Universidad de La Laguna

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Section of Physics

Final Degree Project:

Introduction to String Theory

Author: Carlos Ferrera González

Supervisor: José María Gómez Llorente

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Objetivo: *El objetivo de este trabajo es, mediante la aplicación y la ampliación el conocimiento adquirido durante el grado, entender y explicar las ideas fundamentales en las que se basa la Teoría de Cuerdas, así como obtener o presentar algunos de sus resultados.*

Objetive: The objective of this work is to, through the application and expansion of the knowledge acquired during the degree, understand and explain the fundamental ideas on which String Theory is based, as well as to obtain some of its results.

Abstract

El siguiente trabajo comienza, en su primer capítulo, tratando de enfatizar por qué se ha desarrollado la Teoría de Cuerdas, explicando a qué cuestiones busca dar solución, para ello se exponen una serie de ideas que giran entorno a la búsqueda histórica de la unificación en la física. Tras esta breve discusión, se repasa en el siguiente capítulo la cuerda clásica, dado que es el elemento del que se derivan e inspiran modelos más complejos que se tratan en sucesivos capítulos, precisamente se utiliza la misma para ilustrar cómo se aplican los conceptos de acción y mecánica lagrangiana al estudio de campos, y dejar así patente la efectividad de estas herramientas. También se derivan, a modo de ejemplo, una serie de expresiones correspondientes a campos complejos tales como la ecuaciones de Schrödinger y Dirac, con ello se persigue realzar el poder de la acción en la física matemática. En el capítulo 4 se estudia la partícula puntual relativista, aquí nos detenemos para repasar las bases de la relatividad especial, explicar las coordenadas del cono de luz y comprender la compactificación como vía para la posibilidad de dimensiones adicionales. Tras ello, se expone como estudiar la partícula relativista empezando por la propuesta de una acción, desgranando a partir de la misma las ecuaciones de movimiento correspondientes. Posteriormente, en el capítulo 6, se utiliza este modelo para mostrar cómo se realiza el proceso de cuantización de una teoría no cuántica. Para ello, se localizan las variables dinámicas del sistema y se transforman en los consiguientes operadores, se obtienen las relaciones de conmutación, se define un hamiltoniano del que se pone a prueba su validez y se construyen el espacio de estados y la ecuación de Schrödinger de la partícula cuántica libre.

En paralelo, se comienza a estudiar la cuerda relativista. Mediante la hoja del mundo generada por la cuerda, que es una idea basada en la línea del universo de la partícula puntual, se justifica la acción de Nambu-Goto, esta acción pasa a ser la base la base de nuestro modelo. A partir de esta y de las condiciones de contorno que imponemos a los extremos de nuestra cuerda, derivamos su ecuación de ondas, la forma de su energía potencial, obtenemos que los extremos de las cuerdas libres se mueven transversalmente a la velocidad de la luz y evaluamos las leyes de conservación de la misma. Con esta información desarrollamos la solución de la ecuación de movimiento, que luego, sin llegar a demostrar explí citamente, presentamos en el formalismo del cono de luz, definiendo en el proceso los modos transversales de Visasoro. Finalmente cuantizamos la cuerda relativista siguiendo un procedimiento similar al de la partícula puntual, y discutimos como la primera se convierte en un oscilador armónico, cuyos modos de vibración marcan la diferencia con segunda. Por último presentamos, sin ánimo de demostrar explícitamente, como la Teoría de Cuerdas bosónica predice la existencia de hasta 26 dimensiones, es decir, 22 dimensiones adicionales.

Contents

1 Introduction[1][2][3]

La Teoría de Cuerdas es una teoría unificadora, y por lo tanto, busca proveer de una explicación común a una amplia variedad de fenómenos de una manera simple y elegante. En esencia, propone que la materia no está compuesta por partículas tal y como normalmente las entendemos, sino por pequeños elementos unidimensionales que recuerdan a las cuerdas, siendo las características de estas cuerdas la causa fundamental de las propiedades de la misma, así como de sus interacciones. Esta teoría ha sido motivo de discusión desde que fue propuesta en la década de 1960, y para entender por qué es merecedora de tanto interés, primero debemos comentar los problemas que pretende resolver.

String Theory is a unifying theory, and therefore, seeks to provide a common explanation to a wide variety of phenomena in a simple and elegant way. In essence, it proposes that matter is not composed by particles as we usually think about them, but by small one-dimensional elements that resemble to strings, and that the characteristics of those strings, are the fundamental cause of the properties of latter, as well as their interactions. This theory has been a topic of discussion since it was first proposed in the 1960s, and for understanding why is worthy of such an interest, we must first comment on the problems it tries to solve.

1.1 The standard model

Before 1936 only three particles of matter (the electron, the proton and the neutron) and one of interaction (the photon) were known, those four particles allowed scientists to explain a wide variety of phenomena related to the composition of matter, and it seemed that not much was left to discover. But with the detection in that same year of the muon by Carl D. Anderson and collaborators, particle physics changed radically. After that discovery, many experiments proved the existence of dozens of new particles, even antiparticles and many of the known particles turned out not to be fundamental (not composed by others) as well. All what we know to the date about the fundamental components of matter and their physical properties is contained into the Standard Model.

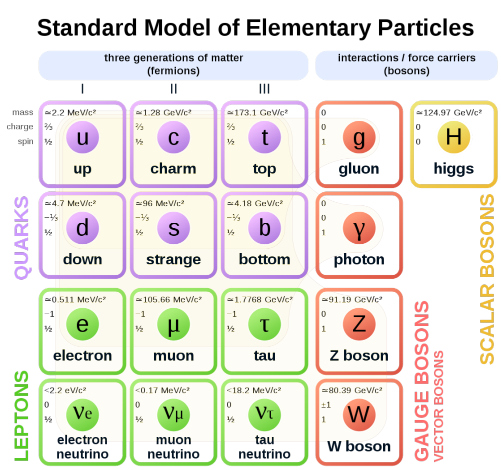

Figure 1: The Standard Model of Particles

The Standard Model (Fig 1) groups the elements in families according to their similarities and classifies them logically. For instance, we can see that the electron, the muon and the tau share the same spin and charge, and both have an associated neutrino, but they differ in their mass, and each one has his own leptonic number. With this in mind, we can think of at least two questions that arise, to which the Standard Model provides no answer. Why do we have the particles we have and not others, and which is the reason for these properties? It looks like the interpretation of particles as extensionless, apart from being already counterintuitive, imposes an intellectual barrier that prevents us from delving into these issues. However, the degrees of freedom that a one-dimensional chain possesses gives us a richer starting point.

1.2 The unification of the fundamental interactions

There are four known fundamental forces or interactions in matter, this forces are; electromagnetism, strong or colour interaction, weak interaction and gravity. The three first, have associated interaction gauge bosons (see Fig 1), which transfer to one particle of matter to the other certain effects due to the corresponding force and vice versa. For the EM interaction we have the photon, the weak interaction is mediated by the W^+, W^- and Z^0 bosons, and the strong force is carried by eight different gluons. It is really interesting to know that trough the Weinberg-Salam model of electroweak interactions, EM and Weak interactions have already been unified, and currently there are several Theories of Great Unification (TGU) that attempt to unify them along with the colour interaction. Although this last step has not been verified yet, it seems that what we understand as different interactions can have a profound, common origin. But, Where's the gravity in all of this?

Because of the success of the theory of general relativity, gravity has been interpreted and understood in a radically different way compared to the other forces. According to general relativity, in principle there is no gravitational gauge boson, and this interaction is a consequence of the deformation of space-time due to bodies with mass, in other words, the dynamics of a body due to gravity are determined by the metric tensor $g_{\mu\lambda}(\mathbf{x})$, where no interaction boson appears. Even thought general relativity works satisfactorily in the prediction of astrophysical and cosmological events, if we take into account how the other three forces work, it is not crazy to think that it must also exist a gauge boson for gravity, that has been named graviton, but due to the relative weakness of this interaction, it has a series of properties that make this hypothetical boson much harder to find. Precisely, one of the vibrational modes obtained in String Theory, predicts the existence of a closed string with the same properties that would need to have the graviton, this result will not be obtained in this work, since we will limit ourselves to the development of open strings, but we will show the general procedure of quantization. Obviously, this mathematical model alone doesn't prove that the graviton has to exist, and that the reductionist perspective of interactions, which assumes that all forces must be related somehow, has to be correct. But there are some good reasons to think so, as it wouldn't be the first great unification in the history of physics.

1.3 Our conception of physics

If we take a look at the history and development of physics, several cases of unification can be found, of which we can mention a couple. By the end of the eighteenth century, Charles-Augustin de Coulomb had provided a consistent theory for electrostatics, shortly after, some subsequent experiments showed that electricity had a sort of relation with a phenomena known since the ancient Greeks, magnetism. After many new observations and experiments made by Biot, Savart, Ampère and others, James Clerk Maxwell constructed a set of equations, known as Maxwell equations or equations of electromagnetism, which collect an inseparable connection between magnetism and electricity. Another important unification was achieved in 1928, when Paul Dirac managed to introduce special relativity in the formalism of the non-relativistic Schrödinger equation applied to an electron, and finding a common expression for two independently-developed heavyweights of physics, special relativity and quantum mechanics.

As we stated in the previous section, general relativity works, and is one of the cornerstones of nowadays physics. But, it is a classical theory, and therefore, we have a framework in which we assume well-defined trajectories, with no uncertainty in the position and/or the momentum. This essentially means that this theory is not governed by the Heisenberg uncertainty principle, something which is completely incompatible with quantum mechanics. This presents an inconsistency in the way we interpret the universe, and at least one of this theories has to be modified if we want to solve this dilemma. The reason why both theories have survived for so long independently, is that usually have very different ranges of application, while the first one focuses on planets, stars and super massive objects at a macroscopic level, the latter studies systems from their smaller components, as they are electrons or molecules. The successive attempts made to quantize general relativity have failed, and it seems necessary to think outside of the box in order to find an answer to this persisting problem. As we will see during the development of this work, String Theory presents itself as an elegant candidate to unify this two conceptions of nature.

2 The classical string

Comenzamos nuestro estudio de la Teoría de Cuerdas repasando brevemente la mecánica de la cuerda clásica homogénea, ya que este es el elemento en el que se inspira dicha teoría. La exposición de estos resultados podría parecer omitible, pero a medida que empecemos a obtener conclusiones significativas en sucesivos capítulos, regresaremos aquí para compararlos.

We start our analysis of String Theory by briefly revising the mechanics of a classical homogeneous string, as it is the element in which this theory is inspired. The exposition of this results could seem omitible, but as we start getting significant conclusions in subsequent chapters we will return here to compare them.

2.1 Review of the results for a classical string

Lets imagine an homogeneous string of length a, where we have two scalar magnitudes of relevance, the mass density μ_0, and the tension T_0. We name x the longitudinal coordinate which follows the length of the string when it is at rest, and y the coordinate in which transverse oscillations may occur, so there is a two-dimensional frame. We also impose the condition of small oscillations around equilibrium, which means that:

$$\frac{\partial y}{\partial x} \ll 1 \tag{1}$$

The condition above ensures that the length of the string along the x coordinate remains approximately constant during motion, and that the tension stays unchanged. When two infinitesimally close points of the string A(x,0) and B(x+dx,0), are displaced from the equilibrium configuration by a small perturbation that meets the condition (1), they shift to new coordinates A(x,y) and B(x+dx, y+dy), and therefore suffer a vertical force due to the tension of the string:

$$dF_v = T_0 \frac{\partial y}{\partial x}|_{x+dx} - T_0 \frac{\partial y}{\partial x}|_{x} \simeq T_0 \frac{\partial^2 y}{\partial x^2} dx \tag{2}$$

Where we have just applied the definition of the derivative in the second equality. By using Newton's second law, we can obtain a relation between this force and the vertical acceleration of our string:

$$T_0 \frac{\partial^2 y}{\partial x^2} dx = (\mu_0 dx) \frac{\partial^2 y}{\partial t^2} \tag{3}$$

And by cancelling the dx at each side, we obtain the equation of motion, that takes the form of a wave equation:

$$\boxed{\frac{\partial^2 y}{\partial x^2} - \frac{\mu_0}{T_0} \frac{\partial^2 y}{\partial t^2} = 0} \tag{4}$$

Where $v_0 = \sqrt{T_0/\mu_0}$ is the velocity of the wave. For solving (4) we need to know some initial and boundary about the string, the two most common boundary conditions are, the Dirichlet boundary condition, in which the extremes of the strings are fixed, and the Neumann boundary condition, in which the extremes are confined into vertical oscillations:

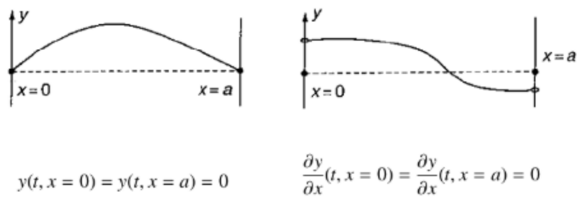

Figure 2: Dirichlet and Neumann conditions applied to a classical string

Both conditions are also relevant in the case of fundamental strings that appear in String Theory, as they can be attached to D-dimensional elements called D-branes that limit their movement, and to which we will refer later. For now, in our still non-relativistic world, we know that if we apply to our wave equation (4) the Dirichlet boundary conditions, we obtain sinusoidal oscillation solutions of the form:

$$y_n(x) = A_n sin\left(\frac{n\pi x}{a}\right); \qquad n = 1, 2, 3, ... \tag{5}$$

Where A_n is the amplitude of the oscillation and n is a natural number. On the other hand, for the Neumann boundary conditions, we obtain a quite similar result:

$$y_n(x) = A_n cos\left(\frac{n\pi x}{a}\right); \qquad n = 1, 2, 3, ... \tag{6}$$

For both conditions (5) and (6) we have the same value of the allowed frequencies:

$$\omega_n = \sqrt{\frac{T_0}{\mu_0}}\left(\frac{n\pi}{a}\right) \qquad n = 1, 2, 3, ... \tag{7}$$

The results that have been enumerated, will serve us as reference, so when we face relativistic, less intuitive strings, we can deal with them mildly. The reader must keep in mind that, in the same way that Rutherford atomic model is taught as a first approximation to more accurate quantum atomic models, the classical string is just a rude, but simple example, that serves us as an starting point in the process of understanding fundamental strings. On the other hand, this set of equations will also serve us as reference for evaluating how useful can lagrangian mechanics be in the study of strings, as we are going to see in the following chapter.

3 The action[3][4]

En este capítulo, vamos a repasar brevemente los fundamentos de la mecánica lagrangiana, así como el concepto de acción S, después explicaremos como la mecánica lagrangiana puede ser adaptada al estudio de campos complejos, y finalmente, vamos a ilustrar la relevancia de este formalismo estudiando algunos ejemplos. En conjunto, el propósito de este análisis es enfatizar el poder de estas herramientas para derivar resultados físicos, y mostrar la variedad de ecuaciones de evolución, tanto dinámicas como cinemáticas, que pueden ser derivadas de una acción definida apropiadamente.

In this chapter, we are going briefly review the basics of lagrangian mechanics, as well as the concept of action S, then we will explain how lagrangian mechanics can be adapted to the study of complex fields, and finally, we are going to illustrate the relevance of this formalism by studying some examples. The overall purpose of this analysis is to highlight the power of this tools for deriving physical results, and to show how many cinematic and dynamic equations of evolution can ultimately be derived from an appropriately defined action.

