquad \theta = \Theta(\mu, r(0), \dot{\theta}(0), t) \tag{2.48}$$

On the other hand, as they are periodic, it must be verified that for a certain instant $t = T + \tau$ is complied as:

$$\theta(T + \tau) = 2\pi, \qquad \dot{r}(T + \tau) = \dot{r}(0) = 0 \tag{2.49}$$

We have expressed the period as $T + \tau$ instead of as T because we will take T as the period of the circular orbit in which the searched orbit becomes when $\mu \to 0$. In addition, we demand that:

$$\lim_{\mu \to 0} R(\mu, r(0), \dot{\theta}(0), t) = R_0(t); \qquad \lim_{\mu \to 0} \Theta(\mu, r(0), \dot{\theta}(0), t) = \Theta_0(t) \tag{2.50}$$

If such a solution exists, it will be the analytical continuation for $\mu \neq 0$ of the circular reference orbit, described above, for the case $\mu = 0$. Poincaré studied this problem from a more general point of view in "New Methods of Celestial Mechanics" [3]

$$\dot{\vec{x}} = \vec{X}(\mu, \vec{x}); \qquad \vec{\xi} = \vec{x}(0) \tag{2.51}$$

Let us suppose that for $\mu = 0$ there are periodic solutions, of period T:

$$\vec{x} = \vec{\phi}(t), \qquad \vec{\phi}(T) = \vec{\phi}(0) = \vec{\xi} \tag{2.52}$$

In the example of having a case with $\mu \neq 0$:

$$\vec{x}(0) = \vec{\phi}(0) + \vec{\beta} \tag{2.53}$$

With a very similar period to the case of $\mu = 0$ but different:

$$\vec{x}(T + \tau) = \vec{\phi}(0) + \vec{\beta} + \vec{\Psi} \tag{2.54}$$

We want to know if for a $\mu \neq 0$ there is a choice of $\vec{\beta}$ and τ that creates a simple orbit, that is to say:

$$\vec{\Psi}(\mu, \vec{\beta}, \tau) = \vec{0} \tag{2.55}$$

For a given value from μ, this expression is a system of n equations with $n + 1$ variables: $\vec{\beta}, \tau$. Therefore, there will be periodic orbits if there are relationships $\vec{\beta} = \vec{\beta}(\mu, \tau)$ that satisfy .

Poincaré showed that if Ψ_i are analytical functions of $\mu, \vec{\beta}$ and τ, a sufficient condition for there to be functions $\vec{\beta} = \vec{\beta}(\mu, \tau)$ to verify (2.55) for small values of μ is that not all the determinants of the matrix:

$$\begin{pmatrix} \frac{\partial \Psi_1}{\partial \beta_1} & \frac{\partial \Psi_1}{\partial \beta_2} & \cdots & \frac{\partial \Psi_1}{\partial \beta_n} & \frac{\partial \Psi_1}{\partial \tau} \\ \frac{\partial \Psi_2}{\partial \beta_1} & \frac{\partial \Psi_2}{\partial \beta_2} & \cdots & \frac{\partial \Psi_2}{\partial \beta_n} & \frac{\partial \Psi_1}{\partial \tau} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \Psi_n}{\partial \beta_1} & \frac{\partial \Psi_n}{\partial \beta_2} & \cdots & \frac{\partial \Psi_n}{\partial \beta_n} & \frac{\partial \Psi_n}{\partial \tau} \end{pmatrix} \tag{2.56}$$

are simultaneously null for $\mu = \tau = 0$ and $\vec{\beta} = \vec{0}$. In this case there are periodic solutions of (2.51)

2.7.1 Poincaré theorem in the restricted three body problem

The restricted problem of the three bodies can comply with the Poincaré theorem. First of all, thera are periodic circular orbits in the case of $\mu = 0$, as we have already seen for $C > 3$. The conditions (2.49) and (2.55) can be expressed as:

$$\Psi_1(\mu, r_0, \dot{\theta_0}, \tau) = 0 \qquad \Psi_2(\mu, r_0, \dot{\theta_0}, \tau) = 0 \tag{2.57}$$

with $r_0 = r(0)$ and $\dot{\theta}_0 = \dot{\theta}(0)$ and being:

$$\Psi_1(\mu, r_0, \dot{\theta_0}, \tau) = (\frac{dr}{dr})_{t=T+\tau}; \qquad \Psi_1(\mu, r_0, \dot{\theta_0}, \tau) = \theta(T + \tau) - 2\pi \tag{2.58}$$

Poincaré's theorem allows us to ensure that periodic solutions exist if the following condition is met:

$$\begin{vmatrix} \frac{\partial \Psi_1}{\partial t} & \frac{\partial \Psi_1}{\partial \dot{\theta_0}} \\ \frac{\partial \Psi_2}{\partial t} & \frac{\partial \Psi_2}{\partial \dot{\theta_0}} \end{vmatrix} \neq 0 \tag{2.59}$$

for $\mu = 0$, that is, with the calculated determinant for the two-body problem. It can be shown that this condition is met by Sharma R.K. [4]

2.8 Summary of the section (Spanish)

En este capítulo se ha hecho una introducción teórica del problema restringido de los tres cuerpos. Se han definido unas ecuaciones de movimiento en un sistema de referencia inercial (SRI) centrado en el centro de masas del sistema sobre el cual se han descrito los vectores de posición de las dos masas (o componentes), llegando a la primera ecuación de movimiento (2.2). Tras esto se ha descrito una fuerza conservativa en una partícula en el sistema, teniendo en cuenta los potenciales que actuan en el sistema, obteniendo la expresión (2.7). A partir de este punto se define el parámetro μ y se deriva la ecuación de movimiento en el sistema corrotante (2.11). Esta ecuación de movimiento se desplaza infinitisimalmente, obteniendo así la expresión de la constante de Jacobi. A partir de este momento se estudia esta constante en el (SRI) y se definen las coordenadas polares para el problema.

Tras esto se hace un profundo estudio del problema de los dos cuerpos, llegando a resultados que nos dan información del tipo de órbita que se puede obtener en función del movimiento medio del sistema (2.25). Tras esto y una derivación de la constante de Jacobi en este problema, se llega a varias expresiones que nos dan la forma de la constante de Jacobi en el caso de órbitas circulares ecuaciones (2.37) y (2.38).

Una vez con estos resultados se estudia la continuidad analítica del problema para $\mu \neq 0$, y finalmente se acaba definiendo el Teorema de Poincaré y la aplicación del mismo al problema restringido de los tres cuerpos.

3 Numerical integration

3.1 Introduction

As described in the previous chapter, there are some systems of differential equations that cannot be solved by any analytical method, therefore it is necessary to resort to a numerical integration method that allows us to obtain useful results. The problem in our work is that it is necessary to solve the system of equations for each orbit individually, which is a significant computational burden. In addition, being a solution of a system of differential equations, this is treated as a problem of initial values, so we will only obtain particular orbit.

It can also be said that as with any numerical method, we will not obtain exact results, but approximate ones, but we will be able to increase the precision by varying the integration step.

3.2 Runge-Kutta method

Our problem (restricted three body problem) can't be resolved by an analitycal method, so we need a numerical method allowing us to integrate the equations of motion.We will use the Runge-Kutta method cause of his efficiency and simplicity.

In what follows, we are going to describe the Runge-Kutta method:

Be the system of N equations of first order:

$$\begin{aligned} \dot{y_1} &= f_1(t, y_1, y_2, ..., y_N) \\ \dot{y_2} &= f_2(t, y_1, y_2, ..., y_N) \\ &... \\ \dot{y_N} &= f_N(t, y_1, y_2, ..., y_N) \end{aligned} \tag{3.1}$$

with initial conditions:

$$y_{i,0} = y_i(t_0); \qquad i = 1, 2, ..., N \tag{3.2}$$

with $y_{i,0}$ and t_0 known parameters.

