# UNIVERSIDAD DE LA LAGUNA

## «Quasi-ordinary singularities via toric geometryc»

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# Contents

# Introducción

En esta memoria empleamos métodos de la geometría tórica para estudiar singularidades casi-ordinarias de variedades analíticas complejas, principalmente en el caso de hipersuperficies.

Una *singularidad casi-ordinaria* de dimensión $d$ es un gérmen de variedad analítica compleja que admite una *proyección casi-ordinaria* (i.e., un morfismo finito sobre $(\mathbb{C}^d, 0)$ no ramificado fuera de un divisor con cruzamientos normales). Por ejemplo, toda singularidad de curva analítica compleja es casi-ordinaria.

Si $f(X,Y) = 0$ es la ecuación de una *rama plana* $C$ (i.e., un germen de curva plana compleja analíticamente irreducible en el origen) diferente de $X = 0$, el Teorema de Newton-Puiseux proporciona de forma constructiva una parametrización de la forma: $X = T^n$, $Y = \zeta(T)$, donde $\zeta(T)$ es una serie de potencias compleja convergente y el entero $n$ es la *multiplicidad de intersección* de la curva y el eje $X = 0$ en el origen, esto es, $n := \dim_{\mathbb{C}} \mathbb{C}\{X,Y\}/(f(X,Y),X)$. Si ponemos $T = X^{1/n}$, se obtiene un desarrollo de $Y$ como serie de potencias fraccionarias en $X$. (Véase [W], o [Z2]).

Una singularidad casi-ordinaria de hipersuperficie irreducible puede ser definida por una ecuación polinomial $F(X_1,\ldots,X_d;Y) = 0$, y es parametrizada por una serie de potencias fraccionarias $\zeta = z(X)$ en las variables $X_1^{1/r_1},\ldots,X_d^{1/r_d}$ verificando algunas propiedades. Este es el aserto del Teorema de Jung-Abhyankar, que generaliza el Teorema de Newton Puiseux (véase [A1], [J], [Zu] and Théorème 1.1). En el primer capítulo de esta memoria damos un método constructivo que determina los términos de una parametrización, $Y = z(X)$, en forma de serie de potencias fraccionarias de carácter "más general", de una singularidad de hipersuperficie definida por una ecuación $F(X;Y) = 0$ polinomial en $Y$. Esto generaliza el método proporcionado por el Teorema de Newton-Puiseux al caso de hipersuperficies y conlleva una ralación entre las parametrizaciones obtenidas y el poliedro de Newton del discriminante de $F$ con respecto de la variable $Y$. Como aplicación mostramos que bajo ciertas hipótesis de *no degeneración*, el poliedro de Newton del discriminante de $F$ está determinado por el poliedro de Newton de $F$, por medio del *poliedro-fibra* de [Bi-St]. Estos resultados son enunciados de forma más general para gérmenes de singularidades de hipersuperficie sumergidos en una variedad tórica afín normal.

En la primera parte del capítulo segundo estudiamos la parametrización de una singularidad casi-ordinaria de hipersuperficie $S$ definida por una serie de potencias fraccionarias $\zeta$. Esta última se denomina *rama casi-ordinaria* y se caracteriza porque en ella aparecen un número finito de términos distinguidos denominados *monomios característicos* cuyos exponentes son vectores con coordenadas racionales que se llaman *exponentes característicos o distinguidos.*

Esta situación generaliza el caso de los clásicos *exponentes característicos de Puiseux* asociados a la parametrización de una rama plana. El *semigrupo* de una rama plana $C$ es el subsemigrupo de los enteros no negativos cuyos miembros son las multiplicidades de intersección en el origen de la rama plana $C$ con las curvas planas que no contienen a $C$ como componente. Este semigrupo es claramente un invariante de la singularidad. Es un resultado conocido que el dato del semigrupo asociado a la rama plana es equivalente al *tipo topológico* de la misma (véase [Z2] and [Re]). El dato de este semigrupo equivale a los exponentes característicos de Puiseux correspondientes a una *proyección transversal*[1]. La parametrización proporciona una inclusión del álgebra analítica de la singularidad en el anillo $\mathbb{C}\{T\}$. La $\mathbb{C}$-álgebra del semigrupo de la rama plana $C$ aparece como el anillo graduado asociado a la filtración de $R$ definida por el ideal maximal del anillo local $\mathbb{C}\{T\}$. Este anillo graduado es el anillo de coordenadas de una curva afín que admite una parametrización monomial (véase [T2]).

En el caso de singularidades casi-ordinarias de hipersuperficie se tiene que el dato proporcionado por el conjunto de monomios característicos de una rama-casi ordinaria $\zeta$ parametrizando $S$ es equivalente al tipo topológico de la singularidad. Este es un resultado de Gau que dice concretamente que el tipo topológico de la singularidad $S$ es equivalente al dato proporcionado por el conjunto de monomios característicos de una rama casi-ordinaria *normalizada* [2] (véase [Gau]). Asociamos un semigrupo $\Gamma \subset \mathbb{Z}^d_{\geq 0}$ a una rama casi-ordinaria $\zeta$ parametrizando una singularidad casi-ordinaria de dimensión $d$, generalizando algunas de las ideas del caso de ramas planas. Definimos una filtración del álgebra analítica $R$ de la singularidad $S$ y probamos que su anillo graduado asociado es la $\mathbb{C}$-álgebra del semigrupo $\Gamma$. Este es el anillo de coordenadas de una

[1] La proyección $(X,Y) \mapsto X$ es tranversal para una curva plana si la multiplicidad en el origen de la curva coincide con la multiplicidad de intersección de la curva con el eje $X=0$

[2] Ésta es una condición técnica significa, en el caso de curvas planas, que la proyección $(X,Y) \mapsto X$ es transversal. El lema de inversión de Lipman garantiza que toda singularidad de hipersuperficie casi ordinaria irreducible puede ser parametrizada por una rama casi-ordinaria *normalizada* y determina los monomios característicos de ésta a partir de los monomios característicos de la rama de partida mediante *fórmulas de inversión* . Este resultado es una generalización del clásico lema de inversión de Zariski y Abhyankar para curvas planas (véase [A2]).

variedad tórica afín de dimensión $d$ parametrizada por monomios, que denominamos la *variedad monomial* siguiendo la terminología de [T2] and [G-T] para ramas planas. El resultado principal de este capítulo, probado utilizando el resultado de Gau, es que el dato del semigrupo $\Gamma$ es equivalente al tipo topológico de $S$.

En el tercer capítulo determinamos para una singularidad de casi-ordinaria de hipersuperficie irreducible $S$ dos procesos de resolución sumergida que dependen sólo del tipo topológico de $S$ (esto es del semigrupo asociado). Este resultado resuelve el problema abierto 5.1 planteado por Lipman en [L5].

Esta memoria es una contribución al estudio de los invariantes de singularides casi-ordinarias, principalmente desde un punto de vista algebraico y del papel desempeñado por éstos en la resolución sumergida.

## Clasificación mediante invariantes

Uno de los principales problemas en el estudio de las singularidades es su clasificación. Los resultados de Gau permiten clasificar por su tipo topológico las singularidades de hipersuperficie casi-ordinaria irreducible mediante los monomios característicos de una rama casi-ordinaria normalizada parametrizando la singularidad. En esta sección damos un recorrido, a modo de motivación, sobre las aproximaciones que se han realizado para abordar este este problema.

### Invariantes y resolución

Este enfoque esta relacionado con la posibilidad de definir un procedimiento "canónico" de resolución[3] (sumergida) de la singularidad en términos de los monomios característicos.

En el caso de ramas planas Zariski prueba geométricamente la unicidad de los monomios característicos de una rama normalizada demostrando que éstos determinan y son determinados por la secuencia de multiplicidades de las curvas obtenidas explotando recursivamente los puntos singulares de cada transformada estricta de la curva de partida. De forma más general, la resolución minimal sumergida de una singularidad de curva plana está determinada por los monomios característicos de sus componentes irreducibles y de los números de intersección de dos cualesquiera de ellas.

[3] Véase la siguiente sección de esta introducción para más detalles acerca de la resolución de singularidades

Lipman demuestra que los monomios característicos de una rama casi-ordinaria normalizada $\zeta$ parametrizando un germen de superficie $(S, 0) \subset (\mathbb{C}^3, 0)$ son invariantes analíticos de la singularidad, esto es dependen únicamente del germen de la superficie $(S, 0)$, (véase [L1], [L3]). Esto lo hace definiendo un proceso de resolución (no sumergida) que determina y está totalmente determinada por los monomios característicos de $\zeta$. Luego proporciona en [Lu] otra prueba de la constancia (analítica) de los monomios característicos de una rama casi-ordinaria normalizada. Asocia a una superficie algebroide un árbol pesado invariante. En el caso de que esta superficie sea casi-ordinaria irreducible prueba que este árbol esta determinado por los monomios característicos de $\zeta$. Ambas pruebas emplean transformaciones cuadráticas (explosiones de puntos) y monoidales (explosiones de curvas lisas y equimúltiples). El hecho importante mostrado por Lipman es la "estabilidad monoidal", que significa que la transformada estricta de la superficie por estas transformaciones (en todos los casos significativos) es casi-ordinaria, irreducible y los correspondientes monomios característicos son determinados a partir de aquéllos de $\zeta$ (véase [L3], página 78). Sin embargo las singularidades casi-ordinarias de dimensión $\geq 3$, no disponen de esta propiedad. Por ejemplo, la transformada estricta de la hipersuperficie $X_4^4 - X_1 X_2 X_3 = 0$ por la transformación cuadrática $X_1 = Z_1, X_i = Z_i Z_1$ para $i = 2, 3, 4$ es $Z_4 Z_1 - Z_2 Z_3$; esta superficie no es casi-ordinaria para ninguna de las proyecciones coordenadas. Este es uno de los obstáculos principales que se presentan para determinar un proceso de resolución sumergida dependiente únicamente de los monomios característicos. En esta dirección Ban-McEwan anuncian en [B-M] una respuesta afirmativa a esta cuestión para singularidades casi-ordinarias de superficies sumergidas, por medio de los algoritmos de resolución sumergida de Bierstone-Milman.

## Invariantes y topología

Dos gérmenes de hipersuperficies $(X, 0)$ y $(X', 0)$ en $\mathbb{C}^{d+1}$ tienen igual *tipo topológico* si y sólo si existe un homeomorfismo $U \to U'$ entre dos entornos abiertos del origen que aplica $X \cap U$ en $X' \cap U'$.

El tipo topológico de una singularidad de una curva plana compleja en el origen está determinado completamente por el *nudo* de la singularidad, esto es la intersección de la curva con una esfera suficientemente pequeña centrada en el origen. El resultado, obtenido mediante teoría de nudos, afirma que el tipo topológico de la singularidad es equivalente al dato de los monomios característicos de cada rama plana componente de la curva y a los números de intersección cada par de componentes distintas (véase [Re]).

Se tiene que los monomios característicos de una rama casi-ordinaria $\zeta$ determinan el tipo topológico de la singularidad que parametriza la rama. Esto es consecuencia de resultados generales de Zariski sobre saturación de anillos locales (véase [Z1], [L3] §2 y [Oh]). Los invariantes topológicos de singularidades casi-ordinarias son estudiados por Lipman (véase [L4]), sus resultados son utilizados por Gau, quien muestra en [Gau] que si dos pares de gérmenes de hipersuperficie casi-ordinaria $(X, 0)$ y $(X', 0)$ en $\mathbb{C}^{d+1}$ tienen igual tipo topológico entonces dos ramas casi-ordinarias normalizadas parametrizando $(X, 0)$ y $(X', 0)$ tienen iguales monomios característicos. (Véase Theorem 2.3).

## Invariantes y álgebra

Para una rama plana $(C, 0) \subset (\mathbb{C}^2, 0)$ se tienen varias formas de probar la constancia de los monomios característicos de correspondientes a una proyección transversal.

- Zariski y Abhyankar dan pruebas explícitas y algebraicas de la constancia de los monomios característicos asociados a una proyección transversal, mediante las denominadas "fórmulas de inversión" que relacionan los monomios característicos correspondientes a parametrizaciones distintas (véase [Z1] Proposition 2.2 y [A2]).

- Al conjunto de monomios característicos de una parametrización de $C$ le podemos asociar un sistema de generadores del semigrupo asociado a la rama que determina los monomios característicos de partida; (en el caso de que la parametrización corresponda a una proyección transversal se obtienen un sistema minimal de generadores del semigrupo de la rama (véase [Z2])). Este sistema de generadores del semigrupo está relacionado con las *raíces aproximadas* de Abhyankar y Moh del polinomio de Weierstrass definiendo la curva $S$ respecto de la parametrización dada, por medio del concepto de *semi-raíz* (véase [PP]). Geométricamente, esta noción se corresponde con las curvas de contacto maximal definidas por Lejeune (véase [LJ]). Las *fórmulas de inversión* que relacionan los exponentes característicos de Puiseux de dos parametrizaciones de la rama $C$ pueden ser deducidas por medio del semigrupo de la rama (véase [PP]).

- La normalización de la rama es una parametrización *primitiva* cualquiera de la misma y está descrita algebraicamente por un homomorfismo $R \to \mathbb{C}\{T\}$, donde $R$ es el álgebra analítica de $C$. El álgebra de la normalización es el *anillo de valoración discreta* $\mathbb{C}\{T\}$ con parámetro uniformizador $T$. La filtración $(T)$-ádica de $R$ no depende de la parametrización elegida porque la normalización es única. El anillo graduado asociado es la $\mathbb{C}$-álgebra del semigrupo de la curva, además este semigrupo queda determinado por la graduación inducida (véase [T2]).

Para singularidades casi-ordinarias de dimensión $d \geq 2$ se han encontrado varias dificultades para extender este enfoque: cómo identificar el conjunto de proyecciones casi-ordinarias, el hecho de que el anillo $\mathbb{C}\{T_1, \ldots, T_d\}$ no sea un anillo de valoración discreta, la ausencia de una noción equivalente a la multiplicidad de intersección para dos gérmenes de hipersuperficies, etc.

## Resolución de singularidades

Introducimos a continuación algunas definiciones y motivaciones básicas acerca de la resolución de singularidades que necesitamos para explicar los resultados obtenidos para singularidades casi-ordinarias.

El problema de de la resolución (sumergida) de singularidades consiste en:

> *Resolución de singularidades*:
>
> *Dada una variedad $\mathcal{X}$ encontrar un morfismo propio $\phi : \mathcal{Y} \to \mathcal{X}$, con $\mathcal{Y}$ variedad lisa tal que $\phi$ es un isomorfismo sobre la parte no singular de $\mathcal{X}$.*
>
> *Resolución sumergida de singularidades (I)* :
>
> *Dada una variedad $\mathcal{W}$ y una subvariedad $\mathcal{X}$ con complementario denso, encontrar un morfismo propio $\phi : \mathcal{W}' \to \mathcal{W}$, donde $\mathcal{W}'$ es una variedad lisa tal que $\phi$ es un isomorfismo sobre $\mathcal{W} - \mathcal{X}$ y tal que $\phi^{-1}(\mathcal{X})$ es un divisor en $W'$ con cruzamientos normales.*

donde *variedad* significa aquí variedad algebraica sobre un cuerpo algebraicamente cerrado, o un espacio analítico complejo o real. Un divisor $D \subset \mathcal{W}$ con *cruzamientos normales* es el tipo más simple de singularidad. La condición es que en todo punto $w \in \mathcal{W}$ el ideal de definición del divisor $D$ está generado por un monomio $X_1^{a_1} \ldots X_d^{a_d}$, donde $a_1, \ldots, a_d \in \mathbb{Z}_{\geq 0}$, $\{X_1, \ldots, X_d\}$ es un sistema de coordenadas locales en $w$ y $d$ es la dimensión de $W$. Asociado a un divisor con cruzamientos normales se tiene una estratificación natural (véase la sección 2.2).

En este trabajo cuando nos referimos a resolución sumergida de singularidades será siempre en el sentido preciso de la definición siguiente salvo que se especifique lo contrario.

> *Resolución sumergida de singularidades (II)* :
>
> *Dada una variedad $\mathcal{W}$, y una subvariedad $\mathcal{X}$ con complementario denso encontrar una modificación $\phi : \mathcal{W}' \to \mathcal{W}$, con lugar discriminante*[4] *nunca denso en $\mathcal{X}$ de forma*

[4] El lugar discriminante es la imagen del lugar crítico

*que el lugar crítico*[5] *de* $\phi$ *sea un divisor con cruzamientos normales, la restricción de* $\phi$ *a la transformada estricta* [6] $\tilde{\mathcal{X}}$ *de* $\mathcal{X}$ *sea una resolución de singularidades de* $\mathcal{X}$ *y la transformada estricta* $\tilde{\mathcal{X}}$ *sea transversal a la estratificación natural del divisor crítico de la modificación* $\phi$ *.*

Hironaka ha dado respuesta positiva a estos problemas para variedades algebraicas definidas sobre cuerpos de característica cero y para espacios analíticos reales y complejos (véase [L2] para referencias precisas). Bierstone-Milman y Villamayor han dado algoritmos canónicos de resolución, basándose en la obra de Hironaka.

Con respecto a la resolución de singularidades de superficies casi-ordinarias el enfoque clásico es normalizar primero y resolver después. La composición de la normalización $\hat{S} \to S$ con una proyección casi-ordinaria es claramente una proyección casi-ordinaria para $\hat{S}$. Es más, la parametrización de $S$ se factoriza a través de la normalización $\hat{S}$ y ésta es una *singularidad cociente* [7]. Una resolución de $\hat{S}$ se puede obtener entonces explotando un ideal cero dimensional (véase [L2]). Es también conocido que la normalización es una superficie tórica afín (véase [B-P-V], Chapter III, Theorem 5.2, para una prueba topológica). La *resolución minimal* de una singularidad de superficie tórica normal se puede calcular por medio de morfismos tóricos (véase [Od]).

## El método de Jung

Las singularidades casi-ordinarias aparecen de forma natural en el enfoque de Jung para abordar la resolución sumergida de singularidades (véase [L2], Lecture 2). La idea de Jung consiste en realizar inducción sobre la dimensión. Supongamos para empezar que tenemos una hipersuperficie $S \subset \mathbb{C}^{d+1}$ y que tomamos una proyección finita de $S$ sobre $\mathbb{C}^d$. El lugar discriminante reducido de la proyección es una hipersuperficie $D \subset \mathbb{C}^d$. Por hipótesis de inducción podemos suponer

[5] El lugar crítico de $\phi$ es el conjunto de puntos donde el morfismo $\phi$ no es un isomorfismo local

[6] La transformada estricta es la clausura de $\phi^{-1}(X - \text{discrim}(\phi))$ en $\mathcal{W}'$.

[7] Un germen $S$ es una singularidad cociente si es normal y el anillo de fracciones $K$ del álgebra analítica $R$ de $S$ admite una extensión Galois finita $K'$ en al que la clausura integral $R_{K'}$ de $R$ es un anillo local regular. De forma equivalente $S$ es normal y $R$ es el anillo de invariantes de un grupo finito de automorfismos del anillo de series $\mathbb{C}\{T_1, \dots, T_d\}$ ( $d = \dim S$ ).

La normalización $\hat{S}$ de una singularidad casi-ordinaria $S$ de dimension $d$ es una singularidad cociente: si $\hat{R}$ es el álgebra analítica de $\hat{S}$, tenemos las extensiones finitas de anillos $R \subset \hat{R} \subset \mathbb{C}\{T_1, \dots, T_d\}$. Tomando el cuerpo de fracciones se define una extensión de Galois finita de cuerpos $K \subset K'$ y la clausura integral de $\hat{R}$ en $K'$ es igual a $\mathbb{C}\{T_1, \dots, T_d\}$ (véase [L4], §2).

que existe una resolución sumergida de $\mathcal{X} \to \mathbb{C}^d$ de $D \subset \mathbb{C}^d$ (con respecto a la definición I). Tomando el *producto fibrado* obtenemos el diagrama conmutativo donde la variedad $\mathcal{W}$ es lisa.

$$\begin{array}{ccc} \mathcal{W} & \longrightarrow & \mathbb{C}^{d+1} \\ \cup & & \cup \\ \tilde{S} & \longrightarrow & S \\ \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & \mathbb{C}^d \\ \cup & & \cup \\ \tilde{D} & \longrightarrow & D \end{array} \tag{1}$$

La proyección izquierda es no ramificada fuera del divisor con cruzamientos normales $\tilde{D}$ que se aplica en $D$. En particular, esto significa que la hipersuperficie $\tilde{S} \subset \mathcal{W}$ tiene sólo singularidades casi-ordinarias. Este enfoque es empleado por Walker, Hirzebruch, Zariski y Abhyankar para probar la existencia de resolución de superficies y variedades de dimensión dos y tres (véase [L2]).

## Métodos de la geometría tórica en la resolución de singularidades

Una variedad $Z$ que contiene al toro $(\mathbb{C}^*)^d$ como abierto denso y que esta provista de una acción de $(\mathbb{C}^*)^d$ que extiende el producto en el toro como grupo algebraico, se denomina una *variedad tórica*. El estudio de este tipo de variedades emerge en la decada de los setenta a través del trabajo de muchos matemáticos. La geometría de las variedades tóricas es muy interesante porque disponemos de un diccionario que "traduce" muchos enunciados geométricos en enunciados de convexidad combinatoria (véase [Od], [KKMS], [Ew], [F1], [St] y [G-K-Z]). Esta interacción ha sido muy constructiva para el desarrollo de ambas teorías.

En particular, las variedades tóricas normales están definidas por abanicos. Un abanico $\Sigma$ es un complejo poliédrico formado por conos en el espacio vectorial $(\mathbb{R}^d)^*$ que verifican algunas propiedades (véase la sección 2.2). Cada cono $\sigma$ del abanico corresponde con una variedad tórica afín $Z(\sigma)$ que es una carta de la variedad $Z(\Sigma)$ definida por $\Sigma$. La variedad $Z(\Sigma)$ es lisa si y sólo si el abanico $\Sigma$ es *regular* (véase la sección 2.2). La resolución de singularidades de una variedad tórica $Z(\Sigma)$ se corresponde con un hecho meramente combinatorio: la existencia de subdivisiones regulares del abanico $\Sigma$ (véase [Ew], [Co], y el teorema 2.1). Se tienen algunos algoritmos canónicos para calcular estas subdivisiones en dimensiones dos (véase [Od]), y tres (véase [Bo-Go] y [Ag]).

Los métodos de la geometría tórica en el estudio de las singularides hacen uso de la noción de *poliedro de Newton*. El poliedro de Newton $\mathcal{N}(f)$ de una serie de potencias $f = \sum c_\alpha X^\alpha$

con $X = (X_1, \ldots, X_d)$, es la envolvente convexa del conjunto $\bigcup_{c_\alpha \neq 0} \alpha + \mathbb{R}^d$. Por ejemplo, Kouchnirenko, prueba que si $f \in \mathbb{C}\{X\}$ define un germen de singularidad aislada $f = 0$, su número de Milnor

$$\dim_{\mathbb{C}} \mathbb{C}\{X\}/(\frac{\partial f}{\partial X_1}, \ldots, \frac{\partial f}{\partial X_d})$$

está acotado inferiormente por un entero determinado a partir del poliedro de Newton de $f$, y se tiene la igualdad si los coeficientes de los términos cuyos exponentes yacen en las caras compactas del poliedro de Newton de $f$ son suficientemente generales. Este tipo de condición se denomina de *no degeneración* (véase la sección 2.2 y [Ko]).

Consideramos los abanicos contenidos en $(\mathbb{R}^d)^*$, el espacio dual al espacio vectorial real engendrado los exponentes de las series de potencias de $\mathbb{C}\{X\}$. Un abanico regular $\Sigma$ con soporte $(\mathbb{R}^d)^*_{\geq 0}$, define una modificación $\pi : Z(\Sigma) \to \mathbb{C}^d$. Si el abanico $\Sigma$ es *compatible* con el poliedro de Newton de $f$ la modificación $\pi$ tiende a reducir la complejidad de las singularidades de la hipersuperficie $f = 0$. Esta idea es útil también para singularidades de intersección completa (véase [Ok2], y Theorem 2.2). Varchenko prueba que si la función $f$ es "no degenerada" y si su poliedro de Newton corta todos los ejes de coordenadas (lo que significa en particular que define una singularidad aislada) entonces la modificación $\pi$ es una resolución sumergida de la variedad $f = 0$ (véase [Me]).

Una de las mayores dificultades en este enfoque es cómo determinar si la restrición del morfismo $\pi$ a la transformada estricta de la variedad considerada induce un isomorfismo fuera del lugar singular de la variedad.

## Resumen de la memoria y resultados que aporta

El primero de los capítulos está escrito en francés. Está publicado en la revista *Canadian Journal of Mathematics* (véase [GP]). Los otros dos capítulos están redactados en inglés.

En el primer capítulo de esta memoria estudiamos los polinomios $F \in \mathbb{C}\{S_\sigma\}[Y]$, con coeficientes en el anillo de gérmenes de funciones holomorfas en el punto especial de una variedad tórica afín normal. Extendemos a este caso, la parametrización clásica de singularidades casi-ordinarias cuya existencia prueba el Teorema de Jung-Abhyankar. Para ello damos un algoritmo generalizando el Teorema de Newton Puiseux que construye los términos de todas las raíces de $F$ como series de potencias fraccionarias. Este método está basado en un trabajo de McDonald para polinomios (véase [McD]). Obtenemos relaciones entre estas parametrizaciones, el poliedro de Newton del discriminante de $F$ con respecto a la variable $Y$, y el poliedro de Newton de $F$

por medio del *poliedro-fibra* de Billera y Sturmfels (véase [Bi-St]) que permiten calcular, bajo hipótesis de *no degeneración*, los vértices del discriminante a partir de los vértices del poliedro de Newton de $F$ y los coeficientes correspondientes a partir de los coeficientes de los términos de $F$ cuyos exponentes yacen en las aristas su poliedro de Newton. Se deducen resultados análogos para la *resultante* de dos polinomios $F$ y $G$ en el anillo $\mathbb{C}\{S_\sigma\}[Y]$.

En el capítulo segundo estudiamos los invariantes de las singularidades casi-ordinarias de hipersuperficie irreducible. Este estudio nos permite describir la normalización y dos procesos de resolución sumergida que sólo dependen de los invariantes.

Comenzamos asociando a una rama casi-ordinaria $\zeta$ (con $g$ monomios característicos) definiendo una singularidad $S$ de dimensión $d$ un sistema de $d+g$ generadores de un subsemigrupo $\Gamma$ de $\mathbb{Z}^d$. Este sistema de generadores puede ser definido empleando las *semi-raíces* asociadas a la rama casi-ordinaria $\zeta$. Se trata de funciones definiendo singularidades casi-ordinarias de hipersuperficie irreducible que se pueden parametrizar mediante ramas casi-ordinarias con unos *órdenes de contacto* determinados con la rama casi-ordinaria $\zeta$. Estas funciones generalizan las propiedades de las *raíces aproximadas* de Abhyankar y Moh (véase [A-M] y [PP]) y definen hipersuperficies que generalizan a este caso las curvas de contacto maximal de Lejeune (véase [LJ]). Probamos en particular que las raíces aproximadas del polinomio mínimo de $\zeta$ proporcionan un conjunto completo de semi-raíces. La relación entre las semi-raíces y el sistema de generadores de $\Gamma$ se obtiene a partir de una "filtración" del anillo de series formales en $d$ variables que tiene índices en el semigrupo de los poliedros de Newton; (esta construcción está lejanamente relacionada con la filtración dada por Kouchnirenko en [Ko]). El anillo $R$ de gérmenes de funciones holomorfas de $S$ en el origen está contenido en el anillo de series de potencias formales en $d$ variables por medio de la parametrización. Mostramos que el anillo "graduado" asociado a la "filtración" inducida de $R$ es igual a la $\mathbb{C}$-álgebra del semigrupo $\Gamma$ y que es independiente de la parametrización. Obtenemos de esta manera el resultado principal de esta sección: la equivalencia entre el tipo topológico de la singularidad $S$ y el dato del semigrupo $\Gamma$ asociado a la rama casi-ordinaria $\zeta$. La variedad tórica afín $\mathcal{S}_0 := \mathrm{Spec}\mathbb{C}[\Gamma]$ admite una parametrización definida por monomios. La denominamos la *variedad monomial* asociada a $S$ siguiendo la terminología de [T2] y de [G-T] para el caso de ramas planas. Como aplicación damos pruebas simples de las "fórmulas de inversión" para hipersuperficies casi-ordinarias. Empleamos esta construcción para determinar de forma explícita la normalización de la singularidad $S$.

En el tercer capítulo nos ocupamos de la determinación de una resolución sumergida de singularidades de un germen de hipersuperficie casi-ordinaria irreducible. Empleamos métodos de la geometría tórica para dar una solución a este problema de dos maneras diferentes. El resultado principal es que estas dos resoluciones sumergidas dependen únicamente del tipo topológico de la singularidad. Esto responde a una pregunta planteda por Lipman en [L5]. Como las singularidades de los gérmenes de hipersuperficie casi-ordinaria no son aisladas en general hemos realizado un estudio detallado del lugar singular y de su transformación mediante estos morfismos.

En primer lugar proporcionamos una resolución sumergida de una singularidad casi-ordinaria de hipersuperficie de dimension $d$ sumergida en $\mathbb{C}^{d+1}$ como composición de aplicaciones toroidales. En el caso, $d=1$, este es un resultado bien conocido y el método ha sido extendido por Lê y Oka a gérmenes de curvas analíticas planas complejas no necesariamente irreducibles (véase [Le-Ok] y [Ok1]). Las aplicaciones toroidales utilizadas se definen a partir de los monomios característicos de una rama casi-ordinaria $\zeta$ parametrizando la singularidad $S$. Estas aplicaciones preservan y simplifican la naturaleza casi-ordinaria de la transformada estricta. Las semi-raíces desempeñan un papel auxiliar al definir coordenadas adecuadas en cada etapa y se tiene que elecciones diferentes de semi-raíces se corresponden con un cambio de coordenadas adecuadas.