3.1 Lagrangian mechanics and equations of motion

First, we state that any action can be defined in terms of the lagrangian of the system by the following expression:

$$S = \int_{\mathscr{P}} L(q_n(t), \dot{q}_n(t), t)dt \qquad n = 1, 2, 3, ... \tag{8}$$

Where $\mathscr{P}$ is the path that follows the element whose movement we are interested on, and q and $\dot{q}$ are the natural variables of the Lagrangian, the Lagrangian can also depend explicitly on t. The only paths with physical meaning, are those that make the action stationary against infinitesimal variations of the q spatial coordinates, so for obtaining those paths, we have to make $\delta S = 0$. If we impose this condition to (8) we derive the well-known Lagrange equations:

$$\frac{\partial L}{\partial q_n} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_n}}\right) = 0 \qquad n = 1, 2, 3, ... \tag{9}$$

If we have non-punctual objects, we can work in terms of the lagrangian density $\mathscr{L}$ too, which is related to the lagrangian by its integration over all spatial coordinates:

$$L = T - V = \int d^3r\mathscr{L} \tag{10}$$

In this situations, is usually easier to work directly with the lagrangian density. For instance, we can return to our classical string (see section 2.1) and derive, in an alternative way, the equation of motion (5). We start by identifying the kinetic energy T, throughout the relation $T = \frac{1}{2}mv^2$, integrated over the points that form a string of length a, we get that:

$$T = \int_0^a \frac{1}{2}(\mu_0 dx)\left(\frac{\partial y}{\partial t}\right)^2 \tag{11}$$

Where remember that μ_0 is the mass density of our homogeneous string. For the potential energy, we know that the work made in stretching an infinitesimal segment of the string is given by $T_0\Delta l$, where the stretch Δl is the change of length suffered by that segment in an oscillation:

$$\Delta l = \sqrt{(dx)^2 + (dy)^2} - dx = \left(\sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2} - 1\right) dx \simeq \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 dx \tag{12}$$

In the last step we performed a Taylor expansion over the square root. So the potential energy V of the whole string due to its stretching is:

$$V = \int_0^a \frac{1}{2}T_0\left(\frac{\partial y}{\partial x}\right)^2 dx \tag{13}$$

Then the lagrangian is easily calculated by its definition L=T-V, which integrated between the initial and final times gives us the action:

$$\boxed{S=\int_{t_i}^{t_f}dt\int_0^a\mathscr{L}dx=\int_{t_i}^{t_f}dt\int_0^a\left[\frac{1}{2}\mu_0\left(\frac{\partial y}{\partial t}\right)^2-\frac{1}{2}T_0\left(\frac{\partial y}{\partial x}\right)^2\right]dx}\tag{14}$$

Now, we can check the equation of motion (4) by calculating δS and making it zero. We start doing so by performing the change y(t,x) $\longrightarrow$ y(t,x) + $\delta y(t,x)$ in the expression of the action(14):

$$S\left[y+\delta y\right]=\int_{t_i}^{t_f}dt\int_0^a\left[\frac{1}{2}\mu_0\left(\frac{\partial\left(y+\delta y\right)}{\partial t}\right)^2-\frac{1}{2}T_0\left(\frac{\partial\left(y+\delta y\right)}{\partial x}\right)^2\right]dx=$$

$$=\int_{t_i}^{t_f}dt\int_0^a\left[\frac{1}{2}\mu_0\left(\left(\frac{\partial y}{\partial t}\right)^2+\left(\frac{\partial\delta y}{\partial t}\right)^2+2\frac{\partial y}{\partial t}\frac{\partial\delta y}{\partial t}\right)-\frac{1}{2}T_0\left(\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial\delta y}{\partial x}\right)^2+2\frac{\partial y}{\partial x}\frac{\partial\delta y}{\partial x}\right)\right]dx=$$

We use the condition (1) to neglect a couple of terms, and we regroup the remaining ones:

$$=\int_{t_i}^{t_f}dt\int_0^a\left[\frac{1}{2}\mu_0\left(\frac{\partial y}{\partial t}\right)^2-\frac{1}{2}T_0\left(\frac{\partial y}{\partial x}\right)^2\right]dx+\int_{t_i}^{t_f}dt\int_0^a\left[\mu_0\frac{\partial y}{\partial t}\frac{\partial\delta y}{\partial t}-T_0\frac{\partial y}{\partial x}\frac{\partial\delta y}{\partial x}\right]dx$$

Where the first term is just the action given in (14), and therefore, the second one is δS. We develop it for finding the equations of motion:

$$\delta S=\int_{t_i}^{t_f}dt\int_0^a dx\left[\mu_0\frac{\partial y}{\partial t}\frac{\partial\delta y}{\partial t}-T_0\frac{\partial y}{\partial x}\frac{\partial\delta y}{\partial x}\right]=$$

$$=\int_{t_i}^{t_f}dt\int_0^a dx\left[\frac{\partial}{\partial t}\left(\mu_0\frac{\partial y}{\partial t}\delta y\right)-\mu_0\frac{\partial^2 y}{\partial t^2}\delta y-\frac{\partial}{\partial x}\left(T_0\frac{\partial y}{\partial x}\delta y\right)+T_0\frac{\partial^2 y}{\partial x^2}\delta y\right]=$$

$$=\int_0^a dx\left[\mu_0\frac{\partial y}{\partial t}\delta y\right]_{t=t_i}^{t=t_f}+\int_{t_i}^{t_f}dt\left[-T_0\frac{\partial y}{\partial x}\delta y\right]_{x=0}^{x=a}-\int_{t_i}^{t_f}dt\int_0^a dx\left(\mu_0\frac{\partial^2 y}{\partial t^2}-T_0\frac{\partial^2 y}{\partial x^2}\right)\delta y=0$$

The first term just vanishes when we define the initial and final configurations of the string for a given trajectory. The second term depends on the endpoints of the string x=a and x=0, so with some boundary condition (see Fig (2)), we can get rid of this one too. And, if we equate what is inside the integrals in the last term, we just have obtained the equation of motion of a string (4)! Now that we have contrasted our results for a classical string with the use of an action, is a good moment for introducing two quantities that will be very useful in following chapters and derive from lagrangian mechanics:

$$\mathscr{P}^t\equiv\frac{\partial\mathscr{L}}{\partial\dot{y}}=\mu_0\frac{\partial y}{\partial t};\qquad\qquad\mathscr{P}^x\equiv\frac{\partial\mathscr{L}}{\partial y'}=-T_0\frac{\partial y}{\partial x}\tag{15}$$

Where $\dot{y}=\frac{\partial y}{\partial t}$ and $y'=\frac{\partial y}{\partial x}$. The first quantity $\mathscr{P}^t$ is precisely the momentum density of the string, while the second one has a deeper interpretation in relativistic strings (see chapter 5). By applying the definitions (15), we can rewrite the equation of motion (4) of an one dimensional classical string in a more fashionable way:

$$\boxed{\frac{\partial\mathscr{P}^t}{\partial t}+\frac{\partial\mathscr{P}^x}{\partial x}=0}\tag{16}$$

3.2 Fields and mathematical physics

As we mentioned in the introduction, String Theory is also a theory of fields, and we will have to work with operators and wave functions characteristic of quantum mechanics. For working in this frame by our own, we need a reliable mathematical tool, and with that purpose, we introduce lagrangian mechanics expanded to the study of complex fields. This formalism is oriented towards a lagrangian which is dependent, not on generalised coordinates q and velocities $\dot{q}$, but on fields labelled as $\mathscr{A}_j$ and their associated velocities $\dot{\mathscr{A}_j}$, as well as their complex conjugates $\mathscr{A}_j^*$ and $\dot{\mathscr{A}}_j^*$.

$$L = \int d^3r \mathscr{L}(\mathscr{A}_j, \dot{\mathscr{A}}_j, \partial_i \mathscr{A}_j, \mathscr{A}_j^*, \dot{\mathscr{A}}_j^*, \partial_i \mathscr{A}_j^*) \tag{17}$$

Where the ∂_i symbol refers to a derivation over each spatial coordinate. The derivatives $\partial_i \mathscr{A}_j$ and $\partial_i \mathscr{A}_j^*$ are not considered to be independent variables, as they can be directly derived from the fields. Note that we must obtain two sets of equations, one for the fields and other for the complex conjugated fields, this equations are:

$$\frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{\mathscr{A}}_j} = \frac{\partial \mathscr{L}}{\partial \mathscr{A}_j} - \sum_i \partial_i \frac{\partial \mathscr{L}}{\partial(\partial_i \mathscr{A}_j)} \tag{18}$$

$$\frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{\mathscr{A}}_j^*} = \frac{\partial \mathscr{L}}{\partial \mathscr{A}_j^*} - \sum_i \partial_i \frac{\partial \mathscr{L}}{\partial(\partial_i \mathscr{A}_j^*)} \tag{19}$$

This new equations are quite similar to the classical Lagrange equations (9), which we used for studying the classical string. But, in this case, we have added an extra term that includes the variations of the fields over all spatial coordinates, it kind of make sense, since as it costs energy to have the field vary in time, it must also cost energy to have the field vary in space.

For illustrating how the equations (18) and (19) work, we can apply them to a complex classical field known as the Schrödinger matter field. This field is given by the lagrangian density

$$\mathscr{L} = \frac{i\hbar}{2}\left(\Phi^*\dot{\Phi} - \dot{\Phi}^*\Phi\right) - \frac{\hbar^2}{2m}\nabla\Phi^* \cdot \nabla\Phi - V(r)\Phi^*\Phi \tag{20}$$

to which we apply the equation (19):

$$\frac{\partial \mathscr{L}}{\partial \dot{\Phi}^*} = -\frac{i\hbar}{2}\Phi; \qquad \frac{\partial \mathscr{L}}{\partial \Phi^*} = \frac{i\hbar}{2}\dot{\Phi} - V(r)\Phi; \qquad \frac{\partial \mathscr{L}}{\partial(\partial_j \Phi^*)} = -\frac{\hbar^2}{2m}\partial_j \Phi$$

$$i\hbar\dot{\Phi} - V(r)\Phi + \frac{\hbar^2}{2m}\nabla^2\Phi = 0 \tag{21}$$

We have just obtained the Schrödinger equation for a particle in a central potential! It is worth noting that if we instead apply the equation (18) to the action above

$$\frac{\partial \mathscr{L}}{\partial \dot{\Phi}} = \frac{i\hbar}{2}\Phi^*; \qquad \frac{\partial \mathscr{L}}{\partial \Phi} = -\frac{i\hbar}{2}\dot{\Phi}^* - V(r)\Phi^*; \qquad \frac{\partial \mathscr{L}}{\partial(\partial_j \Phi)} = -\frac{\hbar^2}{2m}\partial_j \Phi^*$$

$$i\hbar\dot{\Phi}^* - V(r)\Phi^* + \frac{\hbar^2}{2m}\nabla^2\Phi^* = 0 \tag{22}$$

and we compare this result to (21), we can check how we have reached the complex conjugate expression of each other. So in fact, we get the same result with both of them.

Another illustrative example, consists in the derivation of the Dirac equation (see introduction). As we previously stated, this equation is an improvement of the Schrodinger equation applied to particles of spin one half since it also takes into account relativistic effects. For this case, our given lagrangian density is:

$$\mathscr{L} = -c\hbar\varphi^*\gamma_\mu \frac{\partial}{\partial x_\mu}\varphi - mc^2\varphi^*\varphi \qquad \mu = 0, 1, 2, 3 \tag{23}$$

In the above, we use the Einstein's convention for indexes, the value $\mu = 0$ refers to the temporal coordinate and the values $\mu = 1, 2, 3$ to the spatial coordinates x,y,z. We must take into account that, as we are working now with special relativity, we are in the four-dimensional Lorentz frame. The symbols γ_μ that appear in (23) are the Dirac matrices, which are simply spinorial extensions of the Pauli matrices. The fist of them γ_0, is defined as:

$$\gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{24}$$

While the others, which we label with the lateen letter i because it is the convention with the spatial components, turn out to be:

$$\gamma_i = \begin{pmatrix} 0 & -i\sigma_i \\ i\sigma_i & 0 \end{pmatrix} \tag{25}$$

Where the σ_i values are the Pauli matrices. We just have to apply the Lagrange equation (19) to our new lagrangian density (23). The required calculations are done below:

$$\frac{\partial}{\partial x_\mu}\left(\frac{\partial \mathscr{L}}{\partial(\frac{\partial \varphi^*}{\partial x_\mu})}\right) = 0 \implies \frac{\partial \mathscr{L}}{\partial \varphi^*} = -c\hbar\gamma_\mu \frac{\partial}{\partial x_\mu}\varphi - mc^2\varphi = 0$$

Where we have just used a simplified version of the equation (19), see how the index μ covers all spatial and time derivatives over the complex variable φ. So in the end, after reordering the terms, we get none other than the Dirac equation:

$$\left(\gamma_\mu \frac{\partial}{\partial x_\mu} + \frac{mc}{\hbar}\right)\varphi = 0 \tag{26}$$

We see how the Dirac equation is in fact a set of equations, one for each value the index μ can take. This expression is of great relevance in quantum electrodynamics, since enables us to study particles of spin one half, which, in fact, are all the fundamental particles of matter (see Fig 1).

A particular set of fields contained in the group of complex conjugated fields, is the set of scalar fields. This ones are characterised by a lagrangian density dependent only of real variables, the general expression of the lagrangian density for this fields is:

$$\mathscr{L} = \mathscr{L}\left(\phi, \dot{\phi}, \partial_i\phi\right) = \frac{1}{2}(\dot{\phi})^2 - \frac{1}{2}\partial_i\phi\partial_i\phi - \frac{1}{2}m^2\phi^2 \tag{27}$$

to which if we apply the still valid equation (18) to our lagrangian density (23), we obtain the Klein-Gordon equation:

$$-\frac{\partial^2\phi}{\partial t^2} + \nabla^2\phi - m^2\phi = 0 \tag{28}$$

This equation is an equivalent of the Dirac equation, but for particles of spin zero as, for example, the pions. We hopefully have highlighted the power that provides to a physicist a properly defined action, and how many formulas and physical results can be derived from one, of which we have seen a few. In fact, even Einstein's field equations of general relativity can be derived from an action (for further reading see Classical field theory Volume 2, Landau and Lifshitz, Chapter 11). As you may have already realised, the main difficulty of this method, usually lies in finding a correct action that allow us to derive the correct evolution equations.

4 The relativistic point particle[3][5][6]

Antes de comenzar con las cuerdas relativistas, vamos a comentar algunos resultados fundamentales de la relatividad especial, y cómo se aplica esto al conjunto de coordenadas conocidas como coordenadas del cono de luz. Después, mostraremos como todo el formalismo de la relatividad especial puede ser fácilmente expandido a un sistema de d dimensiones espaciales, y trataremos de entender como esas dimensiones extra pueden encajar en nuestro universo. Tras esto, combinaremos nuestro conocimiento de la relatividad especial con lo que hemos explicado de la mecánica lagrangiana, al hacerlo, obtendremos las ecuaciones de movimiento para una partícula puntual relativista. Finalmente, los resultados obtenidos con la partícula relativista serán aplicados al gauge de las coordenadas del cono de luz. Este es el último paso previo antes de comenzar con las cuerdas relativistas.