For simplicity we will write the above expression in vectorial notation:

$$\dot{\vec{y}} = \vec{f}(t, \vec{y}); \qquad \vec{y_0} = \vec{y}(t_0) \tag{3.3}$$

The Runge-Kutta method approximates the actual values of the solution $\vec{y}(t)$ evaluated at $t = t_1, t_2, ..., t_N$ $(\vec{y}(t_{n+1}))$ through others, that we will denote like $\vec{y_n}$, obtained from the value obtained in the previous step, applying the following secuence:

$$\vec{y_{n+1}} = \vec{y_n} + \frac{h}{6}(\vec{k_1} + 2\vec{k_2} + 2\vec{k_3} + \vec{k_4}) \tag{3.4}$$

being:

$$\begin{aligned}\vec{k_1} &= \vec{f}(t_n, \vec{y_n}) \\ \vec{k_2} &= \vec{f}(t_n + \tfrac{h}{2}, \vec{y_n} + \tfrac{h}{2}\vec{k_1}) \\ \vec{k_3} &= \vec{f}(t_n + \tfrac{h}{2}, \vec{y_n} + \tfrac{h}{2}\vec{k_2}) \\ \vec{k_4} &= \vec{f}(t_n + h, \vec{y_n} + h\vec{k_3})\end{aligned} \tag{3.5}$$

with h the step:

$$h = t_{n+1} - t_n \tag{3.6}$$

Applying this procedure to our equations (2.20) and (2.21) , we will get a system of four equations of first order (being v_θ the angular velocity of the particle in the IRS):

$$\begin{aligned}\dot{r} &= v_r \\ \dot{\theta} &= v_\theta - 1 \\ \dot{v_r} &= r_\theta^2 + f_r(r, \theta) \\ \dot{v_\theta} &= \tfrac{-2v_r v_\theta + f_\theta(r,\theta)}{r}\end{aligned} \tag{3.7}$$

with:

$$f_r(r, \theta) = -\frac{1-\mu}{r_1^3}(r + \mu - \cos\theta) - \frac{\mu}{r_3^3}[r - (1-\mu)\cos\theta] \tag{3.8}$$

$$f_\theta(r, \theta) = \mu(1-\mu)(\frac{1}{r_1^3} - \frac{1}{r_2^3})\sin\theta \tag{3.9}$$

and (2.19):

$$r_1 = (r^2 + \mu^2 + 2\mu\cos\theta)^{\frac{1}{2}} \tag{3.10}$$

$$r_2 = [r^2 + (1-\mu)^2 - 2(1-\mu)r\cos\theta]^{\frac{1}{2}} \tag{3.11}$$

3.3 Jacobi constant

The value of the Jacobi constant is a parameter that can be used to check that the orbits obtained are correct (an error in the constant of less than 0.01 % is accepted). Later, in the analysis of results, Jacobi's constant will be used to classify the orbits obtained.

The expression used to obtain the numerical values is a result obtained after the development of a restricted three-body system centered on the mass center of the system, and it appears explicitly in the Jacobi integral given by:

$$v^2 = 2\frac{1-\mu}{r_1} + 2\frac{\mu}{r_2} + r^2 - C \tag{3.12}$$

with C the Jacobi constant.

Figure 2 shows an orbit in Cartesian coordinates on the right, and on the left the evolution of Jacobi's constant with time. In this case, the variation of the Jacobi constant is approximately 0.0014 %

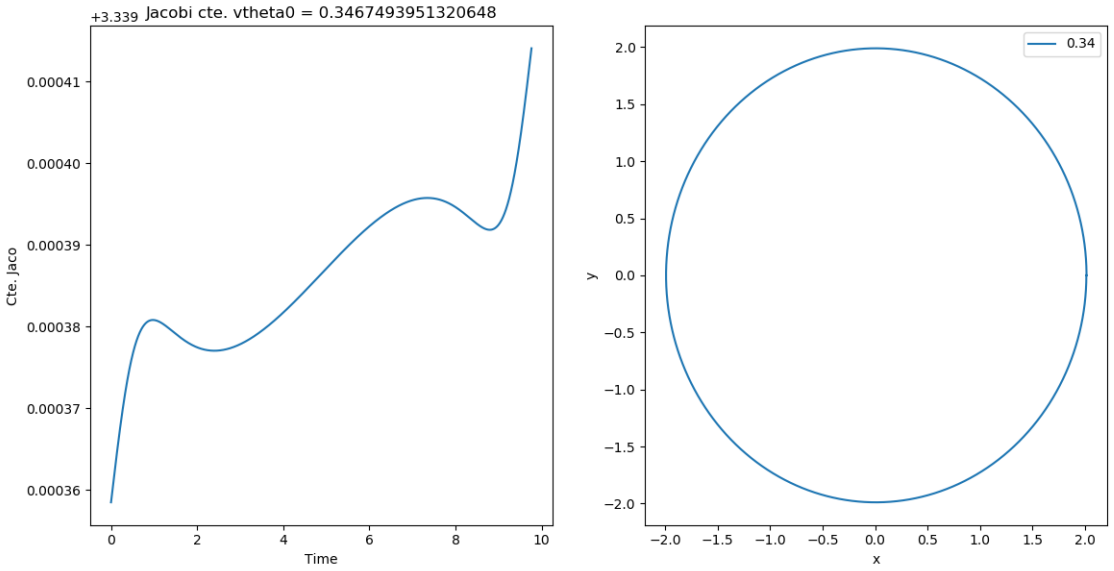


Figure 2: Orbit and Jacobi constant representation for values: $v_\theta(0) = 0.3467, r_0 = 2.013, \mu = 0.05$

In the figure 3 it can be seen a situation in which a lower variation of the constant is archieved, that in this case is 0.00065 %.

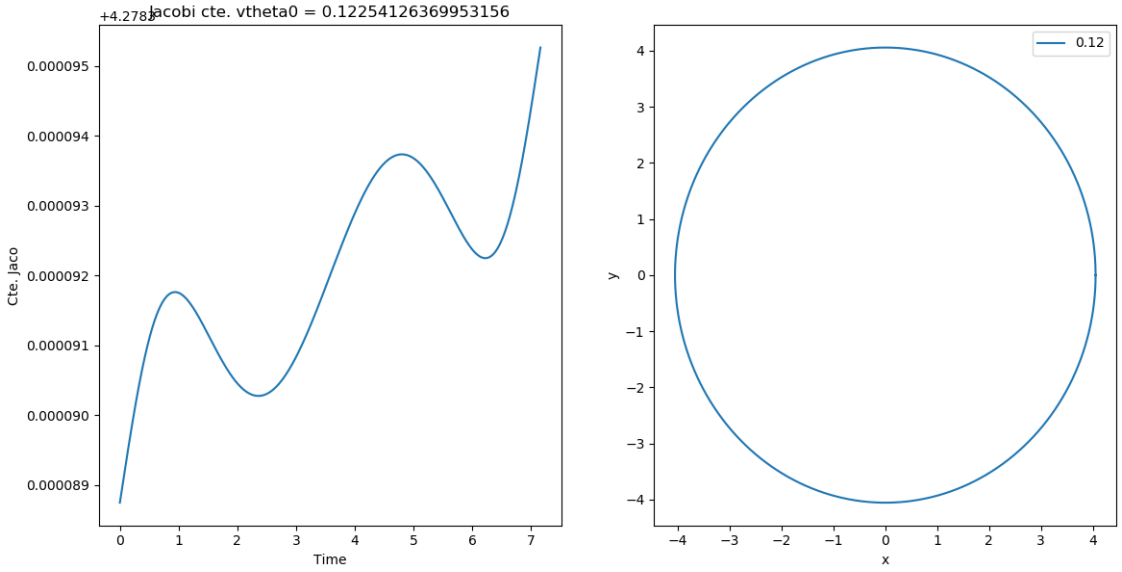


Figure 3: Orbit and Jacobi constant representation for values: $v_\theta(0) = 0.1225, r_0 = 4.055, \mu = 0.05$

3.4 Initial conditions

An important point that we need to talk about is the study of the initial conditions. Firstable we will take a measurement of the angles found in the lane that separates both components. We search for orbits are symmetric with respect to the polar axis since we are interested on pseudo-circular orbits, that is to say, simples and symmetric to that axis. For a given initial radius r_0 we have:

$$\theta(0) = 0; \qquad v_r(0) = 0 \tag{3.13}$$

The second of this conditions is necessary but a not sufficient for get a pseudo-circular orbit. In the order to get the proper orbit, we have to obtain a correct value of the initial velocity $v_\theta(0)$.