El otro método está basado en el trabajo de Goldin y Teissier para gérmenes de curvas analíticas planas complejas irreducibles. Este método ha sido aplicado con éxito por Lejeune y Reguera para estudiar las singularidades de superficies tipo sandwich (véase [LJ-R]). Definimos por medio de las semi-raíces una inmersión de $S$ en el espacio afín complejo $\mathbb{C}^{d+g}$ (siendo el entero $g$ el número de monomios característicos de la rama casi-ordinaria $\zeta$ y $d$ la dimensión de $S$). Se tiene una deformación $d$-paramétrica de $S \subset \mathbb{C}^{d+g}$, que se especializa en la variedad monomial sumergida en $\mathbb{C}^{d+g}$. Esta deformación es una intersección completa de singularidad casi-ordinaria; describimos sus ecuaciones a partir de las ecuaciones de la variedad monomial empleando para ello el graduado asociado a la "filtración" del álgebra analítica de la deformación (definida en la primera sección) y algunas propiedades de platitud en álgebra conmutativa. El resultado principal es que una modificación tórica *compatible* con un conjunto de ecuaciones de la variedad monomial proporciona una resolución sumergida de la deformación y de todas las fibras simultáneamente. En particular, es una resolución sumergida de la variedad monomial y de la singularidad casi-ordinaria $S \subset \mathbb{C}^{d+g}$. Por último añadimos una sección donde se calculan ejemplos explícitos de resoluciones sumergidas de una superficie casi-ordinaria parametrizada por una rama casi-ordinaria con dos monomios característicos mediante los dos métodos indicados.

## Perspectivas y problemas abiertos

La primera pregunta que surge del capítulo segundo es la determinación de las relaciones entre ambos métodos de resolución sumergida. Para ello parece necesario tener una mayor comprensión de qué es lo que permanece invariante bajo las posibles elecciones de morfismos tóricos o toroidales en cada uno de los dos procesos. Sería interesante describir, al menos en en el caso de superficies usando los algoritmos canónicos de subdivisión regular para abanicos de dimensión tres de [Bo-Go] y de [Ag], la estructura de del divisor excepcional y del divisor crítico de morfismo de la resolución sumergida de la singularidad en $\mathbb{C}^3$ (damos un ejemplo de esto en el capítulo tercero). Es posible que la extensión de éstos métodos a las singularidades casi ordinarias tóricas del capítulo primero pueda ayudar a resolver esta cuestión. Como perspectiva, parece razonable el extender el método toroidal de resolución a singularidades casi-ordinarias de hipersuperficie *reducida* (siguiendo el trabajo de Lê y de Oka para curvas planas). Este podría ser un primer paso hacia el desarrollo de una resolución toroidal de singularidades, por medio del enfoque de Jung.

Queda pendiente cómo generalizar la noción del semigrupo asociado para una singularidad casi-ordinaria, no necesariamente de hipersuperficie y encontrar una prueba de su constancia analítica independiente de métodos topológicos. Para esto, podría ser interesante extender el desarrollo de Hamburguer Noether para el caso de singularidades casi-ordinarias (véase [Ca] para el caso de curvas).

Otro problema tiene que ver con la *teoría de la saturación*. Se trata de determinar si es cierto que todas las singularidades casi-ordinarias de hipersuperficie con igual semigrupo se pueden obtener como proyecciones genéricas de una variedad monomial que se pueda asociar al conjunto de monomios característicos de una rama normalizada cualquiera correspondiente a esta clase de singularidad (véase [T1] para el caso de ramas planas).

Sería interesante dar una la descripción de las *variedades polares* de una singularidad casi-ordinaria en función de su tipo topológico. Los avances que se pudiesen hacer en este problema podrían tener aplicaciones en el estudio de la curvatura de las fibras de Milnor de la singularidad (véase [GB-T] para el caso de curvas planas).

Finalmente, se podría estudiar la teoría de valoraciones sobre singularidades casi-ordinarias. El uso de las deformaciones y de los anillos graduados puede servir para entender y determinar la resolución de singularidades casi-ordinarias en característica positiva.

# Introduction

In this work we use tools coming from *toric geometry* to study of *quasi-ordinary singularities* on complex analytic varieties, mainly in the hypersurface case.

A *quasi-ordinary singularity* of dimension $g$ is a germ of complex analytic variety which admits a *quasi-ordinary projection* (i.e., a finite map onto $\mathbb{C}^d$ unramified outside a normal crossing divisor). For instance, the singularities of complex analytic curves are quasi-ordinary.

If $f(X,Y) = 0$ is the equation of a complex analytic plane curve $S$ analytically irreducible at the origin, and different from $X = 0$, then the Newton-Puiseux Theorem constructs a parametrization of the form: $X = T^n, Y = \zeta(T)$ where $\zeta(T)$ is a convergent complex power series and the integer $n$ is the *intersection multiplicity* of the curve and the axis $X = 0$ at the origin, i.e., the number $\dim_{\mathbb{C}} \mathbb{C}\{X,Y\}/(f(X,Y),X)$. If we set $T = X^{1/n}$ then, we obtain an expansion of $Y$ as fractional power in $X$. (See [W], or [Z2]). An irreducible quasi-ordinary $d$ dimensional hypersurface singularity $S$ can be defined by an equation $F(X_1,\dots,X_d;Y) = 0$, and it is parametrized by some special fractional power series $\zeta$ in the variables $X_1^{1/r_1},\dots,X_d^{1/r_d}$, generalizing the classical Puiseux parametrization of plane complex analytic curves. This is the assertion of the Jung-Abhyankar Theorem (see [A1], [J], [Zu] and Théorème 1.1).

In the first chapter of this memoir we give a constructive procedure which constructs the terms of a more general type of fractional power series parametrization $Y = z(X)$ of a germ of hypersurface singularity defined by an equation $F(X;Y) = 0$ polynomial in $Y$. This generalizes the Newton-Puiseux procedure for hypersurfaces and involves an interesting relation with the Newton polyhedron of the discriminant of the polynomial $F$ with respect to $Y$. As application we show that under certain "non degeneracy" hypothesis the Newton polyhedron of the discriminant of $F$ its determined from the Newton polyhedron of $F$ by means of the *fiber-polyhedron* of [Bi-St]. This procedure is based on work of McDonald (see [McD]). The results are stated for germs of hypersurface singularities embedded in an affine normal toric variety.

In the second chapter we study the fractional power series parametrization of an irreducible quasi-ordinary hypersurface singularity $S$. A fractional power series $\zeta$ of this type is called a *quasi-ordinary branch.* It has a finite number of special terms called *characteristic monomials* and their vector exponents are called *characteristic exponents.*

This situation generalizes the classical characteristic Puiseux exponents of a Puiseux parametrization of a plane branch (i.e., an irreducible complex plane curve germ). The semigroup of a plane branch $C$ is the semigroup of intersection multiplicities at the origin of $C$ and the plane curves not containing $C$ as a component. This is clearly an invariant of the curve singularity $C$ and its data is equivalent to the data of the characteristic Puiseux exponents corresponding to a *transversal projection* [8]. Its well known that the data of the semigroup is equivalent to the *topological type* of the curve singularity (see [Z2] and [Re]). The parametrization gives an inclusion of the analytic algebra $R$ of the singularity in the ring $\mathbb{C}\{T\}$. Then the $\mathbb{C}$-algebra of the semigroup of the plane branch appears as the graded ring associated to the filtration of $R$ defined by the maximal ideal of the local ring $\mathbb{C}\{T\}$. This is the ring of coordinates of a monomially parametrized curve (see [T2]).

The data of the characteristic monomials of a quasi-ordinary branch parametrizing a $d$-dimensional hypersurface singularity $S$ is equivalent to the *topological type* of $S$. This is done by Gau, who shows that the *topological type* of the quasi-ordinary hypersurface singularity is equivalent to the set of characteristic monomials of any *normalized* [9] quasi-ordinary branch parametrizing it (see [Gau]).

In the second chapter we associate a semigroup $\Gamma \subset \mathbb{Z}^d_{\geq 0}$ to any quasi-ordinary branch $\zeta$ parametrizing a hypersurface singularity $S$ of dimension $d$ by generalizing some of the ideas for the plane branch case. We show that the graded ring associated to the induced $(T)$-adic filtration of the analytic algebra $R$ of the singularity is equal to the $\mathbb{C}$-algebra of the semigroup $\Gamma$. This is the ring of coordinates of a non necessarily normal toric variety, monomially parametrized, that we call the monomial variety following the terminology of Goldin and Teissier. The main result of this part is the invariance of the semigroup $\Gamma$. Precisely, we prove, using Gau's results, that the data of this semigroup is equivalent to the topological type of the singularity $S$.

In the third chapter we determine for an irreducible quasi-ordinary hypersurface singularity

[8]The projection $(X, Y) \mapsto X$ is *transversal* for a plane curve if the multiplicity at the origin of the curve coincides with the intersection multiplicity of the curve with the axis $X = 0$.

[9]This is a technical condition which in the plane curve case means that the projection $(X, Y) \mapsto X$ is *transversal.* Lipman's inversion lemma guarantees that any quasi-ordinary hypersurface is parametrized by a *normalized* quasi-ordinary branch, and determines the its characteristic monomials from the characteristic monomials of the initial quasi-ordinary branch by means of the *inversion formulae* (see [Gau], Appendix). This result is a generalization of the classical inversion lemma of Zariski and Abhyankar for plane curves (see [A2]).

two embedded resolution procedures which depend only on the topological type of $S$ (i.e., on the associated semigroup by the results of chapter two). This result solves the open problem 5.1 in [L5].

The first procedure determines an embedded resolution of the singularity $S$ embedded in $\mathbb{C}^{d+1}$ as a composition of toroidal maps. The procedure generalizes the well known result that the embedded resolution of a plane branch can be factored as composition of $g$ toroidal maps; the number $g$ being equal to the number of characteristic Puiseux exponents of a given parametrization. The second procedure generalizes the work of Goldin and Teissier (see [G-T]). We give an embedding of the hypersurface $S$ in the affine space $\mathbb{C}^{d+g}$ using functions, such that their initial forms define an embedding of the associated monomial variety. This embedding provides a deformation which specializes to the monomial variety. Then we build an embedded resolution of the monomial variety using a toric map and we prove that it is also an embedded resolution of the singularity $S$.

This memoir is a contribution to the study of invariants of quasi-ordinary hypersurface singularity, mainly from the algebraic viewpoint, and their role in the embedded resolution of singularities.

## Classification by invariants

One of the main problems in the study of singularities is their classification. Gau's result gives a finite encoding of the topological type of the singularity by means of characteristic monomials of a normalized quasi-ordinary branch parametrizing it. We give in this section, as a motivation, a reminder of different approaches to this problem.

### Invariants and resolution

This approach is related with the possibility of constructing a "canonical" (embedded) resolution[10] procedure for the singularity in terms of the characteristic exponents.

In the case of plane branches Zariski proved that the characteristic monomials of a normalized plane branch determine and are determined by the sequence of multiplicities of the curves obtained by blowing up recursively the singular point of each strict transform of the curve. More generally, the minimal embedded resolution of a plane curve singularity is determined by the characteristic

[10] See the next section of this introduction

exponents of its irreducible components (branches) and the intersection numbers of any pair of them.

Lipman proved that the characteristic exponents of any normalized quasi-ordinary branch parametrizing a germ of surface, $(S,0) \subset (\mathbb{C}^3, 0)$, are analytical invariants of the singularity, (see [L1], [L3]), which means that they only depend on the germ $(S,0)$. This is done by constructing a resolution procedure which determines and is completely determined by the characteristic exponents of a normalized quasi-ordinary branch. Luengo gives in [Lu] another proof of the (analytical) invariance of the characteristic exponents of a normalized quasi-ordinary branch. He constructs an invariant weighted tree associated to an algebroid surface. If this surface is quasi-ordinary and irreducible he shows that this tree is determined by the characteristic exponents. Both proofs used quadratic transformations (blowing ups of points) and monoidal transformations (blowing ups of smooth equimultiple curves in the singular locus). The important fact shown by Lipman is monoidal stability, i.e., the strict transform of the surface by these operations is (in all the important cases) is quasi-ordinary, irreducible and the corresponding characteristic exponents are determined by those of $\zeta$ (see [L3], page 78). Quasi-ordinary hypersurface singularities of dimension $\geq 3$ do not have this property. As example, the strict transform of the hypersurface $X_4^4 - X_1X_2X_3 = 0$ by the quadratic transformation $X_1 = Z_1, X_i = Z_iZ_1$ for $i = 2,3,4$ is $Z_4Z_1 - Z_2Z_3$ is not quasi-ordinary for any of the coordinate projections. This is the main obstacle to understand if in general a resolution procedure can be determined from the characteristic exponents using this method. Ban-McEwan use the canonical resolution of Bierstone-Milman in the surface case, to determine if the embedded resolution of an irreducible quasi-ordinary surface depends only on the (normalized) characteristic exponents (see [B-M]). See also [Vi] for another approach to this problem.

## Invariants and topology

Two hypersurfaces germs $(X,0)$ and $(X',0)$ in $\mathbb{C}^{d+1}$ have the same *embedded topological type* if and only if there is an homeomorphism $U \to U'$ between two open neighborhoods of the origin, which maps $X \cap U$ to $X' \cap U'$.

The topological type of a plane curve singularity is completely determined by the link of the singularity, i.e., the intersection of the curve with a sufficiently small sphere centered at the origin. The result, obtained using knot theory, states that the topological type is equivalent to the datum of the characteristic exponents of any irreducible component and the intersection numbers of any two different components (see [Re]).

For quasi-ordinary hypersurfaces, we have that the characteristic exponents of a quasi-ordinary branch $\zeta$ determine the embedded topological type of the hypersurface it defines. This can be deduced using results of Zariski on saturation of local rings (see [Z1], [L3] §2 and also [Oh] for another proof). Topological invariants of quasi-ordinary singularities are studied by Lipman (see [L4]). Lipman's results are used by Gau to prove that if two pairs of quasi-ordinary hypersurface germs $(X, 0)$ and $(X', 0)$ in $\mathbb{C}^{d+1}$ have the same topological type then any two normalized quasi-ordinary branches parametrizing $(X, 0)$ and $(X', 0)$ have the same characteristic exponents. (See Theorem 2.3).

## Invariants and algebra

For the plane branch $(S, 0) \subset (\mathbb{C}^2, 0)$ there are various ways to prove the invariance:

- Zariski and Abhyankar proved explicitly and algebraically the invariance of the characteristic exponents associated to a transversal parametrization by showing explicit "inversion formulae" to relate the characteristic exponents of different parametrizations (see [Z1] Proposition 2.2 and [A2]).

- The set of intersection multiplicities of plane curves not containing $S$ is a sub-semigroup $\Gamma$ of $\mathbb{Z}_{\geq 0}$, which only depends on the curve. The characteristic exponents of a parametrization of $S$ determine a set of generators of $\Gamma$ and if the parametrization is transversal if and only of this set is the minimal set of generators of $\Gamma$. Conversely the minimal set of generators of $\Gamma$ determines the characteristic exponents corresponding to a transversal parametrization (see [Z2]). We can recover the inversion formulae using the semigroup $\Gamma$. The set of generators of the semigroup are related with the Abhyankar's *approximate roots* of the equation defining the curve with respect to the given parametrization (see [PP]).

- The normalization is defined by any primitive parametrization of the curve and is described algebraically by the homomorphism, $R \rightarrow \mathbb{C}\{T\}$, where $R$ is the analytic algebra of the plane branch. The ring of the normalization $\mathbb{C}\{T\}$ is a discrete valuation ring with uniformazing parameter $T$. The $(T)$-adic filtration of $R$ does not depend on the choice of the parametrization by the unicity of the normalization. The associated graded ring is the $\mathbb{C}$-algebra $\mathbb{C}[\Gamma]$ of the semigroup of the curve. We can recover the semigroup $\Gamma$ from the graduation (see [T2]).

For quasi-ordinary singularities of dimension $d \geq 2$ there are several difficulties to extend this sort of ideas: how identify the set of quasi-ordinary projections, the fact that the ring $\mathbb{C}\{T_1, \ldots, T_d\}$ is not a discrete valuation ring, the absence of a generalization of the intersection multiplicity, etc.

## Resolution of singularities

We introduce some basic definitions and motivations about resolution of singularities which are needed to explain our results for quasi-ordinary hypersurface singularities.

The problem of resolution of singularities is to answer the following questions:

> *Resolution of singularities* :
>
> *Given any variety $\mathcal{X}$ does there exist a proper map $\phi : \mathcal{Y} \to \mathcal{X}$, with $\mathcal{Y}$ non singular such that $\phi$ is an isomorphism over the non singular part of $\mathcal{X}$?*
>
> *Embedded resolution of singularities (I)* :
>
> *Given a non singular variety $\mathcal{W}$, and a closed subvariety $\mathcal{X}$ with a dense complement, does there exist a proper map $\phi : \mathcal{W}' \to \mathcal{W}$, with $\mathcal{W}'$ non singular, such that $\phi$ is an isomorphism over $\mathcal{W} - \mathcal{X}$, and $\phi^{-1}(\mathcal{X})$ is a divisor on $W'$ having only normal crossings?*

where *variety* means here an algebraic variety over an algebraically closed field or a complex (or real) analytic space. We say that a divisor $D$ in a smooth space $\mathcal{W}$ has only *normal crossing* singularities if for any $w \in \mathcal{W}$ the defining ideal of $D$ at this point is generated by a "monomial", $X_1^{a_1} \dots X_d^{a_d}$, where $\{X_1, \dots, X_d\}$ is a local coordinate system at $w$ and $d$ is the dimension of $W$. A normal crossing divisor has a natural stratification (see [KKMS]).

In this work we will always refer to the following definition of embedded resolution unless the contrary is specified.

> *Embedded resolution of singularities (II)* :
>
> *Given a non singular variety $\mathcal{W}$, and a closed subvariety $\mathcal{X}$ with a dense complement, does there exist a modification $\phi : \mathcal{W}' \to \mathcal{W}$, with discriminant locus*[11] *rare in $\mathcal{X}$ such that the critical locus*[12] *of $\phi$ is a normal crossing divisor, the restriction of $\phi$ to the strict transform*[13] *$\tilde{\mathcal{X}}$ of $\mathcal{X}$ is a resolution of singularities of $\mathcal{X}$ and such that $\tilde{\mathcal{X}}$ is transversal to the natural stratification of the critical divisor of $\phi$?*

Hironaka has given positive answers for these questions for algebraic varieties defined over fields of characteristic zero, and for complex (or real) analytic space (see [L2] for references).

[11] The discriminant locus is the image of the critical locus

[12] The critical locus is the set of points where the morphism $\phi$ fails to be a local isomorphism

[13] The strict transform is the closure of the set $\phi^{-1}(X - \text{discrim}(\phi))$ in $\mathcal{W}'$.

Bierstone-Milman and Villamayor have given canonical algorithms for resolution, based on the work of Hironaka.

Concerning the resolution of two dimensional quasi-ordinary singularities the classical approach is to normalize them. If $\hat{S} \to S$ is the *normalization* map, then the composition with the quasi-ordinary projection is clearly quasi-ordinary, and moreover the parametrization of $S$ factors through the normalization $\hat{S}$ and $\hat{S}$ is a *quotient singularity* [14]. A desingularization of $\hat{S}$ can be obtained by blowing up a zero dimensional ideal (see [L2]). It is also known that the normalization is a toric affine surface (see [B-P-V], Chapter III, Theorem 5.2, for a topological proof). The minimal resolution of affine normal toric surfaces can be computed using toric methods (see [Od]).

## Jung's approach

Quasi-ordinary singularities appears naturally in *Jung's approach to resolution of singularities* (see [L2], Lecture 2). Jung's idea to approach embedded resolution is to use induction on the dimension. We begin with, say, an hypersurface $S \subset \mathbb{C}^{d+1}$, then we take a finite projection of $S$ onto $\mathbb{C}^d$. The reduced *discriminant locus* of this projection is an hypersurface $D \subset \mathbb{C}^d$, hence by induction there is an embedded resolution $\mathcal{X} \to \mathbb{C}^d$ of $D \subset \mathbb{C}^d$. Taking the fiber product leads to a commutative diagram, where $\mathcal{W}$ is smooth,

$$\begin{array}{ccc} \mathcal{W} & \longrightarrow & \mathbb{C}^{d+1} \\ \cup & & \cup \\ \tilde{S} & \longrightarrow & S \\ \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & \mathbb{C}^d \\ \cup & & \cup \\ \tilde{D} & \longrightarrow & D \end{array} \tag{2}$$

the left projection is unramified over the normal crossing divisor $\tilde{D}$ which maps to $D$. In particular, this implies that the hypersurface $\tilde{S} \subset \mathcal{W}$ has at worst quasi-ordinary singularities.

[14] A germ $S$ is a quotient singularity if it is normal, and the ring of fractions $K$ of the analytic algebra $R$ of $S$ has a finite Galois extension $K'$ in which the integral closure $R_{K'}$ of $R$ is a regular local ring. Equivalently, $S$ is normal and $R$ is the ring of invariants of a finite group of automorphisms of the power series ring $\mathbb{C}\{T_1, \dots, T_d\}$ ($d = \dim S$). The normalization $\hat{S}$ of $S$ is a quotient singularity: if $\hat{R}$ is the analytic algebra of $\hat{S}$, then we have the finite ring extensions. $R \subset \hat{R} \subset \mathbb{C}\{T_1, \dots, T_d\}$. Taking the ring of fractions defines a finite Galois extension $K \subset K'$ and the integral closure of $\hat{R}$ in $K'$ is equal to $\mathbb{C}\{T_1, \dots, T_d\}$ (see [L4], §2).

This approach is used by Walker, Hirzebruch, Zariski and Abhyankar to prove the existence of a resolution for surfaces and varieties of dimension two and three (see [L2]).

### Toric methods in the resolution of singularities

A variety $Z$ containing the torus $(\mathbb{C}^*)^d$ as an open dense subset, and provided with an action of $(\mathbb{C}^*)^d$ that extends the product of the torus as an algebraic group, is called a *toric variety.* The study of this type of varieties arise at the beginning of the seventies by the work of many mathematicians. The geometry of toric varieties is very interesting because there exist a dictionary translating many geometric statements about toric varieties into statements of combinatorial convexity, (see [Od], [KKMS], [Ew], [F1], [St] and [G-K-Z]). This interaction has been very useful for the developpement of both theories, algebraic geometry and combinatorial convexity.

In particular normal toric varieties are defined by fans. A fan $\Sigma$ is a polyhedral complex of cones contained in $(\mathbb{R}^d)^*$ verifying some properties. Each cone $\sigma$ of the fan corresponds to an affine toric variety $Z(\sigma)$, which is a chart of the variety $Z(\Sigma)$ defined by $\Sigma$. The variety $Z(\Sigma)$ is smooth if and only if the fan $\Sigma$ is *regular* (see section 2.2). The resolution of singularities of the variety $Z(\Sigma)$ corresponds to a purely combinatorial fact, the existence of regular subdivision of the fan $\Sigma$ (see [Ew], [Co], and theorem 2.1). There are some canonical algorithms to do this in dimensions 2 (see [Od]) and 3 (see [Bo-Go] and [Ag]).

The methods of toric geometry to study singularities make use of the notion of *Newton polyhedron* [15]. For instance, Kouchnirenko shows that if $f \in \mathbb{C}\{X\}$ defines a germ of isolated singularity $f = 0$, its Milnor number

$$\dim_{\mathbb{C}} \mathbb{C}\{X\}/(\frac{\partial f}{\partial X_1}, \dots, \frac{\partial f}{\partial X_d})$$

is bounded below by a number defined from the Newton polyhedron of $f$, and we have the equality if the coefficients of the terms with exponents lying on the compact faces of the Newton polyhedron are sufficiently general. This is an example of *non degeneracy conditions.* See section 2.2 and [Ko].

We consider the fans contained in $(\mathbb{R}^d)^*$, the dual space of the space spanned by the exponents of the power series in $\mathbb{C}\{X\}$. If $\Sigma$ is a regular fan supported on $(\mathbb{R}^d)^*_{\geq 0}$, it defines a proper modification $\pi : Z(\Sigma) \to \mathbb{C}^d$. If the fan $\Sigma$ is "compatible" with the Newton polyhedron of $f$, the modification above "reduce the complexity" of the singularities of the hypersurface $S$

[15] the Newton polyhedron $\mathcal{N}(f)$ of a series $f = \sum c_\alpha X^\alpha$ with $X = (X_1, \dots, X_d)$, is the convex hull of the set $\bigcup_{c_\alpha \neq 0} \alpha + \mathbb{R}^d$

defined by $f=0$. This ideas works well also for complete intersections (see [Ok2], and Theorem 2.2). For instance, Varchenko showed that if the function $f$ is is non degenerate and its Newton polyhedron of cuts all coordinate axis (which implies that $f$ has an isolated singularity), then the toric map $\pi$ is an embedded resolution of the germ $f=0$, (see [Me]).

One of the main difficulties of this approach is to determine when the restriction of the map $\pi$ to the strict transform $\tilde{S}$ of $S$ induces an isomorphism

$$\tilde{S} - \pi^{-1}(\text{ Sing } (S)) \to S - \text{ Sing } (S).$$

In general we know only that this map provides an isomorphism outside the intersection of $S$ with the discriminant locus of $\pi$.

## Description of contents and main results

Now we describe the main results of each chapter: The first chapter is written in French. It is a paper of the author in *Canadian Journal of Mathematics*. The other two are written in English.

In the first chapter of this memoir we study polynomials $F \in \mathbb{C}\{S_\sigma\}[Y]$ with coefficients in the ring of germs of holomorphic functions at the special point of an affine normal toric variety. We generalize to this case the classical parametrization of quasi-ordinary singularities provided by the Jung-Abhyankar Theorem. To do this we generalize the Newton-Puiseux Theorem, giving an algorithm to construct all the roots of $F$ as fractional power series. This procedure is based on work of McDonald for polynomials (see [McD]). On the other hand, this involves also a relation between the Newton polyhedron of the discriminant of the polynomial $F$ with respect to $Y$ and the Newton polyhedron of $F$ by means of the fiber-polyhedron of Billera and Sturmfels (see [Bi-St]). This relation allows us to compute, under a “non degeneracy” hypothesis, the vertices of the Newton polyhedron of the discriminant from the Newton polyhedron of $F$ and the corresponding coefficients from the coefficients of the terms of $F$ having exponents on the edges of its Newton polyhedron. We give some analogous results for the *resultant* of two polynomials $F$ and $G$ in the ring $\mathbb{C}\{S_\sigma\}[Y]$. We obtain, using a very different method, results of the same nature as those obtained by Gel’fand, Kapranov and Zelevinski in [G-K-Z].

In the second chapter we associate to a quasi-ordinary branch $\zeta$ with $g$ characteristic monomials parametrizing a singularity $S$ of dimension $d$ a set of generators of a sub-semigroup $\Gamma$ of $\mathbb{Z}^d_{\geq 0}$. We characterize all sub-semigroups of $\mathbb{Z}^d_{\geq 0}$ arising in this way. This set of generators can be defined using the *semiroots* associated to the quasi-ordinary branch $\zeta$ (we follow the terminology of [PP] for plane branches); the semi-roots are functions parametrized by quasi-ordinary branches with $0 \leq j \leq g$ characteristic monomials, having maximal *order of coincidence* with the series $\zeta$. These functions generalize by their properties the *approximate roots* of Abhyankar and Moh (see [A-M] and [PP]) and define hypersurfaces which generalizes to the quasi-ordinary case, the curves of maximal contact of Lejeune (see [LJ]). In particular we prove that the approximate roots of the minimal polynomial of $\zeta$ provide a complete set of semi-roots. The relation between the semi-roots and the generators of the semigroup $\Gamma$ is obtained by analyzing the inclusion of the ring $R$ in the ring of power series in $d$ variables with the help of some suitable filtrations. We show that the graded ring associated to the induced filtration of $R$ is equal to the $\mathbb{C}$-algebra of the semigroup $\Gamma$ and that it is independent of the parametrization. The affine variety $\mathcal{S}_0 := \text{Spec}\mathbb{C}[\Gamma]$ is monomially parametrized and we call it the *monomial variety* associated to the singularity $S$, following the terminology introduced in [T2] and [G-T] for plane branches. The main result of this part, obtained using Gau's Theorem, is that the semigroup $\Gamma$ does not depend on the quasi-ordinary branch parametrizing $S$ and its data is equivalent to the topological type of the singularity $S$. We give an application, showing that the *normalization* of the quasi-ordinary hypersurface is an affine toric variety determined the saturation of the semigroup $\Gamma$.

In the third chapter we address the problem of determining an embedded resolution procedure of a quasi-ordinary hypersurfaces. We use a toric approach to solve this problem in two different ways. The main result is that the two resolution procedures we find depend only on the topological type of the singularity, and it gives an answer to a question of Lipman (see [L5]). Since quasi-ordinary hypersurfaces do not have isolated singularities in general and we must study carefully the singular locus to prove that the restriction of the resolution map to the strict transform of the singularity is an isomorphism outside the singular locus of the singularity.

In the first procedure we construct an embedded resolution of a quasi-ordinary hypersurface of dimension $d$ in embedded in $\mathbb{C}^{d+1}$ as a composition of toroidal maps. If $d = 1$, this is well known, Lê and Oka have extended this method to the case of *reduced* plane curves (see [Le-Ok] and [Ok1]). These maps are defined by the characteristic monomials of a quasi-ordinary branch parametrizing the hypersurface $S$. They preserve and simplify the quasi-ordinary nature of the strict transforms and are compatible with the semi-roots, which are needed to define *good* coordinates at each stage. Different choices of semi-roots corresponds to a change of good coordinates.