Before starting with relativistic strings, we are going to comment on some fundamental aspects about special relativity, and how they apply to a particular set of coordinates called light-cone coordinates. After that, we will show how all the formalism of special relativity can be easily expanded to a system of d spatial dimensions, and then try to understand how can extra dimensions fit in our universe. Then, we will combine special relativity with what we have explained about the action and lagrangian mechanics, by doing so, we will obtain the equations of motion for a relativistic particle. Finally, the results obtained for the relativistic point particle will be applied to the light-cone coordinates gauge. This analysis is the final step in our path towards the relativistic strings.

4.1 Special relativity

In our experience, for locating any punctual event in space and time we need a maximum of four coordinates, three spatial coordinates and the time. With this four values, we can define the following vector, that represents a point in a four dimensional space:

$$x^{\mu} = (x^0, x^1, x^2, x^3) \equiv (ct, x, y, z) \tag{29}$$

Where c is the constant value of the speed of light in the vacuum. By defining the Minkowski metric tensor $\eta_{\mu\nu}$

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{30}$$

we can set that the distance or interval between any two events in space-time is:

$$-ds^2 = \eta_{\rho\mu} x^{\rho} x^{\nu} = x_{\mu} x^{\nu} \tag{31}$$

Where $\eta_{\rho\mu}$ defines the scalar product between the two vectors x^{ρ} and x^{ν}, which are four-dimensional as defined by (29), and correspond to two arbitrary events according to a certain inertial system or frame. Since the times of Galileo, we know that two (or more) observers located in different inertial frames can, and will usually, differ about the value of this vectors, independently of how good their respective measurements are. But, on the other hand, and according to the theory of special relativity, both of the observers must always agree about the value of the interval (31) between these two events:

$$ds^2 = ds'^2; \qquad \text{For any reference system} \tag{32}$$

Any magnitude that meets the condition of showing the same value in all inertial reference systems is called a Lorentz invariant. According to the value of this interval, we have three possibilities, if the interval fulfils that $ds^2 > 0$ we have timelike separated events, which can be causally related, if $ds^2 < 0$ we have spacelike separated events, which cannot be causally related, and if $ds^2 = 0$ we have light-like separated events. The important stuff about this property of the interval, will be exploited in the study of relativistic strings. Returning to our dilemma of the observers in disagreement, the relation between their observations, even though they may be different, is given by the Lorentz transformations, which ensure that the condition (32) is satisfied. This transformations can be written in a very simple way, without loosing any important information, for the particular case in which both observers share the same origin of their frames (O=O') at t=t'=0, and the relative spacial movement between them is along the x-axis. By doing so, we express the Lorentz transformations in their matricial form:

$$L^{\mu}_{\nu} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{33}$$

Where $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ and $\beta = \frac{v}{c}$, being v the velocity of one observer according to the other. Henceforth, when we talk about boosts or Lorentz transformations, we will assume this particular configuration. Two very important physical magnitudes are the energy and the momentum. The relativistic energy and momentum can be obtained in terms of the rest mass of a body and the factor γ:

$$E = \gamma mc^2; \qquad \vec{p} = \gamma m\vec{v} \tag{34}$$

With this two quantities we define the following four-vector called four-momentum:

$$p^{\mu} = (p^0, p^1, p^2, p^3) \equiv \left(\frac{E}{c}, \vec{p}\right) \tag{35}$$

The four-momentum will be very useful in our study of relativistic elements, since is intrinsically related to some important conservation laws. This four-vector also contains an important Lorentz scalar:

$$p^2 \equiv p \cdot p = -mc^2 \tag{36}$$

We could extend talking about many other concepts and results concerning special relativity, but is not the purpose of this dissertation, and we have already reviewed almost all the basic relativistic fundamentals that we need to handle for studying the relativistic phenomena in which we are interested. Now, we proceed to rewrite all of this concepts in terms of a new more interesting coordinates, the light-cone coordinates.

4.2 The world line and light-cone coordinates

The line generated by all the points covered by particle along its trajectory in a D-dimensional frame, is called the world line. We must not mistake the world line of a particle with the 3-dimensional conventional trajectory, since the latter one doesn't even takes into account the time as a coordinate. For showing intuitively how the world line works, we can limit ourselves to a simple 2-dimensional world, with only one spatial coordinate $x^1 = x$ and one time coordinate $x^0 = ct$. The following picture is a representation of a generic world line in a plane:

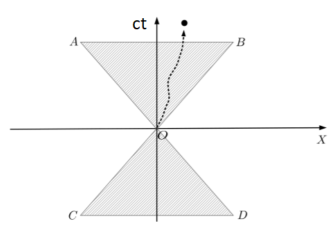

Figure 3: Spacetime diagram of a world line

In the figure, we have represented two lines DA and CB, of slope c and -c respectively, this two lines are the physical limits in which the world-line of a particle that goes through the origin O is confined, since the trespassing of this limits, would mean that we have a particle faster than light! Because of this theoretically absolute frontier, we define the light-cone coordinates as:

$$x^+ \equiv \frac{1}{\sqrt{2}}(x^0 + x^1) \tag{37}$$

$$x^- \equiv \frac{1}{\sqrt{2}}(x^0 - x^1) \tag{38}$$

You can check how, trough this new definitions, we have located our new axis in the limits of the light-cone. With the new coordinates (37) and (38) we can define the new vectors $x^\mu = (x^+, x^-, x^2, x^3)$, were we have left the rest of the coordinates, which we will call transverse coordinates, untouched. While working with light-cone coordinates, one must take into account that the concept of velocity becomes a little tricky, since time itself is no longer a coordinate, as well as x. Although both x^+ and x^- are defined in terms of both a temporal and a spacial coordinate, by convention, is usual to establish x^- as the spacial coordinate and x^+ as the temporal coordinate. Since we have performed a change of coordinates, we must rewrite the Minkoski metric tensor:

$$\eta_{\mu\nu} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{39}$$

Another relevant consequence, is that the light-cone four-momentum new values are:

$$p^+ = \frac{1}{\sqrt{2}} = (p^0 + p^1) \tag{40}$$

$$p^- = \frac{1}{\sqrt{2}} = (p^0 - p^1) \tag{41}$$

Finally, we can obtain the new form of the Lorentz transformations matrix for a boost with the conditions previously presented in (33). For our old coordinates (29), we have that the equations contained in the matrix (33) essentially mean that:

$$x^{0\prime} = \gamma\left(x^0 - \beta x^1\right); \quad x^{1\prime} = \gamma\left(-\beta x^0 + x^1\right); \quad x^{2\prime} = x^2; \quad x^{3\prime} = x^3$$

So, if we apply the definition of light-cone coordinates (37) (38) we get that

$$x^{+\prime} = \frac{1}{\sqrt{2}}\left[x^{0\prime} + x^{1\prime}\right] = \frac{1}{\sqrt{2}}\left[\gamma\left(x^0 - \beta x^1\right) + \gamma\left(-\beta x^0 + x^1\right)\right] = \frac{\gamma}{\sqrt{2}}\left[x^0(1-\beta) + x^1(1-\beta)\right] =$$
$$= \frac{\gamma(1-\beta)}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\left(x^+ + x^-\right) + \frac{1}{\sqrt{2}}\left(x^+ - x^-\right)\right] = \gamma(1-\beta)x^+$$

if we follow the same procedure for x^-

$$x^{-\prime} = \frac{1}{\sqrt{2}}\left[x^{0\prime} - x^{1\prime}\right] = \frac{1}{\sqrt{2}}\left[\gamma\left(x^0 - \beta x^1\right) - \gamma\left(-\beta x^0 + x^1\right)\right] = \frac{\gamma}{\sqrt{2}}\left[x^0(1-\beta) - x^1(1-\beta)\right] =$$
$$= \frac{\gamma(1-\beta)}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\left(x^+ + x^-\right) - \frac{1}{\sqrt{2}}\left(x^+ - x^-\right)\right] = \gamma(1-\beta)x^-$$

The coordinates x^2 and x^3 remain the same, so no more calculations are needed. We find that the expression for the Lorentz transformations matrix in light-cone coordinates is:

$$L^\mu_\nu = \begin{pmatrix} \gamma(1-\beta) & 0 & 0 & 0 \\ 0 & \gamma(1-\beta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{42}$$

Another curious result because, according to the above, we can define the light-cone coordinates as those in which the Lorentz transformations for inertial systems are diagonal. As an advance, the most interesting property of this coordinates will show up in the quantization of relativistic strings, where the process turns out to be very natural.

Until now, when we have talked about special relativity, we have always assumed a four-dimensional frame. But as we will see in one the most astonishing results of String Theory, there must exist several extra spatial dimensions in our universe for this theory to be correct, let's take a look at how can this be possible

4.3 Extra dimensions: The concept of compactification

As we previously stated, we have never required of more than four coordinates or dimensions for locating a punctual event in any experiment ever. But, on the other hand, and as we will prove, String Theory predicts the existence of several extra dimensions. On paper, this doesn't represent a big issue for special relativity, which can be easily expanded to any dimensional space-time of range D:

$$x^{\mu} = (x^{+}, x^{-}, x^{2}, x^{3}, x^{4}, ..., x^{d}) \tag{43}$$

Where D = d+1, being D the number of total dimensions, while d is the number of spacial dimensions of the system. Note that we are using light-cone coordinates, so our new D-dimensional metric tensor is:

$$\eta_{\mu\nu} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & ... & ... & ... \\ 0 & 0 & 0 & 0 & ... & ... & ... \\ 0 & 0 & 0 & 0 & ... & ... & 1 \end{pmatrix} \tag{44}$$

The same procedure must be followed for safely expanding all the elements of special relativity in as many dimensions as we want. The problem here is that, although the paper holds it, How can extra-dimensions merge with observation? The first thought that comes to mind is that it is not possible, but theoretical physics suggests a pretty reasonable alternative answer, which relies on the concept of compactification. First of all, a dimension is said to be compactified when the values that the coordinate related to that dimension takes are periodic, for understanding how this works, we can imagine a simplified example.

Imagine that somebody is in a world where you can only move forward and backward, in other words, you only have one spatial dimension. Now, imagine that this person does the only thing he or she can, start walking aimlessly hoping for something to happen. After a while, he or she realises that every time a certain distance is covered, he or she returns to the starting point, and that this distance is a constant, which we can call it as we want, for example, $2\pi R$. This experience shows our observer a periodicity in the dimension, which we can expressed by writing:

$$w \sim w + 2\pi R \tag{45}$$

Mathematicians call a definition of this kind an identification, and it means that our dimension has a periodicity, hence it is compactified. There are many possible kinds of compactifications, and our current example is just the simplest of them all. In fact, the form of the identification constant provides us a clue about the mathematical shape, or orbifold, of the compactification, since it takes the form of the perimeter of a circle. See how by knowing the interval $0 \leq w \leq 2\pi R$, which is called fundamental domain, and the identification (45), we have completely defined the properties of the dimension w. In principle, the constant the value of R can be as big or as small as we want, but for being compatible with the real world, it must be extremely short, according to many experts in String Theory, we could be talking about the scale of magnitude of the Planck length $l_P = 1.616255(18)x10^{-35}$ m, this ridiculously tiny size would be the reason of why we have not detected any extra dimension yet. By combining a compactified dimension w, as defined by (45), with the four usual space-time dimensions, we can build up a five-dimensional world, this has been already proposed and is known as the Kaluza-Klein theory:

$$x^{\mu} \equiv (t, x, y, z, w); \qquad w \sim w + 2\pi R \tag{46}$$

In the figure (4), we have that for every point in space, there is a fifth dimension which is compactified and takes the form of a circle, this dimension is what we have represented by w in (46). This means that even if a

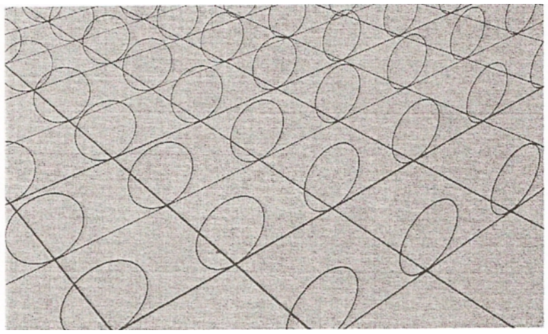

Figure 4: Representation on the plane of the Kaluza-Klein extra dimension

particle (or a string) seemed to be static in the traditional dimensions x, y and z, it could be moving extremely fast along the w dimension without us noticing it.

4.4 Action and equations of motion of a relativistic point particle

In chapter 3, we showed how powerful the action is in the obtainment of physical results, now we are going to obtain the equations of motion of a relativistic point particle using this tool. First, the action of a non-relativistic free particle is given by its kinetic energy, so we have the following non-relativistic action:

$$S_{nr} = \int_{\mathscr{P}} L dt = \int_{t_i}^{t_f} \frac{1}{2} m v^2 dt \tag{47}$$

After analysing the above, we can quickly check that this action allows particles to move at any velocity, but according to special relativity this situation is forbidden, so therefore (47) cannot be correct. A second condition arises from the fact that, for finding an appropriate action which satisfies special relativity, we must ensure that all the observers in the different Lorentz frames derive the same equations of motion. Our last condition, which any relativistic or non-relativistic action must satisfy, is to ensure that the action shows the correct units of $[Energy] \cdot [Time]$. Due to all the above, we propose the following action:

$$\boxed{S = -mc \int_{\mathscr{P}} ds = -mc^2 \int_{t_i}^{t_f} dt \sqrt{1 - \frac{v^2}{c^2}}} \tag{48}$$

For the last equality we have just applied the definition of the interval (31) and the Lorentz transformations (33). In the right term, we have a factor mc^2, which has units of energy, multiplying something that has units of time (the square root is adimensional), so we get the correct magnitude of the action. The square root also ensures that $v < c$, since the opposite would lead to non-physical results. See also how the equations of motion will be the same in all reference systems, because we have used only Lorentz scalars! Due to the definition of the action itself, we can inmediately derivate the relativistic lagrangian, and by performing an expansion up to first order of the latter, we can check how, for small velocities, it is completely compatible with the non-relativistic Lagrangian given by the action (47):

$$L = -mc^2 \sqrt{1 - \frac{v^2}{c^2}} \simeq -mc^2 + \frac{1}{2} m v^2; \qquad v << c \tag{49}$$

The first term is constant, so it doesn't affect any derivation, while the second one is exactly the same that we found in in the non-relativistic scenario. We can continue checking the validity of (48) by calculating the canonical momentum:

$$\vec{p} = \frac{\partial L}{\partial \vec{v}} = -mc^2 \left(-\frac{\vec{v}}{c^2} \right) \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{m\vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{50}$$

This momentum is just the relativistic momentum for a free particle, and in the particular case of $v \ll 1$ we return to the the classical non-relativistic expression.

Another interesting property about the action of a free point particle is the reparameterization invariance. The world line (see fig (3)), which remember is the trajectory of the particle in space-time, can be defined at any point by giving a single parameter, as for example, the proper time τ. The parameterization, is in fact a common property of any line, lets use it to to rewrite the interval (31):

$$ds^2 = -\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} (d\tau)^2 \tag{51}$$

We utilised a trick above by multiplying and dividing by the same thing, so if we substitute it in the action (48) we get:

$$S = -mc \int_{t_i}^{t_f} \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}} d\tau \tag{52}$$

The results derived from the action cannot depend on the choice of parameter, the proper time is obviously a parameter with an important physical relevance, but we can choose any other and obtain the same equations. Lets see how this can be checked for an arbitrary parameter σ. We perform a change of variable using the chain rule:

$$\frac{dx^\mu}{d\tau} = \frac{dx^\mu}{d\sigma} \frac{d\sigma}{d\tau}$$

If we perform this change of variable in (52), we get that

$$S = -mc \int_{t_i}^{t_f} \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma}} \frac{d\sigma}{d\tau} d\tau = -mc \int_{t_i'}^{t_f'} \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma}} d\sigma$$

The above result takes the same form as (52), but with different values of the integration limits, this is logical since the τ and σ are not identical, we say then that the mentioned action is manifestly reparameterization invariant. The previous conclusion is relevant, because it means that we can choose the parameter that suits us better for our calculations and also preserve the action unchanged, this is the same idea behind the canonical transformations of classical Hamiltonian mechanics.