The initial value of v_θ will not be very different from v_c (the initial velocity needed to obtain a circular orbit in the two body problem), so we know that the correct value is around this. v_c is the initial velocity necessary for a two-body system to generate a circular orbit, which is given by the expression:

$$v_c = r(0)^{\frac{-3}{2}} \tag{3.14}$$

in the case of orbits around the center of mass.

With this value of v_c, the velocity $v_\theta(0)$ will be changed between $0.1v_c$ and $2v_c$ until we obtain the wanted orbit (one which is pseudo-circular, simple, periodic and symmetrical).

3.5 Poincaré maps

Using the method described above, we will never get a perfect orbit that closes with infinite precision, that's why we introduce the Poincaré maps.

It is a concept that will able us to determinate the perfect initial conditions for our problem and study the stability of the orbits.

Firstable we define the "Poincaré section":

Be a n-dimensional dynamical system:

$$\dot{\vec{x}} = f(\vec{x}) \tag{3.15}$$

Poincaré section is a subspace of dimension $n-1$, which is transversal to the flux, i.e, such that all the orbits of the system cross it.

Let's consider any orbit that at a certain moment cuts the section at the point $\vec{x_0}$. The next few times it cuts it moving in the same direction it passes through the points $\vec{x_1}, \vec{x}, \vec{x_3}$.... The Poincaré map P is a map or a transformation defined as:

$$P : \Sigma \to \Sigma \tag{3.16}$$

So that:

$$\vec{x}_{i+1} = P(\vec{x_i}) \tag{3.17}$$

that is, it transforms the point $\vec{x_i}$ into $\vec{x_{i+1}}$

On this kind of map, the orbits that meet $P(\vec{x}) = \vec{x}$ are periodical simple orbits. Periodic orbits will comply that $P^m(\vec{x}) = \vec{x}$, because after m steps through the section, the orbit repeats. Consequently, in general, each orbit is represented on a Poincaré map by a set of points. If this point is single, the orbit is periodic and simple, if the number of points is finite, the orbit is periodic and not simple, and if the orbit is not periodic, it will be given by an infinite number of points.

In our case, we are interested in building a Poincaré map whose section is the axis connecting the two components, i.e $\theta = 0$. Since the system has dimension 4, the Poincaré section will have dimension 3 but since we have $r = r_0$ and $\dot{r}(0) = v_r(0) = 0$, the actual dimension will be 2 $(r, \dot{\theta})$

To calculate the Poincaré map, it will be necessary to integrate the equation of motion from $(\theta = 0, r = r_0)$ with $\dot{r_0} = 0$ to $\theta = 2\pi$, trying different $\dot{\theta_0}$ values, and check if the final $(\theta = 2\pi)$ obtained r value matches with r_0. If it matches, we have the correct orbit, if not, we will have to try with another $\dot{\theta_0}$ value.

It's very important that the orbit is calculated very accurately, because we want it to close completely, this means that the initial radius r_0 is the same as the final r when $\theta = 2\pi$. By integrating with a time step h, it's impossible to close the orbit in all as the point is likely to be passed or shortened, thus having a different initial and final radius . Therefore, we will apply a new method with which when the value of θ passes 2π a perfect result can also be achieved (Henon M. [6]).

In order to obtain $r(\theta = 2\pi)$ once we have exceeded the value in $\theta = 2\pi$, deriving the iteration process, we return to the previous point, the last one for which $\theta = \theta_k < 2\pi$, and the system of equations is replaced by:

$$\begin{array}{l} \frac{dr}{d\theta} = \frac{\dot{r}}{\dot{\theta}} = \frac{v_r}{v_\theta - 1} \\ \frac{dv_r}{d\theta} = \frac{\dot{v}_r}{\dot{\theta}} = \frac{r v_\theta^2 + f_r(r,\theta)}{v_\theta - 1} \\ \frac{dv_\theta}{d\theta} = \frac{\dot{v}_\theta}{\dot{\theta}} = \frac{-2 v_r v_\theta + f_\theta(r,\theta)}{r(v_\theta - 1)} \\ \frac{dt}{d\theta} = \frac{1}{v_\theta - 1} \end{array} \tag{3.18}$$

That is, taking as an independent variable θ instead of t, and we solve this system, equivalent to the previous one, by taking a step $\Delta\theta = 2\pi - \theta_k$. This way, the last step will give exactly a point for which $\theta = 2\pi$.

3.6 Orbit stability

Once the periodical orbit has been found, it is also interesting to study whether it is a stable or not. There is a procedure that allows to do this study in a simple way and taking advantage of the properties of the Poincare map (Hénon M. [6])we will work in the usual rotating system, but in Cartesian coordinates. The method studies orbits close to the periodic one with the same value of Jacobi's constant using a theorem that says that "for a simple periodical orbit in the restricted circular problem of the three bodies to be stable, it is necessary and sufficient that the point associated with that orbit on the Poincare map be stable".

We express the transformation represented by the Poincare map as:

$$\begin{aligned} x_1 &= f(x_0, \dot{x}_0, C) \\ \dot{x}_1 &= g(x_0, \dot{x}_0, C) \end{aligned} \tag{3.19}$$

being C the Jacobi constant.

The equations of motion have the property that they remain invariant if the sign of y and the sign of time are changed at the same time, so:

$$\begin{aligned} x_0 &= f(x_1, -\dot{x}_1, C) \\ -\dot{x}_0 &= g(x_1, -\dot{x}_1, C) \end{aligned} \tag{3.20}$$

Be the representative point of the periodic orbit $x_1 = x_0, \dot{x}_1 = \dot{x}_0$. Let's consider now a close point to $(x_0, \dot{x}_0)$: $(x_0 + \Delta x_0, \dot{x}_0 + \Delta \dot{x}_0)$; his transformation will be $(x_1 + \Delta x_1, \dot{x}_1 + \Delta \dot{x}_1)$ with:

$$\begin{aligned} \Delta x_1 &= a\Delta x_0 + b\Delta \dot{x}_0 \\ \Delta \dot{x}_1 &= c\Delta x_0 + d\Delta \dot{x}_0 \end{aligned} \tag{3.21}$$

being:

$$a = \frac{\partial f}{\partial x}, \qquad b = \frac{\partial f}{\partial \dot{x}}, \qquad c = \frac{\partial g}{\partial x}, \qquad d = \frac{\partial g}{\partial \dot{x}} \tag{3.22}$$

The transformation has the property of conserving areas:

$$\frac{\partial(f, g)}{\partial(x_0, \dot{x}_0)} = 1, \tag{3.23}$$

meaning:

$$ad - bc = 1. \tag{3.24}$$

The orbits that interest us are symmetrical, therefore for them $\dot{x}_0 = \dot{x}_1 = 0$. Deriving (3.20):

$$\begin{aligned} \Delta x_0 &= a\Delta x_1 - b\Delta \dot{x}_1 \\ -\Delta \dot{x}_0 &= c\Delta x_1 - d\Delta \dot{x}_1 \end{aligned} \tag{3.25}$$

reversing (3.21):

$$\begin{aligned} \Delta x_0 &= d\Delta x_1 - b\Delta \dot{x}_1 \\ \Delta \dot{x}_0 &= -c\Delta x_1 + a\Delta \dot{x}_1 \end{aligned} \tag{3.26}$$

Comparing both system:

$$a = d \tag{3.27}$$

let's study the stability of the point. In order to do this, it is necessary to obtain the eigenvalues of the linear transformation 3.21: if its real part has an absolute value less than 1, the orbit is stable, and if it is greater than 1, it is unstable. Be $(\Delta x_0)_p$ and $(\Delta \dot{x}_0)_p$ the eigenvectors of the transformation, then:

$$\begin{array}{l}(\Delta x_1)_p = \lambda(\Delta x_0)_p \\ (\Delta \dot{x}_1)_p = \lambda(\Delta \dot{x}_0)_p\end{array} \tag{3.28}$$

and the eigenvalues will be given by the equation:

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0 \tag{3.29}$$

considering (3.24):

$$\lambda^2 - (a+d)\lambda + 1 = 0 \tag{3.30}$$

This equation has two real roots if $\mid a+d \mid > 2$, one of which has an absolute value greater than 1, whereas if $\mid a+b \mid < 2$, it has two complex roots, so that it's parts will have an absolute value of lees than 1. In the first case, the invariant point is unstable, and in the second it is stable.