The other method is based on work of Goldin and Teissier for complex plane branches. It has been used by Lejeune and Reguera to study to study sandwiched surface singularities. We use a toric modification of the complex affine space to construct a toric embedded resolution of the germ $S$ embedded in a particular way in the affine complex space $\mathbb{C}^{d+g}$ (the integer $g$ being the number of characteristic monomials of the given parametrization). This embedding is defined using the semi-roots, and it provides a natural $d$-parameter deformation of the quasi-ordinary hypersurface which specializes to the monomial variety embedded in $\mathbb{C}^{d+g}$. The deformation space is a complete intersection. We describe its equations from the equations of the monomial variety using some of the results on the graded ring of the filtration of a quasi-ordinary singularity and some properties of flatness. The main result is that a toric morphism which is *compatible* with a set of equations defining the monomial variety, provides an embedded resolution of singularities of all the fibers of the deformation. In particular, of the monomial variety and of the quasi-ordinary hypersurface $S$. Finally we add a section with an example of toric resolutions using these two methods of a singularity with two characteristic exponents.

## Open questions and perspectives

The first question which arises from Chapter 2 is the existence of a natural relation between the two procedures of embedded resolution. It seems necessary to have a better understanding of what stays invariant under the possible choices of regular subdivisions in any of the two procedures. It would be nice to describe, at least in the case of surfaces, using the known algorithms to produce regular subdivisions of fans in $\mathbb{R}^3$ (see [Bo-Go] and [Ag]), the structure of the critical divisor of the map giving the embedded resolution of $S$ in $\mathbb{C}^3$ (we give an example at the end of the third chapter). The possible extension of these methods to the "toric quasi-ordinary singularities" of Chapter 1, might help to answer these two questions.

It is interesting to generalize the notion of the semigroup associated to any quasi-ordinary singularity (not necessarily an hypersurface), and to find a proof of the analytical invariance of the semigroup not depending on topological methods. For this purpose, it could be interesting to extend the Hamburger-Noether expansions for quasi-ordinary singularities (see [Ca] for the plane case).

Another problem is related with *saturation theory*: Is it true that all quasi-ordinary hypersurface singularities with the same semigroup are all general projections of some monomial variety determined by the characteristic exponents of a normalized quasi-ordinary branch (see [T1] for the plane branch case).

As a perspective, it seems reasonable to try to extends the toroidal method of resolution, to *reduced* quasi-ordinary hypersurface singularities (following the works of Lê and Oka for curves). This could be a first step to build a toroidal resolution of singularities, using Jung's approach.

It would be interesting to give some examples and applications to the theory of *polar varieties* in the quasi-ordinary case. The results in this direction might have applications to study the curvature of the Milnor fibers of the singularity (see [GB-T] for the case of plane curves).

Finally, the use of deformations and graded rings might help to understand and determine the resolution of quasi-ordinary singularities in positive characteristic.

# Chapitre 1

# Singularités quasi-ordinaires toriques et polyèdre de Newton du discriminant

**Paru dans Canadian Journal of Mathematics (voir [GP])**

**Résumé.** [1] Nous étudions les polynômes $F \in \mathbb{C}\{S_\tau\}[Y]$ à coefficients dans l'anneau de germes de fonctions holomorphes au point spécial d'une variété torique affine. Nous généralisons à ce cas la paramétrisation classique des singularités quasi-ordinaires. Cela fait intervenir d'une part une généralization de l'algorithme de Newton-Puiseux, et d'autre part une relation entre le polyèdre de Newton du discriminant de $F$ par rapport à $Y$ et celui de $F$ au moyen du polytope-fibre de Billera et Sturmfels ([Bi-St]). Cela nous permet enfin de calculer, sous des hypothèses de non dégénérescence, les sommets du polyèdre de Newton du discriminant a partir de celui de $F$, et les coefficients correspondants à partir des coefficients des exposants de $F$ qui sont dans les arêtes de son polyèdre de Newton.

[1]Numéros de classification : principal 14M25, secondaire 32S25.

## 1.1 Introduction

Le sujet de la première partie de ce travail est la représentation des racines $Y(X)$ d'une équation polynôme $F(X_1, \dots, X_d; Y) = 0$ par des séries à exposants fractionnaires en les variables $X = (X_1, \dots, X_d)$. Il s'agit de généraliser le théorème de Newton-Puiseux. Nous poursuivrons dans une direction inaugurée par McDonald dans [McD], et précisons ses résultats.

Notre approche est d'étudier d'abord le problème dans le cas d'un polynôme $F \in \mathbb{C}\{S_\tau\}[Y]$, où $\mathbb{C}\{S_\tau\}$ est l'anneau des germes des fonctions holomorphes au point spécial d'une variété torique affine correspondant à un cône rationnel strictement convexe, $\tau \subset (\mathbb{R}^d)^*$, de dimension d. Nous résolvons le probleme, lorsque le *discriminant* $\Delta_Y F$ de $F$ par rapport à $Y$ est de la forme $X^u \epsilon$ où $\epsilon$ est une unité dans $\mathbb{C}\{S_\tau\}$ et $u$ appartient au semigroupe $S_\tau := \tau^\vee \cap \mathbb{Z}^d$ des élements de $\mathbb{Z}^d$ qui appartiennent au cône dual $\tau^\vee := \{w \in \mathbb{R}^d / \langle w, u \rangle \geq 0, \ \forall u \in \tau\}$. Ceci est en fait une généralisation de l'étude classique des singularités quasi-ordinaires, qui correspondent au cas où $\tau$ est le quadrant positif.

La réduction du cas général à ce cas fait appel à des constructions combinatoires sur le polyèdre de Newton $\mathcal{N}(F) \subset \mathbb{R}^{d+1}$ de $F$. La plus importante, déjà utilisée dans [McD] est celle du *polyèdre-fibre* $\mathcal{Q}(F) \subset \mathbb{R}^d$ de $\mathcal{N}(F)$ par rapport à sa projection $\mathcal{N}(F) \to \mathbb{R}^d$ sur l'espace des exposants des monômes en $X$. Les points extrêmes de $\mathcal{Q}(F)$ correspondent à certains chemins dans les arêtes de $\mathcal{N}(F)$.

Le polyèdre-fibre est également relié au polyèdre de Newton du discriminant de $F$ par rapport à $Y$. Si $F = a_0(X) + \cdots + a_r(X) Y^r$, on a l'inclusion de polyèdres de Newton

$$\mathcal{N}(\Delta_Y F) + \mathcal{N}(a_0) + \mathcal{N}(a_r) \subseteq \mathcal{Q}(F)$$

(où la somme est la somme de Minkowski), avec l'égalité sous des hypothèses de non-dégénérescence des coefficients de $F$ par rapport à $\mathcal{N}(F)$, (théorème 1.4).

Un cône $\tau \subset (\mathbb{R}^d)^*$ est compatible avec des polyèdres $\mathcal{P}_1, \dots, \mathcal{P}_s \subset \mathbb{R}^d$ s'il est constitué des fonctions linéaires qui prennent toutes leur valeur minimale sur $\mathcal{P}_1, \dots, \mathcal{P}_s$ en des point fixés $p_1 \in \mathcal{P}_1, \dots, p_s \in \mathcal{P}_s$. On décrit le résultat principal, le théorème 1.3, dans le cas où $F$ est un polynôme réduit dans l'anneau $\mathbb{C}\{X\}[Y]$. Si $\tau \subset (\mathbb{R}^d_+)^\vee$ est un cône rationnel strictement convexe de dimension $d$ compatible avec les polyèdres $\mathcal{N}(\Delta_Y F)$, $\mathcal{N}(a_r)$ et $\mathcal{Q}(F)$, alors l'homomorphisme $\mathbb{C}\{X\} \to \mathbb{C}\{S_\tau\}$ étendant l'inclusion des algèbres $\mathbb{C}[\mathbb{Z}^d_+] \to \mathbb{C}[S_\tau]$ transforme $F$ en un polynôme $F_\tau \in \mathbb{C}\{S_\tau\}[Y]$ dont toutes les racines sont de la forme $X^u \varepsilon(X)$ où $u \in \frac{1}{k}\mathbb{Z}^d$, $\varepsilon(X)$ est une unité dans l'anneau $\mathbb{C}\{\frac{1}{k} S_\tau\}$ et $k$ est un entier positif. La construction des racines est donnée par un algorithme qui généralise celui du théorème de Newton-Puiseux, (théorème 1.2), et qui pourrait se développer à l'aide d'un logiciel de calcul formel. On peut comparer l'algorithme obtenu avec celui de [A-L-R], developpé pour le cas quasi-ordinaire.

Nous utilisons ces résultats pour donner une description des sommets du polyèdre de Newton du *résultant* $\mathrm{Res}_Y(F,G)$ des polynômes $F,G \in \mathbb{C}\{S_\tau\}[Y]$ à partir des polyèdres-fibres $\mathcal{Q}(F)$, $\mathcal{Q}(G)$ et $\mathcal{Q}(FG)$ ; nous calculons enfin les coefficients des monômes correspondants aux sommets du polyédre de Newton de $a_0 a_r \Delta_Y F$ et de $\mathrm{Res}_Y(F,G)$ sous des hypothèses de non dégénérescence. Ces coefficients dépendent entre autres des résultants des paires de polynômes à une variable obtenus en-regardant de $F$ et de $G$ que les termes donts les exposants appartiennent à des paires parallèles d'arêtes de $\mathcal{N}(F)$ et $\mathcal{N}(G)$. On montre un résultat analogue pour le discriminant. Nous trouvons aussi, avec une méthode très différente, des résultats de même nature que ceux de Gel'fand, Kapranov et Zelevinski dans [G-K-Z].

## 1.2 Paramétrisation de singularités quasi-ordinaires toriques

### 1.2.1 L'algèbre des germes de fonctions holomorphes au point distingué d'une variété torique affine

Soit $\tau$ un cône convexe rationnel de dimension $d$ dans $(\mathbb{R}^d)^*$. Cette condition garantit que le cône dual $\tau^\vee := \{w \in \mathbb{R}^d / \langle w,u\rangle \geq 0,\ \forall u \in \tau\}$ est un cône rationnel strictement convexe. Pour cette raison, chaque élément de $S_\tau$ peut s'exprimer commme somme d'éléments du semigroupe $S_\tau$ d'un nombre fini de manières. L'ensemble des séries formelles à exposants dans $S_\tau$ est un anneau, que nous notons par $\mathbb{C}[[S_\tau]]$. La propriété de finitude précédente permet de garantir que les coefficients de la série produit sont des fonctions polynomiales des coefficients des facteurs. Ces anneaux sont définis dans [McD], pour construire des racines d'un polynôme $F \in \mathbb{C}[X_1,\ldots,X_d][Y]$, à la Newton-Puiseux.

On va donner une interprétation géométrique de ces anneaux au moyen de la variété torique associée au cône $\tau$ dans le cas où $\tau$ est un cône rationnel strictement convexe de dimension $d$ dans $\mathbb{R}^d$.

Soit $N \subset (\mathbb{R}^d)^*$ un réseau de dimension $d$, de réseau dual $M \subset \mathbb{R}^d$. Le cône $\tau$ définit le semi-groupe de type fini $S_\tau :=\tau^\vee \cap M$. Soit $\mathbb{C}[S_\tau]$ l'algèbre du semi-groupe $S_\tau$ à coefficients dans $\mathbb{C}$. Associons à $(\tau,N)$ la variété torique affine $Z_\tau :=$ Spec $\mathbb{C}[S_\tau]$. Chaque point fermé de $Z_\tau$ est définit par un homomorphisme de semi-groupes $S_\tau \to \mathbb{C}$. La valeur de la fonction $X^u \in \mathbb{C}[S_\tau]$ au point $x$ est $x(u)$. L'orbite de dimension 0 de la variété $Z_\tau$ est le point spécial

$z_\tau$ défini par l'homomorphisme de semi-groupes $S_\tau \to \mathbb{C}$ qui applique $0 \mapsto 1$ et $u \mapsto 0$ si $u \neq 0$. (Pour tout ceci voir [F1] ou [Od]).

*L'anneau des séries convergentes à exposants dans* $S_\tau$ , que nous notons par $\mathbb{C}\{S_\tau\}$, est l'ensemble des séries de $\mathbb{C}[[S_\tau]]$ qui sont absolument convergentes dans un voisinage du point spécial $z_\tau$ de la variété torique $Z_\tau$. Si $\tau$ est un cône convexe rationnel de dimension $d$, on défini $\mathbb{C}\{S_\tau\} = \cap\, \mathbb{C}\{S_\sigma\}$ où $\sigma$ parcourt les cônes rationnels strictement convexes de dimension $d$ contenus dans $\tau$.

**Lemme 1.1** *L'algèbre locale des germes de fonctions holomorphes au point* $z_\tau$ *de* $Z_\tau$ *est isomorphe à* $\mathbb{C}\{S_\tau\}$.

*Preuve.* Soient $u_1, \dots, u_s$ des générateurs du semi-groupe $S_\tau$. L'homomorphisme $\mathbb{C}[U_1, \dots, U_s] \to \mathbb{C}[S_\tau]$ défini par $U_i \mapsto X^{u_i} \in \mathbb{C}[S_\tau]$ est surjectif. Son noyau est un idéal premier $I$. Ce morphisme définit un plongement $Z_\tau \subset \mathbb{C}^s$ de la variété torique affine $Z_\tau :=$ Spec $\mathbb{C}[S_\tau]$ définie par le cône $\tau$, et l'image du point distingué $z_\tau$ est l'origine de $\mathbb{C}^s$. Soit $R$ l'algèbre des germes fonctions holomorphes dans un voisinage de $z_\tau$ dans $Z_\tau$.

Remarquons que l'homomorphisme composé $\mathbb{C}[U_1, \dots, U_s] \to \mathbb{C}[S_\tau] \hookrightarrow \mathbb{C}[[S_\tau]]$ s'étend en un homomorphisme $\chi : \mathbb{C}\{U_1, \dots, U_s\} \to \mathbb{C}[[S_\tau]]$ dont l'image est $\mathbb{C}\{S_\tau\}$. En effet, l'image du monôme $U^\lambda$, où $\lambda = (\lambda_1, \dots, \lambda_s) \in \mathbb{N}^s$, est le monôme $X^{u(\lambda)}$, où $u(\lambda) = \sum \lambda_i u_i \in S_\tau$. Donc, l'image de $\phi$ est la série $\chi(\phi) = \sum_{u \in S_\tau} (\sum_{u(\lambda)=u} c_\lambda) X^u$, qui est bien définie parce que $\tau$ est strictement convexe. Supposons que $\phi$ est absolument convergente au point $x' = (x'_1, \dots, x'_s)$ correspondant au point $x \in Z_\tau$ par le plongement torique $Z_\tau \subset \mathbb{C}^s$. La valeur de la fonction $X^u \in \mathbb{C}[S_\tau]$ au point du $Z_\tau$, ne dépend pas de l'immersion, donc si $u = u(\lambda)$, on a $x(u) = {x'}^\lambda$ et la série $\chi(\phi)$ est absolument convergente au point $x$. Ceci implique que la série $\chi(\phi) \in \mathbb{C}[[S_\tau]]$ est convergente. Avec un raisonnement analogue, on peut montrer que l'homomorphisme d'algèbres $\chi : \mathbb{C}\{U_1, \dots, U_s\} \to \mathbb{C}\{S_\tau\}$ est surjectif.

Par ailleurs, on a montré aussi que $\chi(\phi)$ est une fonction holomorphe dans un voisinage de $z_\tau$ dans $Z_\tau$, définissant un unique élément de $R$. Clairement, tous les éléments de $R$ sont obtenus de cette forme. Si la fonction $\chi(\phi)$ est nulle dans un voisinage de $z_\tau$ dans $Z_\tau$, la série $\phi$ est dans l'ideal engendré par $I$ dans $\mathbb{C}\{U_1, \dots, U_s\}$ donc $\chi(\phi) = 0$. ◇

Comme conséquence de ce lemme, on déduit que l'anneau $\mathbb{C}\{S_\tau\}$ est noethérien et intégralement clos parce que $Z_\tau$ est une variété normale (voir [K], §71).

### 1.2.2 Extensions galoisiennes

Soit $k \in \mathbb{Z}$ un entier positif fixé. Considérons les réseaux $N' = kN \subset N$. Leurs réseaux duaux respectifs sont $M' = \frac{1}{k}M \supset M$. Un cône $\tau$ strictement convexe dans $(\mathbb{R}^d)^*$ est rationnel pour les deux réseaux en même temps. Nous notons $Z_\tau$ (resp. $Z'_\tau$) la variété torique associée à $(\tau, N)$ (resp. à $(\tau, N')$). Le semi-groupe associé à $(\tau, N')$ est $S'_\tau := \frac{1}{k}S_\tau \subset M'$. L'homomorphisme de semi-groupes $M \supset S_\tau \hookrightarrow S'_\tau \subset M'$ définit un morphisme torique $f_k : Z'_\tau \to Z_\tau$. L'image du point distingué de $Z'_\tau$ est le point distingué du $Z_\tau$, donc on obtient un morphisme de germes irréductibles $(Z'_\tau, z'_\tau) \to (Z_\tau, z_\tau)$. En utilisant le lemme 1.1 on vérifie que l'homomorphisme des algèbres intègres associées est $\mathbb{C}\{S_\tau\} \hookrightarrow \mathbb{C}\{S'_\tau\}$.

L'homomorphisme des semi-groupes $M \hookrightarrow M'$ définit le morphisme $f_k : T' \to T$ obtenu en restreignant $f_k$ aux tores respectifs de $Z'_\tau$ et $Z_\tau$ On peut vérifier directement que le noyau de ce morphisme $f_{k\,|T'}$, comme morphisme de groupes algébriques, est le sous-groupe fini $H$ de $T'$, formé des éléments $(w_1, \dots, w_d)$ tels que $w_i^k = 1$, pour $i = 1, \dots, d$. Ce morphisme est un revêtement galoisien à $k^d$ feuilles de la variété $T$, parce que le groupe $H$ agit transitivement sur les fibres. Donc on a une extension galoisienne des corps des fonctions rationnelles $\mathbb{C}(T) \hookrightarrow \mathbb{C}(T')$. On va montrer qu'on a une situation analogue pour les corps des fonctions méromorphes aux points distingués des variétés toriques correspondantes.

Soit $L$ (resp. $L'$) le corps des fractions de $\mathbb{C}\{S_\tau\}$, (resp. de $\mathbb{C}\{S'_\tau\}$). L'homomorphisme $\mathbb{C}\{S_\tau\} \hookrightarrow \mathbb{C}\{S'_\tau\}$ définit une extension de corps $L \hookrightarrow L'$.

**Lemme 1.2** *L'extension de corps $L \hookrightarrow L'$ est galoisienne. Soit $G$ son groupe de Galois. L'action de $H$ sur les monômes définit un épimorphisme de groupes $H \to G$ et l'ensemble des éléments $G$-invariants de l'anneau $\mathbb{C}\{S'_\tau\}$ est $\mathbb{C}\{S_\tau\}$.*

*Preuve.* Clairement, $L \hookrightarrow L'$ est une extension normale finie. A chaque $w \in H$ est associé l'homomorphisme d'algèbres $\mathbb{C}\{S'_\tau\} \to \mathbb{C}\{S'_\tau\}$ qui applique $X^{\frac{u}{k}} \mapsto w(u)X^{\frac{u}{k}}$. Cela définit un homomorphisme de groupes $H \to G$.

Remarquons que $X^{\frac{u}{k}} \mapsto w(u)X^{\frac{u}{k}}$ définit l'action de l'élément $w \in H$ sur un monôme de $\mathbb{C}[S'_\tau]$. Le corollaire 1.16 de [Od] garantit que le morphisme $Z'_\tau \to Z_\tau$ coïncide avec la projection du quotient de $Z_\tau$ par rapport à l'action du groupe $H$. C'est-à-dire que $\mathbb{C}[S_\tau]$ est l'ensemble des éléments invariants de l'algèbre $\mathbb{C}[S'_\tau]$ par l'action de $H$.

Si $H'$ est l'image de $H$ dans $G$ on a montré que le sous-corps fixe de $L'$ par $H'$ coïncide avec $L$, donc $(L')^G \subset L$, c'est-à-dire que $L \subset L'$ est une extension galoisienne et donc $H' = G$.
$\diamond$

### 1.2.3 Paramétrisation de singularités quasi-ordinaires toriques

Supposons que $F \in \mathbb{C}\{X_1, \ldots, X_d\}[Y]$ est un polynôme réduit tel que $0 \in \mathbb{C}$ est une racine de multiplicité $r \geq 1$ du polynôme $F(0, Y)$ et que le discriminant de $F$ soit de la forme $X^q \varepsilon$ où $\varepsilon$ est une unité de $\mathbb{C}\{X_1, \ldots, X_d\}$. D'après le théorème de préparation de Weierstrass, il existe un pseudo-polynôme à la Weierstrass $H$ de degré $r$ en $Y$, et une unité $\epsilon$ dans $\mathbb{C}\{X_1, \ldots, X_d, Y\}$ tels que $F = \epsilon H$. Par définition, la projection du germe $(\mathcal{X}, 0) \subset \mathbb{C}^d \times \mathbb{C}$ défini par le polynôme $H \in \mathbb{C}\{X_1, \ldots, X_d\}[Y]$ sur $(\mathbb{C}^d, 0)$ est quasi-ordinaire. D'après [A1], Theorem 3, il existe $k \in \mathbb{N}$ tel que $H$ ait ses $r$ racines dans l'anneau $\mathbb{C}\{X_1^{1/k}, \ldots, X_d^{1/k}\}$.

On va généraliser la construction de racines associées à une projection quasi-ordinaire, au cas où le germe $(\mathbb{C}^d, 0)$ est remplacé par un germe de variété torique affine $(Z_\tau, z_\tau)$ au point distingué.

**Théorème 1.1** *Pour tout polynôme $F \in \mathbb{C}\{S_\tau\}[Y]$ réduit tel que le discriminant de $F$ soit de la forme $X^{u_0}\varepsilon$, où $\varepsilon$ est une unité dans l'anneau $\mathbb{C}\{S_\tau\}$ et que $0 \in \mathbb{C}$ soit une racine de multiplicité $r \geq 1$ du polynôme $F(z_\tau, Y)$ il existe $k \in \mathbb{N}$ tel que $F$ ait $r$ racines sans terme constant dans l'anneau $\mathbb{C}\{\frac{1}{k} S_\tau\}$.*

*Preuve.* Nous fixons un nombre fini de générateurs du semi-groupe $S_\tau$. Cela permet de définir un plongement de la variété torique affine $Z_\tau \subset \mathbb{C}^s$. Il lui est associé un épimorphisme d'algèbres $\chi : \mathbb{C}\{U_1, \ldots, U_s\} \to \mathbb{C}\{S_\tau\}$ (voir le lemme 1.1). Considérons un polynôme $G \in \mathbb{C}\{U_1, \ldots, U_s\}[Y]$ tel que $G^\chi = F$. On a $G(0, Y) = F(z_\tau, Y)$. D'après le théorème de préparation de Weierstrass il existe un pseudo-polynôme à la Weierstrass $H$ de degré $r$ en $Y$, et une unité $\varepsilon$ dans $\mathbb{C}\{U_1, \ldots, U_s, Y\}$ tels que $G = \varepsilon H$. Clairement, les germes définis au point $(z_\tau, 0)$ par $F$ et par $H^\chi$ coïncident. Donc, on peut supposer que $F$ est un polynôme réduit de degré $r$ tel que $F(z_\tau, Y) = Y^r$.

Soit $L$ le corps de fractions de l'anneau intégre $\mathbb{C}\{S_\tau\}$. Les facteurs $F_i$ de la factorisation de $F$ en polynômes irréductibles dans $L[Y]$ sont dans $\mathbb{C}\{S_\tau\}[Y]$ parce que, d'après le lemme 1.1, l'anneau $\mathbb{C}\{S_\tau\}$ est intégralement clos et le coefficient de $Y^r$ est une unité (voir le théorème 5, §3, chap V, [Z-S]). De plus, le discriminant de $F_i$ divise le discriminant de $F$ donc $\Delta_Y F_i$ est de la forme $X^u \epsilon$ où $\epsilon \in \mathbb{C}\{S_\tau\}$ est une unité. On peut donc supposer que $F$ est irréductible, engendrant un ideal premier $(F)$ dans $\mathbb{C}\{S_\tau\}[Y]$. Considérons le germe de variété analytique irréductible $(\mathcal{X}, x) \subset (Z_\tau \times \mathbb{C}, x)$ au point $x$ correspondant à l'algèbre intègre $R = \mathbb{C}\{S_\tau\}[Y]/(F)$.

Soit $(\mathcal{X}, x) \to (Z_\tau, z_\tau)$ la projection des germes, et choisissons un représentant fini $\pi : \mathcal{X} \to Z_\tau$ tel que $\pi^{-1}(z_\tau) = \{x\}$. L'hypothèse sur le discriminant implique qu'il existe un voisinage $W$ du point $z_\tau$ dans $Z_\tau$ tel que $\pi$ est non ramifié sur $W^* := W \cap T$. Par continuité, comme $\pi^{-1}(z_\tau) = \{x\}$, on peut supposer que $\pi^{-1}(W)$ est un sous-ensemble relativement compact de $\mathbb{C}^{s+1}$.

Comme $\pi$ est un morphisme fini, l'intersection de l'image inverse du lieu discriminant de $\pi$ avec $\mathcal{X}$ est une sous-variété analytique fermée propre de $\mathcal{X}$. Son complémentaire est un ouvert $\mathcal{X}^* \subset \mathcal{X}$, connexe parce que $\mathcal{X}$ est analytiquement irréductible. Ceci montre que $\pi : \mathcal{X}^* \to W^*$ est un revêtement connexe à $r$ feuillets.

On peut supposer que l'ouvert $W^*$ du tore $T$ est $W^* = (\mathbb{D}^*)^d$ où $\mathbb{D}^* = D(0,1) \setminus \{0\} \subset \mathbb{C}^*$. Soit $J$ le sous-groupe du groupe fondamental $\pi_1(W^*, w) \cong \mathbb{Z}^d$ associé au revêtement $\pi : X^* \to W^*$. Puisque $J$ est d'indice fini, il existe $k \in \mathbb{N}$ tel que $k\mathbb{Z}^d \subset J$. Le revêtement $f_k : W^* \to W^*$, défini par $x \mapsto (x_1^k, \dots, x_d^k)$, est associé au sous-groupe $k\mathbb{Z}^d$ du $\mathbb{Z}^d$. Donc, il existe un revêtement $p : W^* \to X^*$ tel que $\pi \circ p = f_k$, (voir [F2], chap. 13).

Clairement, $p$ est holomorphe, et borné dans le complémentaire dans $W$ d'un ensemble analytique fermé, c'est-à-dire que $p$ est une fonction faiblement holomorphe dans $W$. Comme $W \subset Z_\tau$ est une variété normale, toute fonction faiblement holomorphe est holomorphe, (voir [K], §71). Donc, $p$ s'étend en un morphisme $W \to \mathcal{X}$. La fonction holomorphe $\pi \circ p$ coïncide sur $W^*$ avec le morphisme torique $f_k : Z'_\tau \to Z_\tau$ (où on considère $W^* \subset Z'_\tau$ et aussi $W^* \subset Z_\tau$). Donc, elle est égale à la restriction du morphisme $f_k$ à $W$. Nous remarquons que $p(z'_\tau) = x$ parce que $f_k(z'_\tau) = z_\tau$ et $\pi^{-1}(z_\tau) = \{x\}$.

En utilisant le lemme 1.1, on voit que l'homomorphisme d'algèbres intègres associé au morphisme $f_k$ aux points distingués est $\mathbb{C}\{S_\tau\} \to \mathbb{C}\{S'_\tau\}$. Considérons le monomorphisme d'algèbres $R \to \mathbb{C}\{S'_\tau\}$ correspondant au morphisme de germes $p : (W, z'_\tau) \to (X, x)$. L'algèbre $R$ est une sous-$\mathbb{C}\{S_\tau\}$-algèbre de $\mathbb{C}\{S'_\tau\}$ parce que $\pi \circ p = f_k$. Nous avons donc un diagramme :

$$\begin{array}{ccc} R & \rightarrow & \mathbb{C}\{S'_\tau\} \\ \uparrow & \nearrow & \\ \mathbb{C}\{S_\tau\} & & \end{array}$$

Soit $L$ (resp. $K, L'$) le corps des fractions de $\mathbb{C}\{S_\tau\}$, (resp. de $R, \mathbb{C}\{S'_\tau\}$). Par construction, $L \subset K \subset L'$ et $K = L(\zeta)$ où $\zeta$ est l'image de $Y \in R$ dans $\mathbb{C}\{S'_\tau\}$. D'après le lemme 1.2, l'extension de corps $L \hookrightarrow L'$ est galoisienne, et la série $\zeta$ a ses $r$ conjugués dans l'anneau $\mathbb{C}\{S'_\tau\}$. Ces conjugués sont les racines de $F$ dans $\mathbb{C}\{S'_\tau\}$ qui paramétrisent $(\mathcal{X}, x)$. $\diamond$

**Remarque 1.3** *Si $F$ est irréductible, on peut prendre $k = r$ dans le théorème 1.1.*

En effet, puisque le polynôme $F$ est irréductible, l'indice du sous-groupe $J$ (dans la preuve du théorème 1.1), est égal à $r$. L'ordre du sous-groupe engendré par l'image d'un vecteur de la base canonique de $\mathbb{Z}^d$ dans $\mathbb{Z}^d/J$ est un diviseur de $r$. Donc, on a $r\mathbb{Z}^d \subset J$.