Once the preparations are finished, is the time for deriving the equations of motion for a relativistic point particle. We start doing so by calculating the δS

$$\delta S = -mc \int_{t_i}^{t_f} \delta(ds) \tag{53}$$

by using the definition of the interval (31) and the symmetry of the metric tensor (30) we get that

$$\delta(ds) = -\eta_{\mu\nu} \delta(dx^\mu) \frac{dx^\nu}{ds} \tag{54}$$

so we substitute (54) in (53)

$$\delta S = -mc \int_{t_i}^{t_f} -\eta_{\mu\nu} \delta(dx^\mu) \frac{dx^\nu}{ds} = mc \int_{t_i}^{t_f} \eta_{\mu\nu} \frac{d(\delta x^\mu)}{d\tau} \frac{dx^\nu}{ds} d\tau \tag{55}$$

We need to rewrite the above so we have something multiplying the variation of the coordinates δx^μ. For obtaining so, we use the property of the derivation of the product:

$$\delta S = mc \int_{t_i}^{t_f} d\tau \frac{d}{d\tau} \left(\eta_{\mu\nu} \delta x^\mu(\tau) \frac{dx^\nu}{ds} \right) - \int_{t_i}^{t_f} d\tau \delta x^\mu(\tau) \left(mc\ \eta_{\mu\nu} \frac{d}{d\tau} \frac{dx^\nu}{ds} \right) \tag{56}$$

As it is usual in this cases, and we already saw for the non-relativistic string, we have a first term that disappears when we set the initial conditions, and a second term which is the one that really interests us. In the example of a classical string (14), we had a third term related with the extremes of that object, but it doesn't show up here since we have a punctual particle. Because of this reasons, we overview the study of the first term, which we know will disappear, and we focus on the latter:

$$\delta S = -\int_{t_i}^{t_f} d\tau \delta x^\mu(\tau) \left(mc\, \eta_{\mu\nu} \frac{d}{d\tau} \frac{dx^\nu}{ds} \right) = -\int_{t_i}^{t_f} d\tau \delta x^\mu(\tau) \eta_{\mu\nu} \frac{dp^\nu}{d\tau} = -\int_{t_i}^{t_f} d\tau \delta x^\mu(\tau) \frac{dp_\mu}{d\tau} \tag{57}$$

After this calculations, it becomes clear that for the variation of the action to be zero, we need our equations of motion to be:

$$\boxed{\frac{dp_\mu}{d\tau} = 0} \tag{58}$$

An important aspect about this result, is that we have not required either the momentum p_μ or its counterpart p^μ to be of a certain dimension. It is true that in (35) we initially defined this vector as a four-dimensional vector, taking the index μ the values $\mu = 0, 1, 2, 3$, but as we explained in section 4.3, there is no impediment in theory to expand this definition to D dimensions. Other fundamental conclusion, which supports our previous assumption of defining the action as (48), is that (58) means that the energy and classical momentum are conserved, we could have not expected otherwise for a free particle. Also, due to reparameterization invariance, the equations of motion must hold for other arbitrary parameters than the proper time τ, which can be defined from any reference system, this means that we have obtained a Lorentz invariant equations of motion.

4.5 Light-cone point particle

To finish with our analysis of the relativistic point particle, we are going to apply the light-cone gauge to this object. This formalism will result to be very useful when we quantize this model, which will be the previous step in the quantization of the relativistic string. We begin expressing the action of the relativistic particle (52) more compactly by simplifying the term inside the square root

$$\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = \eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = \dot{x}^2 \tag{59}$$

so our action for the point particle (48) in natural units (c=e=$\hbar$=1), takes the new form

$$S = -m \int_{\tau_i}^{\tau_f} \sqrt{-\dot{x}^2} d\tau \tag{60}$$

and therefore the equations for conjugated momentum components (35) are:

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = \frac{m \dot{x}_\mu}{\sqrt{-\dot{x}^2}} \tag{61}$$

Remember that the components of the momentum are conserved according to our equation of motion (58). Now, we define the light-cone gauge of the particle, by defining a gauge, what we are doing is stating a condition that a free relativistic point particle fulfils. This condition is satisfied in a certain reference system, which is precisely the one we impose, choosing a certain gauge doesn't alter the equations of motion, and it is only a help for simplifying the process:

$$x^+(\tau) = \frac{1}{m} p^+ \tau \tag{62}$$

This gauge is very logical, since the momentum divided by the mass has units of velocity and we set the parameter τ to have units of time. If we derive with respect to τ the gauge expression above, substitute it in the equation for the momentum (61) applied to the p^+ coordinate and we square it, we obtain the following constraint:

$$\dot{x}^2 = -1 \longrightarrow p_\mu = m \dot{x}_\mu \tag{63}$$

Where we have applied the equation (61) for the momentum to the second expression, this relation between the momentum and the velocity of the string allows us to rewrite the constraint of the light-cone gauge (63):

$$p^2 + m^2 = 0 \tag{64}$$

See how the imposition of the light-cone gauge reduces in one the number of independent components of the momentum that a free point particle can hold. This constraint can be developed by expanding into light-cone components throughout the metric tensor (44). We do so by labelling the transverse coordinates $x^I = (x^2, x^3, .., x^d)$:

$$-2p^+p^- + p^Ip^I + m^2 = 0 \longrightarrow p^- = \frac{1}{2p^+}\left(p^Ip^I + m^2\right) \tag{65}$$

So we get that one of the components of the momentum is not independent from the others due to the imposed constraint. The solution for the other coordinates turns out to be:

$$x^-(\tau) = x_0^- + \frac{1}{m}p^-\tau \tag{66}$$

$$x^I(\tau) = x_0^I + \frac{1}{m}p^I\tau \tag{67}$$

Where the x_0^- and $x_0^I(\tau)$ are constants, check how a term of this kind does not appear in (62), but it is the only difference between them. After this results of the light-cone gauge, we conclude that we only need to know the value of the coordinates $x^+(\tau)$, $x^-(\tau)$ and $x^I(\tau)$ for being able to fully describe the movement of the relativistic point particle. If we take a look at their expressions (72), (76) and (77), we can find in their dependence, the following dynamical variables that we ought to know for describing the relativistic point particle:

$$\left(x_0^-, p^+, x^I, p^I\right) \tag{68}$$

5 The relativistic string[3]

Ahora vamos a construir un modelo para las cuerdas relativistas, que son aquellas cuerdas que vibran tan rápidamente que no pueden ser estudiadas con métodos no relativistas. Para ello, nos basaremos en la partícula puntual relativista y en el enfoque de parametrización del capítulo anterior, de esta forma obtendremos y justificaremos la acción de Nambu-Goto. Desde esta acción seremos capaces de derivar y entender, aplicando la mecánica lagrangiana, las sorprendentes conclusiones que surgen de estas velocidades tan extremas, también a su vez obtendremos las cantidades conservadas en estos objetos matemáticos. Al final, recuperaremos las coordenadas del cono de luz para obtener, mediante la aplicacion de la invariancia frente a reparametrizaciones de la hoja del mundo, los modos de oscilación normales de Visasoro, estos modos se mostrarán como una representación sencilla desde la que podremos cuantizar las cuerdas relativistas.

We are now going to build a model for relativistic strings, which are those strings that vibrate so fast that they cannot be studied with non-relativistic methods. For doing so, we will base on the relativistic point particle and the parameterization approach of the previous chapter, thus, in this fashion we will obtain and justify the Nambu-Goto action. From the Nambu-Goto action, we will be able to derive and understand, by applying lagrangian mechanics, the surprising results that arise from this extreme velocities, as well as obtaining the conserved quantities present in this mathematical objects. In the end, we will recover the light-cone coordinates for obtaining, trough applying the reparameterization invariance of the world sheet, the Visasoro transverse modes, this modes will naturally show up as a simple representation from where we can quantize relativistic strings.

5.1 Parameterization of the world sheet

We explained in the previous chapter how a point particle describes a line in the space-time called the world line. In the same fashion, the set of points that compose a string, generate a surface in space-time, this surface is conventionally called the world sheet of a string. If the string is open, the world sheet will have a folded plane, and if the string is closed we will have a cylindrical-like form.

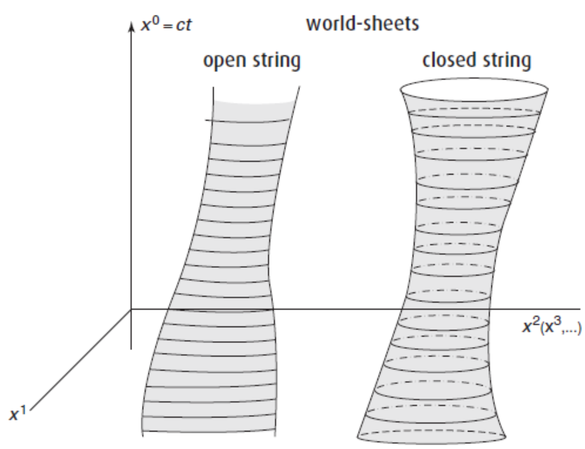

Figure 5: Spacetime diagram of two world sheets, at the left and open string and at the right a closed string

We must be aware that, from now on, the strings with which we are going to deal now are not made by rope, or by any other day-to-day material, what we are going to approach are mere mathematical one-dimensional elements with no inner structure whatsoever. To parameterize the world sheet surface created by this strings we will need two parameters instead of one, so any point of the world sheet in a D-dimensional world can be located knowing only this two values, we set this parameters to be dimensionless (in contrast to the point particle) and we call them τ and σ :

$$X(\tau,\sigma) = \Big(x^0(\tau,\sigma), x^1(\tau,\sigma), x^2(\tau,\sigma), ..., x^d(\tau,\sigma)\Big) \tag{69}$$

Hereinafter, we will reserve the capital letter X to coordinates in the world sheet, that is, in space-time. On the other hand, in the target space there is also generated another surface, composed by all the positions that the string takes with time, for this vectors we will use the classic $\vec{x}$. This surface is a projection of the world sheet in space, and can be parameterized with two parameters too, we call them ξ^1 and ξ^2:

$$\vec{x}(\xi^1, \xi^2) = \Big(x^1(\xi^1, \xi^2), x^2(\xi^1, \xi^2), ..., x^d(\xi^1, \xi^2)\Big) \tag{70}$$

Note that the above vector is purely spatial, while (69) has also a temporal component $x^0(\tau, \sigma)$. We will see first how to parameterize the surface created in the target space, and then we will apply this results to the world sheet. For calculating the area of a small two-dimensional element in the target space surface, we rely on the fact that, for an infinitesimally small region, this surface is equivalent to a parallelogram. The sides of this parallelogram will be:

$$d\vec{v_1} = \frac{\partial \vec{x}}{\partial \xi^1} d\xi^1; \qquad d\vec{v_2} = \frac{\partial \vec{x}}{\partial \xi^2} d\xi^2 \tag{71}$$

Where we are using the chain rule to write the sides in terms of the parameters. The reason why each of the vectors depends only on one of the parameters is simply because it is the easier to define the parameters this way. This simplification has an important foundation since, as we proved in the case of a point particle, the election of parameters is arbitrary, so we can define them as we wish. The differential of area represented by the two vectors above is:

$$dA = |d\vec{v_1}||d\vec{v_2}||sin\theta| \tag{72}$$

Where θ is the angle between the two vectors. By applying some basic trigonometric properties, and the definition of the Euclidean scalar product, we can work out a little bit the above:

$$dA = |d\vec{v_1}||d\vec{v_2}|\sqrt{1 - cos^2\theta} = \sqrt{|d\vec{v_1}|^2|d\vec{v_2}|^2 - |d\vec{v_1}|^2|d\vec{v_2}|^2 cos^2\theta} = \sqrt{(d\vec{v_1} \cdot d\vec{v_1})(d\vec{v_2} \cdot d\vec{v_2}) - (d\vec{v_1} \cdot d\vec{v_2})^2} \tag{73}$$

Then we substitute the value of the vectors as given by (71)

$$dA = d\xi^1 d\xi^2 \sqrt{\left(\frac{\partial \vec{x}}{\partial \xi^1} \cdot \frac{\partial \vec{x}}{\partial \xi^1}\right)\left(\frac{\partial \vec{x}}{\partial \xi^2} \cdot \frac{\partial \vec{x}}{\partial \xi^2}\right) - \left(\frac{\partial \vec{x}}{\partial \xi^1} \cdot \frac{\partial \vec{x}}{\partial \xi^2}\right)^2} \tag{74}$$

So the total area that a string generates in the target space is given by the integration of the above:

$$A = \int d\xi^1 d\xi^2 \sqrt{\left(\frac{\partial \vec{x}}{\partial \xi^1} \cdot \frac{\partial \vec{x}}{\partial \xi^1}\right)\left(\frac{\partial \vec{x}}{\partial \xi^2} \cdot \frac{\partial \vec{x}}{\partial \xi^2}\right) - \left(\frac{\partial \vec{x}}{\partial \xi^1} \cdot \frac{\partial \vec{x}}{\partial \xi^2}\right)^2} \tag{75}$$

See how this area is in fact the trajectory followed by the string in space, it would be the result of taking photos of the string at many consecutive times and putting them together. Now we define the following matrix, in which are involved the derivatives above:

$$g_{ij} = \begin{bmatrix} \frac{\partial \vec{x}}{\partial \xi^1} \cdot \frac{\partial \vec{x}}{\partial \xi^1} & \frac{\partial \vec{x}}{\partial \xi^1} \cdot \frac{\partial \vec{x}}{\partial \xi^2} \\ \frac{\partial \vec{x}}{\partial \xi^2} \cdot \frac{\partial \vec{x}}{\partial \xi^1} & \frac{\partial \vec{x}}{\partial \xi^2} \cdot \frac{\partial \vec{x}}{\partial \xi^2} \end{bmatrix} \tag{76}$$

You can check how it resembles the form of a metric tensor in a two-dimensional space, something that is more clear when we write the area (75) in the more elegant way:

$$A = \int d\xi^1 d\xi^2 \sqrt{g} \tag{77}$$

Where g is the determinant of the matrix g_{ij}. We will now check how the reparameterization invariance that we supposed to hold in this case, for doing so, we know that if we have a set of variables which describe a particular system, we can perform a change of the variables of integration throughout the Jacobian matrix:

$$d\xi^1 d\xi^2 = \left| det\left(\frac{\partial \xi^i}{\partial \tilde{\xi^j}}\right)\right| d\tilde{\xi^1} d\tilde{\xi^2} = |detJ|\, d\tilde{\xi^1} d\tilde{\xi^2} \tag{78}$$