Since in the case of a simple periodical orbit $a = b$, the stability condition for it is:

$$\mid a \mid < 1 \tag{3.31}$$

To obtain the numerical value of a, an orbit very close to the one studied aws calculated with a small Δx_0 and $\Delta \dot{x}_0$ null, and if Δx_1 is the deviation for the next intersection with the Poincaré section, we have:

$$a = \frac{\Delta x_1}{\Delta x_0} \tag{3.32}$$

In the case applied to our problem:

$$a = \frac{\Delta r_1}{\Delta r_0} \tag{3.33}$$

3.7 Summary of the section (Spanish)

En esta sección se hace un estudio general del método numérico usado para resolver nuestro sistema (ecuación (3.5)), el método de Runge-Kutta, y la aplicación del mismo para resolver las ecuaciones del problema restringido de los tres cuerpos (ecuación (3.7)).

Tras esto se profundiza un poco en la constante de Jacobi (C), así los como indicadores mediante los cuales podemos saber si las órbitas son válidas o no, como que la varianza máxima de la constante de Jacobi (C) debe tener un valor máximo del 0.01%. Además se representan dos ejemplos prácticos del mismo figura 2 y 3.

Las condiciones iniciales son necesarias para resolver nuestro problema numérico, y en el resto de la sección se estudia el cómo encontrarlas, con especial interés en las de la velocidad angular v_θ. Se presenta la ecuación de la velocidad inicial necesaria para que un sistema de dos cuerpos tenga órbitas circulares (3.14) y a partir de la misma se establece un rango de posibles valores, los cuales son dados por válidos tras aplicar la definición de los Mapas de Poincaré, y el sistema de ecuaciones que se define del mismo (3.18). A partir de los mapas de Poincaré se puede introducir el concepto de estabilidad de una órbita, usando el teorema: "Para que una órbita periódica simple en el problema circular restringido de los tres cuerpos sea estable, es necesario y suficiente que el punto asociado con esa órbita en el mapa de Poincare sea estable." Llegando tras un análisis de las condiciones iniciales y valores finales a que la estabilidad se da cuando se cumple la condición (3.31) en la ecuación (3.33)

4 Results

4.1 Choice of initial values

We calculated different orbits for μ values between 0 and 1, in step of 0.05, so we get a total of 19 mass values. For the radius, we have taken the values corresponding to the logarithm of r_0 equal to values from 0.1 to 2.1 with a step of 0.1, where r_0 is the initial radius. And finally the initial angular velocity has been obtained by the method defined in the section 3.5, using in particular the expression (3.14), which gives us the initial angular velocity to obtain a circular orbit and which is applicable to our problem because we are working with pseudo-circular orbits; the way this definition is applied is as follows: Values of the angular velocity will be taken from $0.2v_\theta$ to $1.5v_\theta$ with a step of $\frac{v_\theta}{50}$.

With these definitions of initial values we obtain a large number of orbits, many of which were not either closed or simple or periodic, that is to say, cases that have no interest in our study, therefore it is necessary to eliminate them. Orbits have been removed for cases where $\Delta\theta < 0$ at some point, which means that there is a retrose (there are loops or attempts at loops)

4.2 Obtaining of periodical and simple closed orbits

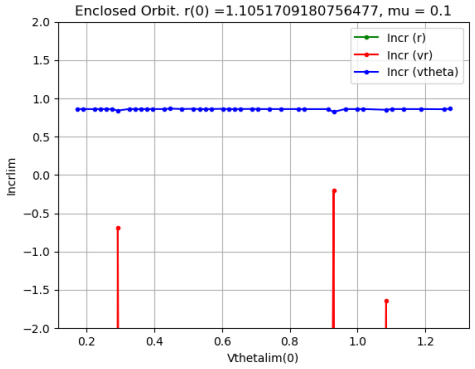
Once the orbits were calculated correctly, it is necessary to begin to rule out some types of orbits. A simple orbit is one in which the trajectory does not cut itself in the same cycle, i.e. that it has no ties or attempts at ties. A closed orbit could be described mathematically as one in which at the starting point, after a cycle the radius is the same and the condition that the angle is equal to 2π is already met thanks to the application of the Poincaré map. And to be periodic, the orbit must have the same velocity value, both angular and radial, which assures us that the orbit will repeat itself.

The method used to obtain the correct orbits has been to calculate for the same radius and the same mass all the possible orbits with all the possible initial angular velocities $v_\theta(0)$. Once calculated, a graph is displayed (figure 4) for each case in which the radius , radial velocity and angular velocity increments (the difference of the parameter values between $\theta = 2\pi$ and $\theta = 0$) are drawn. The interest of studying these increments is that for the orbits that interest us (pseudocircular, symmetric, simple and periodical), the final and initial radius must be equal (condition that the orbit is closed), and the velocities, both radial and angular, must be equal too for $\theta = 0$ and $\theta = 2\pi$, if this is true ($\Delta r = \Delta v_r = \Delta v_\theta = 0$), we could ensure that the orbits we are obtaining are both simple (there are no speed increases or decreases, so the orbit has no loops or irregularities) and periodic (if you have the same velocity and radius at the beginning and end, it means that the orbit will always be completed with the same period)

Therefore, the following criterion will be followed: there will be a single, closed and periodic orbit if the zeros of all the increments coincide at the same point.

In some cases there are some orbits with a large number of loops or imperfections, which causes the representations of the increments to present large jumps and divergences, which makes the graphs dirty in a significant way. It was necessary to eliminate points that are too far from the curve thus obtaining a cleaner and more homogeneous graphics. This is why the graphs of these increments show some abrupt changes, that is, there are points at irregular distances.