## 1.3 Racines à la Newton Puiseux

### 1.3.1 Valuation induite par un vecteur irrationnel

Soit $R$ un anneau commutatif et $\Gamma$ un groupe totalement ordonné. Une valuation $\omega$ de $R$ dans $\Gamma$ est une application $\omega : R\backslash\{0\} \to \Gamma$ tel que

(*i*) $\omega(ab) = \omega(a) + \omega(b)$ pour $0 \neq a, b \in R$,

(*ii*) $\omega(a - b) \geq \inf\ (\omega(a), \omega(b))$ pour $0 \neq a, b \in R$ et $a \neq b$, avec égalité si $\omega(a) \neq \omega(b)$.

On associe à chaque $\lambda \in \Gamma$ l'ensemble $\mathcal{I}_\lambda = \{a \in R/\omega(a) > \lambda\}$. L'ensemble $\mathcal{I}_\lambda$ est un ideal de $R$ et on a $\lambda > \beta \Rightarrow \mathcal{I}_\lambda \subset \mathcal{I}_\beta$. La topologie $\omega$-adique sur $R$, est la topologie qui fait de $R$ un groupe topologique dans lequelle l'ensemble des ideaux $\{\mathcal{I}_\lambda\}_{\lambda\in\Gamma}$ est un système fondamental de voisinages de $0 \in R$.

La valuation $\omega$ est *archimédienne* si $\Gamma$ est isomorphe comme groupe totalement ordonné à un sous-groupe de $\mathbb{R}$. Soit $(R, \mathfrak{M})$ est un anneau local de corps de fractions $L$. La valuation $\omega$ de $L$ est *centrée sur* $R$ si $\omega(a) \geq 0$ pour $a \in R$ et $\omega(a) > 0$ pour $a \in \mathfrak{M}$.

**Lemme 1.4** *Soient $(R, \mathfrak{M})$ un anneau local noethérien de corps de fractions $L$, et $\omega$ une valuation archimédienne de $L$ centrée sur $R$ Alors, la topologie $\mathfrak{M}$-adique coïncide avec la topologie $\omega$-adique de $R$.*

*Preuve.* Puisque $R$ est noethérien, le semi-groupe $\omega(R\backslash\{0\})$ est bien ordonné, et il existe un plus petit élément $\lambda$ de l'ensemble $\omega(\mathfrak{M}\backslash\{0\})$. Si $\beta \in \omega(\mathfrak{M}\backslash\{0\})$, il existe $n \in \mathbb{N}$ tel que $n\lambda > \beta$ donc $\mathfrak{M}^n \subset \mathcal{I}_\beta \subset \mathfrak{M}$. ◇

On appelle un vecteur $w \in \mathbb{R}^d$ *irrationnel* si ses coordonneés sont linéairement indépendantes sur $\mathbb{Q}$. Associé à un vecteur irrationnel $w \in \mathbb{R}^d$ nous définissons un ordre total sur $\mathbb{Q}^d$ par :

$$u <_w u' \Leftrightarrow \langle u, w\rangle \ < \ \langle u', w\rangle.$$

**Remarque 1.5** *Soit* $\tau$ *un cône rationnel strictement convexe de dimension* $d$ *dans* $\mathbb{R}^d$ *. Un vecteur irrationnel* $w \in \tau^\vee$ *définit une valuation archimédienne de l'anneau local complet* $\mathbb{C}[[S_\tau]]$ *par* $w(\sum_{u \in S_\tau} c_u X^u) = min_{c_u \neq 0}(\langle u, w \rangle)$ *. Cette valuation vérifie les hypothèses du lemme 1.4. Si* $(\phi_j) \subset \mathbb{C}[[S_\tau]]$ *vérifie que la suite* $(w(\phi_j)) \subset \mathbb{R}$ *est strictement croissante alors* $\phi_j$ *tend vers* $0 \in \mathbb{C}[[S_\tau]]$ *.*

On appelle *l'exposant initial* d'une série $\phi \in \mathbb{C}[[S_\tau]]$ par rapport a $w$, l'exposant $u$ de $\phi$ tel que $w(X^u) = w(\phi)$. L'exposant initial est le plus petit, pour l'ordre $<_w$, parmi les exposants de $\phi$.

### 1.3.2 Chemins monotones dans le polyèdre de Newton

Un polyèdre $\mathcal{N}$ dans $\mathbb{R}^d$ est l'intersection d'une famille de demi-espaces d'équation $\langle \omega, u \rangle \geq \lambda_\omega$, pour $\omega \in \Xi \subset (\mathbb{R}^d)^*$. On dira que le polyèdre $\mathcal{N}$ est *rationnel* si ses sommets sont dans le réseau $\mathbb{Z}^d$ et si ses faces ont des équations à coefficients dans $\mathbb{Q}$. Le cône $\tau$ associé au sommet $u$ d'un polyèdre rationnel $\mathcal{N}$ dans $\mathbb{R}^d$ est l'ensemble des fonctions linéaires qui atteignent leur valeur minimale sur $\mathcal{N}$ au sommet $u$. Le cône $\tau$ est rationnel de dimension $d$, et $\tau$ est strictement convexe si et seulement si le polyèdre $\mathcal{N}$ est de dimension $d$. Dans ce cas, l'ensemble des cônes associés au polyèdre $\mathcal{N}$ forme un éventail $\Sigma$ dans $(\mathbb{R}^d)^*$ avec un nombre éventuellement infini de cônes ; le support $|\Sigma| = \cup_{\sigma \in \Sigma} \sigma$ de l'éventail associé à $\mathcal{N}$ n'est pas nécessairement fermé. Si $\tau \subset |\Sigma|$ est un cône rationnel strictement convexe de dimension $d$, l'ensemble des cônes $\tau \cap \sigma$, pour $\sigma \in \Sigma$ est une subdivision de $\tau$. En particulier, c'est l'éventail associé à la somme de Minkowski $\tau^\vee + \mathcal{N}$. Cette subdivision est finie parce que si $S^{d-1}$ est l'sphère unité $\{\tau \cap \sigma \cap S^{d-1}\}$ est un complexe polyèdral de support l'ensemble compact $\tau \cap \sigma \cap S^{d-1}$.

Considérons $\mathbb{R}^d \times \mathbb{R}$ avec des coordonnées fixées $(u, v)$. On dira qu'une arête bornée $e$ d'un polyèdre $\mathcal{N} \subset \mathbb{R}^d \times \mathbb{R}$ est *admissible* si elle n'est pas parallèle à l'hyperplan $v = 0$. Une arête admissible est de la forme $[p_{v_1}, p_{v_2}]$ où $p_{v_i} = (u_{v_i}, v_i), \in \mathbb{R}^d \times \mathbb{R}$ avec $v_1 < v_2$. Nous appellerons le vecteur $q_e := \frac{u_{v_1} - u_{v_2}}{v_2 - v_1}$ l'*inclinaison* , et le nombre $l_e = v_2 - v_1 \in \mathbb{N}$ la *longueur* de l'arête. Considérons la projection $\pi_e : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d \times \{0\}$ parallèlement à l'arête $e$, définie par $\pi_e(u, v) = u + vq_e$. Le cône $\sigma(e) \subset (\mathbb{R}^d)^*$, associé au sommet $\pi_e(e)$ du polyèdre $\pi_e(\mathcal{N})$ et de dimension $d$ et on a :

**Lemme 1.6** *Pour* $w \in (\mathbb{R}^d)^*$ *, les propriétés suivantes sont équivalentes :*

1. $w \in \sigma(e)$ .

*2. La fonction lineaire $w$ atteint sa valeur minimale sur chaque section $v = \lambda$ de $\mathcal{N}$ au point $(u(\lambda), \lambda)$ de l'arête $e$.* ◇

On dit qu'un chemin $\gamma$ dans les arêtes de $\mathcal{N} \subset \mathbb{R}^d \times \mathbb{R}$ est *monotone* si on peut le paramétriser par $\gamma(\lambda) = (u(\lambda), \lambda)$. Supposons que le chemin $\gamma$ a pour sommets $\{p_0, p_{i_1}, \dots, p_{i_t}, p_n\}$ avec $p_j = (u_j, j)$ pour $j \in \{i_0, i_1, \dots, i_t, i_{t+1}\}$ avec $0 = i_0 < \cdots < i_{t+1} = n$. Nous notons $q_r$ l'inclinaison du segment $e_r = [p_{i_r}, p_{i_{r+1}}]$, pour $r = 0, \dots, t$. Le chemin monotone $\gamma$ est *cohérent* si il existe $w \in (\mathbb{R}^d)^*$ tel que $\gamma(\lambda)$ est l'unique point de la section $v = \lambda$ du polyèdre $\mathcal{N}$ en lequel $w$ atteint sa valeur minimale sur cette section, pour $\lambda \in [0, n]$.

**Lemme 1.7** *Avec les notations précedentes, si $w \in (\mathbb{R}^d)^*$ est un vecteur irrationnel définissant le chemin monotone cohérent $\gamma$ dans les arêtes du polyèdre rationnel $\mathcal{N}$, alors les inclinaisons des arêtes de $\gamma$ vérifient :*

$$q_t <_w q_{t-1} \cdots <_w q_0$$

*Preuve.* Notons $\pi_s$ la projection parallèlement au segment $e_s$. Le vecteur $\pi_{s-1}(e_{s-1}) - \pi_s(e_s)$ a le même sens que le vecteur $\pi_s(p_{i_{s-1}}) - \pi_s(e_s)$. Par le lemme précedent $w$ appartient à $\sigma(e_s)$, donc on a $\langle w, \pi_{s-1}(e_{s-1}) - \pi_s(e_s) \rangle = i_s \langle w, q_{s-1} - q_s \rangle \geq 0$, pour $s = 1, \dots t$. ◇

Soit $\rho \subset (\mathbb{R}^d)^*$ un cône strictement convexe de dimension $d$. Nous notons $\mathbb{C}((S_\rho))$ (resp. $\mathbb{C}\{\{S_\rho\}\}$) l'anneau de fractions de l'anneau $\mathbb{C}[[S_\rho]]$ (resp. $\mathbb{C}\{S_\rho\}$) pour l'ensemble multiplicativement fermé correspondant aux monômes $X^u$ pour $u \in S_\rho$.

**Définition 1.1** *Le $\rho$-polyèdre de Newton d'une série $\phi \in \mathbb{C}((S_\rho))$ non nulle est la somme de Minkowski de l'enveloppe convexe de ses exposants et du cône $\rho^\vee$. Le $\rho$-polyèdre de Newton d'un polynôme $F \in \mathbb{C}((S_\rho))[Y]$, est la somme de Minkowski de l'enveloppe convexe de ses exposants et du cône $\rho^\vee \times \{0\}$.*

Le $\rho$-polyèdre de Newton de $\phi$, que nous notons $\mathcal{N}_\rho(\phi)$, est un polyèdre de dimension $d$ ayant un nombre fini de sommets. L'éventail associé est la subdivision du cône $\rho$ induite par l'éventail associé à l'enveloppe convexe des exposants de $\phi$.

Nous notons $\mathcal{N}_\rho(F)$ le polyèdre de Newton d'un polynôme $F \in \mathbb{C}((S_\rho))[Y]$. Remarquons que le $\rho$-polyèdre de Newton de $F$ ne dépend que des exposants de $F$, il dépend aussi de l'anneau dans lequel on considère que se trouvent les coefficients de $F$. Pour tout cône $\tau \subset (\mathbb{R}^d)^*$ rationnel strictement convexe de dimension $d$, l'inclusion d'algèbres $\mathbb{C}[X_1, \dots, X_d] \to \mathbb{C}\{\{S_\tau\}\}$, permet de considérer un polynôme $F \in \mathbb{C}[X_1, \ldots, X_d][Y]$ comme élement de $\mathbb{C}\{\{S_\tau\}\}[Y]$. L'enveloppe convexe des exposants de $F$ est un polyèdre compact, $\mathcal{P}(F)$, mais le polyèdre $\mathcal{N}_\rho(F)$ n'est pas compact.

On définit la relation suivante parmi les vecteur irrationnels du cône $\rho$ : $w \sim w'$, si et seulement si, ils définissent le même chemin polygonal dans les arêtes du polyèdre $\mathcal{N}_\rho(F)$. Par le lemme 1.6, cette relation définit un éventail qui subdivise le cône $\rho$. Cet éventail est défini par un polyédre que nous allons décrire maintenant.

### 1.3.3 Le polyèdre-fibre de la projection du polyèdre de Newton

Soient $\mathcal{P} \subseteq \mathbb{R}^N$ un polytope et $\pi : \mathbb{R}^N \to \mathbb{R}^M$ une application affine surjective, l'image de $\mathcal{P}$ est un polytope $\mathcal{Q}$. L'*intégrale de Minkowski* de l'application $\pi : \mathcal{P} \to \mathcal{Q}$ est l'ensemble des intégrales $\int_{\mathcal{Q}} \gamma \in \mathbb{R}^N$ lorsque $\gamma$ parcourt l'ensemble des sections Borel-mesurables $\gamma : \mathcal{Q} \to \mathcal{P}$ de $\pi$. D'après [Bi-St], l'intégrale de Minkowski est un polytope convexe de dimension égale à $\dim \mathcal{P} - \dim \mathcal{Q}$.

Si $F$ est un polynôme dans $\mathbb{C}[X_1, \ldots, X_d][Y]$, son polytope de Newton $\mathcal{P}(F) \subset \mathbb{R}^{d+1}$ est l'enveloppe convexe de ses exposants. Considérons un polynôme de la forme $F = \sum_{k=0}^{n} a_k Y^k$, où les $a_k$ sont des polynômes dans $\mathbb{C}[X_1, \ldots, X_d]$ avec $a_0 a_n \neq 0$. Soit $\mathcal{P}(F)$ le polytope de Newton de $F$ ; nous allons décrire l'intégrale de Minkowski de la projection $\pi : \mathcal{P}(F) \subset \mathbb{R}^d \times \mathbb{R} \to [0, n] \subset \mathbb{R}$. Ceci est un cas particulier du théorème 7.3 de [Bi-St]. Une section de $\pi$ est une application monotone de la forme $t \mapsto (\gamma(t), t) \in \mathcal{P}(F)$, pour $t \in [0, n]$, et il lui est associé le point $\int_{[0,n]} \gamma$ dans l'intégrale de Minkowski de $\pi$. Il est montré en [Bi-St] que les sommets de l'intégrale de Minkowski correspondent à des intégrales des chemins monotones cohérents dans les arêtes de $\mathcal{P}(F)$.

Ces chemins sont décrits par une collection $\{p_0, p_{i_1}, \ldots, p_{i_s}, p_n\}$ de sommets de $\mathcal{P}(F)$, avec $p_j = (u_j, j)$, où $u_j$ est un sommet de $\mathcal{P}(a_j)$ pour $j \in \{0, i_1, \ldots, i_s, n\}$, avec $0 < i_1 < \cdots < i_s < n$. Si $v < v'$ on peut paramétriser le segment $[(u, v), (u', v')] \subset \mathbb{R}^d \times \mathbb{R}$, par $\gamma(\lambda) = u + \frac{\lambda - v}{v' - v}(u' - u)$, avec $\lambda \in [v, v']$, donc $\int_{[v,v']} \gamma = \frac{v' - v}{2}(u + u')$.

L'intégrale du chemin $\gamma$ correspondant à la collection de sommets $\{p_0, p_{i_1}, \ldots, p_{i_s}, p_n\}$ est :

$$\begin{aligned}
\int_{[0,n]} \gamma &= \int_{[0,i_1]} \gamma + \cdots + \int_{[i_s,n]} \gamma \\
&= \frac{1}{2}(i_1(u_0 + u_{i_1}) + \sum_{k=2}^{s}(i_k - i_{k-1})(u_{i_k} + u_{i_{k-1}}) + (n - i_s)(u_{i_s} + u_n)) \\
&= \frac{1}{2}(i_1 u_0 + i_2 u_{i_1} + \sum_{k=2}^{s-1}(i_{k+1} - i_{k-1})u_{i_k} + (n - i_{s-1})u_{i_s} + (n - i_s)u_n) \qquad (1.1)
\end{aligned}$$

Ces considérations motivent la définition suivante :

**Définition 1.2** *Soient un cône strictement convexe $\rho \subset \mathbb{R}^d$ de dimension $d$ et $F \in \mathbb{C}((S_\rho))[Y]$ un polynôme de degré $n$ de terme constant non nul. Soit $\mathcal{Q}$ l'enveloppe convexe des intégrales $\int \gamma_w$ des chemins monotones $\gamma$ dans le polyèdre $\mathcal{N}_\rho(F)$ définis par des vecteurs irrationnels $w \in \rho$. Le $\rho$-polyèdre-fibre de $F$ est la somme de Minkowski $\mathcal{Q}_\rho(F) := 2(\mathcal{Q} + \rho^\vee)$.*

Le $\rho$-polyèdre-fibre $\mathcal{Q}_\rho(F)$ est un polyèdre rationnel. Il dépend du $\rho$-polyèdre de Newton de $F$. L'éventail $\Sigma$ associé au polyèdre-fibre $\mathcal{Q}_\rho(F)$ est une subdivision rationnelle finie du cône $\rho$. Si $w, w'$ sont des vecteurs dans l'interieur d'un cône de dimension $d$ de $\Sigma$, ils définissent le même chemin polygonal dans les arêtes du polyèdre $\mathcal{N}_\rho(F)$.

Dans le cas où $F$ est un polynôme dans l'anneau $\mathbb{C}[X_1, \ldots, X_d][Y]$ on appelle *polytope-fibre* de $F$ le polytope $\mathcal{Q}(F) := 2\mathcal{Q}$ où $\mathcal{Q}$ est l'intégrale de Minkowski de la la projection du polytope de Newton $\pi : \mathcal{P}(F) \subset \mathbb{R}^d \times \mathbb{R} \to [0, n] \subset \mathbb{R}$.

### 1.3.4 Théorème de Newton-Puiseux

On va généraliser un résultat de [McD].

**Théorème 1.2** *Soient $\rho$ un cône rationnel strictement convexe de dimension $d$ et $F \in \mathbb{C}((S_\rho))[Y]$ un polynôme non constant. Pour tout vecteur irrationnel $w \in \rho$ il existe un cône rationnel strictement convexe $\sigma_w$ de dimension $d$, et $k \in \mathbb{N}$ tels que $w \in \sigma_w \subset \rho$ et que $F$ se décompose dans l'anneau $\mathbb{C}((\frac{1}{k} S_{\sigma_w}))[Y]$.*

*Preuve.* Elle est essentiellement la même que celle de [McD]. Un vecteur irrationnel $w \in \rho \subset (\mathbb{R}^d)^*$ définit un chemin monotone cohérent $\gamma$ dans les arêtes du polyèdre rationnel $\mathcal{N}_\rho(F)$. Fixons une arête $e = [(u, v), (u', v')] \subset \mathbb{R}^d \times \mathbb{R}$ du chemin $\gamma$, avec $v < v'$. L'inclinaison de $e$ est un vecteur

$q \in \frac{1}{l}\mathbb{Z}^d$ où $l$ est la longueur $v'-v$. La restriction de $F$ à l'arête $e$ est le polynôme $F_{|e} = \sum_{I\in e} \alpha_I X_1^{i_1} \dots X_d^{i_d} Y^{i_{d+1}}$. On associe à l'arête $e$ le polynôme $f_e \in \mathbb{C}[t]$ par $F_{|e}(1,\dots,1,t) = t^v f_e$ où $f_e(0) \neq 0$. Le polynôme $f_e$ est de degré $l$ et toutes ses racines sont non nulles.

Soit $c$ une racine de $f_e$, définissons le polynôme $F_2 = F(Y+cX^q) \in \mathbb{C}((\frac{1}{l}S_\rho))[Y]$. Clairement, $F_2$ est un polynôme de degré $r$ et on a :

$$F_2 = \sum_I \alpha_I \sum_{j=0}^{i_{d+1}} \binom{i_{d+1}}{j} c^j X_1^{i_1+jq_1} \dots X_d^{i_d+jq_d} Y^{i_{d+1}-j}.$$

On en déduit que :

1. Les exposants de $F_2$ sont de la forme $I + j(q,-1)$ où $I = (i_1,\dots,i_{d+1})$ est un exposant de $F$ et $j \in \{0,\dots,i_{d+1}\}$, donc $\pi_e(\mathcal{N}_\rho(F_2))$ est contenu dans $\pi_e(\mathcal{N}_\rho(F))$.
2. Le coefficient du terme constant de $F_2$ d'exposant dans la droite $E$ défini par l'arête $e$ est, $\sum_{I\in e} \alpha_I c^{i_{d+1}} = f_e(c)$, nul par construction.
3. Le coefficient du terme de $F_2$ d'exposant $p_{v'}$ coïncide avec celui de $F$.
4. Si $Y$ ne divise pas $F_2$, le polyèdre de $F_2$ a toujours des points dans l'hyperplan $v=0$.
5. L'exposant de $F_2$ dans la droite $E$ correspondant au terme de plus petit degré en $Y$ est un sommet du $\rho$-polyèdre de Newton de $F_2$. Cet exposant est de la forme $(u,m)$ où $m$ est la multiplicité de $c$ comme racine de $f_e$. En effet, la plus petite ordonnée des exposants de $F_2$ dans la droite $E$ est le nombre $m$ de fois qu'il faut dériver pour que $\frac{\partial^m}{\partial Y^m}(F_{2\,|E})$, ait un terme constant non nul. Comme $F_{2\,|E} = F_{|e}(Y+cX^q)$ le coefficient du terme constant de $Y$ de $\frac{\partial^k}{\partial Y^k}(F_{2\,|E})$ est égal à $\sum_{I\in e} \alpha_I i_{d+1} \dots (i_{d+1}-k+1) c^{i_{d+1}-k} = \frac{d^k f_e}{dt^k}(c)$.

Le sommet $(u,m)$ du polyèdre $\mathcal{N}_\rho(F_2)$ défini par 5. est un sommet du chemin monotone défini par $w$ dans les arêtes de $\mathcal{N}_\rho(F_2)$. Si $Y$ ne divise pas $F_2$ on va considérer la partie finale du chemin entre le sommet $(u,m)$ et l'hyperplan $v=0$.

Parmi les segments de cette partie finale du chemin polygonal on choisit une arête $e_2$ d'inclinaison $q_2$ et de longueur $l_2$. On choisit une racine $c_2$ du polynôme associé $f_{e_2}$ de multiplicité $m_2$ et on définit $F_3 := F_2(Y + c_2X^{q_2})$. On continue par récurrence. Le polynôme $F_n$ est un élément de l'anneau $\mathbb{C}((\frac{1}{l_1 \dots l_{n-1}}S_\rho))[Y]$.

On obtient une suite décroissante de nombres entiers positifs : $l \geq m \geq l_2 \geq m_2 \geq \dots > 0$, qui est donc stationnaire ; il existe $n_0 \in \mathbb{N}$ tel que pour tout $n \geq n_0$ on a $l_n = m_n = m_{n_0} = m$. Ceci implique que $f_{e_n} = \theta(t-c_n)^m$, et aussi que la partie finale du chemin défini par $w$ dans les arêtes de $\mathcal{N}_\rho(F_n)$ est le segment $e_n$. De plus les sommets de $e_n$ et $e_{n+1}$ qui ne sont pas dans l'hyperplan $v=0$ coïncident, pour $n > n_0$.

Pour $n > n_0$, on a le segment $e_n = [(u_n, 0), (u_0, m)]$ d'inclinaison $q_n = \frac{1}{m}(u_n - u_0)$. L'intersection de la droite définie par $e_n$ avec l'hyperplan $v = 0$ est le point $p_n := u_0 + mq_n$. Par définition du $\rho$-polyèdre de Newton, le cône $\sigma(e_{n_0})$ associé au sommet $p_{n_0}$ du polyèdre $\pi_{e_{n_0}}(\mathcal{N}_\rho(F_{n_0}))$ est contenu dans $\rho$. Le lemme 1.6 implique que $w \in \sigma(e_{n_0})$.

On vérifie qu'il existe $k \in \mathbb{N}$ tel que les inclinaisons construites sont dans un réseau $\frac{1}{k}\mathbb{Z}^d$. On sait que $u_0 = \frac{\beta_0}{k}$, $u_{n_0} = \frac{\beta_{n_0}}{k}$ où $\beta_0, \beta_{n_0} \in \mathbb{Z}^d$ et $k = l_1 l_2 \dots l_{n_0 - 1}$ est un entier. L'inclinaison de $e_{n_0}$ est $q_{n_0} = \frac{\beta_{n_0} - \beta_0}{km} = \frac{\beta}{k\lambda}$ où $\beta \in \mathbb{Z}^d$ et $\lambda \in \mathbb{N}$ est premier avec une coordonnée de $\beta$. Si $(\frac{\beta'}{k}, h)$ est un exposant de $F^{n_0}_{|e}$ on a $q_{n_0} = \frac{\beta_{n_0} - \beta'}{kh} = \frac{\beta}{k\lambda}$. Comme $\lambda(\beta_{n_0} - \beta') = h\beta$, on déduit que $\lambda$ divise $h$ donc $f_{e_{n_0}}$ est un polynôme en $t^\lambda$. Par ailleurs $f_{e_n} = \theta(t - c_n)^m$ et comme la caractéristique de $\mathbb{C}$ est zéro, on a $\lambda = 1$. Par récurrence on obtient que $q_n \in \frac{1}{k}\mathbb{Z}^d$, pour $n \geq n_0$ et donc pour $n \in \mathbb{N}$.

Montrons que les inclinaisons $q_j$ sont dans un cône affine strictement convexe.

Le $\rho$-polyèdre de Newton de $F_{n_0}$ est contenu dans le cone affine :

$$W(e_{n_0}) := \{\lambda(u - u') \,/ u \in \mathcal{N}_\rho(F_{n_0}),\ u' \in e_{n_0},\ \lambda \geq 0\},$$

associé à l'arête $e_{n_0}$. Comme $u_{n_0+1} \in W(e_{n_0})$, par construction on a l'inclusion $\mathcal{N}_\rho(F_{n_0+1}) \subset W(e_{n_0})$. Par récurrence, en utilisant que le sommet $(u_0, m)$ de $e_n$ est sur la droite qui contient le segment $e_{n_0}$, on montre que $u_n \in W(e_{n_0})$ et que $\mathcal{N}_\rho(F_n) \subset W(e_{n_0})$, pour $n > n_0$. Ceci implique pour tout $w' \in \sigma(e_{n_0})$ que $\langle w', q_n - q_{n_0} \rangle = \frac{1}{m}\langle w', p_n - p_{n_0} \rangle$ est $\geq 0$. Donc les exposants construits sont dans le cône rationnel affine $q_{n_0} + \sigma(e_{n_0})^\vee$ pour $n \geq n_0$.

Notons $\sigma$ pour $\sigma(e_{n_0})$. Il existe $u_0 \in \frac{1}{k}\mathbb{Z}^d$ tel que les inclinaisons $q_n$ appartiennent à $u_0 + \frac{1}{k}S_\sigma$. Définissons les sommes partielles, $\phi_n = \sum_{j=1}^n c_j X^{q_j}$ pour $n \in \mathbb{N}$. On a $\phi_n \in \mathbb{C}((\frac{1}{k}S_\sigma))$ et $X^{-u_0}\phi_n \in \mathbb{C}[[\frac{1}{k}S_\sigma]]$. Par construction, et par le lemme 1.7, on sait que $q_j <_w q_{j+1}$ pour $j \in \mathbb{N}$. Par la remarque 1.5, ceci implique que la série formelle $\phi := \sum c_j X^{q_j}$ est égale à $X^{u_0} \lim_{n \to \infty} X^{-u_0}\phi_n$ où la limite est dans l'anneau complet $\mathbb{C}[[\frac{1}{k}S_\sigma]]$.

Comme $w \in \sigma \subset \rho$, on peut considérer $F$ comme élément de $\mathbb{C}((\frac{1}{k}S_\sigma))[Y]$. La série formelle $\phi$ est une racine de $F$. En effet, si $n \geq n_0$ la série $F(\phi_{n-1}) = F_n(0)$ a tous ses exposants dans le cône rationnel affine $p_{n_0} + \sigma(e_{n_0})^\vee$. L'égalité suivante $F(\phi) = X^{p_{n_0}} \lim_{n \to \infty} F(\phi_{n-1})X^{-p_{n_0}}$ est clair. Si $n \geq n_0$ l'exposant initial de $F(\phi_{n-1})X^{-p_{n_0}}$ par rapport à $w$ est $p_{n+1} - p_n = m(q_{n+1} - q_{n_0})$ et on a $\lim_{n \to \infty} F_n(0)X^{-p_{n_0}} = 0$.

On vérifie que la multiplicité de $\phi$ comme racine de $F$ est $\geq m$. La multiplicité de $c_n$ comme racine de $f_{e_n}$ est $\geq m$, donc $c_n$ est une racine de $\frac{d^s f_{e_n}}{dt^k}$ pour $1 \leq s \leq m-1$. Le polynôme $\frac{d^s f_{e_n}}{dt^s}$ est le polynôme de l'arête $e_n^s$ du polyèdre $\mathcal{N}_\rho(\frac{\partial^s F}{\partial Y^s})$ qui est sur le segment $-(0, \dots, 0, s) + e_n$.

L'arête $e_n^s$ est déterminée par le vecteur irrationnel $w$. Comme $\frac{\partial^s F_n}{\partial Y^s} = \frac{\partial^s F}{\partial Y^s}(Y+\phi_{n-1})$ on obtient que $\phi$ est une racine de $\frac{\partial^s F}{\partial Y^s}$ pour $1 \leq s \leq m-1$.