Where $(\tilde{\xi^1}, \tilde{\xi^2})$ are two new arbitrary parameters and J is the Jacobian. We also know that the metric tensor g_{ij} must have a mathematical relationship with the metric tensor corresponding to the new parameters $\tilde{g_{ij}}$, and is that both of them must hold the same interval (see (31)). This is expressed mathematically such as:

$$g_{ij}(\xi^1, \xi^2) d\xi^i d\xi^j = \tilde{g}_{km}(\tilde{\xi^1}, \tilde{\xi^2}) d\tilde{\xi}^k d\tilde{\xi}^m \tag{79}$$

If we make use of the chain rule at the right side of the equation we obtain that:

$$g_{ij}(\xi^1, \xi^2) = \tilde{g}_{km}(\tilde{\xi^1}, \tilde{\xi^2}) \frac{\partial \tilde{\xi}^k}{\partial \xi^i} \frac{\partial \tilde{\xi}^m}{\partial \xi^j} \tag{80}$$

Which, after applying the definition of the Jacobian and taking the determinant at both sides leads us to:

$$\sqrt{g} = \sqrt{\tilde{g}} \left| det\tilde{J} \right| \tag{81}$$

Here $\left| det\tilde{J} \right|$ is the Jacobian correspondent to the change of variables from the ones with an accent to those without it. By combining the result above with (78), and substituting in there the definition of the area in the target space (77), we derive a quite expected result:

$$A = \int d\xi^1 d\xi^2 \sqrt{g} = \int d\xi^1 d\xi^2 \sqrt{\tilde{g}} \left| det\tilde{J} \right| = \int |detJ|\, d\tilde{\xi^1} d\tilde{\xi^2} \sqrt{\tilde{g}} \left| det\tilde{J} \right| = \int d\tilde{\xi^1} d\tilde{\xi^2} \sqrt{\tilde{g}}$$

Proving the reparameterization invariance of the area! Now that we have obtained the area of the surface generated by the string in the target space, and we have successfully proven its invariance, we can approach the problem of the area of the world sheet. This is the surface which is equivalent to the world line that we studied in the point particle chapter, and therefore, also has a deep physical meaning, narrowly related with the action. In this case the reasoning is very similar, we define the area vectors of the world sheet in analogy with (72):

$$dv_1^\mu = \frac{\partial X^\mu}{\partial \tau} d\tau \qquad\qquad dv_2^\mu = \frac{\partial X^\mu}{\partial \sigma} d\sigma \tag{82}$$

So we would expect the area of the world sheet in terms of the parameters to be:

$$dA = \sqrt{(d\vec{v_1} \cdot d\vec{v_1})(d\vec{v_2} \cdot d\vec{v_2}) - (d\vec{v_1} \cdot d\vec{v_2})^2} \tag{83}$$

Now we have supposed a similar expression to the one we obtained for the target space (73), where the dot refers to the relativistic product as defined by the metric tensor in (31). Due to this different scalar product, unlike in the purely spatial case, it turns out that the products on the left side of the square root can be negative, meaning that we could eventually have an imaginary number, something that we cannot allow in the physical concept of area. In the case of a world sheet, we have that either $d\vec{v_1}$ or $d\vec{v_2}$ will be timelike vectors, meaning that the norm of one of them is negative. Subsequently, we have to perform a change of sign inside the the whole square root, by substituting (82) and integrating over the range of the parameters we get the following area:

$$A = \int d\tau d\sigma \sqrt{\left(\frac{\partial X}{\partial \tau} \cdot \frac{\partial X}{\partial \sigma}\right)^2 - \left(\frac{\partial X}{\partial \tau}\right)^2 \left(\frac{\partial X}{\partial \sigma}^2\right)} \tag{84}$$

The physical paths that a string will cover between two different configurations are those that make this area minimal, this is of course a key concept behind the whole idea of the action. A very important concept that we must really beware of in the case of open strings, is that the world sheet only allows us to keep track of the endpoints of itself. Now we are going to derive the relationship between the area of the world sheet and the action of the string, this action is called the Nambu-Goto string action.

5.2 The Nambu-Goto action: Equations of motion and boundary conditions

After revising the derivation of the action for the particular case of a relativistic point particle (52), and taking a look at the expression of the area of the world sheet generated by a string (84). One can figure out that a logical action for a relativistic string could be defined as proportional to the area of the world sheet, we must keep in mind that this area is a Lorentz scalar and invariant under reparameterization, how it could not be otherwise. We must also not forget that the action has units of $[Energy] \cdot [Time]$. Due to all the above, the Nambu-Goto string action is proposed:

$$\boxed{S = \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_*} d\sigma \mathscr{L}\left(\dot{X}^\mu, X'^\mu\right) = -\frac{T_0}{c}\int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_*} d\sigma \sqrt{\left(\dot{X}\cdot X'\right) - (\dot{X})^2 (X')^2}} \tag{85}$$

Where remember that $\mathscr{L}\left(\dot{X}^\mu, X'^\mu\right)$ is the lagrangian density. The parameter σ has been defined within the limits $\sigma \in [0, \sigma^*]$, this two values denote the endpoints of the string. We have also made a change in the notation in the sake of simplification:

$$\dot{X}^\mu \equiv \frac{\partial X^\mu}{\partial \tau} \qquad\qquad X'^\mu \equiv \frac{\partial X^\mu}{\partial \sigma} \tag{86}$$

See how we have used two Lorentz scalars, the speed of light c and the tension of the string T_0 to ensure that we obtain the right units, this action is for a free relativistic string, meaning by free that it is not affected by any interaction. In the same fashion as we did in the case of a surface in the target space, we can use with a matrix to simplify the expression above:

$$\gamma_{\alpha\beta} = \begin{bmatrix} (\dot{X})^2 & \dot{X}\cdot X' \\ \dot{X}\cdot X' & (X')^2 \end{bmatrix} \tag{87}$$

And by taking the determinant γ, we can rewrite the Nambu-Goto action in a manifestly reparameterization invariant form:

$$S = -\frac{T_0}{c}\int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_*} d\sigma \sqrt{-\gamma} \tag{88}$$

If you remember how we derived the momentum densities for a non-relativistic string (15), you saw that we only needed of two conjugated momentum densities, $\mathscr{P}^t$ and $\mathscr{P}^x$, this was due to fact that we could confine our oscillations to a single coordinate. For relativistic strings we will require of up to $2\cdot\mu$ different components, this momentum densities can be calculated from lagrangian mechanics:

$$\mathscr{P}^\tau_\mu \equiv \frac{\partial \mathscr{L}}{\partial \dot{X}^\mu} = -\frac{T_0}{c}\frac{2\left(\dot{X}\cdot X'\right)X'_\mu - 2(X')^2\dot{X}_\mu}{2\sqrt{\left(\dot{X}\cdot X'\right) - (\dot{X})^2(X')^2}} = -\frac{T_0}{c}\frac{\left(\dot{X}\cdot X'\right)X'_\mu - (X')^2\dot{X}_\mu}{\sqrt{\left(\dot{X}\cdot X'\right) - (\dot{X})^2(X')^2}} \tag{89}$$

$$\mathscr{P}^\sigma_\mu \equiv \frac{\partial \mathscr{L}}{\partial X'^\mu} = -\frac{T_0}{c}\frac{2\left(\dot{X}\cdot X'\right)\dot{X}_\mu - 2(X')^2 X'_\mu}{2\sqrt{\left(\dot{X}\cdot X'\right) - (\dot{X})^2(X')^2}} = -\frac{T_0}{c}\frac{\left(\dot{X}\cdot X'\right)\dot{X}_\mu - (X')^2 X'_\mu}{\sqrt{\left(\dot{X}\cdot X'\right) - (\dot{X})^2(X')^2}} \tag{90}$$

The momentum components obtained are extremely complex, since they depend on up to $2\cdot\mu$ different derivatives each. Due to this complexity, choosing the correct parameterization that allow us to eliminate as much components as possible will be crucial. Returning to the action as written in (85), we derive the equations of motion of a relativistic string by calculating δS:

$$\delta S = \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_*} d\sigma \left[\frac{\partial \mathscr{L}}{\partial \dot{X}^\mu}\frac{\partial(\delta X^\mu)}{\partial \tau} + \frac{\partial \mathscr{L}}{\partial X'^\mu}\frac{\partial(\delta X^\mu)}{\partial \sigma}\right] = \tag{91}$$

We introduce the results obtained above:

$$\delta S = \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_*} d\sigma \left[\mathscr{P}^\tau_\mu \frac{\partial(\delta X^\mu)}{\partial \tau} + \mathscr{P}^\sigma_\mu \frac{\partial(\delta X^\mu)}{\partial \sigma}\right] =$$

$$= \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_*} d\sigma \left[\frac{\partial}{\partial\tau}(\delta X^\mu \mathscr{P}^\tau_\mu) + \frac{\partial}{\partial\sigma}(\delta X^\mu \mathscr{P}^\sigma_\mu) - \delta X^\mu \left(\frac{\mathscr{P}^\tau_\mu}{\delta\tau} + \frac{\mathscr{P}^\sigma_\mu}{\delta\sigma} \right) \right] \tag{92}$$

You can check how if we develop the derivatives that appear in the last equality and sum, we basically return to the previous one but, by writing them in this way, we can separate some terms. This is useful for knowing the equations of motion, that appear by making $\delta S = 0$.

$$\delta S = \int_{\tau_i}^{\tau_f} d\tau \left[\delta X^\mu \mathscr{P}^\sigma_\mu \right]_0^{\sigma_1} - \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_*} d\sigma \delta X^\mu \left(\frac{\mathscr{P}^\tau_\mu}{\delta\tau} + \frac{\mathscr{P}^\sigma_\mu}{\delta\sigma} \right) \tag{93}$$

So finally, the equations of motion derived from the Nambu-Goto action for a relativistic string are:

$$\boxed{\frac{\mathscr{P}^\tau_\mu}{\partial\tau} + \frac{\mathscr{P}^\sigma_\mu}{\partial\sigma} = 0} \tag{94}$$

This equations are quite similar to the one obtained for a classical string (16) but, of course, the momentum densities take notoriously different forms! We must also comment on the fact that we have not set the range of values that takes the index μ, so the current analysis is not conditioned by, or requires of a particular number of dimensions. Now, as we did with the classical string, lets evaluate the boundary conditions. As we have already commented on, strings can be attached to D-branes, and this elements are responsible of the boundary conditions, this surfaces can be of various dimensions, conditioning the vibrations of the strings.

As we know, the Dirichlet boundary condition fixes the position endpoints of the string, in String Theory this is equivalent to fixing the endpoints of the string to the surface of a D-brane. So basically, we have that the position of the endpoints fulfils the following condition:

$$\boxed{\frac{\partial X^\mu(\tau,0)}{\partial\tau} = \frac{\partial X^\mu(\tau,\sigma_*)}{\partial\tau} = 0} \qquad \mu \neq 0 \tag{95}$$

Obviously, fixing the position of some points in space doesn't mean that the time has to be fixed too, so $\mu \neq 0$. On the other hand, we have the free point boundary condition, which means that the string is not attached to any object. this is equivalent to having a space-filling D-brane:

$$\boxed{\mathscr{P}^\sigma_\mu(\tau,0) = \mathscr{P}^\sigma_\mu(\tau,\sigma_*) = 0} \tag{96}$$

In this case we have that the index μ can also take the value $\mu = 0$, this is neccesary for the conservation of the energy

5.3 The Nambu-Goto action: Some fundamental results

With all the results that we have derived, we can start doing some calculations. For instance, we can calculate the rest energy of an stretched relativistic string of length a, which we assume is extended along the x^1 coordinate, so we have the following boundary conditions:

$$X^1(\tau,0) = 0 \qquad X^1(\tau,\sigma^*) = a; \qquad\qquad X^\mu(\tau,\sigma) = 0 \qquad \mu \neq 0,1 \tag{97}$$

Remember that when we talk about the parameter $\sigma = 0$, we refer to one of the endpoints, and when we write $\sigma = \sigma^*$, we refer to the other. All the other points of the string at a particular time will be also a funcion of the parameter σ, so we have that the position X^1 is a function of the parameter σ and then $X^1 = f(\sigma)$. For the x^0 coordinate, we can simply choose a reference system that moves with the same inertia as the string, so $X^0 = c\tau$, see how this can be done because the string is not vibrating, when all the points share the same inertia and therefore, all share the same proper time. Because of all the above, our string is represented by the vector:

$$X^\mu(\tau,\sigma) = (c\tau, f(\sigma), \vec{0}) \tag{98}$$

Where $\vec{0}$ refers to the remaining spatial coordinates. This is in fact all what we need for calculating the string action (85):

$$\dot{X}^\mu = (c, 0, \vec{0}); \quad X'^\mu = (0, f'(\sigma), \vec{0}) \quad \longrightarrow \quad (\dot{X})^2 = -c^2; \quad (X')^2 = (f'(\sigma))^2; \quad \dot{X} \cdot X' = 0 \tag{99}$$

We substitute the values above in the expression of the action, and we get that:

$$S = -\frac{T_0}{c} \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_*} d\sigma \sqrt{0 - (-c^2)(f'(\sigma))^2} = -T_0 \int_{\tau_i}^{\tau_f} d\tau \left(f(\sigma_*) - f(0)\right) = \int_{\tau_i}^{\tau_f} d\tau(-T_0 a) \tag{100}$$

We have stated that the string is at rest, so the energy is purely potential and as a consequence L = -V = -$T_0 a$, which thereforr means that what we have inside the integral is just potential energy of a string at rest. This conclusion tells us that the length of the string is proportional to the potential energy. We can find the relationship between the length and the mass of the string by a simple calculation:

$$\boxed{\mu_0 c^2 = \frac{E}{a} = \frac{V}{a} \longrightarrow \mu_0 = \frac{T_0}{c^2}} \tag{101}$$

So basically, this conclusion leads us to thinking that a longer string will have a bigger potential energy, which means a bigger mass. This result is in fact very logical, and applies to many objects, but we are going to see now how open relativistic strings show up other non-intuitive properties, especially when they have free endpoints that show the boundary condition (96).