The following graphs (figure 4) are attached which, for different radius values, represent the increase in the parameters (incrlim) compared to the initial angular velocity ($v_\theta(0)$).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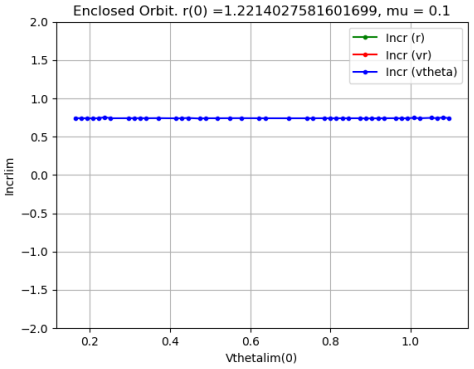
Incr (vr)
Incr (vtheta)
Incrlim
Vthetalim(0)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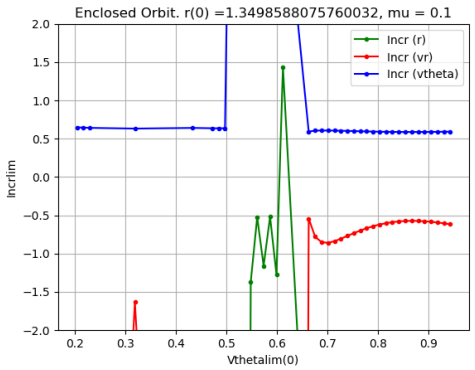
Vthetalim(0)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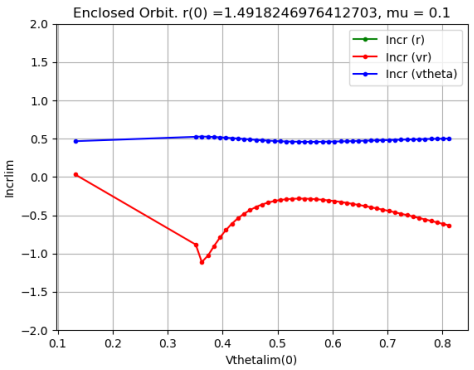
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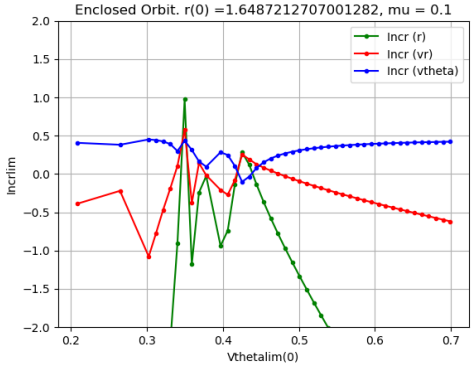
Incr (vtheta)
Incrlim
Vthetalim(0)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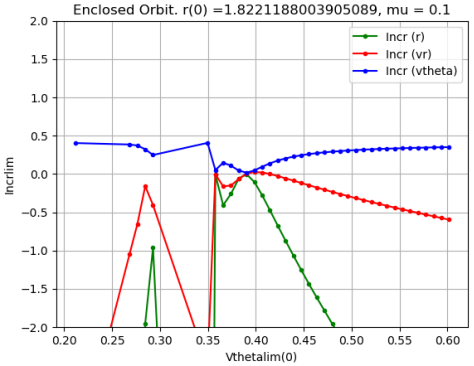
Incr (vr)
Incr (vtheta)
Incrlim
Vthetalim(0)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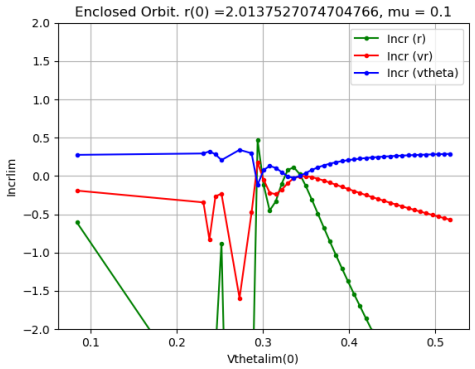
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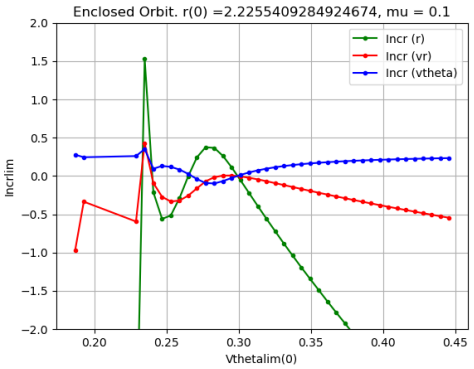
Incrlim
Vthetalim(0)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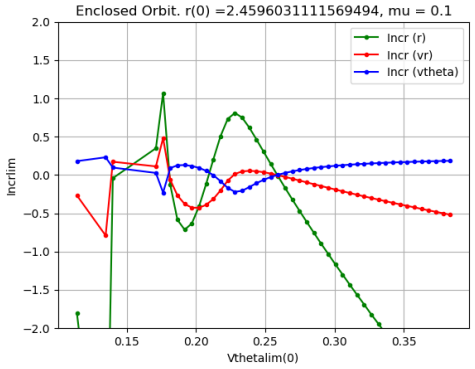
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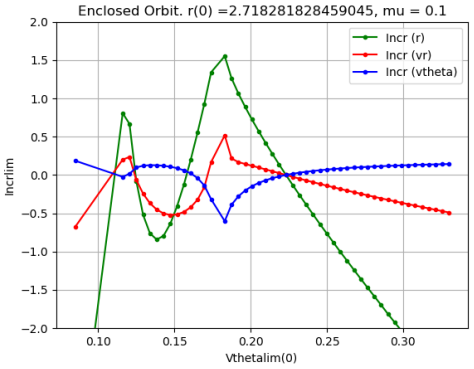
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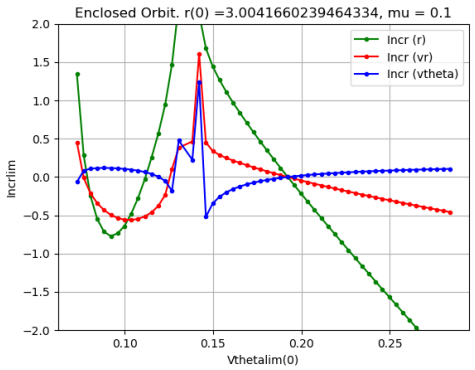
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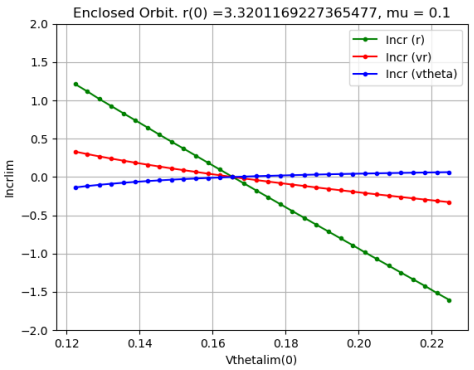
Incrlim
Vthetalim(0)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Incr (vr)
Incr (vtheta)
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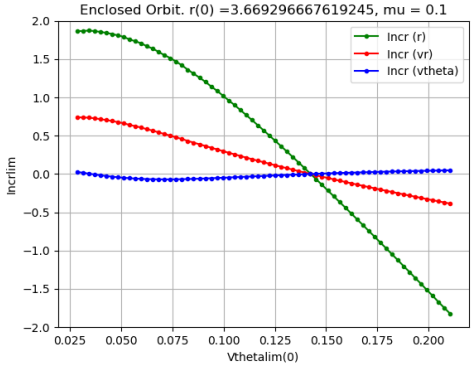
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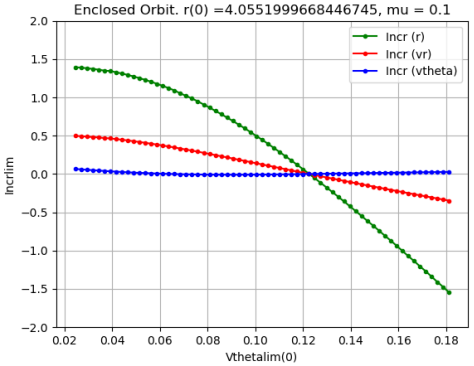
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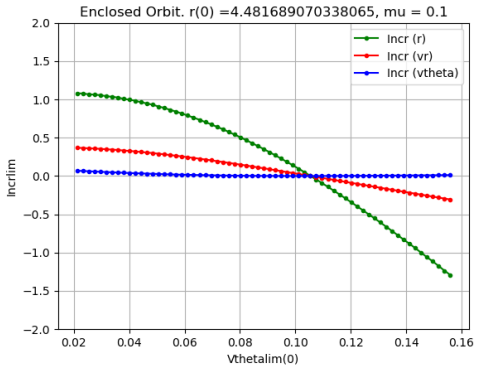
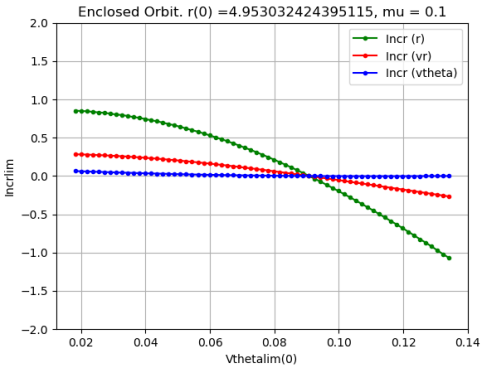
Incr (vr)
Incr (vtheta)
Incrlim
Vthetalim(0)