On a montré que, associés à chaque arête $e$ du chemin monotone $\gamma$, il existe $k \in \mathbb{N}$ et un cône rationnel $\sigma_e$ strictement convexe de dimension $d$ tel que $w \in \sigma_e \subset \rho$, tels que $F$ ait au moins $l_e$ racines à la Newton-Puiseux dans $\mathbb{C}[[\frac{1}{k}S_{\sigma_e}]]$. On peut choisir $k \in \mathbb{N}$ valable pour toutes les arêtes de $\gamma$. Comme le vecteur $w$ est irrationnel, le cône rationnel $\tau = \cap_{e\in\gamma}\sigma_e$ est de dimension $d$. L'existence d'un homomorphisme d'algèbres $\mathbb{C}[[\frac{1}{k}S_{\sigma_e}]] \hookrightarrow \mathbb{C}[[\frac{1}{k}S_\tau]]$, pour chaque arête $e$ de $\gamma$, garantit que $F$ se décompose dans $\mathbb{C}[[\frac{1}{k'}S_\tau]]$, parce que les exposants initiaux par rapport à $<_w$ des séries correspondantes à segments différents de $\gamma$ sont différents. ⋄

**Remarque 1.8** *Soient $F \in \mathbb{C}[[S_\rho]]$ un polynôme de degré $\geq 1$ et $w \in \rho$ un vecteur irrationnel, définissant un chemin polygonal $\gamma$ dans les arêtes de $\mathcal{N}_\rho(F)$. La démonstration du théorème 1.2 montre que, associées à chaque arête $e$ de $\gamma$, il existe $l_e$ racines de $F$ telles que leur exposant initial par rapport à $w$ est l'inclinaison $q_e$.*

### 1.3.5 Rapport avec les paramétrisations des singularités quasi-ordinaires

**Théorème 1.3** *Soit $F = \sum_{j=0}^{n} a_j Y^j$ un polynôme réduit de degré $n \geq 1$ avec $a_j \in \mathbb{C}\{\{S_\rho\}\}$.*

1. *Pour tout cône $\tau$ de dimension $d$ de l'éventail associé au polyèdre $\mathcal{N}_\rho(a_n\Delta_Y F)$, il existe $k \in \mathbb{N}$ tel que $F$ se décompose dans l'anneau $\mathbb{C}\{\{\frac{1}{k}S_\tau\}\}$.*
2. *Si $a_0 \neq 0$, pour tout cône $\tau$ de dimension $d$ de l'éventail associé au polyèdre $\mathcal{N}_\rho(a_n\Delta_Y F)+\mathcal{Q}_\rho(F)$ il existe $k \in \mathbb{N}$ tel que $F$ se décompose dans l'anneau $\mathbb{C}\{\{\frac{1}{k}S_\tau\}\}$, et de plus les racines de $F$ sont des unités.*

**Preuve.** Soit $\tau$ un cône de dimension $d$ de l'éventail associé au polyèdre $\mathcal{N}_\rho(a_n\Delta_Y F)$. Par définition de $\rho$-polyèdre de Newton, le cône $\tau$ est contenu dans $\rho$, et on a l'homomorphisme d'algèbres $\mathbb{C}\{S_\rho\} \hookrightarrow \mathbb{C}\{S_\tau\}$ qui permet de considérer $F$ comme élément de $\mathbb{C}\{\{S_\tau\}\}[Y]$.

Tout vecteur irrationnel $w \in \tau$ atteint sa valeur minimale sur les exposants de $a_n$ au même point $u_n$. On montre par récurrence sur $n$ qu'il existe $q_0 \in \mathbb{Z}^d$ tel que le polyèdre $\mathcal{N}_\tau(F)$ est contenu dans le cône affine :

$$W := \{(u_n, n) + \lambda(q_0, -1) + (u', 0) \ / \ \lambda \in [0, n], \ u' \in \tau^\vee\}$$

Si $n = 1$, il suffit de prendre $q \in \mathbb{Z}^d$ tel que $\mathcal{N}_\tau(a_0)$ soit contenu dans le cone affine $u_n+q+\tau^\vee$. Si $n > 1$, par récurrence on a construit $q$ pour le polynôme $(F-a_0)Y^{-1}$. Il suffit de prendre $q_0 \in \mathbb{Z}^d$ tel que le polyèdre $u_n+nq+\tau^\vee+\mathcal{N}_\tau(a_0)$ soit contenu dans le cone affine $u_n+nq_0+\tau^\vee$.

Nous notons $p_0$ le point $u_n + nq_0$, $e_0$ le segment $[(u_n, n), (p_0, 0)]$ et $\pi_{e_0} : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d \times \{0\}$ la projection parallèlement à l'arête $e_0$. On a $\pi_{e_0}(u, v) = u + vq_0$.

On définit le changement :

$$G = X^{-p_0} F(X^{q_0} Y).$$

On en déduit :

1. Si $F = \sum_{j=0}^{n} a_j Y^j$ avec $a_i \in \mathbb{C}\{\{S_\tau\}\}$ on obtient que $G = \sum_{j=0}^{n} a_j X^{jq_0 - p_0} Y^j$, et donc l'exposant de $G$ qui correspond à l'exposant $(u, j)$ de $F$ est $(\pi_{e_0}(u) - p_0, j)$. Par construction, comme $\mathcal{N}_\tau(F) \subset W$, le vecteur $\pi_{e_0}(u) - p_0$ appartient au cône $\tau^\vee$. Ceci implique que $G$ est un polynôme dans l'anneau $\mathbb{C}\{S_\tau\}$. De plus, l'exposant de $G$ qui correspond à l'exposant $(u_n, n)$ de $F$ est $(0, n)$, donc $G(z_\tau, Y) \in \mathbb{C}[Y]$ est un polynôme de degré $n$.
2. La quasi-homogénéité et l'homogénéité du discriminant générique impliquent que le discriminant de $G$ par rapport à $Y$ est de la forme $\Delta_Y G = X^{u_0} \varepsilon$ où $\varepsilon$ est une unité dans l'anneau $\mathbb{C}\{S_\tau\}$.

En appliquant le théorème 1.1, on voit qu'il existe $k \in \mathbb{N}$ tel que $G$ se décompose dans l'anneau $\mathbb{C}\{\frac{1}{k} S_\tau\}$. Les racines correspondantes de $F$ sont dans $\mathbb{C}\{\{\frac{1}{k} S_\tau\}\}$.

Soit $\tau$ est un cône de dimension $d$ de la subdivision finie de $\rho$ induite par le polyèdre $\mathcal{N}_\rho(a_n \Delta_Y F) + \mathcal{Q}_\rho(F)$ ; vérifions que les racines construites sont des unités dans $\mathbb{C}\{\{S_\tau\}\}$.

Soit $w \in \tau$ un vecteur irrationnel. D'après le théorème 1.2, il existe un cône rationnel strictement convexe $\sigma$ qui contient $w$, et $k \in \mathbb{N}$ tels que le polynôme $F$ se décompose sur l'anneau $\mathbb{C}((\frac{1}{k} S_\sigma))$.

Puisque le vecteur $w \in \tau \cap \sigma$ est irrationnel, le cône rationnel $\tau \cap \sigma$ est nécessairement un cône de dimension $d$, et son cône dual $\tau^\vee + \sigma^\vee$ est strictement convexe. L'anneau intègre $\mathbb{C}((\frac{1}{k} S_{\tau\cap\sigma}))$ contient $\mathbb{C}((\frac{1}{k} S_\sigma))$ et $\mathbb{C}((\frac{1}{k} S_\tau))$ comme sous-anneaux.

D'abord, les racines de $F$ obtenues par le théorème 1.1 sont dans l'anneau $\mathbb{C}\{\{\frac{1}{k} S_\tau\}\}$, et donc elles doivent coïncider avec les racines obtenues à la Newton-Puiseux. Nous affirmons que ces racines sont des éléments inversibles dans l'anneau $\mathbb{C}\{\{\frac{1}{k} S_\tau\}\}$. Par hypothèse, chaque élément irrationnel $w \in \tau$ définit le même chemin $\gamma$ dans les arêtes de $\mathcal{N}_\tau(F)$. Donc l'exposant initial $u$ par rapport à $w$ d'une racine $\phi$ ne dépend pas de $w \in \tau$ (par la remarque 1.8). On obtient que $\phi = X^u \varepsilon$ où $\varepsilon$ est une unité de $\mathbb{C}\{\frac{1}{k} S_\tau\}$, c'est-à-dire que $\phi$ est une unité dans $\mathbb{C}\{\{\frac{1}{k} S_\tau\}\}$. ⋄

On obtient aussi la version polynomiale du théorème précedent :

**Corollaire 1.9** *Soit un polynôme* $F = \sum a_j Y^j \in \mathbb{C}[X_1, \ldots, X_d][Y]$ *réduit de degré* $n \geq 1$.

1. *Si le polytope $\mathcal{P} = \mathcal{P}(a_n\Delta_Y F)$ est de dimension $d$, pour tout cône $\tau$ de dimension $d$ de l'éventail associé à $\mathcal{P}$ il existe $k \in \mathbb{N}$ tel que $F$ se décompose dans l'anneau $\mathbb{C}\{\{\frac{1}{k}S_\tau\}\}$.*
2. *Si $a_0 \neq 0$ et si $\tau$ est un cône associé à un sommet du polytope $\mathcal{P}(\Delta_Y F) + \mathcal{Q}(F)$, toutes les racines de $F$ sont des unités dans $\mathbb{C}\{\{\frac{1}{k}S_\tau\}\}$.*

*Preuve.* Si le polytope $\mathcal{P}(a_n\Delta_Y F)$ est de dimension $d$, le cône $\tau$ associé à un sommet de $\mathcal{P}(a_n\Delta_Y F)$ est de dimension $d$. On applique le théorème 1.3 à $F$ vu comme élément de $\mathbb{C}\{\{S_\tau\}\}[Y]$.

Si le polytope $\mathcal{P} := \mathcal{P}(\Delta_Y F) + \mathcal{Q}(F)$ est de dimension $< d$, le cône $\tau$ associé à un sommet de $\mathcal{P}$ est rationnel de dimension $d$ mais il n'est pas strictement convexe. Le cône $\tau^\vee$ est strictement convexe et $\tau$ définit l'algèbre $\mathbb{C}[[S_\tau]]$. Si $\sigma \subset \tau$ est un cône strictement convexe de dimension $d$, on considére $F$ comme élément de $\mathbb{C}\{\{S_\sigma\}\}[Y]$ et on obtient que il existe $k \in \mathbb{N}$ tel que $F$ se décompose dans l'anneau $\mathbb{C}\{\{\frac{1}{k}S_\sigma\}\}$.

On peut recouvrir le cône $\tau$ par un nombre fini de cônes rationnels strictement convexes de dimension d, $\{\sigma_i\}_{1\geq i\geq s}$, tels que $\sigma_i \cap \sigma_{i+1}$ soit d'intérieur non vide, pour $i = 1, \ldots, s-1$. Ceci implique que les racines de $F$ obtenues par le théorème 1.1 correspondant à $\sigma_i$ et à $\sigma_{i+1}$ vont coïncider, et le terme initial d'une racine par rapport à la valuation induite par un vecteur irrationel $w$ ne dépend pas de $w \in \sigma_i \cup \sigma_{i+1}$. Donc toutes les racines de $F$ sont des séries à exposants dans un translaté du cône $\cap_{i=1}^s \sigma_i^\vee = \tau^\vee$. ◇

**Remarque 1.10** *Soit $F \in \mathbb{C}[X_1, \ldots, X_d][Y]$ un polynôme de degré $n$ tel que $0$ soit une racine simple de $F(0, Y)$, le théorème des fonctions implicites garantit qu'il existe une unique série $\phi \in \mathbb{C}\{X_1, \ldots, X_d\}$ telle que $F(\phi) = 0$. Si on a $a_n\Delta_Y F = X^u\varepsilon$ où $\varepsilon(0) \neq 0$ le théorème 1.3 montre que les exposants de $\phi$ sont dans un translaté entier du cône dual associé au sommet $u$ de $\mathcal{P}(a_n\Delta_Y F)$ lorsque ce polytope est de dimension $d$. (Voir l'exemple 1).*

## 1.4 Application aux polyèdres de Newton du discriminant et du résultant

### 1.4.1 Les conditions discriminantales pour le polyèdre de Newton

Suivant [McD], on dit que un polynôme $F \in \mathbb{C}((S_\rho))[Y]$ vérifie la *condition discriminantale* si pour toute arête admissible $e$ de son $\rho$-polyèdre de Newton, le polynôme $f_e$ n'a que des racines simples.

**Théorème 1.4** *Soit* $F = \sum_{k=0}^{n} a_k Y^k$ *un polynôme à coefficients dans* $\mathbb{C}((S_\rho))$ *tels que* $a_0 a_n \neq 0$. *On a l'inclusion de polyèdres*

$$\mathcal{N}_\rho(a_0) + \mathcal{N}_\rho(a_n) + \mathcal{N}_\rho(\Delta_Y F) \subseteq \mathcal{Q}_\rho(F)$$

*où* $\Delta_Y F$ *est le discriminant de* $F$ *par rapport à* $Y$. *On a l'égalité si* $F$ *vérifie la condition discriminantale.*

*Preuve.* On va trouver les conditions génériques que doivent vérifier les coefficients des termes qui apparaissent dans $F$, pour garantir l'égalité dans le théorème.

Soit $w \in \rho$ un vecteur irrationnel, et soit $\gamma$ le chemin monotone défini par $w$ dans les arêtes du $\rho$-polyèdre de Newton. Le chemin $\gamma$ a des sommets $\{p_0, p_{i_1}, \ldots, p_{i_t}, p_n\}$ dans $\mathcal{N}_\rho(F)$, avec $p_j = (u_j, j)$ pour $j \in \{0 = i_0, i_1, \ldots, i_t, i_{t+1} = n\}$ avec $i_0 < \cdots < i_{t+1}$. Nous notons $q_r := \frac{-u_{i_r} + u_{i_{r-1}}}{i_r - i_{r-1}} \in \mathbb{Q}^d$ et $l_r := i_r - i_{r-1}$ l'inclinaison et la longueur du segment $e_r = [p_{i_{r-1}}, p_{i_r}]$ du chemin $\gamma$, pour $r = 1, \ldots, t+1$.

D'après le théorème 1.2, il existe un cône rationnel strictement convexe $\sigma_w$, et $k \in \mathbb{N}$ tels que $F$ se décompose dans l'anneau $\mathbb{C}((\frac{1}{k} S_{\sigma_w}))[Y]$. A chaque segment $e_r = [p_{i_{r-1}}, p_{i_r}]$ du chemin $\gamma$ sont associées $i_r - i_{r-1}$ racines de $F$ de la forme :

$$\phi_j = c_j X^{q_r} + \ldots,$$

où $q_r$ est l'inclinaison du segment $e_r$, et $c_j$ parcourt les racines de $f_{e_r}$ comptées avec leur multiplicité. De plus $q_r$ est l'exposant initial par rapport à $w$ des termes qui apparaissent dans $\phi_j$. On indexe les racines $\phi_j$ correspondant à $e_r$, par $j \in \mathcal{A}_r := \{i_{r-1} + 1, \ldots, i_r\}$.

En appliquant le lemme 1.7, on voit que parmi les termes qui peuvent apparaître dans $\phi_k - \phi_j$, celui d'exposant de le plus petit par rapport à $<_w$ est égal à :

$$\begin{cases} (c_k - c_j) X^{q_r} & \text{si } k, j \in \mathcal{A}_r \\ c_j X^{q_m} & \text{si } k \in \mathcal{A}_r,\ j \in \mathcal{A}_m \text{ et } r < m. \end{cases}$$

Comme,

$$\Delta_Y F = (-1)^{\frac{1}{2} n(n-1)} a_n^{2(n-1)} \prod_{k<j} (\phi_k - \phi_j)^2,$$

le terme d'exposant le plus petit par rapport à $<_w$ qui peut apparaître dans $a_0 a_n \Delta_Y F$ est égal a $ABC$ où :

$$\begin{aligned} A &= (-1)^{\frac{1}{2}n(n-1)} \alpha_{p_0} \alpha_{p_n}^{2n-1} X^{u_0+(2n-1)u_n} \\ B &= \prod_{r=1}^{t+1} \prod_{i_{r-1}<k_r<j_r\leq i_r} (c_{k_r} - c_{j_r})^2 X^{2q_r} \\ C &= \prod_{r=1}^{t} \prod_{k_r\in\mathcal{A}_r} \prod_{m=r+1}^{t+1} \prod_{j_m\in\mathcal{A}_m} c_{j_m}^2 X^{2q_m}. \end{aligned}$$

L'exposant correspondant à $B$ est :

$$\begin{aligned} & 2\Big(\binom{i_1}{2} q_1 + \binom{i_2 - i_1}{2} q_2 + \cdots + \binom{i_{t+1} - i_t}{2} q_{t+1}\Big) \\ = \;& (i_1 - 1)(u_0 - u_{i_1}) + (i_2 - i_1 - 1)(u_{i_1} - u_{i_2}) + \cdots + (i_{t+1} - i_t - 1)(u_{i_t} - u_{i_{t+1}}) \\ = \;& (i_1 - 1)u_0 + (i_2 - 2i_1)u_{i_1} + (i_3 - 2i_2 + i_1)u_{i_2} + \cdots + (n - 2i_t + i_{t-1})u_{i_t} + (-n + i_{t-1})u_n \end{aligned}$$

L'exposant correspondant à $C$ est :

$$\begin{aligned} 2(\sum_{r=1}^{t}(i_r - i_{r-1}) \sum_{m=r+1}^{t+1} q_m(i_m - i_{m-1})) &= 2(\sum_{r=1}^{t}(i_r - i_{r-1})(-u_n + u_{i_r})) \\ &= 2(i_1 u_{i_1} + (i_2 - i_1)u_{i_2} + \cdots + (i_t - i_{t-1})u_{i_t} - i_t u_n) \end{aligned}$$

L'exposant $u$ correspondant à $ABC$ coïncide avec $2\int\gamma$, (voir la formule (1.1), §1.3.3). Clairement, le coefficient correspondant à $ABC$ est non nul si et seulement si les segments du chemin $\gamma$ vérifient la condition discriminantale. Ceci termine la preuve, parce que le vecteur irrationnel $w$ est arbitraire. ◇

**Corollaire 1.11** *Avec les notations précedentes, le coefficient du terme de la série* $a_0 a_n \Delta_Y F$ *d'exposant égal à* $2\int\gamma$ *est :*

$$c(\gamma) := (-1)^k \alpha_{p_{i_0}} \alpha_{p_{i_1}}^2 \ldots \alpha_{p_{i_t}}^2 \alpha_{p_{i_{t+1}}} \Delta f_{e_1} \ldots \Delta f_{e_{t+1}},$$

*où* $f_{e_r} = \alpha_{p_{i_{r-1}}} + \cdots + \alpha_{p_{i_r}} t^{l_r}$ *est le polynôme de l'arête* $e_r = [p_{i_{r-1}}, p_{i_r}]$ *du chemin* $\gamma$*, son discriminant est* $\Delta f_{e_r}$ *et* $k = \frac{1}{2}\big(n(n-1) + \sum_{r=1}^{t+1} l_r(l_r - 1)\big)$*.*

*Preuve.* En utilisant que $\Delta f_{e_r} = (-1)^{\frac{1}{2}l_r(l_r-1)} \alpha_{p_{i_r}}^{2(l_r-1)} \prod_{i_{r-1}<k_r<j_r\leq i_r} (c_{k_r} - c_{j_r})^2$ on obtient que le coefficient de $B$ est

$$\prod_{r=1}^{t+1} (-1)^{\frac{1}{2}l_r(l_r-1)} \alpha_{p_{i_r}}^{-2(l_r-1)} \Delta f_{e_r}.$$

Comme le produit des racines de $f_{e_m}$ est égal à $(-1)^{l_m}\frac{\alpha_{p_{i_{m-1}}}}{\alpha_{p_{i_m}}}$, on déduit que le le coefficient de $C$ est

$$\prod_{r=1}^{t}\prod_{k_r\in\mathcal{A}_r}\prod_{m=r+1}^{t+1}\Big(\frac{\alpha_{p_{i_{m-1}}}}{\alpha_{p_{i_m}}}\Big)^2=\prod_{r=1}^{t}\prod_{k_r\in\mathcal{A}_r}\Big(\frac{\alpha_{p_{i_r}}}{\alpha_{p_{i_{t+1}}}}\Big)^2=\prod_{r=1}^{t}\Big(\frac{\alpha_{p_{i_r}}}{\alpha_{p_{i_{t+1}}}}\Big)^{2l_r}$$

Donc le coefficient de $ABC$ est :

$$(-1)^k\Delta f_{e_1}\ldots\Delta f_{e_{t+1}}\alpha_{p_{i_0}}\alpha_{p_{i_{t+1}}}^{2n-1-2(n-l_{t+1})-2(l_{t+1}-1)}\prod_{r=1}^{t}\alpha_{p_{i_r}}^{2l_r-2(l_r-1)}=c(\gamma)$$

◇

On déduit des théorèmes 1.3 et 1.4 :

**Corollaire 1.12** *Soit le polynôme $F=\sum_{k=0}^{n}a_kY^k$, où $a_k$ sont des séries dans $\mathbb{C}((S_\rho))$ telles que $a_0a_n\neq 0$. Si $F$ vérifie la condition discriminantale, pour tout cône $\tau$ de dimension $d$ de l'éventail associé au polyèdre $\mathcal{Q}_\rho(F)$, il existe $k\in\mathbb{N}$ tel que $F$ se décompose dans l'anneau $\mathbb{C}((\frac{1}{k}S_\tau))$, et de plus les racines de $F$ sont des unités.* ◇

**Remarque 1.13** *Le corollaire 4.1 de [McD] énonce une "version polynomiale" incorrecte du corollaire précedent. Il est dit que des racines de $F$ correspondants aux cônes associés aux sommets différents du polytope-fibre sont différentes. Supposons que le polytope $\mathcal{P}:=\mathcal{P}(\Delta_YF)+\mathcal{P}(a_n)$ soit de dimension $d$, et que l'éventail $\Sigma$ associé au polytope-fibre soit une sous-division stricte de l'éventail $\Sigma'$ associé à $\mathcal{P}$. Par le corollaire 1.9, les racines de $F$ correspondants aux cônes de $\Sigma$ qui subdivisent un cône $\tau\in\Sigma'$ de dimension $d$ vont coïncider dans l'anneau $\mathbb{C}\{\{\frac{1}{k}S_\tau\}\}$. Elles ne seront pas toutes des unités dans cet anneau. (Voir l'exemple 1).*

**Corollaire 1.14** *Soit le polynôme $F=\sum_{i=1}^{n}a_iY^i$ où $a_i$ sont des polynômes dans $\mathbb{C}[X_1,\ldots,X_d]$ tels que $a_0a_n\neq 0$. On a l'inclusion de polytopes*

$$\mathcal{P}(a_0)+\mathcal{P}(a_n)+\mathcal{P}(\Delta_YF)\subseteq\mathcal{Q}(F)$$

*et on a l'égalité si et seulement si le polynôme $F$ vérifie la condition discriminantale.*

*Preuve.* Soit $\rho$ un cône de dimension $d$ strictement convexe. On va considérer le polynôme $F$ comme un élément de l'anneau $\mathbb{C}[[S_\rho]][Y]$. En appliquant le théorème 1.4 pour chaque $w\in\rho$, on voit que $\mathcal{Q}(F)+\rho^\vee\supseteq\mathcal{P}(a_0)+\mathcal{P}(a_n)+\mathcal{P}(\Delta_YF)+\rho^\vee$, et que l'on a l'égalité si et seulement si toutes les arêtes admissibles du polyèdre $\mathcal{P}(F)+\rho^\vee\times\{0\}$ vérifient la condition discriminantale. Ceci termine la preuve parce que $\rho$ est arbitraire. ◇

**Remarque 1.15** *En utilisant le corollaire 1.14 et le théorème 7.3 de [Bi-St], on peut déduire de ce qui précède les théorèmes 2.2, et 2.3, Chp. 12, de [G-K-Z]. Ces résultats donnent le polytope de Newton du discriminant générique (c'est-à-dire le discriminant du polynôme* $F = X_n Y^n + \dots X_1 Y + X_0 \in \mathbb{C}[X_0, X_1, \dots, X_n][Y]$ *par rapport à* $Y$ *) et les coefficients des termes correspondant aux sommets du polytope.*

En effet, le polytope $\mathcal{P}(F)$ est un simplexe de dimension $n$, de sommets $(u_j, j) \in \mathbb{R}^{n+1} \times \mathbb{R}$, où $\{u_j\}_{j=0}^n$ sont les vecteurs de la base canonique dans $\mathbb{R}^{n+1}$. Comme le polynôme $F$ vérifie la condition discriminantale on a $\mathcal{Q}(F) = \mathcal{P}(X_0 X_n \Delta_Y F)$. Chaque sous-ensemble $\{i_1, \dots, i_s\}$ de $\{1, \dots, n-1\}$ correspond de manière unique à un chemin monotone $\gamma_{\{i_1,\dots,i_s\}}$ dans les arêtes de $\mathcal{P}(F)$. Comme $\mathcal{P}(F)$ est un simplexe, il existe un vecteur irrationnel $w \in (\mathbb{R}^{n+1})^*$ définissant le chemin $\gamma_{\{i_1,\dots,i_s\}}$. Le sommet de l'intégrale de Minkowski $\int \gamma_{\{i_1,\dots,i_s\}}$ est décrit par la formule (1.1). En appliquant le corollaire 1.11, on obtient aussi le coefficient correspondants aux sommets du polytope de Newton du discriminant générique.

### 1.4.2 Application au polyèdre de Newton du résultant

On dira que deux polynômes $F, G \in \mathbb{C}((S_\rho))[Y]$ vérifient la *condition résultante* si pour toute paire d'arêtes $e$ de $\mathcal{N}_\rho(F)$ et $e'$ de $\mathcal{N}_\rho(G)$ ayant la même inclinaison, les polynômes des arêtes respectives $f_e, g'_e \in \mathbb{C}[t]$ n'ont pas de racines en commun.

Soit $w$ un vecteur irrationnel dans un cône $\tau$ de dimension $d$ de l'éventail associé au polyèdre $\mathcal{Q}_\rho(F) + \mathcal{Q}_\rho(G)$. Le vecteur $w$ détermine des chemins monotones uniques $\gamma_F$ et $\gamma_G$ dans les arêtes des $\rho$-polyèdres de Newton de $F$ et de $G$. Le chemin $\gamma_F$ a des arêtes $e_i$ d'inclinaisons $q_i$, pour $i = 1, \dots, m$ et par le lemme 1.7 on a $q_m <_w \dots <_w q_1$. Le chemin $\gamma_G$ a des arêtes $e'_j$ d'inclinaisons $q'_j$, pour $j = 1, \dots, m'$ tels que $q'_{m'} <_w \dots <_w q'_1$.

Par contre, l'ordre défini par $w$ dans $q_1, \dots q_m, q'_1, \dots q'_{m'}$ peut varier lorsque $w$ parcourt $\tau$. Nous considérons la subdivision finie rationnelle la moins fine de $\tau$ possédant la propriété suivante : des vecteurs irrationnels qui sont dans le même cône de la subdivision définissent le même ordre sur l'ensemble des inclinaisons $q_1, \dots q_m, q'_1, \dots q'_{m'}$. On définit de cette manière une subdivision $\Sigma$ de l'éventail associé à $\mathcal{Q}_\rho(F) + \mathcal{Q}_\rho(G)$.

**Proposition 1.16** *L'éventail associé à l'intégrale de Minkowski* $\mathcal{Q}_\rho(FG)$ *est égal à* $\Sigma$.

*Preuve.* Soit $\gamma_{FG}$ le chemin monotone dans le polyèdre $\mathcal{N}_\rho(FG) = \mathcal{N}_\rho(F) + \mathcal{N}_\rho(G)$ défini par un vecteur irrationnel $w \in \tau \in \Sigma$. Chaque point $\gamma(t)$ est la somme de deux points situés dans les chemins $\gamma_F$ et $\gamma_G$ définis par $w$ dans les polyèdres respectifs. Clairement on a $\gamma_{FG}(n+n') = \gamma_F(n) + \gamma_G(n')$. Si $q_m \geq_w q'_m$, le segment $l_{m+m'} := \gamma_G(n') + e_m$ est contenu dans $\gamma_{FG}$. Ce segment n'est pas une arête de $\gamma_{FG}$ si et seulement si, on a $q_m = q'_m$. Par récurrence, on subdivise $\gamma_{FG}$ en $m+m'$ segments, $l_1, \dots, l_{m+m'}$, tels qu'il existe une bijection $\{l_1, \dots, l_{m+m'}\} \to \{e_1, \dots, e_m, e'_1, \dots, e'_{m'}\}$ qui préserve l'inclinaison et la longueur. De plus, $\gamma_{FG}$ est complètement déterminé par $\gamma_F$, $\gamma_G$ et l'ordre des inclinaisons. Ceci implique que $\Sigma$ est un éventail plus fin que l'éventail associé à $\mathcal{Q}_\rho(FG)$.

Réciproquement, si $w, w'$ sont des vecteurs irrationnels dans un cône de l'éventail associé au polyèdre $\mathcal{Q}_\rho(FG)$, ils définissent un unique chemin monotone $\gamma$ et par le lemme 1.7 les inclinaisons de ses arêtes ont le même ordre par rapport à $<_w$ et $<_{w'}$, donc $w, w'$ sont dans le même cône de $\Sigma$. $\diamond$

**Proposition 1.17** *Soient $F, G \in \mathbb{C}((S_\rho))[Y]$ des polynômes de degrés $n, n' \geq 1$ ayant des termes constants non nuls. Si $F, G$ vérifient la condition résultante, alors :*

1. *L'éventail $\Sigma$ associé au polyèdre-fibre $\mathcal{Q}_\rho(FG)$ est une subdivision de l'éventail du $\rho$-polyèdre de Newton du résultant de $F$ et $G$.*
2. *Soit $\tau \in \Sigma$ un cône de dimension $d$, définissant les chemins monotones $\gamma_F$, $\gamma_G$ et $\gamma_{FG}$ dans les polyèdres de Newton $\mathcal{N}_\rho(F)$, $\mathcal{N}_\rho(G)$ et $\mathcal{N}_\rho(FG)$. Le sommet du $\rho$-polyèdre de Newton du résultant de $F$ et $G$ associé a $\tau$ est $\int \gamma_{FG} - \int \gamma_F - \int \gamma_G$.*

*Preuve.* Soit $w \in \tau \in \Sigma$ un vecteur irrationnel, on montre d'abord que l'exposant le plus petit par rapport à $<_w$ qui peut apparaître dans $\mathrm{Res}\,(F, G)$ est le même pour tout $w \in \tau$.