Now, lets see what happens when the string vibrates. In this case, for discerning between the overall movement of string endpoints and the vibrational movement, we have to differentiate between the overall velocity of the string endpoints $\vec{v}$, and the transverse velocity $\vec{v}_\perp$, which is the projection of the velocity perpendicular to the world-sheet. For doing so, we have that for any vector $\vec{u}$, its component perpendicular to a unit vector $\vec{n}$, is given by the mathematical expression:

$$\vec{u}^\perp = \vec{u} - (\vec{u} \cdot \vec{n})\vec{n} \tag{102}$$

If we define the variable s to be the length along the world sheet, and therefore the normal vector to the latter is $\frac{\partial X}{\partial s}$, we get that the transverse velocity of the string is:

$$\vec{v}_\perp = \frac{\partial X}{\partial t} - \left(\frac{\partial X}{\partial t} \cdot \frac{\partial X}{\partial s}\right) \frac{\partial X}{\partial s} \tag{103}$$

Here t is the variable of time, which must correspond, as well as s, to a particular reference system. This variables can be written in terms of parameters as we have been doing lately, we can set the relationship between the length of the string and the parameter σ trough the chain rule of derivation:

$$\frac{\partial X}{\partial s} = \frac{\partial X}{\partial \sigma} \frac{d\sigma}{ds} \tag{104}$$

So if we again set $t = \tau$, we find for the Minkowski vector (29):

$$(\dot{X})^2 = -c^2 + \left(\frac{\partial \vec{x}}{\partial t}\right)^2 \qquad (X')^2 = \left(\frac{\partial \vec{x}}{\partial \sigma}\right)^2 \qquad \dot{X} \cdot X' = \frac{\partial \vec{x}}{\partial t} \cdot \frac{\partial \vec{x}}{\partial \sigma} \tag{105}$$

Take into account that the vectors $\vec{x}$ always refer to spatial coordinates. If we substitute the above, along with the transverse velocity, in the expression for the conjugated relativistic momentum at the endpoints (90), we can rewrite it like:

$$\mathscr{P}^{\sigma\mu} = -\frac{T_0}{c^2} \frac{\left(\frac{\partial \vec{x}}{\partial s} \cdot \frac{\partial \vec{x}}{\partial t}\right) \dot{X}^\mu + \left(c^2 - \left(\frac{\partial \vec{x}}{\partial t}\right)^2\right) \frac{\partial X'^\mu}{\partial s}}{\sqrt{1 - \frac{v_\perp^2}{c^2}}} \tag{106}$$

Remember that according to the boundary condition for free endpoints (96), the above must be zero at $\sigma = 0$ for all values of μ. In particular, for the $\mu = 0$ component we obtain that:

$$\mathscr{P}^{\sigma\mu} = -\frac{T_0}{c}\frac{\left(\frac{\partial\vec{x}}{\partial s}\cdot\frac{\partial\vec{x}}{\partial t}\right)}{\sqrt{1-\frac{v_\perp^2}{c^2}}} = 0 \tag{107}$$

For this momentum density to be zero, we need the numerator to be null, which means that:

$$\boxed{\frac{\partial\vec{x}}{\partial s}\cdot\frac{\partial\vec{x}}{\partial t} = 0} \qquad \textit{At the endpoints} \tag{108}$$

Here we have that the product of two derivatives is zero, this derivatives over the spatial vectors only satisfy the above condition at the endpoints. If you remember that s is the spacial coordinate along the string, the first derivative must be the tangent vector of the string, while the second derivative is basically the velocity. With this information in mind, we conclude that the endpoints move transversely to the string.

$$\boxed{v_\perp = v} \qquad \textit{At the endpoints} \tag{109}$$

($v_\perp = v$ at the endpoints), so any other movement of this points does not have any physical meaning, this conclusion shows that talking about free endpoints is equivalent to talking about the Neumann boundary condition.

Now, we have to return to the boundary condition (96), but applied to the rest of the components of the relativistic momentum density (106) at the endpoints, this are the values where the index $\mu \neq 0$. We can immediately simplify this expression by applying the above condition (108) and substituting the total spatial velocity $\vec{v} = \frac{\partial\vec{x}}{\partial t}$:

$$\mathscr{P}^{\sigma\mu} = \frac{T_0}{c^2}\frac{\left(c^2 - v^2\right)X'^{\mu}}{\sqrt{1-\frac{v^2}{c^2}}} = -T_0\sqrt{1-\frac{v^2}{c^2}}\frac{\partial X^\mu}{\partial s} = 0 \tag{110}$$

The only option for this to be null for all values of X^μ, is that v=c !
‘

$$\boxed{\left(\frac{\partial\vec{x}}{\partial t}\right)^2 = c^2} \qquad \textit{At the endpoints} \tag{111}$$

This is our first major result in String Theory. The reason why the free endpoints can move at the speed of light, is because they do not have a mass in the classical sense, the string as a whole is the one that actually has a mass due to the tension of the points that compose it. We must also understand that saying that the endpoints can vibrate at the speed of light, is not the same as saying they can move straightforward at this velocity. In this case we would have that the whole string would move at this velocity, and therefore, what we rudely see as a particle, would move at the speed of light, this only happens in the case of the massless particles (See Fig (1)).

We have learnt how to calculate the energy of an static string, and we have also studied the cinematic of the endpoints of a free string, lets find now a form of the equations of motion more familiar than the expression (94). For this purpose, we write the relativistic momentum densities (89) and (90) in terms of the transverse velocity (103):

$$\mathscr{P}^{\tau\mu} = \frac{T_0}{c^2}\frac{\frac{ds}{d\sigma}}{\sqrt{1-\frac{v_\perp^2}{c^2}}}\frac{\partial X^\mu}{\partial t} \tag{112}$$

$$\mathscr{P}^{\sigma\mu} = -T_0\sqrt{1-\frac{v_\perp^2}{c^2}}\frac{\partial X^\mu}{\partial s} \tag{113}$$

See how with the use of transverse velocity, the momentum densities take a more manageable shape which resembles to special relativity formulas. After substituting the above in the equation of motion (94), we obtain, for the spatial components ($\mu \neq 0$) the following relation:

$$\frac{\vec{\mathscr{P}}^\tau}{\partial\tau} = -\frac{\vec{\mathscr{P}}^\sigma}{\partial\sigma} \longrightarrow \frac{\partial}{\partial\sigma}\left[T_0\sqrt{1-\frac{v_\perp^2}{c^2}}\frac{\partial\vec{x}}{\partial s}\right] = \frac{T_0}{c^2}\frac{\frac{ds}{d\sigma}}{\sqrt{1-\frac{v_\perp^2}{c^2}}}\frac{\partial\vec{v_\perp}}{\partial t} \tag{114}$$

Which can be simplified a little bit if we apply the chain rule for getting rid of the σ derivative at both sides:

$$\frac{\partial}{\partial s}\left[T_0\sqrt{1-\frac{v_\perp^2}{c^2}}\frac{\partial \vec{x}}{\partial s}\right] = T_0\sqrt{1-\frac{v_\perp^2}{c^2}}\frac{\partial \vec{v_\perp}}{\partial s} = \mu_0\frac{1}{\sqrt{1-\frac{v_\perp^2}{c^2}}}\frac{\partial \vec{v_\perp}}{\partial t} \tag{115}$$

In the last identity, we just applied the previously discovered formula that relates the rest mass and tension of a string (101), we have also used that $v_\perp = v$. At each side of the above equation we can find a Lorentz factor γ, we know from special relativity that some properties or magnitudes in matter are subjected to cinematic motion, being it specially notable at close to light-speed velocities. Among these magnitudes are of course included the mass, and therefore the tension of the strings. For aesthetic purposes, we rewrite this two magnitudes like:

$$T_{eff} = T_0\sqrt{1-\frac{v_\perp^2}{c^2}} = T_0\gamma^{-1}; \qquad \mu_{eff} = \mu_0\frac{1}{\sqrt{1-\frac{v_\perp^2}{c^2}}} = \mu_0\gamma \tag{116}$$

For finally obtaining:

$$\mu_{eff}\frac{\partial^2 \vec{x}}{\partial t^2} = T_{eff}\frac{\partial^2 \vec{x}}{\partial s^2} \longrightarrow \boxed{\frac{\partial^2 \vec{x}}{\partial s^2} - \frac{\mu_{eff}}{T_{eff}}\frac{\partial^2 \vec{x}}{\partial t^2} = 0} \tag{117}$$

This is a pretty wave equation that we know pretty well, you can check how similar it is to the classical wave equation by taking a look at it in chapter 2 (4). From this result one can figure out the resemblance between relativistic strings in String Theory and the classical strings, in fact, in the non-relativistic limit where $v << c$, we have basically the same equation, as it would be expected.

5.4 Conserved charges and currents

In all physical problems, there exist some conservation laws that allow us to understand the dynamics of physical systems, knowing the invariance and symmetry properties in each situation usually simplifies the labour when analysing complex systems, and String Theory is not exception. First, we will say that we have a conserved D-dimensional current j^α in the Minkowski space when it complies that:

$$\partial_\alpha j^\alpha = 0 \tag{118}$$

Every current is intrinsically tied to a charge, in our case, by a charge we refer to any magnitude to which we can assign a value that encompasses the totality of the string. For instance, we have that if we apply the conservation rule above to the electromagnetic current, which is defined as $j^\alpha_{EM} = (c\rho, \vec{j})$, we obtain the continuity equation for the electric charge:

$$\partial_\alpha j^\alpha_{EM} = \frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0 \tag{119}$$

In lagrangian mechanics, we have that charges are associated to lagrangians L, while currents are associated to lagrangian densities $\mathscr{L}$. In this formalism we have that, if a given coordinate q_k does not explicitly appear in the lagragian (or lagrangian density), we can assure that the conjugated momentum p_k (or conjugated momentum density $\mathscr{P}_k$) associated to this coordinate is conserved. In this case we say that p_k is a conserved charge (or that $\mathscr{P}_k$ is a conserved current):

$$Q_k = \frac{\partial L}{\partial \dot{q}_k} = p_k; \qquad j^k = \frac{\partial \mathscr{L}}{\partial \dot{q}_k} = \mathscr{P}^k \tag{120}$$

In the world-sheet, which is the case that interests us, we have a charge associated with each of the components of the momentum of the string p_μ, this charges are conserved in the case of a free string, as we imposed in the boundary condition (96). On the opposite, this is not necessarily true for the Dirichlet boundary condition (95). Without pretending it, we have already been working with two relevant currents of the world sheet, that are given by the two lagrangian momentum densities with which we have been working on:

$$j^\tau_\mu = \mathscr{P}^\tau_\mu; \qquad j^\sigma_\mu = \mathscr{P}^\sigma_\mu \tag{121}$$

We have above the currents associated with the two different conjugated momenta in the world sheet, so the total momentum current is then a composition of both of them $j = (j^{\tau}_{\mu}, j^{\sigma}_{\mu})$. If we apply the definition of the conservation of the current to j we obtain that:

$$\partial_\alpha j^{\alpha}_{\mu} = \partial_\tau j^{\tau}_{\mu} + \partial_\sigma j^{\sigma}_{\mu} = \frac{\partial \mathscr{P}^{\tau}_{\mu}}{\partial \tau} + \frac{\partial \mathscr{P}^{\sigma}_{\mu}}{\partial \sigma} = 0 \tag{122}$$

We recover the equations of motion of a relativistic string (94)! This result shows that the current j is conserved in the world sheet, which also means that the total momentum p_μ of the string is conserved so we have that:

$$\boxed{\frac{dp_\mu}{dt} = 0} \tag{123}$$

This is the same result we obtained for the relativistic particle (58), something that is not surprising at all since the conservation of the momentum for an object moving freely is an already well-known result. But, the conclusion (123) reinforces our election of the Nambu-Goto action (85) as the action of a relativistic string.

Apart from the momentum density components (remember that the zeroth component of this vector is the energy density, which therefore is conserved), other interesting set of currents can be defined by:

$$\mathscr{M}^{\alpha}_{\mu\nu} = X_\mu \mathscr{P}^{\alpha}_{\nu} - X_\nu \mathscr{P}^{\alpha}_{\mu} \tag{124}$$

We have to be careful about the meaning of the indexes here. The index α takes the values of the parameters τ and σ with which we are working with, this is done in the same fashion as in (122). While the parameters μ and ν simply cover the D dimensions of our frame. By integrating over the contour of the world-sheet, we can calculate the charges associated to this currents:

$$M_{\mu\nu} = \int_{\mathscr{S}} \left(\mathscr{M}^{\tau}_{\mu\nu} d\sigma - \mathscr{M}^{\sigma}_{\mu\nu} d\tau\right) = \int \left(X_\mu \mathscr{P}^{\tau}_{\nu} - X_\nu \mathscr{P}^{\tau}_{\mu}\right) d\sigma \tag{125}$$

Here we have that the components M_{0i} (or $-M_{j0}$ since the matrix is antisymmetric) correspond to the boosts along the three spatial directions, whereas the components M_{ij} are associated with the three basic rotations. In fact, we have that the three components of the angular momentum are $L_1 = M_{23}$, $L_2 = M_{31}$ and $L_3 = M_{12}$. At the end of this dissertation, we will retake the definition above for justifying the dimensionality of spacetime that String Theory imposes.

5.5 General solution of the equations of motion for an open string

We have now collected the main conclusions that result from relativistic strings, that provide us with the relevant information we need to understand for describing the mean characteristics of this elements. What we are going to do in this section, is to obtain the general equations of motion for an open relativistic string with free endpoints, from now on, as long as we don't say otherwise we are only going to focus on open strings, although many of the properties of open strings are shared by their closed partners. If we had applied the methodology used in (115) to all the components, including the temporal one, of the momentum densities in (112) and (113), we would have obtained the the same equations of motion but extended to X^μ:

$$\frac{\partial^2 X^\mu}{\partial s^2} - \frac{\mu_{eff}}{T_{eff}} \frac{\partial^2 X^\mu}{\partial t^2} = 0 \tag{126}$$

This result can be expressed in a more elegant way if we take into account the relation between mass density and tension of the string given by (101), this, units (c=$\hbar$=e=1) is:

$$\ddot{X}^\mu - \frac{\mu_{eff}}{T_{eff}} X''^\mu \longrightarrow \boxed{\ddot{X}^\mu - X''^\mu = 0} \tag{127}$$

We must keep in mind that the above is simply a wave equation, so it has a general solution of the form:

$$X^\mu(\tau, \sigma) = \frac{1}{2}\left(f^\mu(\tau + \sigma) + g^\mu(\tau - \sigma)\right) \tag{128}$$

By applying the Neumann boundary conditions that we deduced (96), we can restrict the above to a solution which fullfils that:

$$\frac{\partial X^{\mu}}{\partial \sigma}\left(\tau, \sigma=\sigma^{*}=\pi\right)=\frac{1}{2}\left(f'^{\mu}\left(\tau+\pi\right)+f'^{\mu}\left(\tau-\pi\right)\right)=0 \tag{129}$$

See how we have called $\sigma^{*}=\pi$ one of the endpoints of the strings, this is a very common convention in String Theory. In the case of closed strings we would have had that $\sigma^{*}=2\pi$, this election of the endpoint parameter value marks the fact that we have a closed surface. For the above boundary condition to be true, we have that f'^{μ} must be a function of periodicity 2π, so we can perform a Fourier series to the derivative $f'^{\mu}(\tau \pm \sigma)$:

$$f'^{\mu}(\tau \pm \sigma)=f_{1}^{\mu}+\sum_{n=1}^{\infty}\left(a_{n}^{\mu} cos\ n(\tau \pm \sigma)+b_{n}^{\mu} sen\ n(\tau \pm \sigma)\right) \tag{130}$$

After integrating the above and substituting it into the general solution for a free relativistic string we obtain that:

$$X^{\mu}\left(\tau, \sigma\right)=f_{0}^{\mu}+f_{1}^{\mu} \tau+\sum_{n=1}^{\infty}\left(A_{n}^{\mu} cos\ n\tau+B_{n}^{\mu} sen\ n\tau\right) cos\ n\sigma \tag{131}$$

As we are approaching quantum formalism, we can start writing our results in such a way. With this purpose, we manipulate a little bit the term between the parenthesis:

$$A_{n}^{\mu} cos\ n\tau+B_{n}^{\mu} sen\ n\tau=-\frac{i}{2}\left(\left(B_{n}^{\mu}+i A_{n}^{\mu}\right) e^{i n \tau}-\left(B_{n}^{\mu}-i A_{n}^{\mu}\right) e^{-i n \tau}\right)=-i \frac{\sqrt{2 \alpha'}}{\sqrt{n}}\left(a_{n}^{\mu *} e^{i n \tau}-a_{n}^{\mu} e^{-i n \tau}\right) \tag{132}$$

We have introduced in the last equality the slope parameter α', which is very common in String Theory, this parameter is inversely proportional to the tension of the string and is defined as $\alpha'=\frac{1}{2\pi T_{0}\hbar c}$. Look how the factors $a_{n}^{\mu *}$ and a_{n}^{*} resemble to anihilation and creation operators. If we declare the first term in the general equations of motion (131) to be the initial position, or initial configuration or the string, and the second one to be related with the momentum of the string we obtain that:

$$X^{\mu}(\tau, \sigma)=x_{0}^{\mu}+2 \alpha' p^{\mu} \tau-i \sqrt{2 \alpha'} \sum_{n=1}^{\infty}\left(a_{n}^{\mu *} e^{i n \tau}-a_{n}^{\mu} e^{-i n \tau}\right) \frac{cos\ n\sigma}{\sqrt{n}} \tag{133}$$

This is an interesting equation with a lot of physics behind, by the way, if we eliminate the terms inside the sum, we obtain the movement of a free particle with initial position x_{0}^{μ} and momentum p^{μ}. On the other hand, the right side sum terms are the ones which contain the possible oscillation modes of the relativistic string, this oscillations are determined by the value of the coefficients $a_{n}^{\mu *}$ and a_{n}^{*} and the tension of the string T_{0}, which is implicit in the slope parameter α'. The presence of the imaginary number i does not make this terms imaginary, as the oscillation coefficients cancel this due to the way they were defined (132). It is important to remark that this oscillations happen in as many dimensions as values the index μ takes, this oscillations in various dimensions are expected to explain the different properties that the so called fundamental particles show.