Enclosed Orbit. r(0) =4.481689070338065, mu = 0.1
Incr (r)
Incr (vr)
Incr (vtheta)
Incrlim
Vthetalim(0)


Enclosed Orbit. r(0) =4.953032424395115, mu = 0.1
Incr (r)
Incr (vr)
Incr (vtheta)
Incrlim
Vthetalim(0)

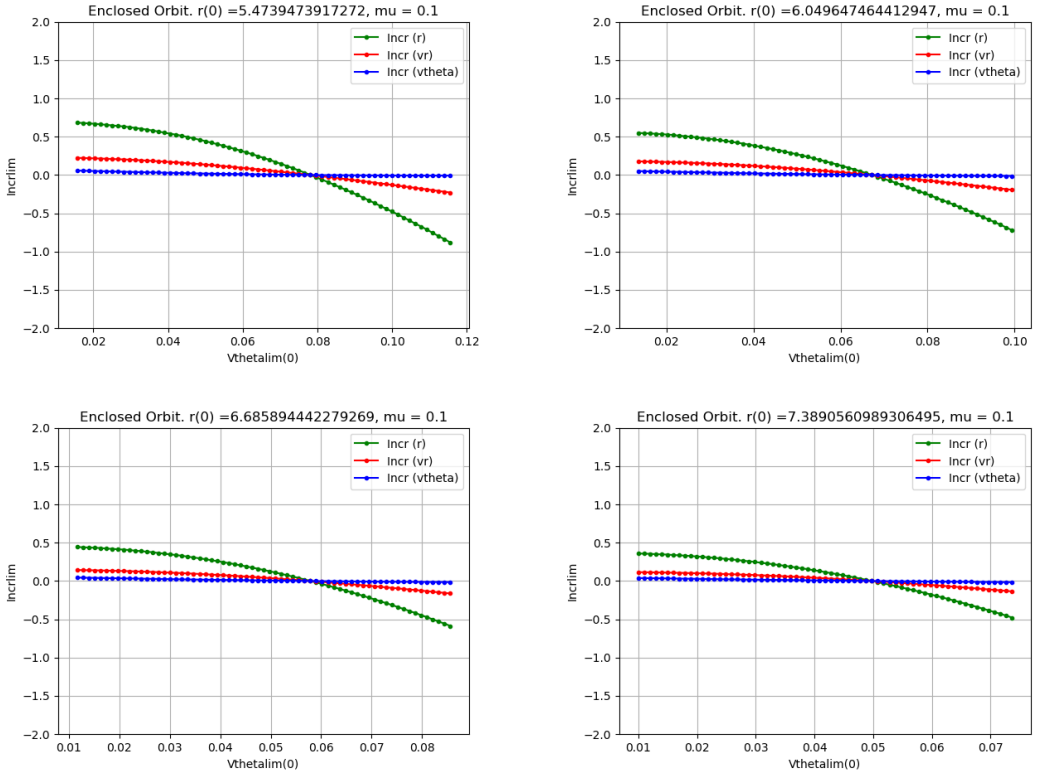


Figure 4: Representation of the increments of r, v_r and v_θ compared to the initial angular velocity of all the possible orbits with a $\mu = 0.1$. The initial radius r_0 were chosen by the method explained in the section 4.1

In the graphs of the figure 4 it is possible to observe the following; for small radius ($r < 1.5$), the graph is very dirty and the curve of Δr dont't even cut with the zero axis, which means that there are no orbits. In the case of intermediate radius ($1.5 < r < 3$), the r increment usually cuts several times the zero, but as mentioned, the only possible orbits are those in which the point is cut to zero by all the increments ($\Delta r, \Delta v_r, \Delta v_\theta$), so they are not periodic orbits. And for higher values of r ($r > 3$), the curves behaves very well, since it tends to circular orbit of the problem of two bodies.

If these graphs are analysed, one condition is met for all cases, and that is that only more than one orbit is generated for $r < 2.5$ because as the radius increases, the three-body system tends to behave more and more like a common two-body system, being a case in which only one result is generated.

In the figure 4, the value of the increments of r, v_r and v_θ presented versus the initial angular velocity. The value of the angular velocity for each orbit is necessary to know exactly the initial conditions of the good orbits. The way to calculate these angular velocity values is with a second-order fit. The three points closest to the $\Delta r = 0$ were chosen to make a second order fit, so we obtain a fitted curve that goes through zero and at that point the angular velocity is chosen. The way to do this is to detect a change of sign of the increments, which means that it has passed through zero, and therefore has found a possible orbit of interest, once this is done the adjustment is made and the value of the initial velocity is obtained.

4.3 Analysis of the searched orbits

Once the initial conditions of the orbits of interest (pseudocircular, symmetric, periodical and simple) are obtained , they are represented in the figure 5, making a graph for different μ values. These μ values have been selected because they show the general characteristics of diagrams corresponding to similar values of μ. Two small masses ($\mu = 0.05, 0.1$) have been chosen, as these generate multiple orbits for a single radius, which is of utmost importance in our problem.

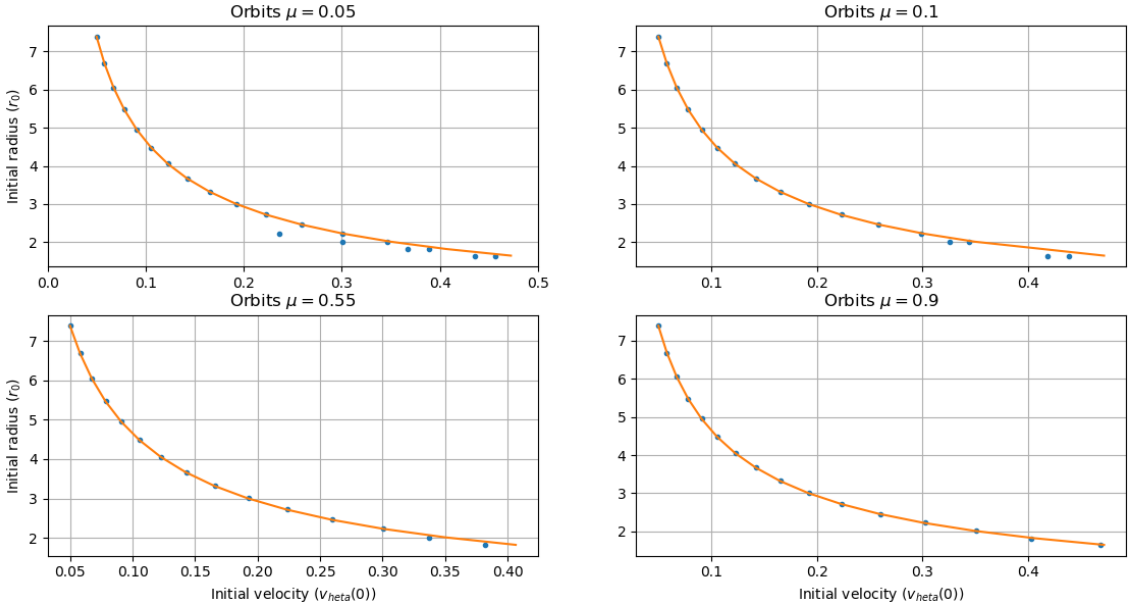


Figure 5: Representation of orbits for values of $\mu = 0.05, 1, 0.55, 0.9$. The graphs draw the initial radius versus the initial angular velocity calculated by the method explained in the section 4.3, also adding a representation of the equation (3.14)

It is possible to study these representations (figure 5) to arrive at several conclusions, among which are found: There are no orbits for values of $r < 1.6$ in neither case. As for the different values of μ, no changes have been observed in the minimum values of r_0 so that some orbit exists, since we have sampled our radius with a precision that does not allow us to discern more possible values.