Le chemin $\gamma_F$ a des arêtes $e_r = [p_{r-1}, p_r]$ de pente $q_r$ est de longueur $l_r$ pour $r = 1, \dots, m$. Le polynôme associé à l'arête $e_r$ est $f_r = \alpha_{p_{r-1}} + \cdots + \alpha_{p_r} t^{l_r}$. Nous notons $\{c_r^1, \dots, c_r^{l_r}\}$ ses racines comptées avec multiplicité.

Le chemin $\gamma_G$ a des arêtes $e'_s = [p'_{s-1}, p'_s]$ de pente $q'_s$ est de longueur $l'_s$ pour $s = 1, \dots, m'$. Le polynôme associé à l'arête $e'_s$ est $g_s = \beta_{p'_{s-1}} + \cdots + \beta_{p'_s} t^{l'_s}$. Nous notons $\{d_s^1, \dots, d_s^{l'_s}\}$ ses racines comptées avec multiplicité.

Par le théorème 1.2, le terme initial par rapport à $<_w$, d'une racine $\phi_r^i$ de $F$ correspondant au segment $e_r$ est $c_r^i X^{q_r}$, et celui d'un racine $\psi_s^j$ de $G$ correspondant au segment $e'_s$ est $d_s^j X^{q'_s}$.

Si $a_n$ et $b_{n'}$ sont les coefficients des termes de degré $n$ et $n'$ de $F$ et $G$ respectivement, on a $\mathrm{Res}\,(F, G) = a_n^{n'} b_{n'}^n \prod (\phi_r^i - \psi_s^j)$. Le coefficient du terme d'exposant le plus petit par rapport à $<_w$ qui peut apparaître dans $\mathrm{Res}\,(F, G)$ est le produit $ABCD$ où le facteur $A$ correspond à $a_n^{n'} b_{n'}^n$ :

$$A = \alpha_{p_m}^{n'} \beta_{p'_{m'}}^{n},$$
$$B = \prod_{q_r = q'_s}^{r,s} \prod_{i=1,\ldots,l_r} \prod_{j=1,\ldots l'_s} (c_r^i - d_s^j),$$
$$C = \prod_{q_r >_w q'_s}^{r,s} \prod_{j=1,\ldots l'_s} \prod_{i=1,\ldots,l_r} c_r^i,$$
$$D = \prod_{q_r <_w q'_s}^{r,s} \prod_{j=1,\ldots l'_s} \prod_{i=1,\ldots,l_r} -d_s^j.$$

Comme $F$ et $G$ vérifient la condition résultante, on a $ABCD \neq 0$, et le terme obtenu ne varie pas lorsque $w$ parcourt $\tau$.

Nous notons $u_{\Delta(F)}$, $u_{D(G)}$ et $u_{\Delta(FG)}$ l'exposant le plus petit par rapport à $<_w$ qui peut apparaître parmi les exposants du discriminant de $F$, $G$ et $FG$ respectivement. Par le théorème 1.4 on a :

$$\begin{array}{lcl} 2\int \gamma_F & = & \gamma_F(0) + \gamma_F(n) + u_{\Delta(F)} \\ 2\int \gamma_G & = & \gamma_G(0) + \gamma_G(n') + u_{\Delta(G)} \\ 2\int \gamma_{FG} & = & \gamma_{FG}(0) + \gamma_{FG}(n+n') + u_{\Delta(FG)} \end{array}$$

On considère l'expression

$$\Delta_Y(F)\,\Delta_Y(G)(\mathrm{Res}\ (F,G))^2 = \Delta_Y(FG) \tag{1.2}$$

en fonction des racines de $F$ et de $G$ et on déduit que si $u_0$ est l'exposant initial par rapport à $<_w$ de Res $(F,G)$ :

$$2u_0 = u_{\Delta(FG)} - u_{\Delta(F)} - u_{\Delta(G)} = 2\int \gamma_{FG} - 2\int \gamma_F - 2\int \gamma_G,$$

parce que l'on a $\gamma_{FG}(0) = \gamma_F(0) + \gamma_G(0)$ et $\gamma_{FG}(n+n') = \gamma_F(n) + \gamma_G(n')$. ◇

**Corollaire 1.18** *Dans les hypothèses de la proposition 1.17, chaque cône $\tau \in \Sigma$ de dimension $d$ définit un sommet du $\rho$-polyèdre de Newton du résultant de $F$ et $G$ de coefficient :*

$$\alpha_{p_m}^{n'} \beta_{p'_m}^{n} \Big( \prod_{q_r = q'_s}^{r,s} \alpha_{p_r}^{-l'_s} \beta_{p'_s}^{-l_r} \mathrm{Res}\,(f_r, g_s) \Big) \Big( \prod_{q_r >_w q'_s}^{r,s} (-1)^{l_r l'_s} \Big(\frac{\alpha_{p_{r-1}}}{\alpha_{p_r}}\Big)^{l'_s} \Big) \Big( \prod_{q_r <_w q'_s}^{r,s} \Big(\frac{\beta_{p'_{s-1}}}{\beta_{p'_s}}\Big)^{l_r} \Big)$$

*Preuve.* Comme $\mathrm{Res}\,(f_r, g_s) = \alpha_{p_r}^{l'_s} \beta_{p'_s}^{l_r} \prod_{i,j} (c_r^i - d_s^j)$ on a :

$$B = \prod_{q_r = q'_s}^{r,s} \alpha_{p_r}^{-l'_s} \beta_{p'_s}^{-l_r} \mathrm{Res}\,(f_r, g_s).$$

En utilisant que $\prod_{i=1,\ldots,l_r} c_r^i = (-1)^{l_r} \frac{\alpha_{p_{r-1}}}{\alpha_{p_r}}$, et que $\prod_{j=1,\ldots,l'_s} -d_s^j = \frac{\beta_{p'_{s-1}}}{\beta_{p'_s}}$ on déduit :

$$C = \prod_{q_r >_w q'_s}^{r,s} (-1)^{l_r l'_s} \Big(\frac{\alpha_{p_{r-1}}}{\alpha_{p_r}}\Big)^{l'_s}, \qquad D = \prod_{q_r <_w q'_s}^{r,s} \Big(\frac{\beta_{p'_{s-1}}}{\beta_{p'_s}}\Big)^{l_r}.$$

◇

**Exemple 1.1** *Considérons le polynôme* $F = U^4 V^2 Y^5 + U^3 V^2 Y^2 - Y + U^2 V + V^2$. *Le discriminant du polynôme* $F$ *par rapport à* $Y$ *est*

$$\begin{aligned}\Delta_Y F &= 108\, U^{25} V^{15} + 3125\, U^{24} V^{12} + 108\, U^{23} V^{16} + 12500\, U^{22} V^{13} \\ &- 2250\, U^{22} V^{12} + 18750\, U^{20} V^{14} - 4500\, U^{20} V^{13} - 27\, U^{20} V^{12} \\ &+ 12500\, U^{18} V^{15} - 2250\, U^{18} V^{14} + 1600\, U^{17} V^{9} + 3125\, U^{16} V^{16} \\ &+ 1600\, U^{15} V^{10} - 256\, U^{12} V^{6} \\ &= U^{12} V^6 \varepsilon\end{aligned}$$

*où* $\varepsilon$ *est une unité de l'anneau* $\mathbb{C}\{S_\Delta\}$, *où* $\Delta = pos\{(-2,5),(2,-1)\}$. *Nous allons montrer de deux manières qu'il existe* $k \in \mathbb{N}$ *tel que* $F$ *se décompose dans l'anneau* $\mathbb{C}\{\{\frac{1}{k} S_\Delta\}\}$.

**1.** D'abord, on définit $G := U^6 V^8 F(U^{-2} V^{-2}) = Y^5 + U^5 V^6 Y^2 - U^4 V^6 Y + U^8 V^9 + U^6 V^{10}$ et on vérifie que $G \in \mathbb{C}\{S_\Delta\}[Y]$ et que $G(z_\Delta, Y) = Y^5$. Comme le discriminant de $F$ est une unité dans $\mathbb{C}\{\{S_\Delta\}\}$ par le théorème 1.1, il existe $k \in \mathbb{N}$ tel que $G$ se décompose dans $\mathbb{C}\{\frac{1}{k} S_\Delta\}$ et donc $F$ se décompose dans $\mathbb{C}\{\{\frac{1}{k} S_\Delta\}\}$.

**2.** Le polynôme $F$ vérifie la condition discriminantale donc on a $\mathcal{N}_\Delta(U^4 V^2 (U^2 V + V^2) \Delta_Y F) = \mathcal{Q}_\Delta(F)$. Le $\Delta$-polyèdre-fibre $\mathcal{Q}_\Delta(F)$ a deux sommets correspondant aux chemins monotones cohérents $(\gamma_i)_{i=1,2}$. Le premier, $\gamma_1$, correspondant aux termes $U^4 V^2 Y^5$, $Y$, $V^2$ et définissant le sommet $2 \int \gamma_1 = (8,10)$ du polyèdre $\mathcal{Q}_\Delta(F)$. Le deuxième $\gamma_2$, correspondant à $U^4 V^2 Y^5$, $Y$, $U^2 V$ et définissant le sommet $2 \int \gamma_2 = (8,9)$. L'éventail associé est la subdivision de $\Delta$ par les cônes $\sigma_1 = \text{pos}\{(2,-1),(1,2)\}$, et $\sigma_2 = \text{pos}\{(-2,5),(1,2)\}$. Fixons un vecteur irrationnel $w \in \Delta$. Si $w \in \sigma_1$, (resp. $w \in \sigma_2$), il détermine le chemin $\gamma_1$, (resp. $\gamma_2$).

Les 4 racines de $F$ correspondant au segment $e = [(4,2,5),(0,0,1)]$ de $\gamma_1$ (resp. de $\gamma_2$) par le théorème 1.2 ont un terme d'exposant $(-1, -\frac{1}{2})$. On définit $F_2 = F(Y + \lambda U^{-1} V^{-1/2})$ où $\lambda^4 = 1$.

$$\begin{aligned} F_2 := \;& Y^5 U^4 V^2 + 5\lambda Y^4 U^3 V^{3/2} + 10 \lambda^2 Y^3 U^2 V + Y^2 U^3 V^2 \\ & + 10 \lambda^3 Y^2 U V^{1/2} + 2 \lambda Y U^2 V^{3/2} + 5\, Y - Y \\ & + U^2 V + \lambda^2 U V + \lambda U^{-1} V^{-1/2} - \lambda U^{-1} V^{-1/2} + V^2 \end{aligned}$$

Par la preuve du théorème 1.2 les exposants des racines correspondant à $e$ sont dans le cône affine de sommet $\pi_e(e) = (-1, \frac{-1}{2})$ qui contient les exposants de $\pi_e(\mathcal{N}_\Delta(F_2))$ cette-à-dire le cône $(-1, \frac{-1}{2}) + \text{pos}\{(2,3),(2,1)\}$. Comme le cône $\text{pos}\{(2,3),(2,1)\}$ est contenu dans $\Delta^\vee$, ces racines sont des éléments de $\mathbb{C}\{\{\frac{1}{2}S_\Delta\}\}$.

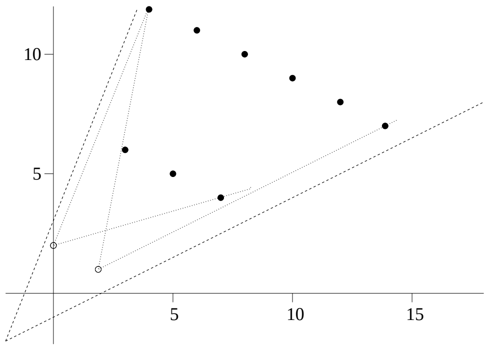


FIG. 1.1 – Les sommets noirs sont les exposants de $F_3^{(1)}(U, V, 0)$

Par ailleurs, $F(0, Y) = -Y$, et en appliquant le théorème des fonctions implicites, il existe un unique $\phi \in \mathbb{C}\{U, V\}$ tel que $F(\phi) = 0$. Clairement, $\phi$ doit coïncider avec la série correspondant au segment $[(0,0,1),(0,2,0)]$ déterminé par $w \in \sigma_1$, (resp. au segment $[(0,0,1),(0,1,2)]$ déterminé par $w \in \sigma_2$).

Pour $w \in \sigma_1$, on définit $F_2^{(1)} := F(Y + V^2)$ et on remarque que le terme initial par la valuation $w$ de $F_2^{(1)}(0) = U^4V^{12} + U^3V^6 + U^2V$ est $U^2V$. On défini $F_3^{(1)} := F_2^{(1)}(Y + U^2V) = F(Y + V^2 + U^2V)$. On sait que les exposants de la série $\phi - V^2$ sont dans le cône affine de sommet $(2,1)$ qui contient les exposants de

$$\begin{aligned} F_3^{(1)}(0) = \quad & U^3\,V^6 + 2\,U^5\,V^5 + U^7\,V^4 + \\ & + U^4\,V^{12} + 5\,U^6\,V^{11} + 10\,U^8\,V^{10} + 10\,U^{10}\,V^9 + 5\,U^{12}\,V^8 + U^{14}\,V^7 \end{aligned}$$

C'est-à-dire le cône $(2,1) + \text{pos}\{(1,5),(2,1)\}$.

(Pour $w \in \sigma_2$, on définit $F_2^{(2)} = F(Y + U^2V)$, on vérifie que $F_3^{(2)} = F_2^{(2)}(Y + V^2) = F(Y + V^2 + U^2V) = F_3^{(1)}$, et que les exposants de la série construit $\phi - U^2V$ sont dans le cône affine $(0,2) + \text{pos}\{(2,5),(7,2)\}$.)

On obtient que les exposants de $\phi$ sont dans le cône affine $(-2,-2) + \text{pos}\{(2,5),(1,2)\}$. Comme le cône $\Delta^\vee = \text{pos}\{(2,5),(1,2)\}$ la série $\phi$ est dans l'anneau $\mathbb{C}\{\{S_\Delta\}\}$ (voir la figure 1.1).

# Chapter 2

# Quasi-ordinary singularities, their semigroups and normalizations

## 2.1 Introduction

In this chapter we study invariants of quasi-ordinary singularities, in particular in the hypersurface case. The results obtained are used to describe the normalization and two embedded resolution procedures that depend only on the invariants.

We begin by a reminder of toric geometry, conditions of non degeneracy and related topics. We describe the parametrizations of quasi-ordinary singularities, i.e., the fractional power series $\zeta$ in the variables $X = (X_1, \ldots, X_d)$ that are roots of a polynomial $f \in \mathbb{C}\{X\}[Y]$ that defines a quasi-ordinary hypersurface germ $S$; these series are called quasi-ordinary branches. The quasi-ordinary branch $\zeta$ has a finite set of distinguished exponent vectors $\lambda_1, \ldots, \lambda_g$ called *characteristic exponents*. We will study the case $f$ irreducible. We introduce the notion of *semi-roots* $q_i$ associated to $f$ (following the terminology of [PP] for plane branches). In the plane branch case, this notion corresponds to the singular curves with maximal contact defined by Lejeune in [LJ]. We show that some of the Abhyankar-Moh *approximate roots* (see [A2] and [PP]) of $f$ are semi-roots, and that they are irreducible quasi-ordinary polynomials. The parametrization of the germ $S$ defined by $\zeta$ corresponds to an extension of the analytic algebra $R$ of $S$ in a regular local ring:

$$R = \mathbb{C}\{X\}[Y]/(f) = \mathbb{C}\{X\}[\zeta] \subset \mathbb{C}\{T\} \text{ where } T = X^{1/n} \text{ and } n = \deg f.$$

We show that the graded algebra $\mathrm{gr}_{(T)} R \subset \mathbb{C}[T]$ induced $(T)$-adic filtration of $R$ is generated by monomials $T^{\gamma_1}, \ldots, T^{\gamma_{d+g}}$ determined by the coordinate functions $\xi_1(T) := X_1, \ldots, \xi_d(T) := X_d$ together with the image of a complete set semiroots: $\xi_{d+1}(T) := q_0(\zeta), \ldots, \xi_{d+g}(T) := q_{g-1}(\zeta)$. We can recover the characteristic exponents of $\zeta$ from the vectors $\gamma_1, \ldots, \gamma_{d+g}$ and conversely. We characterize all subsemigroups of $\mathbb{Z}^d_{\geq 0}$ arising from a quasi-ordinary branch. The generators $(\gamma_1, \ldots, \gamma_{d+g})$ of the semigroup $\Gamma_\zeta$ depends on the choice of the quasi-ordinary branch $\zeta$ parametrizing $S$. The main result of this part is Theorem 2.5, which states that the semigroup $\Gamma_\zeta$ does not depend on the choice of the quasi-ordinary branch $\zeta$ parametrizing $S$ and its datum is equivalent to that of the topological type of the quasi-ordinary singularity. For this purpose, we use the characterization of the topological type given by Gau (see [Gau]). We show that this set is the minimal set of generators of the semigroup if the quasi-ordinary branch $\zeta$ is *normalized*[1]. We give as an application, the proof of the inversion formulae relating the characteristic exponents of a quasi-ordinary branch and a normalized quasi-ordinary branch parametrizing the same hypersurface. These results leads us to study the *monomial variety*, $\mathcal{S}_0 :=$ Spec $\mathbb{C}[\Gamma]$. We show that it is an affine (non necessarily normal) toric variety of dimension $d$, defining a germ quasi-ordinary singularity at its special point. We determine its singular locus, we compute its normalization, and we show that it coincides with the normalization of the hypersurface $S$.

## 2.2 Toric maps, Newton polyhedra and conditions of non degeneracy

In this section we recall some basic notions of toric (and toroidal) geometry that are needed in the sequel.

### 2.2.1 A reminder of toric geometry

A *fan* $\Sigma$ is familly of *convex rational polyhedral cones* in $(\mathbb{R}^{d+1})^*$ such that any face of such a cone is in the family, the intersection of any two of them is a face of each, and all cones in $\Sigma$ are *strictly convex* (i.e the cone contains no linear subspace of dimension $> 0$). The *support* of the fan $\Sigma$ is the set $\bigcup_{\sigma \in \Sigma} \sigma \subset (\mathbb{R}^{d+1})^*$. A cone $\sigma \in \Sigma$ is *regular* if it is simplicial (i.e., the number of $1$-dimensional faces is equal to the dimension of $\sigma$) and the primitive integral vectors defining the $1$-dimensional faces belong to a basis of $\mathbb{Z}^{d+1}$. The fan $\Sigma$ is regular if all the cones

[1] This is a technical condition which in the plane curve case means that the projection $(X, Y) \mapsto X$ is *transversal*

of maximal dimension are regular. We will denote $\Sigma^{(i)}$ the $i$-skeleton of the fan $\Sigma$, (see [F1], [Ew], [Od] for toric varieties, convex polyhedral cones, fans, etc.).

The *dual cone* of a convex polyhedral cone $\sigma$ is $\sigma^\vee := \{w \in \mathbb{R}^{d+1} / \langle w, u \rangle \geq 0, \ \forall u \in \sigma\}$. If the cone $\sigma$ belongs to a fan $\Sigma$, then the cone $\sigma^\vee$ is of dimension $d+1$. The semigroup defined by $\sigma$ is $S_\sigma = \sigma^\vee \cap \mathbb{Z}^{d+1}$. If $\sigma$ is a *rational cone* its semigroup is finitely generated and we can define the associated variety $Z(\sigma) = \text{ Spec } \mathbb{C}[S_\sigma]$. If $\sigma \subset \sigma'$ are cones in $\Sigma$ then we have an open immersion $Z(\sigma) \subset Z(\sigma')$; the affine varieties $Z(\sigma)$ corresponding to cones in a fan $\Sigma$ glue up to define the *toric variety* $Z(\Sigma)$. The torus $T := (\mathbb{C}^*)^{d+1}$ is equal to $Z(\{0\})$ and is contained as an open dense subset of $Z(\Sigma)$, it is proved in the above references that there is an action of the torus $T$ on the variety $Z(\Sigma)$, which restricted to $T$ gives the product on the algebraic group $T$. General toric varieties are defined by this property (see [St], [G-K-Z]). The toric varieties which can be defined using fans are precisely the *normal* ones (see [KKMS]).

We say that a fan $\Sigma'$ is a *subdivision* of the fan $\Sigma$, if both fans have the same support and if any cone of $\Sigma'$ is contained in a cone of $\Sigma$. Associated to a subdivision of fans one can define a *modification*, i. e., a proper birational map $Z(\Sigma') \to Z(\Sigma)$ between the toric varieties.

The toric variety $Z(\Sigma)$ is non singular if and only if the fan $\Sigma$ is regular. A *resolution of singularities* of a variety $Z$ is a modification $Z' \to Z$ which is an isomorphism outside the singular locus of $Z$. The resolution of singularities of normal toric varieties, is reduced to a combinatorial property. This result is due to Kempf, Knudsen, Mumford and Saint-Donat (see [KKMS]); more precisely we have:

**Theorem 2.1** *(see [Co], Theorem 5.1) Given any fan $\Sigma$, there is a regular fan $\Sigma'$ subdividing $\Sigma$, which does not modify the regular faces of $\Sigma$. The associated toric morphism $Z(\Sigma') \to Z(\Sigma)$ is a resolution of singularities of the variety $Z(\Sigma)$.* ◇

We say that the fan $\Sigma'$ in the theorem above is a *regular subdivision* of the fan $\Sigma$.

A primitive vector $a$ in $\mathbb{Z}^{d+1}$ is *essential* for the fan $\Sigma$ if it defines a cone which belongs to any regular subdivision of $\Sigma$ (see [Ag] and [Bo-Go]). In particular, if a 2-dimensional cone $\sigma \in \Sigma$ is not regular, it admits a unique minimal regular subdivision (see [Od] Prop. 1.19). Any primitive vector defining an edge in this subdivision is essential since any regular subdivision of $\Sigma$ provides a regular subdivision of the cone $\sigma$.

If the fan $\Sigma$ is regular and supported on $\mathbb{R}^{d+1}_{\geq 0}$, then we have a subdivision of the fan defined by the cone $\mathbb{R}^{d+1}_{\geq 0}$, defining a modification $\pi(\Sigma) : Z(\Sigma) \to \mathbb{C}^{d+1}$ which we describe in detail:

The variety $Z(\Sigma)$ is non singular, for each cone $\sigma$ of maximal dimension the variety $Z(\sigma)$ is isomorphic to $\mathbb{C}^{d+1}$ and the restriction $\pi(\sigma) : Z(\sigma) \to \mathbb{C}^{d+1}$ of the morphism $\pi(\Sigma)$ to the chart $Z(\sigma)$ is induced by the semigroup inclusion $S_{\mathbb{R}^{d+1}_{\geq 0}} \to S_\sigma$. The primitive vectors in the 1-skeleton of each regular $(d+1)$-dimensional $\sigma$ are a basis of $\mathbb{Z}^{d+1}$, and its dual basis is a minimal set of generators of the semigroup $S_\sigma$. These generators give us coordinates to describe the map $\pi(\sigma) : Z(\sigma) \to \mathbb{C}^{d+1}$ in the form:

$$\begin{array}{rcl} X_1 & = & U_1^{a_1^1} U_2^{a_1^2} \cdots U_{d+1}^{a_1^{d+1}} \\ X_2 & = & U_1^{a_2^1} U_2^{a_2^2} \cdots U_{d+1}^{a_2^{d+1}} \\ & \cdots & \\ X_{d+1} & = & U_1^{a_{d+1}^1} U_2^{a_{d+1}^2} \cdots U_{d+1}^{a_{d+1}^{d+1}} \end{array} \tag{2.1}$$

where $(a_1^i, a_2^i, \ldots, a_{d+1}^i)$ are the coordinates of the primitive vectors $a^i$ in the 1-skeleton of $\sigma$, for $i = 1, \ldots, d+1$. Since the fan $\Sigma$ is regular, it is easy to see directly from formula (2.1) that the map $\pi(\Sigma)$ is an isomorphism over the torus $X_1 \cdots X_{d+1} \neq 0$ of $\mathbb{C}^{d+1}$.

We associate to the vector $a^i$ the hypersurface defined in $Z(\sigma)$ by $U_i = 0$. When we consider all $(d+1)$-dimensional cones containg $a^i$ these hypersurfaces glue up to define a divisor $D(a^i)$. The *critical locus* of the morphism $\pi(\Sigma)$ is the union of the divisors $D(a^i)$ as $a^i$ runs through the primitive vectors of the 1-skeleton of $\Sigma$ which are not in the canonical basis. The *discriminant locus* of the toric morphism is the image of the critical locus. It is a union of codimension $\geq 2$ coordinate subspaces. The image of a divisor $D(a^i)$ in the critical locus is the coordinate subspace defined by $X_j = 0$, for $j$ such that $a^i_j \neq 0$.

### 2.2.2 A reminder on toroidal embeddings

The divisor $D = \sum_{a \in \Sigma^{(1)}} D(a)$ in the non singular variety $\mathcal{X} = Z(\Sigma)$ is a normal crossing divisor with smooth components. The torus $T := (\mathbb{C}^*)^d$ is embedded in the variety $\mathcal{X}$ as the open set $\mathcal{X} - D$. This framework can be generalized as follows. Let $\mathcal{X}$ be a non singular variety and let $\mathcal{U}$ be a Zariski open set embedded in $\mathcal{X}$, such that $\mathcal{X} - \mathcal{U}$ is a finite union of non singular hypersurfaces $E_i$ intersecting transversally. The variety $\mathcal{X}$ is naturally stratified, with strata $\bigcap_{i \in K} E_i - \bigcup_{i \notin K} E_i$ and the open stratum $\mathcal{U}$. We will call this a *regular toroidal embedding*. Usually, a *toroidal embedding* is defined by requiring the varieties $E_i$ and $\mathcal{X}$ to be normal and the triple $(\mathcal{X}, \mathcal{U}, x)$ at any point $x \in \mathcal{X}$ to be formally isomorphic to $(Z(\sigma), T, z)$ for $z$ a point in some toric variety $Z(\sigma)$. This means that there is a formal isomorphism between the completions of the local rings at respective points which sends the ideal of $\mathcal{X} - \mathcal{U}$ into the ideal of $Z(\sigma) - T$; (see [KKMS]).

The *star of a stratum* $\mathfrak{S}$, $\operatorname{star}\mathfrak{S}$, is the union of the strata containing $\mathfrak{S}$ in their closure. We associate to the stratum $\mathfrak{S}$ the set $M^{\mathfrak{S}}$ of Cartier divisors supported on $\operatorname{star}\mathfrak{S} - \mathcal{U}$. We denote by $N^{\mathfrak{S}}$ the dual group $\operatorname{Hom}(M^{\mathfrak{S}}, \mathbb{Z})$. The semigroup of effective divisors defines in the real vector space $M^{\mathfrak{S}}_{\mathbb{R}} := M^{\mathfrak{S}} \otimes \mathbb{R}$ a rational convex polyhedral cone, and we denote its dual cone in $N^{\mathfrak{S}}_{\mathbb{R}} := N^{\mathfrak{S}} \otimes \mathbb{R}$ by $\rho^{\mathfrak{S}}$. If $\mathfrak{S}'$ is a stratum in $\operatorname{star}\mathfrak{S}$, we have a projection $M^{\mathfrak{S}} \to M^{\mathfrak{S}'}$ which is onto; by duality we obtain an inclusion $N^{\mathfrak{S}'} \to N^{\mathfrak{S}}$ and the cone $\rho^{\mathfrak{S}'}$ is mapped onto a face of $\rho^{\mathfrak{S}}$, (see [KKMS]). We can associate in this way to a toroidal embbeding, a *conic polyhedral complex with integral structure* (see [KKMS]). This generalizes the way of recovering from a normal toric variety the associated fan. This complex is *combinatorially isomorphic* to the cone over the dual graph of intersection of the divisors $E_i$. We have that the strata of the stratification are in one-to-one correspondence with the faces of the conic polyhedral complex.

We can define, in an analogous manner to the case of a fan, a regular subdivision of a conic polyhedral complex. Associated to a regular subdivision we have an induced *toroidal modification* (see [KKMS] Th. 6* and 8*), i.e, a non singular variety $\mathcal{X}'$ with a regular toroidal embedding $\mathcal{U} \subset \mathcal{X}'$ and a modification $\mathcal{X}' \to \mathcal{X}$ provided with a commutative diagram:

$$\begin{array}{ccc} \mathcal{U} & \to & \mathcal{X} \\ \downarrow & \nearrow & \\ \mathcal{X}' & & \end{array}$$

**Remark 2.1** *The* $0$*-strata of* $(\mathcal{X},\mathcal{U})$ *are in one-to-one correspondence with the* $(d+1)$*-dimensional cones in the polyhedral complex. We take local equations* $X_i = 0$ *for the codimension one subvarieties* $E_i$*, at the* $0$*-stratum* $0$ *of* $(\mathcal{X},\mathcal{U})$ *(corresponding to the cone* $\sigma$*) for* $i = 1, \dots, d+1$*. Then, the toroidal modification on a neighborhood of* $0$ *is obtained by gluing maps of the form (2.1), defined by the subdivision of* $\sigma$*.*

This is a consequence of the proof of Theorem 1* in [KKMS].