Before ending this section, we perform another change in the notation, to write the equations of motion in terms of a single set of parameters:

$$\alpha_{0}^{\mu}=\sqrt{2 \alpha'} p^{\mu}; \qquad \alpha_{n}^{\mu}=a_{n}^{\mu} \sqrt{n}; \qquad \alpha_{-n}^{\mu}=a_{n}^{\mu *} \sqrt{n} \tag{134}$$

So our equation of motions (133) takes the form of:

$$\boxed{X^{\mu}(\tau, \sigma)=x_{0}^{\mu}+\sqrt{2 \alpha'} \alpha_{0}^{\mu} \tau+i \sqrt{2 \alpha'} \sum_{n \neq 1}^{\infty} \frac{1}{n} \alpha_{n}^{\mu} e^{-i n \tau} cos\ n\sigma} \tag{135}$$

Until now, we have obtained the general solution of the equation of motion for an oscillating relativistic string with free endpoints, and we have written it in a compact form. Now we are going to end with the relativistic particle by applying the light-cone gauge that we utilised for the point particle.

5.6 General solution of the equation of motion in light-cone coordinates

As we commented on in the introduction, for making the relativistic results that we have obtained of physical value, we have to make them compatible with quantum mechanics too. With the purpose of obtaining a model for quantum relativistic strings, we will quantize the relativistic strings that we have developed, by saying so, we mean to quantize the equation of motion of a the relativistic strings. But prior to quantize anything, we must write the solution of the equation of motion in the light-cone coordinates formalism because streamlines the process of quantization.

If you remember from chapter 4, we defined the light-cone coordinates in terms of the Cartesian ones, in equations (37) and (38) when we were dealing with the point particles. We are going to retake them now for the strings, so we have the set of coordinates $X^{\mu} = \left(X^{+}, X^{-}, X^{I}\right)$, where we have named the unchanged coordinates $X^{I} = (X^{2}, X^{3}, ..., X^{d})$. since the transverse coordinates X^{I} remain untouched, they will have the same solution of the equation of motion as obtained in (135):

$$X^{I}(\tau,\sigma) = x_0^I + \sqrt{2\alpha'}\alpha_0^I\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}^{\infty}\frac{1}{n}\alpha_n^I e^{-in\tau} cos\ n\sigma \tag{136}$$

The whole idea behind this change of coordinates relies on the solution for the other two coordinates X^{+} and X^{-}. In fact, the solutions for this two coordinates, once applied the boundary conditions to the wave equation solutions are:

$$X^{+}(\tau,\sigma) = \sqrt{2\alpha'}\alpha_0^{+}\tau \tag{137}$$

$$X^{-}(\tau,\sigma) = x_0^{-} + \frac{1}{p^{+}}L_0^{\perp}\tau + \frac{i}{p^{+}}\sum_{n\neq 0}\frac{1}{n}L_n^{\perp}e^{-in\tau} cos\ n\sigma \tag{138}$$

The mathematical development can be found in the reference, but due to its length and complexity we will omit it. We have above that $L_n^{\perp}$ are the transverse Visasoro modes, which are defined as:

$$\boxed{L_n^{\perp} \equiv \frac{1}{2}\sum_{p\in\mathscr{Z}}\alpha_{n-p}^I\alpha_p^I} \tag{139}$$

We can see how the solution of the component X^{+} (137) is very simple, since we have that it only depends on one of the coefficients, being the rest of them null. The solution for the second component X^{-} (138) seems a little bit more complicated, but it depends on only three things, the α_n^I coefficients of the transverse coordinates X^{I}, the momentum density p^{+}, and the initial condition of the string along that coordinate x_0^{-}. The set of expansions that we have presented will turn out fundamental in the quantum string, were the Visasoro oscillation modes turn to be operators.

The relativistic analysis of the string is now finished, all the results obtained are of great physical relevance, but some of them, as are the generators in (125) and the Visasoro oscillation modes (139), will show up their importance in the process of quantization.

6 The relativistic quantum point particle[3][7]

Hasta ahora, hemos construido un modelo para la cuerda relativista pero, como hemos repetido debidamente durante esta disertación, necesitamos cuantizarlo para hacerlo compatible con la mecánica cuántica. Como paso previo, vamos a aprender cómo se realiza el proceso de cuantización para el caso más simple de una partícula relativista. En este caso, ya hemos encontrado las variables dinámicas que definen el sistema en el capítulo 4, así que la cuantización consistirá simplemente en la transformación de las variables clásicas en operadores, que definiremos tanto en la representación de Schrödinger como en la Heisenberg. Tras este proceso, construiremos el espacio de estados de este sistema.

So far, we have built up a model for the relativistic string but, as it has been duly repeated over this dissertation, we need to quantize it in order to make it compatible with quantum mechanics. Before doing it, we are going to learn how the process of quantization is done for the simpler case of a relativistic point particle. In this case, we previously found the relevant dynamical variables that define the system in chapter 4, so the quantization simply consists of the transformation of this classical variables into operators, which we will define both in the Schrödinger and Heisenberg pictures. After this process, we will finally we construct the state space of the system.

6.1 The Schrödinger and Heisenberg pictures

In quantum mechanics, the uncertainty principle prevents us from defining the position or the momentum of an object as a well-defined variable, instead, it tells us that that all objects have an inherent uncertainty that can only be cleared up by making a measurement. Our approach so far, both with the relativistic point particle and the relativistic string, has consisted in the derivation of physical results regardless of this property of matter, this is an approximation that can be taken in many situations, but it is practically forbidden when studying phenomena at the Planck scale, since this scale is precisely the range in which fundamental strings are found. The so called first quantization, which is the one that we are going to perform, consists in transforming a given non-quantum theory into a quantum one by transforming the dynamical variables of a system into operators, the effect of this operators on the wave function of the system will provide us with the physical information about the state of the system at a given situation.

There are two mean pictures in quantum mechanics, the Schrödinger picture and the Heisenberg picture. The Schrödinger picture, regards the wave functions as entities that change in time following the famous Schrödinger equation (See (21))

$$i\hbar \frac{d}{dt} \left|\Psi\right\rangle_S = \hat{H} \left|\Psi\right\rangle_S \tag{140}$$

while the operators can only depend on time explicitly. On the other hand, the Heisenberg picture construes that the wave functions remain unchanged, whilst the operators are the ones that, implicitly and/or explicitly, evolve in time, the evolution of a generic Heisenberg operator is governed by its commutator with the Hamiltonian H of the system:

$$\frac{d\hat{O}_H}{dt} = \frac{\partial \hat{O}}{\partial t} - i\left[\hat{O}_H, \hat{H}\left(p(t), q(t); t\right)\right] \tag{141}$$

These two pictures are completely equivalent and always show the same physical results, but sometimes, during what we have left of work, the use of one the pictures will show up to be more useful than the use of the other. For differentiating between both of them, we will add the time label to the Heisenberg operators and states. The relation between the operators in the two different pictures will be given by the propagator of the system

$$U(t) = e^{-i\hat{H}t} \tag{142}$$

by the following relation:

$$\hat{O}t) = \hat{U}^\dagger(t)\ \hat{O}\ U(t) \tag{143}$$

The operator on the left hand is in the Heisenberg picture, while the operator on the right side is in the Schrödinger picture. We also have that $\hat{U}^\dagger(t)$ is the complex conjugate of $U(t)$, we must remember that the

Hamiltonian H is an observable, so we have that $H^{\dagger} = H$. Finally, the commutation relations between the operators in both representations do not change from one picture to the other, for instance, the commutator between the position and the momentum operators is, in natural units:

$$[q, p] = [q(t), p(t)] = i \tag{144}$$

6.2 Quantization of the point particle

In chapter 4, while deriving the solutions of the equation of motion for a relativistic particle under the light-cone gauge (62), (66) and (67), we concluded that the dynamical variables of the system where:

$$\left(x_0^-, p^+, x^I, p^I\right) \tag{145}$$

This are the variables required for knowing the evolution of a point particle classically, so in quantum mechanics, we should expect the operators related with the variables above to be the CSCO of the free quantum point particle. Therefore we define the following Schrödinger picture operators:

$$\left(\hat{x}_0^-, \hat{p}^+, \hat{x}^I, \hat{p}^I\right) \tag{146}$$

On the opposite, the Heisenberg picture operators, where we highlight the temporal dependency, turn out to be:

$$\left(\hat{x}_0^-(\tau), \hat{p}^+(\tau), \hat{x}^I(\tau), \hat{p}^I(\tau)\right) \tag{147}$$

The commutation relations between the different operators, which work equally for Schrödinger and Heisenberg operators, are just the usual. so according to (144) we have that:

$$\left[\hat{x}^I, \hat{p}^J\right] = i\eta^{IJ}, \qquad \left[\hat{x}^-, \hat{p}^+\right] = i\eta^{-+} = -i \tag{148}$$

Remember that $\eta^{\mu\nu}$ is the Minkowski metric tensor, which we have adapted to light-cone coordinates and expandend to extra dimensions (44). All the other commutation relations result to be null or can be derived from the ones above.

So far, defining our momentum and position operators has not been a very difficult task. But now we have to define the appropriate Hamiltonian of the system, since it is the operator that really allows us to evaluate the time evolution of any system in both pictures, instead of trying to derive it, we propose the following Hamiltonian of the system:

$$\boxed{\hat{H}\tau) = \frac{1}{2m^2}\left(\hat{p}^I(\tau)\hat{p}^I(\tau) + m^2\right)} \tag{149}$$

We can check the validity of this Hamiltonian by testing it with the evolution equation (141). If we obtain the same conservation rules which were obtained in the non-quantic situation, it will mean that we are heading in the right direction. For the dynamical momentum operators p^+ and p^I, the commutators vanish quickly:

$$i\frac{d\hat{p}^+(\tau)}{d\tau} = \left[\hat{p}^+(\tau), \hat{H}\tau)\right] = 0 \tag{150}$$

$$i\frac{d\hat{p}^I(\tau)}{d\tau} = \left[\hat{p}^I(\tau), \hat{H}\tau)\right] = 0 \tag{151}$$

Very logical, since the conservation of the momentum is also present in quantum mechanics. For the third operator of our CSCO (147) the result is quite similar:

$$i\frac{d\hat{x}_0^-(\tau)}{d\tau} = \left[\hat{x}_0^-(\tau), \hat{H}\tau)\right] = 0 \tag{152}$$

The last commutator, which in fact is a set of commutators corresponding to the position of the particle along the transverse coordinates, is calculated in the same fashion, but with a different result:

$$i\frac{d\hat{x}^I(\tau)}{d\tau} = \left[\hat{x}^I(\tau), \hat{H}\tau)\right] = \left[\hat{x}^I(\tau), \frac{1}{2m^2}\left(\hat{p}^J(\tau)\hat{p}^J(\tau) + m^2\right)\right] = \frac{1}{2m^2}\left[\hat{x}^I(\tau), \hat{p}^J(\tau)\hat{p}^J(\tau)\right] =$$

$$= \frac{1}{2m^2}\left(\left[\hat{x}^I(\tau), \hat{p}^J(\tau)\right]\hat{p}^J(\tau) + \hat{p}^J(\tau)\left[\hat{x}^I(\tau), \hat{p}^J(\tau)\right]\right) = i\frac{\hat{p}^I(\tau)}{m^2}$$

So we have that:

$$\frac{d\hat{x}^I(\tau)}{d\tau} = \frac{\hat{p}^I(\tau)}{m^2} \tag{153}$$

This conclusion is equivalent to the one which would be obtained by deriving the classical equation of motion (67). So now, we got that the evolution of the four operators that conform the CSCO of the relativistic point particle, is in line with their classic homologues. All the other operators can be constructed in terms of these four, so their commutators can be calculated from the latter. This results reinforce our ansatz (149) of the Hamiltonian.

To finish with the point particle, we develop the momentum space. This can be done by labelling with the continuous indexes P^+ and $\vec{P}^I$ the different states that the point particle can show, so we have then the following set of kets:

$$|P^+, \vec{P}^I\rangle \tag{154}$$

In this base, the eigenvalues of the different momentum operators on this states are:

$$\hat{p}^+ |P^+, \vec{P}^I\rangle = P^+ |P^+, \vec{P}^I\rangle\,; \qquad \hat{p}^I |P^+, \vec{P}^I\rangle = P^I |P^+, \vec{P}^I\rangle\,; \tag{155}$$

$$\hat{p}^- |P^+, \vec{P}^I\rangle = \frac{1}{2P^+}\left(P^I P^I + m^2\right)|P^+, \vec{P}^I\rangle \tag{156}$$

In the last equation, the relation between the different momentum components (65) was applied. Finally, the spectrum of energies of the quantum point particle is:

$$H|P^+, \vec{P}^I\rangle = \frac{1}{2m^2}\left(\hat{p}^I\hat{p}^I + m^2\right)|P^+, \vec{P}^I\rangle \tag{157}$$

So the Schrödinger equation is:

$$i\frac{\partial}{\partial\tau}|\Psi\rangle = \frac{1}{2m^2}\left(P^I P^I + m^2\right)|\Psi\rangle \tag{158}$$

By obtaining the Schrödinger equation, we have concluded the study of the quantum point particle. This example has allowed us to explain the procedure of quantization in a very simple fashion, so now we can finally focus on the quantum string, and achieve the main objectives of this project.

7 The open bosonic string [3]

Para cerrar nuestro análisis, vamos a obtener nuestros resultados más interesantes, estos requieren de unos cálculos ciertamente complicados, y para tratar de seguirlos vamos a apoyarnos en el desarrollo de la partícula cuántica. Esta guía nos permitirá establecer las relaciones de conmutación entre los diferentes operadores de la cuerda cuántica, así como proponer un Hamiltoniano, desarollaremos este último para comprobar como la cuantización convierte a la cuerda relativista en un oscilador armónico en las coordenadas transversales de la misma. Finalmente, aplicaremos la conservación de la carga a los generadores $M^{\mu\nu}$ para imponer la dimensionalidad del espacio tiempo D.