More than one possible orbit corresponding to the same radius appears for radii always smaller than 3, since the system begins to behave as the common problem of both bodies when the radius increases. In addition, it has been observed that for $\mu > 0.15$, no more than two orbits are generated for a single radius.

In addition to the representation of the orbits, the equation (3.14) has been drawn, which gives us the initial values of the angular velocity to generate circular orbits in the problem of the two bodies, so it is supposed to match well with the orbits we have calculated in the restricted problem of the three bodies; and that is what is observed for values of r_0 small enough. Also, as is to be expected, the curve coincides less with small values of the radius ($r_0 < 2.5$), since the movement is more complicated in that range.

4.4 Jacobi's constant in the chosen orbits

In the figures 6, 7, 8 and 9 the Jacobi constant is shown versus the initial radius. The relationship between the two is practically linear, and very similar for all the μ values. The biggest difference is found for small radii ($r_0 < 2.5$) where it is possible to find several orbits for a single radius value (figure 6 and 7).

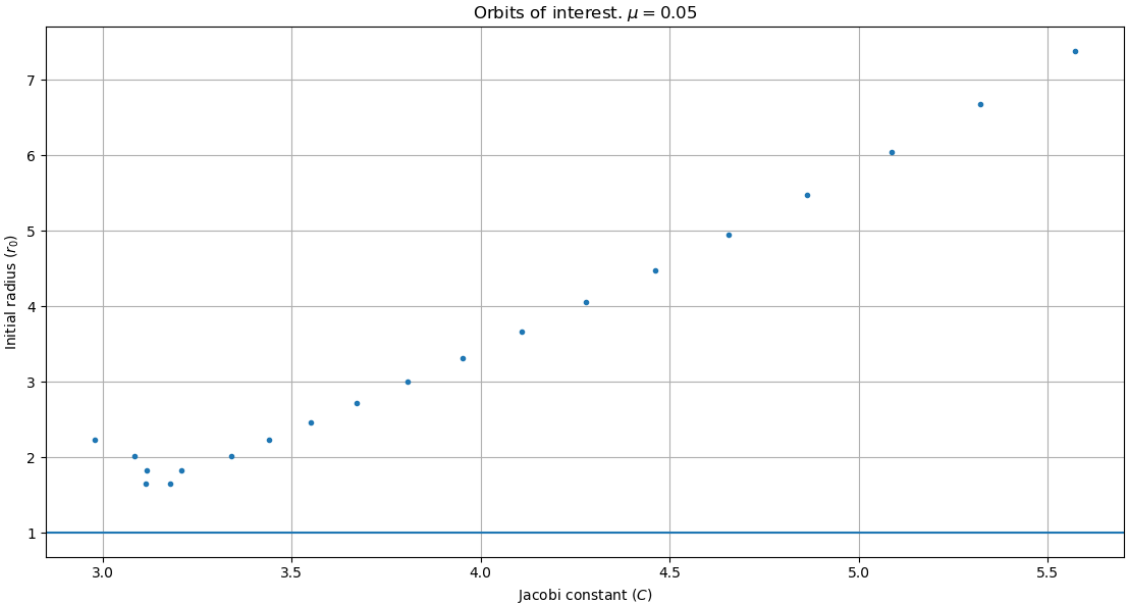


Figure 6: Representation of orbits for value of $\mu = 0.05$. The graphs draw the initial radius versus the Jacobi constant

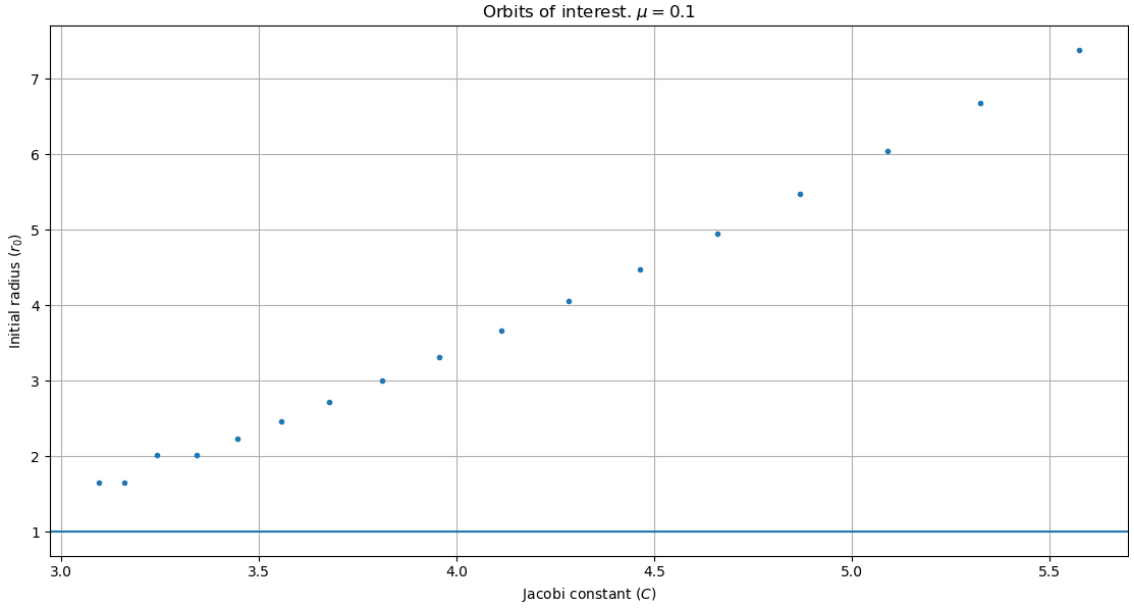


Figure 7: Representation of orbits for value of $\mu = 0.1$. The graphs draw the initial radius versus the Jacobi constant

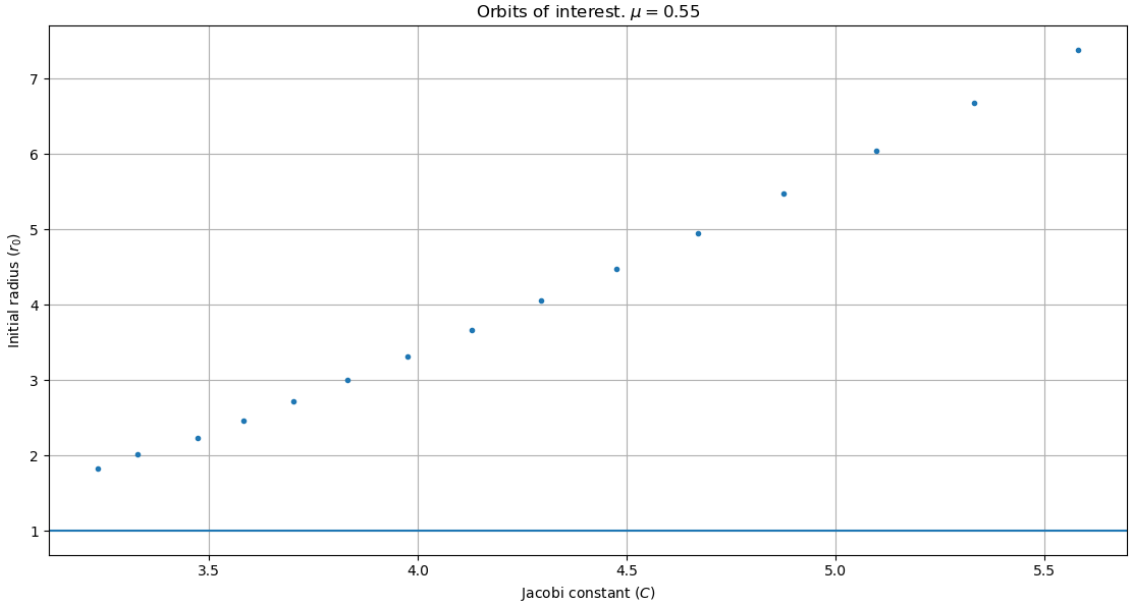


Figure 8: Representation of orbits for value of $\mu = 0.55$. The graphs draw the initial radius versus the Jacobi constant