### 2.2.3 Newton polyhedron and conditions of non degeneracy

The *Newton polyhedron*, $\mathcal{N}(f)$, of a series $0 \neq f = \sum c_v X^v \in \mathbb{C}\{X\}$ is the convex hull of the set $\bigcup_{c_v \neq 0} v + \mathbb{Z}^{d+1}_{\geq 0}$. The *face* $\mathcal{F}_u$ *of the polyhedron* $\mathcal{N}(f)$ *defined by a vector* in $u \in (\mathbb{R}^{d+1})^*_{\geq 0}$ is the set of vectors $v \in \mathcal{N}(f)$ such that

$$\langle u, v \rangle = \inf_{v' \in \mathcal{N}(f)} \langle u, v' \rangle$$

It is easy to see that the face of $\mathcal{N}(f)$ defined by $u$ is compact if and only if the vector $u$ has no zero coordinate. All faces of the polyhedron $\mathcal{N}(f)$ can be recovered in this way.

The *cone* $\sigma(\mathcal{F},\mathcal{N}(f)) \subset (\mathbb{R}^{d+1})^*_{\geq 0}$ *associated to the face* $\mathcal{F}$ of the polyhedron $\mathcal{N}(f)$ is

$$\sigma(\mathcal{F},\mathcal{N}(f)) := \{u \in (\mathbb{R}^{d+1})^*_{\geq 0} \;/\; \forall v \in \mathcal{F}, \text{ we have } \langle u, v\rangle = \inf_{v' \in \mathcal{N}(f)} \langle u, v'\rangle\}.$$

We have that $\dim \mathcal{F} + \dim \sigma(\mathcal{F},\mathcal{N}(f)) = d+1$.

The cones $\sigma(\mathcal{F},\mathcal{N}(f))$, for $\mathcal{F}$ running through the set of faces of the polyhedron $\mathcal{N}(f)$, define a subdivision $\Sigma(\mathcal{N}(f))$ of the cone $(\mathbb{R}^{d+1})^*_{\geq 0}$ (which is not regular in general). We can define this subdivision in a different way. First, we define an equivalence relation; we say that two vectors $u, u' \in (\mathbb{R}^{d+1})^*_{\geq 0}$, are related if they define the same face of the polyhedron $\mathcal{N}(f)$. Then, we find that the classes of this relation are the relative interiors of the cones in the fan $\Sigma(\mathcal{N}(f))$, (see [G-T]).

We say that a fan $\Sigma$ supported on $(\mathbb{R}^{d+1})^*_{\geq 0}$ is *compatible* with the Newton polyhedron of the function $f$ if it subdivides the fan $\Sigma(\mathcal{N}(f))$. In this case, we associate to a cone $\sigma \in \Sigma$ the face $\mathcal{F}_\sigma$ of the polyhedron $\mathcal{N}(f)$ defined by any vector in the relative interior of the cone $\sigma$. We have the inequality $\dim \mathcal{F}_\sigma + \dim \sigma \leq d+1$, which implies that a face associated to a cone of maximal dimension is a vertex of $\mathcal{N}(f)$.

If the fan $\Sigma$ is compatible with $0 \neq f = \sum c_v X^v \in \mathbb{C}\{X\}$ then the transform of $f$ by $\pi(\sigma)$ is

$$f \circ \pi(\sigma) = \sum_v c_v U_1^{\langle a^1, v\rangle} \dots U_{d+1}^{\langle a^{d+1}, v\rangle}.$$

If $m(a^i) := \inf_{v \in \mathcal{N}(f)} \langle a^i, v\rangle$, we find that

$$f \circ \pi(\sigma) = U_1^{m(a^1)} \dots U_{d+1}^{m(a^{d+1})} \tilde{f}_\sigma,$$

where the *strict transform* $\tilde{f}_\sigma$ of $f$ in this chart is a unit in $\mathbb{C}\{U\}$, since the face associated to $\sigma$ is a vertex of $\mathcal{N}(f)$. The *strict transform* $\tilde{S}$ of the hypersurface $S$ is defined on each $Z(\sigma)$ by $\tilde{f}_\sigma = 0$. The strict transforms $\tilde{f}_\sigma$ and $\tilde{f}_{\bar{\sigma}}$ of the function $f$ define the same function on $Z(\sigma \cap \bar{\sigma})$ if and only if the $(d+1)$-dimensional cones $\sigma$ and $\bar{\sigma}$ of $\Sigma$ define the same vertex of $\mathcal{N}(f)$; in other cases they are related by an invertible monomial on $Z(\sigma \cap \bar{\sigma})$. The *symbolic restriction* of $f$ to the compact face $\mathcal{F}$ of the Newton polyhedron is the polynomial $f_{|\mathcal{F}} = \sum_{v \in \mathcal{F}} c_v X^v$. If the compact face $\mathcal{F}$ is associated to the cone $\langle a^{i_1}, \dots, a^{i_s}\rangle$, the intersection of the strict transform of $f = 0$ on $Z(\sigma)$, with $\bigcap_{j \in \{i_i, \dots, i_s\}} D(a^j)$, has equation equal to $\widetilde{f_{|\mathcal{F}_\sigma}} = 0$.

The toric map $\pi(\Sigma) : Z(\Sigma) \to \mathbb{C}^{d+1}$ is a *toric embedded pseudo-resolution* of the germ $f = 0$, if the strict transform of $f = 0$ by the proper map $\pi(\Sigma)$ is non singular and transverse to a stratification of the critical locus of $\pi(\Sigma)$. If, in addition the restriction of the map $\pi(\Sigma)$ to the strict transform $\tilde{S} \to S$ is an isomorphism outside the singular locus of $S$, then the modification $\pi(\Sigma)$ is a *toric embedded resolution* of $S$, (see [G-T]). If $\pi(\Sigma)$ is only a pseudo-resolution we can only guarantee that the map $\tilde{S} \to S$ is an isomorphism outside the intersection of $S$ with the discriminant locus of $\pi(\Sigma)$. In this case, this set contains the singular locus of $S$ but it is not necessarily equal to it.

We say that a set of elements $f_1, \dots, f_s \in \mathbb{C}\{X\}$ is *non degenerate* if for any regular fan $\Sigma$ compatible with the polyhedron $\mathcal{N}(f_1 \cdots f_s)$, and any cone $\sigma \in \Sigma$ not contained in a coordinate hyperplane we have that the Jacobian matrix of $(f_{1|\mathcal{F}_\sigma}, \dots, f_{s|\mathcal{F}_\sigma})$ is of maximal rank $s \leq d+1$ on any point on any point of the torus $(\mathbb{C}^*)^{d+1}$. We have the following result:

**Theorem 2.2** *(see Theorem 5.1 of [G-T]) Let $(V, 0) \subset (\mathbb{C}^{d+1}, 0)$ be a $(d+1-s)$ dimensional germ of complete intersection defined by a non degenerate set of equations $f_1 = 0, \dots, f_s = 0$. Suppose that $\Sigma$ is a regular fan supported on $\mathbb{R}^{d+1}_{\geq 0}$ and compatible with the Newton polyhedra of $f_1, \dots, f_s$. Then the map $\pi(\Sigma)$ induces a toric embedded pseudo-resolution of V.*

Varchenko showed that if the function $f$ is non degenerate and *commode*, i.e., if its Newton polyhedron meets all coordinate axis (which implies that $f$ has an isolated singularity), then a toric map $\pi(\Sigma)$ defined by a regular fan $\Sigma$ compatible with $\mathcal{N}(f)$ is an embedded resolution of the germ $f = 0$ (see [Me]).

## 2.3 Quasi-ordinary singularities

Let $(S, 0)$ be a germ of analytically irreducible complex variety of dimension $d$, and let $R$ be the ring of germs of holomorphic functions on $S$ in a neighborhood of the origin.

A *finite map germ* $(S, s) \to (\mathbb{C}^d, 0)$, is determined by any set of elements $r_1, \dots, r_d$ in $R$, such that $R/(r_1, \dots, r_d)R$ is a finite dimensional complex vector space. A sufficiently small representative $S \to \mathbb{C}^d$ of this map has finite fibers, and its image is an open neighborhood $G$ of the origin; its degree $n$ is the maximal cardinality of the fibers. The *discriminant locus*, i.e. the set of points of $G$ having fibers of cardinality less than $n$, is an analytical subvariety of $\mathbb{C}^d$. Outside this set, the map is *unramified* covering map of degree $n$. We can think of the discriminant locus as an analytic space or as a germ at $0$.

A *parametrization* [2] of $(S, 0)$ is a *finite map germ* $(S', s') \to (S, s)$, where $(S', s')$ is smooth at

[2] We use a restrictive notion of parametrization, the common definition is a generically finite morphism

$s'$. Using the standard equivalence of categories between analytic algebras and germs of complex varieties (see [K]), this finite map germ corresponds to an integral ring extension $R \subset R'$, where $R'$ is isomorphic to $\mathbb{C}\{T_1, \ldots, T_d\}$.

**Definition 2.1** *A germ of complex analytic variety* $(S, 0)$ *is a quasi-ordinary singularity if there exist a finite morphism* $(S, 0) \to (\mathbb{C}^d, 0)$ *(called a quasi-ordinary projection ) and some analytical coordinates* $(X_1, \ldots, X_d)$ *at* $0$*, such that the discriminant locus is contained (germ wise) in the union of coordinate hyperplanes:* $X_1 \ldots X_d = 0$.

For instance, the class of quasi-ordinary singularities contains all curve singularities. Another, example is the hypersurface of equation $Y^k - X_1^{b_1} \cdots X_d^{b_d} = 0$, with respect to the projection $(X, Y) \to X$.

The Jung-Abhyankar Theorem (see [A1], Th. 3 and also Théorème 1.1 of [GP] and [Zu]) guarantees the existence of a *parametrization* $(\mathbb{C}^d, 0) \to (S, 0)$ of a quasi-ordinary singularity $(S, 0)$. Furthermore, there is a finite map $(\mathbb{C}^d, 0) \to (\mathbb{C}^d, 0)$ of the form: $(X_1, \ldots, X_d) = (T_1^{r_1}, \ldots, T_d^{r_d})$ (for some positive integers $r_1, \ldots, r_d$) making the following diagram commutative:

$$\begin{array}{ccc} (\mathbb{C}^d, 0) & \rightarrow & (S, 0) \\ & \searrow & \downarrow \\ & & (\mathbb{C}^d, 0) \end{array}$$

These morphism gives rise algebraically to the integral ring extensions $\mathbb{C}\{T_1^{r_1}, \ldots, T_d^{r_d}\} \subset R \subset \mathbb{C}\{T\}$, where we are denoting $T = (T_1, \ldots, T_d)$. We will consider only parametrizations of this form for quasi-ordinary singularities.

More generally, a *quasi-ordinary hypersurface* germ $(S, 0)$ is defined in suitable local coordinates by an equation of the form:

$$f(X, Y) = Y^n + g_1(X)Y^{n-1} + \cdots + g_n(X) = 0$$

where the $g_i$ are convergent power series in $X = (X_1, \ldots, X_d)$, we have that $f(0, 0) = 0$, and the discriminant $\Delta_Y f$ of $f$ with respect to $Y$ is of the form:

$$\Delta_Y f = X^\eta H(X), \text{ with } H(0) \neq 0.$$

We will say in this case that $f$ is a *quasi-ordinary polynomial.*

We analyze the irreducible hypersurface case in detail. If $f$ is a quasi-ordinary polynomial, the ring $R$ is of the form $R = \mathbb{C}\{X\}[Y]/(f)$. The Jung-Abhyankar Theorem guarantees the existence of a fractional power series $\zeta \in \mathbb{C}\{X^{1/r}\}$ which is a root of $f$. The roots of quasi-ordinary polynomials are called *quasi-ordinary branches* .

**Remark 2.2** *We can take $r_i = n = \deg f$, thus we have $\zeta \in \mathbb{C}\{X^{1/n}\}$ (see remark 1.3 of [GP], and [L4] page 52).*

This choice of $r_i$ is not necessarily the minimal one, for example the quasi-ordinary branch $\zeta = X_1^{3/2} + X_1^{7/4} + X_1^{9/4} X_2^{1/4} + X_1^{19/8} X_2^{11/8} \in \mathbb{C}\{X_1^{1/8}, X_2^{1/8}\}$ has a minimal polynomial of degree 32 (see [L3]).

We set $T_i := X_i^{1/n}$ for $1 \leq i \leq d$. The parametrization defined by the quasi-ordinary branch $\zeta$ is defined by the integral ring extension:

$$R = \mathbb{C}\{T_1^n, \dots T_d^n\}[\zeta(T_1, \dots, T_d)] \subset \mathbb{C}\{T\} \tag{2.2}$$

If $L$ (resp. $L_n$) is the field of fractions of $\mathbb{C}\{X\}$ (resp. of $\mathbb{C}\{X^{1/n}\}$), the field extension $L \subset L_n$ is finite and Galois. Its Galois group is obtained from the action of $d$-tuples $(\eta_1, \dots, \eta_d)$ of $n^{th}$-roots of unity given by $X_i^{1/n} \mapsto \eta_i X_i^{1/n}$, for $i = 1, \dots, d$. Therefore the irreducible polynomial $f$ splits over $L_n$, and the other roots of $f$ are obtained from $\zeta$ by the above action.

**Remark 2.3** *The existence of a parametrization with fractional power series does not imply that the germ $(S, 0)$ is quasi-ordinary.*

For instance, the surface of equation $(Y^2 - X_1 - X_2)^2 - 4X_1X_2 = 0$ is parametrized by $\tau = X_1^{1/2} + X_2^{1/2}$, and the discriminant locus of the projection contains the curve $X_1 - X_2 = 0$.

A quasi-ordinary branch has a finite set of "special" fractional exponents[3] defined in the following way. The difference $\zeta^{(s)} - \zeta^{(t)}$ of two distinct conjugates of $\zeta$ divides the discriminant in the unique factorization domain $\mathbb{C}\{X^{1/n}\}$, thus it is necessarily of the form $X^{\lambda_{st}} H_{st}(X^{1/n})$ where $H_{st}$ is a unit in $\mathbb{C}\{X^{1/n}\}$. The fractional monomials $X^{\lambda_{st}}$ so obtained are called *characteristic monomials of $\zeta$* and the vector exponents $\lambda_{st}$ are called the *characteristic exponents of $\zeta$*.

The partial order in $\mathbb{Q}^d$ defined by $\lambda \leq \lambda'$ if and only if we have $\leq$ coordinate-wise, induces a total order in the set of characteristic exponents (see [L4], lemma 5.6). We can relabel them in a unique form $\lambda_1 < \lambda_2 < \cdots < \lambda_g$, where $\lambda_i$ has coordinates $(\lambda_{i,1}, \dots, \lambda_{i,d}) \in \mathbb{Q}^d_{>0}$. Thus the characteristic monomials $X^{\lambda_i}$ are totally ordered by divisibility. In fact, we have the following conditions for a fractional power series to be a quasi-ordinary branch.

[3] they correspond in the case of plane branches to the classical Puiseux characteristic exponents in general coordinates

**Lemma 2.4** *([L3], prop. 1.5, [Gau], prop 1.3) Let* $\zeta = \sum c_\lambda X^\lambda$ *be a non unit in* $\mathbb{C}\{X^{1/n}\}$. *Then* $\zeta$ *is a quasi-ordinary branch if and only if there exists elements* $\lambda_i \in \frac{1}{n}\mathbb{Z}^d_{\geq 0}$ $(1 \leq i \leq g)$ *such that*

1. $\lambda_1 < \lambda_2 < \cdots < \lambda_g$, *and* $c_{\lambda_i} \neq 0$ *for* $1 \leq i \leq g$.
2. *If* $c_\lambda \neq 0$ *then* $\lambda$ *is in the subgroup of* $\mathbb{Q}^d$ *given by* $\mathbb{Z}^d + \sum_{\lambda_i \leq \lambda} \mathbb{Z}\lambda_i$.
3. $\lambda_j$ *is not in the subgroup of* $\mathbb{Q}^d$ *given by* $\mathbb{Z}^d + \sum_{\lambda_i < \lambda_j} \mathbb{Z}\lambda_i$, *for* $j = 1, \ldots, g$.

*If such elements exist, they are uniquely determined by* $\zeta$, *and they are the characteristic exponents of* $\zeta$.

We denote by $\mathbb{Z}^d$ by $Q_0$ and by $Q_j$ the group $\mathbb{Z}^d + \sum_{\lambda_i < \lambda_{j+1}} \mathbb{Z}\lambda_i$, for $j = 1, \ldots, g$ with the convention $\lambda_{g+1} = +\infty$.

**Definition 2.2** *We associate to the quasi-ordinary branch* $\zeta$ *the subgroups* $Q_0 \subset \cdots \subset Q_g$ *of* $\mathbb{Q}^d$ *and the sequences of integers* $n_0 = 1$, $n_j$ *the index of* $Q_{j-1}$ *in* $Q_j$ *for* $j = 1, \ldots, g$ *and* $e_{i-1} = n_i \cdots n_g$, *for* $i = 1, \ldots, g$.

**Lemma 2.5** *( [L4], lemma 5.7) We have the following equality:* $L[\zeta] = L[X^{\lambda_1}, \ldots, X^{\lambda_g}]$. ◇

It is easy to see that the integers $n_j$ and $e_i$ are field extension degrees:

$$\begin{aligned} e_i &:= [L[\zeta] : L[X^{\lambda_1}, \ldots, X^{\lambda_i}]], \text{ for } i = 1, \ldots, g. \\ n_j &:= [L[X^{\lambda_1}, \ldots, X^{\lambda_j}] : L[X^{\lambda_1}, \ldots, X^{\lambda_{j-1}}]], \text{ for } j = 1, \ldots, g. \end{aligned} \tag{2.3}$$

**Remark 2.6** *Lemma 2.4 give us a canonical way of writing the terms of a quasi-ordinary branch:*

$$\zeta = p_0 + p_1 + \cdots + p_g,$$

*where* $p_0$ *is in* $\mathbb{C}\{X\}$ *and* $p_i$ *belongs to* $\mathbb{C}\{X^{1/n}\}$, *for* $i = 1, \ldots, g$, *the monomial* $X^{\lambda_i}$ *appears in* $p_i$ *with non zero coefficient* $\alpha_i \in \mathbb{C}$, *and a monomial* $X^\lambda$ *appears in a term of* $p_j$ *implies that* $\lambda_j \leq \lambda$ *and* $\lambda \in Q_j - Q_{j-1}$.

By change of coordinates of the form $Y = Y' + p_0$ in $\mathbb{C}\{X, Y\}$, we get a quasi-ordinary branch of the form

$$\zeta = p_1 + \cdots + p_g. \tag{2.4}$$

We say then that we have a quasi-ordinary projection (resp. branch) with *good coordinates.*

We say that the quasi-ordinary branch $\zeta$ has *well ordered variables* if the $g$-tuples $(\lambda_{1,i}, \dots, \lambda_{g,i})$ of $i$-coordinates of the characteristic exponents $\lambda_1, \dots, \lambda_g$ are ordered lexicographically, more precisely: we have, for $1 \leq i < j \leq d$ that

$$(\lambda_{1,i}, \dots, \lambda_{g,i}) \geq_{\text{ lexicographically }} (\lambda_{1,j}, \dots, \lambda_{g,j}).$$

It is clear that given a quasi-ordinary branch $\zeta$ we can relabel the variables $X_1, \dots, X_d$ to satisfy this condition. We will suppose without lost of generality that the quasi-ordinary branch appearing in this text from now on have well ordered variables.

**Definition 2.3** *The quasi-ordinary branch $\zeta$, is normalized, if its given with well ordered variables and if it happens that the first characteristic exponent is of the form $\lambda_1 = (\lambda_{1,1}, 0, \dots, 0)$ then we have $\lambda_{1,1} > 1$.*

This condition in the case of curves, means that the projection $(X, Y) \to X$ is *transversal*, i.e., the multiplicity at the origin of the curve is equal to the degree of the projection. Lipman shows that any quasi-ordinary singularity is parametrized by a normalized quasi-ordinary branch (see section 2.5). Moreover, Lipman proved that the characteristic exponents of any normalized quasi-ordinary branch parametrizing a germ of surface, $(S, 0) \subset (\mathbb{C}^3, 0)$, are analytical invariants of the singularity, (see [L1], [L3], [Lu]), which means that they only depend on the germ $(S, 0)$.

**Definition 2.4** *Two hypersurfaces germs $(X, 0)$ and $(X', 0)$ in $\mathbb{C}^{d+1}$ have the same embedded topological type if and only if there is a homeomorphism $U \to U'$ between two open neighborhoods of the origin, which maps $X \cap U$ to $X' \cap U'$.*

The characteristic exponents of a quasi-ordinary branch $\zeta$ determine the embedded topological type of the hypersurface it defines. This can be deduced using results of Zariski on saturation of local rings (see [Z1], and [L3], §2 and also [Oh] for another proof). The converse has been proved by Gau.

**Theorem 2.3** *([Gau], Theorem 1.6) Two pairs of quasi-ordinary hypersurface germs $(X, 0)$ and $(X', 0)$ in $\mathbb{C}^{d+1}$ have the same topological type if and only if any two normalized quasi-ordinary branches parametrizing $(X, 0)$ and $(X', 0)$ have the same characteristic exponents.* ◇

The characteristic exponents of a quasi-ordinary branch determine the structure of the singular locus of the associated quasi-ordinary hypersurface. First, the singular locus is contained in the inverse image of the discriminant of the quasi ordinary projection, given by the equation $\Delta_Y f = 0$. This means that, if $Z_i$ is the intersection of $S$ with the coordinate hyperplane $X_i = 0$ we have that the singular locus of $S$ is contained in $\cup_{i\in I}^d Z_i$, where $I$ is the set of indexes of variables dividing the discriminant $\Delta_Y f$. In [L4] the following theorem is proved by computing the multiplicities of generic points of the subvarieties $Z_i$ and $Z_i \cap Z_j$ of $S$.

**Theorem 2.4** *([L4], Theorem 7.3) The subvariety $Z_i$ is not contained in the singular locus of $(S,0)$ if and only if one of the following conditions holds:*

1. *No characteristic monomial is divisible by $X_i^{1/n}$, i.e. $\lambda_{1,i} = \lambda_{2,i} = \cdots = \lambda_{g,i} = 0$.*
2. *$\lambda_{1,i} = \cdots = \lambda_{g-1,i} = 0$ and $n_g \lambda_{g,i} = 1$.*

*Moreover, the irreducible components of the singular locus are: the subvarieties $Z_k$ for $1 \leq k \leq d$ not satisfying any of the above conditions, and the intersections $Z_s \cap Z_t$ for the indexes $1 \leq s < t \leq d$ satisfying condition 2.* ◇

Let $f, g$ be two polynomials in $\mathbb{C}\{X\}[Y]$ of positive degrees. We say that $f$ and $g$ are *comparable* if the discriminant of $fg$ with respect to the variable $Y$ is of the form:

$$\Delta_Y(fg) = X^{\lambda}\epsilon, \text{ with } \epsilon \text{ a unit in } \mathbb{C}\{X\}.$$

Obviously, this condition implies that both $f$ and $g$ are quasi-ordinary polynomials. Geometrically, $fg = 0$ defines a germ of *reduced* quasi-ordinary hypersurface at the origin. It is equivalent to say that if $\zeta^{(i)}$ and $\tau^{(k)}$ are any roots of $fg$ then their difference is of the form $X^{\alpha_{ik}}\epsilon_{ik}$, where $\epsilon_{ik}$ is a unit in $\mathbb{C}\{X^{1/n}\}$. As in Lemma 5.6 of [L4] we deduce that the set of monomials $\{X^{\alpha_{ik}}\}_{i=1,\ldots,\deg f}^{k=1,\ldots,\deg g}$ is completely ordered by divisibility.

**Definition 2.5** *In this situation, we call order of coincidence of the polynomials $f$ and $g$ the largest exponent of the set $\{\alpha_{ik}\}_{i=1,\ldots,\deg f}^{k=1,\ldots,\deg g}$ (for $\leq$ coordinate-wise).*

We end up this section with some lemmas on quasi-ordinary branches that will be used later on. The first one tell us that some changes of coordinates are well behaved with respect to quasi-ordinary projections.

**Lemma 2.7** *Let* $H_1, \ldots, H_d, h \in \mathbb{C}\{X_1, \ldots, X_d, Y\}$, *be such that the series* $H_i$ *are units and* $h(0,Y)$ *has order one. Define the change of coordinates:* $X_i = U_i H_i(U_1, \ldots, U_d, W)$ *for* $i = 1, \ldots, d$ *and* $Y = h(U_1, \ldots, U_d, W)$. *The projection of germs* $(\mathbb{C}^{d+1}, 0) \supset (S, 0) \to (\mathbb{C}^d, 0)$, *defined by* $(U, W) \mapsto U$, *is quasi-ordinary, and the hypersurface germ* $(S,0)$ *is parametrized by a quasi-ordinary branch* $\tau \in \mathbb{C}\{U^{1/n}\}$ *with the same characteristic exponents.*

*Proof.* Let us denote by $H_k^{1/n}$ a fixed $n^{th}$ root of $H_k$ in $\mathbb{C}\{U, W\}$, for $k = 1, \ldots, d$. The image $g(U_1^{1/n}, \ldots, U_d^{1/n}, W)$ of $Y - \zeta(X^{1/n})$ by the homomorphism $\phi : \mathbb{C}\{X_1^{1/n}, \ldots, X_d^{1/n}, Y\} \to \mathbb{C}\{U_1^{1/n}, \ldots, U_d^{1/n}, W\}$ given by

$$\begin{array}{rcll} X_i^{1/n} & = & U_i^{1/n} H_i^{1/n} & \text{for } i = 1, \ldots, d. \\ Y & = & h(U_1, \ldots, U_d, W) & \end{array}$$

verifies that $g(0, W)$ has order one. By the Weierstrass Preparation Theorem (see [K]) there exists a unit $E$ in $\mathbb{C}\{U^{1/n}, W\}$ such that $Eg = W - \tau(U^{1/n})$.

We denote by $q^{(i)}$ the conjugates of a fractional power series $q$ under the action of the $d$-tuples of $n^{th}$-roots of unity on its exponents. The difference of two conjugates of $\zeta$ is of the form $\zeta^{(j)} - \zeta^{(i)} = X^{\lambda_{ji}}(H^{(j)} - H^{(i)})(X^{1/n})$ where $H^{(j)}$ and $H^{(i)}$ are units with different constant terms. We obtain then that the corresponding difference of conjugates of $\tau(U^{1/n})$ is

$$\tau^{(i)} - \tau^{(j)} = E^{(j)} g^{(j)} - E^{(i)} g^{(i)} = (E^{(j)} - E^{(i)}) q_{ji} + U^{\lambda_{ji}} (E^{(j)} \phi(H^{(j)}) - E^{(i)} \phi(H^{(i)})) \tag{2.5}$$

where $q_{ji}(U^{1/n}, W)$ is the image of the terms which the conjugates $Y - \zeta^{(i)}$ and $Y - \zeta^{(j)}$ of $Y - \zeta$ have in common.

Since the left side of formula (2.5) does not depend on $W$, we can substitute $W$ by the root $p \in \mathbb{C}\{U^{1/n}\}$ of the Weierstrass polynomial associated to $q_{ji}(U^{1/n}, W)$ to deduce that $\tau^{(i)} - \tau^{(j)}$ is of the form

$$\tau^{(i)} - \tau^{(j)} = U^{\lambda_{ji}} \epsilon_{ij}(U^{1/n}) \text{ with } \epsilon_{ij}(0) \neq 0.$$

◇

The next lemma is a version of lemma 2, in [A2], in the quasi-ordinary case.

**Lemma 2.8** *Let* $\zeta$ *be a quasi-ordinary branch of the form* $\zeta = X^{\lambda_0} H(X^{1/n})$, *with* $\lambda_0$ *an integral vector and* $H(0) \neq 0$. *For any positive integer* $k$, *the series* $\zeta^k$ *is a quasi-ordinary branch and its characteristic exponents are* $\lambda'_j = \lambda_j + (k-1)\lambda_0$, *for* $j = 1, \ldots, g$ *where* $\{\lambda_j\}_{j=1}^g$ *are the characteristic exponents of* $\zeta$.

*Proof.* Firstly, we have that $\lambda'_1 < \cdots < \lambda'_g$. The subgroups, $Q_i$ of $\mathbb{Q}^d$ associated to the $\{\lambda_j\}_{j=1}^g$ coincide with the subgroups associated to $\{\lambda'_j\}_{j=1}^g$. We show that $\lambda'_j$ are exponents of $\zeta^k$. We expand $\zeta = p_0 + p_1 + \cdots + p_g$ using remark 2.6. Each summand

$$\begin{pmatrix} k \\ s \end{pmatrix} (p_0 + \cdots + p_{j-1})^{k-s}(p_j + \cdots + p_g)^s$$

appearing in the binomial expansion of $((p_0 + \cdots + p_{j-1}) + (p_j + \cdots + p_g))^k$ is of the form $X^{(k-s)\lambda_0 + s\lambda_j}$. unit , for $s = 0, \ldots, k$, because $\lambda_0 < \lambda_1$ by the hypothesis.

We have $(k-s)\lambda_0 + s\lambda_j \geq (k-1)\lambda_0 + \lambda_j = \lambda'_j$, for $s = 1, \ldots, k$, hence $\zeta^k$ is of the form:

$$\zeta^k = (p_0 + \cdots + p_{j-1})^k + X^{\lambda'_j}. \text{ unit } .$$

The terms appearing in $(p_0+\cdots+p_{j-1})^k$ belongs to the group $Q_{j-1}$ by lemma 2.4. Therefore the monomial $X^{\lambda'_j}$ appears in $\zeta^k$ with non zero coefficient. It also follows from the formula above, that if we have $\lambda_{j-1} \leq \lambda < \lambda_j$, for $\lambda$ the exponent of a term appearing in $\zeta^k$, then this term appears in $(p_0 + \cdots + p_{j-1})^k$, hence $\lambda$ is in the group $Q_{j-1}$. The conditions in lemma 2.4 are satisfied for $\zeta^k$ and $\{\lambda'_j\}_{j=1}^g$. ◇

The following lemma that relates quasi-ordinary singularities and toric morphisms.