For closing our analysis, we are going to obtain our most interesting results, they require some certainly complicated calculations, and for trying to follow them we are going to support ourselves on the quantum particle. This guide will allow us to establish the commutation relations between the different operators of the quantum string, as well as for proposing a Hamiltonian, we will develop the latter for checking how the quantization transforms the relativistic string into a harmonic oscillator in the transverse coordinates of the string. Finally, we will apply the conservation of the charge of the generators $M^{\mu\nu}$ for imposing the dimensionality of the spacetime D

7.1 String operators, commutators and Hamiltonian.

We found, while studying the relativistic string, two different expressions for the same equation of motion, these equations were (94) and (127). By comparing both, we deduce that the momentum densities $\mathscr{P}^{\sigma}\mu$ and $\mathscr{P}^{\tau}\mu$ are just derivatives of the spacetime coordinates:

$$\mathscr{P}^{\sigma\mu} = \frac{1}{2\pi\alpha'} X^{\mu'}; \qquad \mathscr{P}^{\tau\mu} = \frac{1}{2\pi\alpha'} \dot{X}^{\mu} \tag{159}$$

Where we again have introduced the slope parameter (132), which implicitly includes the tension T_0, and secures the units of the overall expression. In line with the point particle (65), we also find that the density momentum components are not independent:

$$\mathscr{P}^{\tau -} = \frac{\pi}{2p^+} \left(\mathscr{P}^{\tau I} \mathscr{P}^{\tau I} + \frac{X^{I'} X^{I'}}{(2\pi\alpha')^2} \right) \tag{160}$$

Remember that the conjugated momentum density $\mathscr{P}^{\tau\mu}$ is the one related to the momentum of the string p^{μ}. So, similarly to what we did with the point particle, we define our Schrödinger operators:

$$\left(\hat{X}^I(\sigma), \hat{x}_0^-, \hat{\mathscr{P}}^{\tau I}(\sigma), \hat{p}^+ \right) \tag{161}$$

See how although the Schrödinger operators don't depend on the parameter τ, which we associated with time, some of them do depend on the parameter $\sigma \in [0, \pi]$, which we associate with the length of the string. The Heisenberg operators are obtained from the Schrödinger ones according to equation (143):

$$\left(\hat{X}^I(\tau, \sigma), \hat{x}_0^-(\tau), \hat{\mathscr{P}}^{\tau I}(\tau\sigma), \hat{p}^+(\tau) \right) \tag{162}$$

The commutation relations are similar to the ones obtained in the previous chapter, but with an important difference, and is that we expect operators dependant on σ acting on different points of the string to commute. Therefore we set:

$$\left[\hat{X}^I(\sigma), \hat{\mathscr{P}}^{\tau J}(\sigma') \right] = i\eta^{IJ} \delta(\sigma - \sigma') \tag{163}$$

There is a Dirac delta instead of a Kronecker delta because σ takes continuous values, this ensures that operators acting on different points of the string commute. The only other not null commutator among our set is:

$$\left[\hat{x}_0^-, \hat{p}^+ \right] = -i \tag{164}$$

Now we must define our Hamiltonian, so we propose the following one:

$$\hat{H} = 2\sigma'\hat{p}^+\hat{p}^- = 2\sigma'\hat{p}^+ \int_0^\pi d\sigma \hat{\mathscr{P}}^{\tau -} \tag{165}$$

Which after applying the density momentum relation (160) becomes:

$$\boxed{\hat{H}(\tau) = \pi\alpha' \int_0^\pi d\sigma \left(\hat{\mathscr{P}}^{\tau I}(\tau,\sigma)\hat{\mathscr{P}}^{\tau I}(\tau,\sigma) + \frac{\hat{X}^{I'}(\tau,\sigma)\hat{X}^{I'}(\tau,\sigma)}{(2\pi\alpha')^2} \right)} \tag{166}$$

For checking the validity of the Hamiltonian above, we can evaluate the time evolution of some of the Heisenberg operators (162). For instance we have that for the transverse coordinates $\hat{X}^I$:

$$i\frac{d\hat{X}^I}{d\tau} = \left[\hat{X}^I(\tau,\sigma), \hat{H}\tau)\right] = \left[\hat{X}^I(\tau,\sigma),\ \ \pi\alpha' \int_0^\pi d\sigma' \left(\hat{\mathscr{P}}^{\tau J}(\tau,\sigma')\hat{\mathscr{P}}^{\tau J}(\tau,\sigma') + \frac{\hat{X}^{J'}(\tau,\sigma')\hat{X}^{J'}(\tau,\sigma')}{(2\pi\alpha')^2} \right)\right] =$$

We have that all the $\hat{X}^\mu$ operators commute between themselves, so the above simplifies to

$$= \left[\hat{X}^I(\tau,\sigma),\ \ \pi\alpha' \int_0^\pi d\sigma' \hat{\mathscr{P}}^{\tau J}(\tau,\sigma')\hat{\mathscr{P}}^{\tau J}(\tau,\sigma')\right] =$$

Which after applying the commutation relation (163) becomes:

$$= 2\pi\alpha' i \int_0^\pi d\sigma' \hat{\mathscr{P}}^{\tau J}(\tau,\sigma')\eta^{IJ}\delta(\sigma - \sigma')$$

Because in the integral we have a Dirac delta, the infinitesimal sum is reduced to a single term ($\sigma' = \sigma$), this is one of the properties of this distribution. The metric tensor η^{IJ}, in practice just acts as a Kronecker delta so, the time evolution of the position operator $\hat{X}^I$ obtained from the Hamiltonian (166) is:

$$\frac{d\hat{X}^I}{d\tau} = 2\pi\alpha' \hat{\mathscr{P}}^{\tau I}(\tau,\sigma) \tag{167}$$

This result is equivalent to expression for the momentum density (159) that we had at the beginning of the chapter! Similar results are obtained by evaluating the time evolution of the other components of the conjugated momentum operators. We conclude this section by quantizing the Neumann boundary condition. This condition was stated for a free relativistic string (96) in chapter 5, and now we must adapt it to quantum mechanics by writing it in terms of the corresponding operator:

$$\partial_\sigma X^I(\tau, \sigma = 0) = \partial_\sigma X^I(\tau, \sigma = \sigma^*) = 0 \tag{168}$$

7.2 The quantum string as a harmonic oscillator

Since there is a new Hamiltonian of the system, given by the formula (166) in terms of the transverse coordinates. It arises a need of defining a new action for the string in terms of this coordinates, because of this, we define the new action in terms of the transverse operators of the system to be:

$$S = \int d\tau d\sigma \mathscr{L} = \frac{1}{4\pi\alpha'} \int d\tau \int_0^\pi d\sigma \left(\dot{\hat{X}}^I \dot{\hat{X}}^I - \hat{X}^{I'}\hat{X}^{I'} \right) \tag{169}$$

As we showed in the derivation of the Schrödinger (21) and Dirac (26) equations, we can proceed with the actions composed by operators (or fields), in a similar way as we proceed with the actions composed by variables. By applying this property we can check that the action (169) produces the already known equivalence between the derivative of the position vector and the momentum density (159):

$$\frac{\partial \mathscr{L}}{\partial \dot{\hat{X}}^I} = \frac{1}{2\pi\alpha'}\dot{\hat{X}}^I = \hat{\mathscr{P}}^{\tau I} \tag{170}$$

In fact, if we combine the above with the relation between the lagrangian density and the hamiltonian density

$$\mathscr{H} = \mathscr{P}^{\tau I}\dot{X}^I - \mathscr{L} \tag{171}$$

we effectively recover our Hamiltonian but in the Schrödinger picture:

$$\hat{H}(\tau) = \int_0^{\pi} d\sigma \left(\pi\alpha'\hat{\mathscr{P}}^{\tau I}\hat{\mathscr{P}}^{\tau I} + \frac{\hat{X}^{I'}\hat{X}^{I'}}{4\pi\alpha'}\right) \tag{172}$$

At this point, and as in the same fashion as we did with the relativistic string, an expansion of the transverse operators $\hat{X}^I$ is required:

$$\hat{X}^I = \hat{q}^I + 2\sqrt{\alpha'}\sum_{n=1}^{\infty}\hat{q}_n^I\frac{cos\ n\sigma}{\sqrt{n}} \tag{173}$$

By deriving with respect to the two parameters τ and σ, we obtain the following derivatives:

$$\hat{\dot{X}}^I = \hat{\dot{q}}^I + 2\sqrt{\alpha'}\sum_{n=1}^{\infty}\hat{\dot{q}}_n^I\frac{cos\ n\sigma}{\sqrt{n}} \tag{174}$$

$$\hat{X}^{I'} = -\sqrt{\alpha'}\sum_{n=1}^{\infty}\hat{q}_n^I\sqrt{n}sen\ n\sigma\sqrt{n} \tag{175}$$

The process of substituting the above in the action (169), and the subsequent application of the relation between the lagrangian and the Hamiltonian ends up in the following expression:

$$\boxed{\hat{H} = \alpha'\hat{p}^I\hat{p}^I + \sum_{n=1}^{\infty}\frac{n}{2}\left(\hat{p}_n^I\hat{p}_n^I + \hat{q}_n^I\hat{q}_n^I\right)} \tag{176}$$

This is a sum over all the oscillation modes of a quantum harmonic oscillator! This result is absolutely fundamental in String Theory, it tells us that the spectrum of energies of the string has a continuum component given by the operators, but also a quantized term given by the oscillation modes above.

7.3 Dimensionality of spacetime

The calculation of the dimensionality of spacetime is on of the most important calculations in String Theory. As we are not going to do it explicitly, we will comment on the fundamentals on which it is based.

We start from Visasoro oscillation modes that we presented for the relativistic string, which allowed us to write the light-cone coordinates as a Fourier expansion over the coefficients α_n^I. In quantum mechanics, they become a set operators called Visasoro operators:

$$\hat{L}_n^{\perp}\sum_{p\in\mathscr{Z}}\hat{\alpha}_{n-p}^I\hat{\alpha}_p^I \tag{177}$$

Where the former coefficients $\hat{\alpha}_n^I$ act now as creation and anihilation operators, which are intrinsically related to the different oscillation modes of the string. If we take a look to $L_0^{\perp}$ in more detail by expanding it:

$$\hat{L}_0^{\perp} = \frac{1}{2}\sum_p\hat{\alpha}_{-p}^I\hat{\alpha}_p^I = \frac{1}{2}\hat{\alpha}_0^I\hat{\alpha}_0^I + \frac{1}{2}\sum_{p=1}^{\infty}\hat{\alpha}_{-p}^I\hat{\alpha}_p^I + \frac{1}{2}\sum_{p=1}^{\infty}\hat{\alpha}_p^I\hat{\alpha}_{-p}^I \tag{178}$$

If we exchanged the order of the operators in the last sum, we would arrive to the following result:

$$\hat{L}_0^{\perp} = \frac{1}{2}\sum_p\hat{\alpha}_{-p}^I\hat{\alpha}_p^I = \hat{\alpha}_0^I\hat{\alpha}_0^I + \frac{1}{2}\sum_{p=1}^{\infty}\hat{\alpha}_{-p}^I\hat{\alpha}_p^I + \frac{1}{2}\sum_{p=1}^{\infty}p\eta^{II} =$$

$$\hat{L}_0^{\perp} = \frac{1}{2}\sum_p\hat{\alpha}_{-p}^I\hat{\alpha}_p^I = \hat{\alpha}_0^I\hat{\alpha}_0^I + \frac{1}{2}\sum_{p=1}^{\infty}\hat{\alpha}_{-p}^I\hat{\alpha}_p^I + \frac{1}{2}(D-2)\sum_{p=1}^{\infty}p \tag{179}$$

Where D is the dimensionality of spacetime. The main problem with the above, is that we have an infinite sum over the natural numbers in the term on the right, this leads to nonphysical divergences. So we must focus on this term since it is the root of the problem. We call it:

$$a = \frac{1}{2}(D-2)\sum_{p=1}^{\infty} p \tag{180}$$

For calculating the value of a, and therefore the dimensionality D, we retake the world-sheet charges that we defined in (125) while studying the relativistic string. This generators, which we said that were related with the angular momentum of the string as well as their boosts, can be simplified by substituting in there the relations between the momentum and the derivatives of the coordinates (159):

$$\hat{M}^{\mu\nu} = \frac{1}{2\pi\alpha'}\int_0^{\pi} d\sigma \left(\hat{X}^{\mu}\dot{\hat{X}}^{\nu} - \hat{X}^{\nu}\dot{\hat{X}}^{\mu}\right) \tag{181}$$

This can be expanded in the same way we did with the Hamiltonian in (172):

$$\hat{M}^{\mu\nu} = x_0^{\mu}p^{\nu} - x_0^{\nu}p^{\mu} - i\sum_{n=1}^{\infty}\left(\hat{\alpha}_{-n}^{\mu}\hat{\alpha}_n^{\nu} - \hat{\alpha}_{-n}^{\nu}\hat{\alpha}_n^{\mu}\right) \tag{182}$$

We expect that the commutators between the different M^{-I} components of the above charges to be zero, the calculation of this commutators leads to the following expression:

$$\left[\hat{M}^{-I}, \hat{M}^{-J}\right] = -\frac{1}{\alpha'\hat{p}^{+2}}\sum_{n=1}^{\infty}\left(\hat{\alpha}_{-n}^{I}\hat{\alpha}_n^{J} - \hat{\alpha}_{-n}^{J}\hat{\alpha}_n^{I}\right)\cdot\left(n\left[1 - \frac{1}{24}(D-2)\right] + \frac{1}{n}\left[\frac{1}{24}(D-2) + a\right]\right) \tag{183}$$

Where there is the number of dimensions D and the parameter a that we defined in (180). For this to be zero, we have to impose the following conditions:

$$\boxed{D = 26} \qquad \boxed{a = -1} \tag{184}$$

We have obtained that there must be 26 dimensions, this means that we have 22 extra rows in our metric tensor (44), as we already noted, in the hypothetical case of the existence of this dimensions, they should be compactified. With this oustanding result we conclude our introduction to String Theory.

8 Conclusions

In first place, we have that a great part of the results obtained in String Theory, as the conservation of the momentum or the potential energy, are completely compatible with those obtained for the point particle, while those who are particular for the string, as the movement of the endpoints and the vibrational modes, are indeed very difficult to check, this is due to the expected Planck scale of both for the string and for the compactified dimensions, which require of an enormuos amount of energy to be detected. This would make, to the eyes of our current technology, both the string and the particle to seem equal but, it is expected that, with the development of new and more powerful laboratories, we will be able to check which of them is the correct interpretation. We must also take into account that the have only commented on the bosonic String Theory, which cannot serve as a model for fundamental particles, since most of them are fermions, in fact there are currently up to five different fermionic String Theories.

In the development of the present work, we have been able to apply a wide variety of mathematical tools, from the differential calculation used to make the action extreme, to the resolution of integrals for solving commutation relations. Along the way, we have also used linear algebra to study both the metric tensor and Lorentz transformations, and we have applied boundary conditions to different equations of motion. From the physical point of view, Lagrangian mechanics has been repeatedly applied, and it also has been expanded to the study of fields, both scalar and complex, special relativity has also been applied along the work, while of quantum mechanics has been used in the final chapters. The use of so many mathematical and physical tools along the present work, makes String Theory a frame in which, even though if it turns out to be an incorrect theory, can be very instructive for thinking outside of the box on the search for solutions to complex problems.

9 Bibliography

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