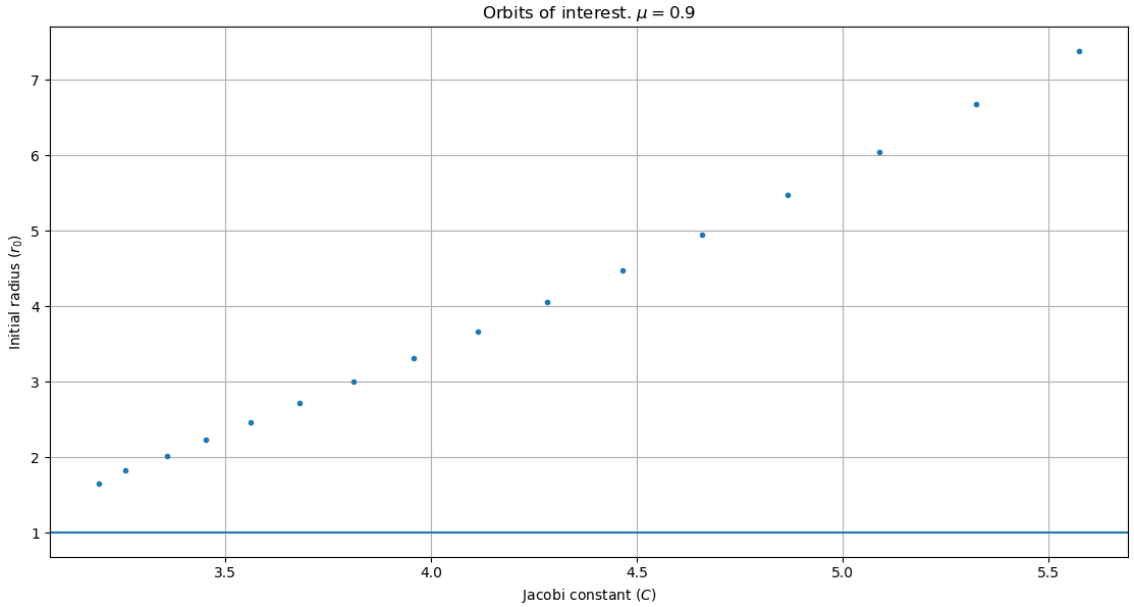


Figure 9: Representation of orbits for value of $\mu = 0.9$. The graphs draw the initial radius versus the Jacobi constant

The individual study of the representation for each μ value is interesting, but the one that encompasses all the masses (figure 10) can give us very interesting information about the groups or families that can be formed.

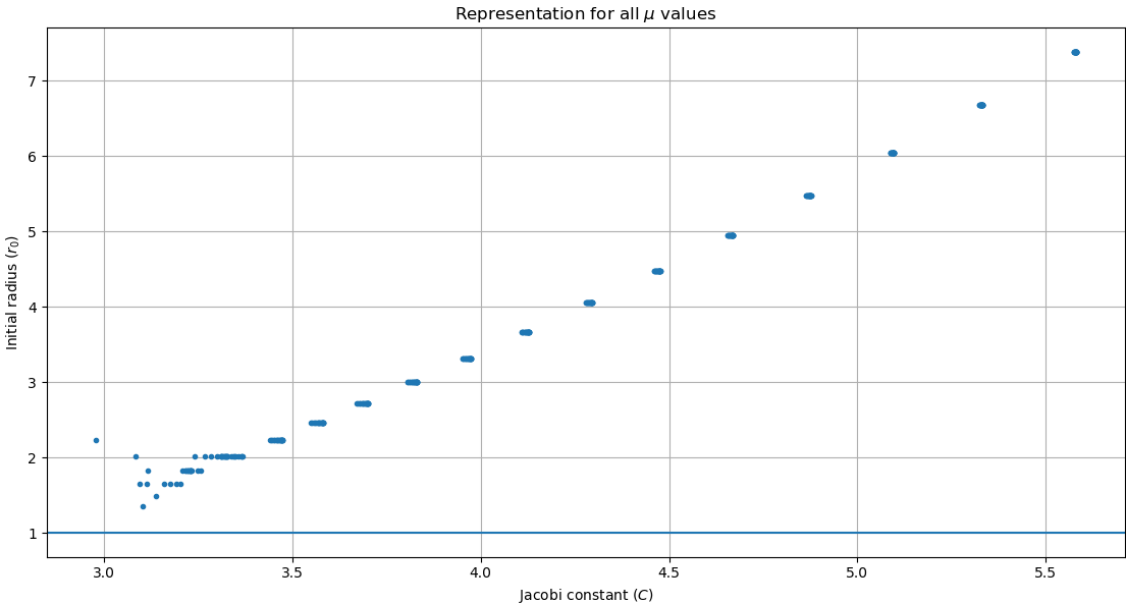


Figure 10: Representation of orbits for all μ values. The graphs draw the initial radius versus the Jacobi constant

The greater the value of the radius, the smaller the difference of the Jacobi constant for different μ values, which is why a displacement is observed on the x-axis (the greater the mass, the greater the value of the Jacobi constant associated with a orbit). As for the distribution of the points: from $r_0 > 2$ the points are grouped in a diagonal, but for $r_0 < 2$ there are more differences; in this range of small radii ($r_0 < 2$), families of orbits are formed in the same radius, which makes it a bit difficult to characterize this area. The most direct way of seeing these groups of orbits is by extending the diagonal, with the orbits falling on this diagonal or its surroundings being orbits, and those that do not coincide are those cases with a double orbit for the same radius value.

There's a part of the range $3.2 < C < 3.4$ that has a big dispersion of points. These points are located mainly in between the radius $r_0 = 1.5$ and $r_0 = 2$, where more orbits for the same radius are located. The points are so scattered because as previously mentioned, when the mass is increased, the constant of Jacobi increases, and the smaller the radius, the greater the difference, which is why the points to the left are so far from the main diagonal, because they belong to the smallest mass ($\mu = 0.05$).

5 Conclusions

- Orbits are only generated for $r_0 > 1.6$
- For $r_0 > 3$ only one orbit per radius value is generated, as the system tends to have the problem of two bodies
- More than one orbit per radius value is generated only when the following conditions are met together: $1.5 < r_0 < 3$ and $\mu < 0.15$
- The greater is the value of the radius, the smaller is the diffetence of the Jacobi constant for different μ values.
- In the range of Jacobi's constant $3 < C < 3.5$ is where the largest number of orbits are generated by the same radius

6 Future work

This work allows us to follow different research paths in the future that may produce very revealing results. An important range of study is that of orbits with radii smaller than 3, with a much higher sampling, which would allow to obtain a much greater precision when it comes to finding minimum radii for the generation of orbits, and the same would happen with very low μ values, since we have focused, in this work, on making a study in a larger range, to better understand the behavior of these systems. The entire study that has been carried out has been for pseudo-circular orbits, so extending the work to elliptical orbits, or with a certain amount of eccentricity, would further generalize the results. Another interesting point is to study orbits outside the plane, or deviated from it.

As has been mentioned, the study of planetary orbits in binary systems is a subject from which many more interesting results can be obtained, since it has not been as worked on as other areas within this type of problem.

References

[1] González Martínez-Pais, Ignacio. Introducción a la Mecánica Celeste (formulación newtoniana), (Servicio de publicaciones Universidad de La Laguna, 2003), 57 - 190.

[2] Szebehely, Victor. Theory of orbits: The restricted problem of three bodies. Defense Technical Information Center, 1967.

[3] Poincaré Henri & Goroff, D. L. New methods of celestial mechanics. Vol 1: Periodic and Asymptotic Solutions, (American Institute of Physics, 1993).

[4] Sharma R.K.: 1976, A&A, 50, 257.

[5] Henon M.: 1982, Physica D, 5, 412

[6] Henon M.: 1965, Ann. Astr., 28, 992