Let $\sigma = \langle a^1, \ldots, a^d \rangle$ be a regular cone contained in $\mathbb{R}^d_{\geq 0}$. It defines the homomorphism $\mathbb{C}\{X_1, \ldots, X_d\} \to \mathbb{C}\{U_1, \ldots, U_d\}$ given by formula (2.1) replacing in it $d+1$ by $d$. Denote by ${}^\sigma b$ the image of a series $b \in \mathbb{C}\{X\}$. If $b = X^u$ is a monomial, then ${}^\sigma b$ is a monomial $U^{u_\sigma}$ and its exponent $u_\sigma = (\langle a^1, v\rangle, \ldots, \langle a^d, v\rangle)$ gives the coordinates of the vector $u$ with respect to the dual basis of $\{a^1, \ldots, a^d\}$. If $h = \sum b_i Y^i$ is a polynomial in $\mathbb{C}\{X\}[Y]$, we denote by ${}^\sigma h \in \mathbb{C}\{U\}[Y]$, the polynomial $\sum {}^\sigma b_i Y^i$.

**Lemma 2.9** *Let $h \in \mathbb{C}\{X\}[Y]$ be an irreducible quasi-ordinary polynomial with characteristic exponents $\{\mu_i\}_{i=1}^s$ then ${}^\sigma h \in \mathbb{C}\{U\}[Y]$ is an irreducible quasi-ordinary polynomial with characteristic exponents $\{(\mu_i)_\sigma\}_{i=1}^s$.*

*Proof.* It is easy to see that ${}^\sigma(\Delta_Y h) = \Delta_Y {}^\sigma h$, so that the polynomial ${}^\sigma h$ is a quasi-ordinary polynomial if $h$ is. If $m = \deg h$, we can extend the homomorphism (2.1) to an homomorphism $\mathbb{C}\{X^{1/m}\} \to \mathbb{C}\{U^{1/m}\}$, dividing by $m$ in all the exponents appearing in formula (2.1). Since $h$ splits over $\mathbb{C}\{X^{1/m}\}$ the roots of ${}^\sigma h$ are the images of the roots of $h$. The characteristic monomials corresponding to an irreducible quasi-ordinary polynomial are the initial forms of the differences of its roots. This implies that the characteristic exponents of any root of ${}^\sigma h$ are of the desired form. ◇

**Remark 2.10** *If* $h \in \mathbb{C}\{X\}[Y]$ *is an irreducible polynomial then* ${}^{\sigma}h \in \mathbb{C}\{U\}[Y]$ *may be reduced as the following example given to me by Patrick Popescu-Pampu shows:*

The polynomial $h = Y^2 + X_1^2 + X_2^3$ defines an irreducible germ since it is of degree two and its discriminant is not an square. If we take the cone $\sigma = \langle (3,2), (1,1) \rangle$ then ${}^{\sigma}h = Y^2 + U_1^6 U_2^2 (1 + U_2)$ is not irreducible over $\mathbb{C}\{U_1, U_2\}$. However its irreducible as a polynomial in $\mathbb{C}[U_1, U_2][Y]$ since the homomorphism defined by (3.1) induces an isomorphism of the fields of fractions, the polynomial $h$ being irreducible over the field of fractions of $\mathbb{C}[X]$ since it is monic (see Theorem 5, §3, Chapter V of [Z-S]). The argument fails when we pass from rational functions to meromorphic functions since $\mathbb{C}\{X\}$ and $\mathbb{C}\{U\}$ do not have the same field of fractions, for instance if we set $X_1 = U_1 U_2, X_2 = U_1$ then $\exp(U_2) \notin \mathbb{C}\{X_1, X_2\}$.

## 2.4 The semigroup associated to a quasi-ordinary hypersurface singularity

### 2.4.1 The definition of the semigroup

We associate a semigroup to a quasi-ordinary branch from the characteristic exponents using the same formulae that appear in the case of plane branches (see [Z2]). We show that the properties characterizing the semigroups of plane branches hold in this case.

From now on, we fix a quasi-ordinary branch $\zeta \in \mathbb{C}\{X^{1/n}\}$, with $g \geq 1$ characteristic exponents $\{\lambda_i\}_{i=1}^{g}$. We will suppose that the variables are well ordered and that $n = \deg f$, where $f \in \mathbb{C}\{X\}[Y]$ is the minimal polynomial of $\zeta$ over $L$ (see Remark 2.2).

Let us define the sequence of vectors in $\mathbb{Z}^d_{\geq 0}$ (following some ideas of Zariski in [Z2] for the one dimensional case).

$$\begin{cases} \gamma_i = (0, \dots, 0, \overset{(i)}{n}, 0 \dots, 0) \text{ for } i = 1, \dots, n \\ \gamma_{d+1} = n\lambda_1, \\ \gamma_{d+j+1} = n_j \gamma_{d+j} + n\lambda_{j+1} - n\lambda_j, \quad \text{for } j = 1, \dots, g-1. \end{cases} \tag{2.6}$$

For $j = 0, \dots, g-1$, we can expand:

$$\begin{aligned} \gamma_{d+j+1} &= n\left((n_1 - 1)n_2 \cdots n_j \lambda_1 + (n_2 - 1)n_3 \cdots n_j \lambda_2 + \cdots + (n_j - 1)\lambda_j + \lambda_{j+1}\right) \\ &\overset{\text{Def. 2.2}}{=} n_1 \dots n_j \left((e_0 - e_1)\lambda_1 + (e_1 - e_2)\lambda_2 + \cdots + (e_{j-1} - e_j)\lambda_j + e_j \lambda_{j+1}\right) \end{aligned} \tag{2.7}$$

If $a_1, \ldots, a_s \in \mathbb{R}^d$, the *semigroup* (resp. the *cone*) $\langle \alpha_1, \ldots, \alpha_s \rangle$ they span is the set of sums $\sum a_i \alpha_i$ for $a_i$ running through $\mathbb{Z}_{\geq 0}$ (resp. $\mathbb{R}_{\geq 0}$). We use the same notations for both concepts. We say then that $\alpha_1, \ldots, \alpha_s$ generate the semigroup; they are a minimal set of generators of the semigroup if it cannot be generated by the elements of any proper subset of $\{\alpha_1, \ldots, \alpha_s\}$.

**Definition 2.6** *The semigroup $\Gamma_\zeta$ associated to the quasi-ordinary branch $\zeta$ is the semigroup spanned by the integral vectors $\gamma_1, \ldots, \gamma_{d+g}$.*

The semigroup $\Gamma_\zeta$ is a sub-semigroup of $\mathbb{Z}^d_{\geq 0}$ that spans a subgroup of $\mathbb{Z}^d$ of finite index. We denote by $\Gamma_j$ the sub-semigroup $\langle \gamma_1, \ldots, \gamma_{d+j} \rangle$ of $\Gamma_\zeta$ for $j = 0, \ldots, g$. The following lemma generalizes results of Zariski and Azevedo for plane branches (see [T2], Chapitre I, Lemma 2.2.1).

**Lemma 2.11**

1. *The subgroup of $\mathbb{Z}^d$ generated by $\gamma_1, \ldots, \gamma_{d+j}$ is equal to $nQ_j$, $0 \leq j \leq g$.*
2. *The order of the image of $\gamma_{d+j}$ in the group $nQ_j/nQ_{j-1}$ is equal to $n_j$ for $j = 1, \ldots, g$.*
3. *We have that $\gamma_{d+j} \gneq n_{j-1}\gamma_{d+j-1}$ for $j = 2, \ldots, g$ (where $\gneq$ means $\neq$ and $\geq$ coordinate-wise).*
4. *If a vector $u_j \in nQ_j$ has non negative coordinates, then $u_j + n_j \gamma_{d+j} \in \Gamma_j$*
5. *The vector $n_j \gamma_{d+j}$ belongs to the semigroup $\Gamma_{j-1}$, for $j = 1, \ldots, g$.*

*Proof.* The first assertion follows from formulae (2.6). and the second assertion follows from the definitions.

We have that:

$$
\begin{aligned}
n_j \gamma_{d+j} - n_{j-1}\gamma_{d+j-1} &= n_{j-1}(n_j - 1)\gamma_{d+j-1} + n_j(n\lambda_j - n\lambda_{j-1}) \\
&\gneq (n_j - 1)(n_{j-1}\gamma_{d+j-1} + n\lambda_j - n\lambda_{j-1}) \\
&= (n_j - 1)\gamma_{d+j},
\end{aligned}
\tag{2.8}
$$

where $\geq$ means for all coordinates simultaneously. This proves *3*.

The assertion *4* is easy for $j = 1$. We suppose it is true for $j - 1 \geq 1$. We take a vector $u_j \in nQ_j$ with non negative coordinates. It is of the form $u_j = \alpha_j \gamma_{d+j} + u'_j$ for a unique $0 \leq \alpha_j < n_j$ and $u'_j \in nQ_{j-1}$. By (2.8), the vector $u_{j-1} := u'_j + n_j\gamma_{d+j} - n_{j-1}\gamma_{d+j-1} \in nQ_{j-1}$

has no negative coordinates. By induction hypothesis, the vector $u_{j-1}+n_{j-1}\gamma_{d+j-1} = u'_j+n_j\gamma_{d+j}$ is in the semigroup $\Gamma_{j-1}$, hence the vector $u_j + n_j\gamma_{d+j} = \alpha_j\gamma_{d+j} + u'_j + n_j\gamma_{d+j}$ belongs to the semigroup $\Gamma_j$.

To prove assertion *5*, we have by formula (2.6): $n_j\gamma_{d+j} = n_j n_{j-1}\gamma_{d+j-1} + n_j(n(\lambda_j - \lambda_{j-1}))$. By the assertions *1* and *2* and the definition, the vector $n_j(n(\lambda_j - \lambda_{j-1}))$ is in the semigroup $7nQ_{j-1}$, and by lemma 2.4 it has non negative coordinates. Now we apply *3*. ⋄

**Remark 2.12** *If the branch $\zeta$ is normalized, then the elements $\gamma_1, \dots, \gamma_{d+g}$, are the unique minimal set of generators for the semigroup $\Gamma_\zeta$.*

Since the semigroup $\Gamma_\zeta$ is contained in the strictly convex cone $\mathbb{R}^d_{\geq 0}$, we can apply lemmas 3.6 and 3.5, chapter V, of [Ew], to proof that $\Gamma_\zeta$ has a unique minimal set of generators. The condition of being normalized implies that the vectors $\gamma_1, \dots, \gamma_d$ are the first elements of the semigroup $\Gamma_\zeta \subset \mathbb{R}^d_{\geq 0}$ appearing on the positive axes of $\mathbb{R}^d$, hence we cannot eliminate any of them while preserving the semigroup. If we have a relation of the form $\gamma_{d+k} = \sum a_j\gamma_j$ with $a_j \in \mathbb{Z}_{\geq 0}$ and $a_{d+k} = 0$, then assertion *3* implies that $a_j = 0$, for $j > d+k$. The relation obtained contradicts assertion *2*.

Let $\Omega$ be a semigroup of rank $d$ admitting a unique minimal set of generators, $\omega_1, \dots, \omega_{d+g}$, this means that they expand a strictly convex cone in $\mathbb{R}^d = M_g \otimes_{\mathbb{Z}} \mathbb{R}$ (where $M_i = \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_{d+i}$, for $i = 1, \dots, g$). We can suppose that $\omega_1, \dots, \omega_d$ are first vectors of $\Omega$ lying on the edges of this cone.

**Proposition 2.13** *The semigroup $\Omega$ is isomorphic to the semigroup associated to a quasi-ordinary branch if we can relabel the generators $\omega_{d+1}, \dots, \omega_{d+g}$ in such a way that*

$$\omega_{d+j+1} \gneqq N_j\omega_{d+j} \text{ for } j = 1, \dots, g-1 \text{ where } N_j := \#M_j/M_{j-1}. \tag{2.9}$$

*Proof.* We embed $\Omega$ in $\mathbb{Z}^d$ by $\omega_i \mapsto Ne_i$, where $e_i$ is the $i^{th}$- canonical basis vector and $N = N_1 \dots N_g$. Then we replace in formulae 2.6, the integers $n$ by $N$ and $n_i$ by $N_i$ and the vectors $\gamma_j$ by $\omega_j$ to define vectors $\lambda_1, \dots, \lambda_g \in \frac{1}{N}\mathbb{Z}_{\geq 0}$ such that $\lambda_1 \lneqq \lambda_2 \lneqq \cdots \lneqq \lambda_g$ by (2.9). The fractional polynomial $\tau := X^{\lambda_1} + \cdots + X^{\lambda_g}$ defines a quasi-ordinary branch by lemma 2.4, and its associated semigroup is equal to the embedding of $\Omega$ in $\mathbb{Z}^d$ by construction. ⋄

### 2.4.2 Semi-roots

We introduce in this section the notion of *semi-root* , which will be very important in what follows, from the algebraic and geometric viewpoints. We follow the terminology of [PP] in the case of plane branches. This notion generalizes the curves of maximal contact with a given plane branch, introduced by Lejeune (see [LJ]). They will play an important role in the toric embedded resolution procedures given in the last section.

We use the parametrization defined by $\zeta$ (see (2.2), to study the class

$$\bar{q} = q(T_1^n, \ldots, T_d^n; z(T_1, \ldots, T_d)) \in \mathbb{C}\{T\}$$

of $q \in \mathbb{C}\{X\}[Y]$ in the analytic algebra $R \subset \mathbb{C}\{T\}$ of $S$.

Since the roots $\{\zeta^{(k)}\}_{k=1}^n$ of $f$ are a complete set of conjugates over the field of fractions of the ring $\mathbb{C}\{X\}$, and the polynomial $q$ has its coefficients on this ring, the series $\{q(\zeta^{(k)})\}_{k=1}^n$ are a complete set of conjugates over the ring $\mathbb{C}\{X\}$. This implies that:

$$\mathcal{N}(\bar{q}) = \mathcal{N}(\prod_{k=1}^{n} q(\zeta^{(k)})) = \deg f \mathcal{N}(q(\zeta)) = \mathcal{N}(\text{ Res}_Y(f, q)) \tag{2.10}$$

where $\text{Res}_Y(f, q)$ is the resultant of the polynomials $f$ and $q$ with respect to $Y$.

**Definition 2.7** *We say that a series $q \in \mathbb{C}\{X, Y\}$ is a $j^{th}$-semi-root of $f$ ($0 \leq j \leq g-1$) if the following two conditions are verified:*

*(a)* $q(0, Y) = Y^{\frac{n}{e_j}} = Y^{n_0 n_1 \ldots n_j}$

*(b)* $q(T_1^n, \ldots, T_d^n; z(T_1, \ldots, T_d)) = T^{\bar{\lambda}_{d+j+1}} \varepsilon_j$, *for some unit* $\varepsilon_j$.

For instance, if we have good coordinates, i.e., $\zeta$ is in the form 2.4, then $q := Y$ is a $0^{th}$-semi-root of $f$. We say that $f$ is equal to its $g^{th}$- semi-root.

**Proposition 2.14** *Let $q \in \mathbb{C}\{X\}[Y]$ be any monic polynomial of degree $n_0 n_1 \ldots n_j$, $0 \leq j \leq g-1$. The polynomial $q$ is a $j^{th}$-semi-root of $f$ if and only if it is is a quasi-ordinary polynomial having order of coincidence $\lambda_{j+1}$ with $f$.*

*Proof.* The result is trivial for $j = 0$. Thus, let us consider the case $j \geq 1$. We will suppose first that $q$ is irreducible and quasi-ordinary. Then condition (b) and (2.10) imply that $f$ and $q$ are comparable quasi-ordinary polynomials with a well defined order of coincidence $\alpha$ (see definition 2.5).

If $\tau$ is any root of the irreducible polynomial $g$ we deduce by symmetry from (2.10) that $\mathcal{N}(\bar{q}) = \deg q \mathcal{N}(f(\tau))$. Take a root $\zeta^{(k)}$ of $f$ such that $\zeta^{(k)} - \tau = X^{\alpha}$. unit. By definition of order of coincidence, the biggest characteristic exponent of $\tau$ that is $< \alpha$, if it exist, is a characteristic exponent $\lambda_i$ of $\zeta^{(k)}$. Any other root of $f$ verifying this property is obtained from $\zeta^{(k)}$ by the action of a $d$-tuple of $n^{th}$-roots of unity that fix $X^{\lambda_1}, \dots, X^{\lambda_i}$ and conversely, thus:

$$\#\{\text{Roots } \zeta^{(l)} \text{ of } f \text{ such that } \zeta^{(l)} - \tau = X^{\alpha}.\text{ unit}\} = [L[\zeta] : L[X^{\lambda_1}, \dots, X^{\lambda_i}]] = e_i$$

Analogously, we obtain that

$$\#\{\text{Roots } \zeta^{(l)} \text{ of } f \text{ such that } \zeta^{(l)} - \tau = X^{\lambda_k}.\text{ unit}\} = e_{k-1} - e_k, \text{ for } k = 1, \dots, i.$$

By (2.10), this implies that the vector $\gamma_{d+j+1}$ is equal to:

$$\gamma_{d+j+1} = n_1 \dots n_j((e_0 - e_1)\lambda_1 + (e_1 - e_2)\lambda_2 + \dots + (e_{i-1} - e_i)\lambda_i + e_i\alpha) \qquad (2.11)$$

If $1 \leq i < j$ or if $\alpha$ is $\leq$ than any characteristic exponent of $\tau$, we deduce from formulae (2.7) and (2.11) that $\alpha \geq \lambda_{j+1}$. This implies that $\lambda_j$ is $< \alpha$ and it is also a characteristic exponent of $\tau$, a contradiction. If $j < i$, we obtain that the value of $\gamma_{d+j+1}$ in (2.11) is $>$ that value of $\gamma_{d+j+1}$ in (2.7). Therefore, we have $j = i$ and this implies that $\alpha = \lambda_{j+1}$. The converse is clear.

We prove now that any $j$-semi-root $q$ of $f$ is quasi-ordinary. We will prove latter, that any $j$-semi-root of $f$ is necessarily irreducible (see remark 2.19). We follow the notations in lemma 2.9. Let $\sigma = \langle a^1, \dots, a^d \rangle$ be a regular cone contained in $\mathbb{R}^d_{\geq 0}$, compatible with the Newton polyhedron of the discriminant $\Delta_Y(q)$. This means that there exist a unique vertex $w$ of $\mathcal{N}(\Delta_Y(q))$ such that

$$\langle a^i, w \rangle = \inf_{v \in \mathcal{N}(\Delta_Y(q))} \langle a^i, v \rangle.$$

The discriminant ${}^{\sigma}(\Delta_Y(g))$ is of the form $U^{w_\sigma} \cdot$ unit, hence the polynomial ${}^{\sigma}g$ is quasi-ordinary by lemma 2.9 and it is irreducible by remark 2.19. By lemma 2.9 and the previous case, the polynomials ${}^{\sigma}f$ and ${}^{\sigma}q$ have order of coincidence $(\lambda_{j+1})_\sigma$, and the characteristic monomials of a root of ${}^{\sigma}q$ are $(\lambda_1)_\sigma, \dots, (\lambda_j)_\sigma$. We find then that the exponent $w$ does not depend on the cone $\sigma$ since it is determined by the characteristic exponents $\lambda_1, \dots, \lambda_j$ of $f$. This implies that there is only one vertex $w$ in the Newton polyhedron of $\Delta_Y(q)$, therefore the polynomial $q$ is quasi-ordinary with order of coincidence $\lambda_{j+1}$ with $f$. ◇

**Remark 2.15** *There exists $j$-semi-roots for $0 \leq j \leq g-1$.*

By remark 2.6 and lemma 2.4 the power series $\zeta_i = p_0 + \dots + p_i$, obtained from $\zeta$ by truncation is a quasi-ordinary branch having a minimal polynomial with order of coincidence $\lambda_{j+1}$ with $f$, for $j = 0, \dots, g-1$. They are semi-roots by proposition 2.14.

### 2.4.3 The graded ring associated to a quasi-ordinary singularity

In this section we compute the graded ring associated to the induced $(T)$-adic filtration of the quasi-ordinary ring $R \subset \mathbb{C}\{T\}$ and we show that its spectrum is a toric variety of dimension $d$.

Recall that a *filtration* of a ring $B$ is a sequence of ideals

$$B = \mathfrak{F}_0 \supset \mathfrak{F}_1 \supset \cdots \supset \mathfrak{F}_j \supset \mathfrak{F}_{j+1} \supset \ldots,$$

such that $\mathfrak{F}_j \mathfrak{F}_r \subset \mathfrak{F}_{j+r}$, for $r, j$ in the semigroup $\mathbb{Z}_{\geq 0}$. The filtration $\{\mathfrak{F}_j\}$ defines a fundamental system of neighborhoods of zero, that gives the structure of topological ring to the ring $B$. The graded ring associated to the filtration is $\mathrm{gr}B := \bigoplus_{j \in \mathbb{Z}_{\geq 0}} \mathfrak{F}_j / \mathfrak{F}_{j+1}$. (See [Bbk] for more details).

We follow some ideas of Zariski and Teissier in [Z2]. We fix $q_0, \ldots, q_{g-1}$ a complete set of semi-roots of $f$, for $i = 0, \ldots, g-1$ (they exist by remark 2.15). Consider the following set of generators of the maximal ideal of the ring $R$

$$\left\{ \begin{array}{l} \xi_1 = T^{\gamma_1}, \ldots, \xi_d = T^{\gamma_d}, \\ \xi_{d+j+1} = q_j(\xi_1, \ldots, \xi_d; z(T_1, \ldots, T_d)) \text{ for } j = 0, \ldots, g-1, \end{array} \right. \tag{2.12}$$

Notice that the elements, $\xi_1, \ldots, \xi_{d+1}$, already define the maximal ideal. Geometrically, the elements $\xi_1, \ldots, \xi_{d+g}$ define an embedding $S \subset \mathbb{C}^{d+g}$, given by $U_i = \xi_i$, for $i = 1, \ldots d+g$. We will suppose, without loss of generality, that the coefficient $c_{d+j+1} \in \mathbb{C}$ of the term $\mathrm{in}(\xi_{d+j+1}) = c_{d+j+1} T^{\gamma_{d+j+1}}$ is equal to one, for $j = 0, \ldots, g-1$.

For any element $h$ in the analytic algebra $R = \mathbb{C}\{X_1, \ldots, X_d\}[Y]/(f)$ of the germ $(S, 0)$ there exist a unique representative $H \in \mathbb{C}\{X_1, \ldots, X_d\}[Y]$ with degree $< n = \deg f$ (by Euclidean division). We have $h = H(\xi_1, \ldots, \xi_d; \xi'_{d+1})$ where $\xi'_{d+1} = z(T_1, \ldots, T_d)$.

Recall that we denote by $\Gamma_j$ the semigroup $\langle \gamma_1, \ldots, \gamma_{d+j} \rangle$ for $j = 0, \ldots, g-1$.

**Proposition 2.16** *(See [Z2], Chapitre II, Th. 3.9, for the one dimensional case).*

1. *If* $\deg H = 0$ *then the initial polynomial* $\mathit{in}(h)$ *belongs to* $\mathbb{C}[\Gamma_0]$.

2. *If* $\deg H < n_1 \ldots n_j$ *then the polynomial* $\mathit{in}(h)$ *belongs to* $\mathbb{C}[\Gamma_j]$, *for* $j = 1, \ldots, g$.

*Proof.* We show the result by induction. If $\deg H = 0$, it is clear that the polynomial $\mathrm{in}(h)$ belongs to $\mathbb{C}[\Gamma_0]$, since any term appearing in $h$ has an exponent in the semigroup $\Gamma_0$. We suppose true for degrees $< n_1 \ldots n_{j-1}$. If $\deg H < n_1 \ldots n_j$, using Euclidean division several times we can expand the polynomial $H$ in a unique way in the form:

$$H = A_0 + A_1 q_{j-1} + \cdots + A_s q_{j-1}^s,$$

where $A_k$ are polynomials in $\mathbb{C}\{X_1,\dots,X_d\}[Y]$ of degree $< n_1\dots n_{j-1} = \deg q_{j-1}$, for $k = 0,\dots,s$ and we have $0 \leq s < n_j$. By definition, (see formula (2.12)), the element of $R$ defined by the polynomial $q_{j-1}$ is $\xi_{d+j}$ and its initial polynomial is the monomial $T^{\gamma_{d+j}}$ since $q_{j-1}$ is a $(j-1)^{th}$-semi-root. Then we have:

$$h = A_0(\xi_1,\dots,\xi_d;\xi'_{d+1}) + A_1(\xi_1,\dots,\xi_d;\xi'_{d+1})\xi_{d+j} + \cdots + A_s(\xi_1,\dots,\xi_d;\xi'_{d+1})\xi^s_{d+j}.$$

By the induction hypothesis, the polynomial $a_k := \operatorname{in}(A_k(\xi_1,\dots,\xi_d;\xi'_{d+1}))$ belongs to the ring $\mathbb{C}[\Gamma_{j-1}]$. Moreover, the terms in the polynomials $a_k T^{k\gamma_{d+j}}$ and $a_l T^{l\gamma_{d+j}}$ cannot cancel each other for $k \neq l$. In fact, their exponents are respectively in the sets $k\gamma_{d+j} + nQ_{j-1}$ and $l\gamma_{d+j} + nQ_{j-1}$ by proposition 2.14; if $k \neq l$, these are disjoint sets since the order of $\gamma_{d+j} + nQ_{j-1}$ is equal to $n_j$, by lemma 2.11, and we have $n_j > |k-l|$.

Hence the polynomial $\operatorname{in}(h)$ is a sum of terms of the polynomials $a_k T^{k\gamma_{d+j}}$, for $k = 0,\dots,s$. and the result follows. ◇

Kouchnirenko (see [Ko]), Lejeune (see [LJ]) and Teissier (see [T2], [G-T], and [T3]) use graded rings with respect to general filtrations to study singularities. Lejeune shows the following corollary in the one dimensional case (see [T2], Chapitre I, 1.2.3).

**Corollary 2.17** *The ring $gr_{(T)}R$ associated to the induced $(T)$-adic filtration of the ring $R$ is equal to the semigroup algebra $\mathbb{C}[\Gamma_\zeta] \subset \mathbb{C}[T]$.*

*Proof.* The parametrization $R \to \mathbb{C}\{T\}$ is a homomorphism compatible with the $(T)$-adic filtration of the ring $\mathbb{C}\{T\}$ if we give the induced filtration to $R$. It induces a homomorphism in the associated rings, $\operatorname{gr}_{(T)}R \to \operatorname{gr}_{(T)}\mathbb{C}\{T\}$, which is injective.

We have that $\operatorname{in}(\xi_{d+j}) = T^{\gamma_{d+j}}$. Since the ring $\operatorname{gr}_{(T)}R$ is the $\mathbb{C}$-algebra generated by the initial polynomials of non zero elements of $R$, we get that $\mathbb{C}[\Gamma_\zeta] \subset \operatorname{gr}_{(T)}R$. By proposition 2.16, the initial polynomials of non zero elements of $R$ are in $\mathbb{C}[\Gamma_\zeta]$. ◇

Any vector irrational $\omega \in \mathbb{R}^d_{>0}$ defines an valuation of the ring $\mathbb{C}\{T\}$ with values in a sub-semigroup of $\mathbb{R}$ which is isomorphic to $\mathbb{Z}_{\geq 0}$. The graded ring associated to the filtration defined by this valuation is isomorphic to $\mathbb{C}[T]$ with the grading $\mathbb{C}[T] = \oplus\mathbb{C}(T^u)$ where $u \leq_\omega u' \Leftrightarrow \langle u,\omega\rangle < \langle u,\omega\rangle$. This defines a total ordering since the vector $\omega$ has coordinates linearly independent over $\mathbb{Q}$ (see [GP], §1.3.1)

**Remark 2.18** *The proposition 2.16 and the Corollary 2.17 holds replacing the $(T)$-adic filtration and its associated graded ring by the filtration and graded ring associated to the valuation defined by an irrational vector $\omega$ in $\mathbb{R}^d_{>0}$ The graduation of $\mathbb{C}[\Gamma_\zeta]$ determines the semigroup $\Gamma_\zeta$.*

**Remark 2.19** *If $q \in \mathbb{C}\{X\}[Y]$ is a $j$-semi-root of $f$, then $q$ is an irreducible polynomial, for* $j = 0, 1, \dots, g-1$.

This is trivial for $j = 0$. If $j > 0$ and the polynomial $g$ can be factored as the product of two polynomials $G$ and $G'$ of degrees $< n_1 \dots n_j$. The polynomials $G$ and $G'$ define the elements $g = G(\xi_1, \dots, \xi_d; \xi'_{d+1})$ and $g = G'(\xi_1, \dots, \xi_d; \xi'_{d+1})$ in the ring $\mathbb{C}\{T\}$ and we have that $\text{in}(G) + \text{in}(G')$ is a term with exponent $\gamma_{d+j+1}$. By proposition 2.16, the polynomials $\text{in}(G)$ and $\text{in}(G')$ belong to $\mathbb{C}[\Gamma_{j-1}]$, hence this implies that $\gamma_{d+j+1} \in \Gamma_{j-1}$, which contradicts lemma 2.11.

### 2.4.4 Approximate roots

We show that Abhyankar's approximate roots of quasi-ordinary polynomial $f$ of appropriate degrees are semi-roots, and therefore are irreducible quasi-ordinary polynomials with a determined order of coincidence with the polynomial $f$. We follow the approach of [PP].

Let $A$ be a $\mathbb{Q}$-algebra. *Approximate roots* are defined by Abhyankar and Moh, (see [A-M], and [PP] for a survey on the subject). If $p$ is any monic polynomial and $k$ divides the degree of $p$, there is a unique monic polynomial $r$ in $A[Y]$ of degree $\frac{\deg p}{k}$, such that $\deg(p - r^k) < \deg p - \frac{\deg p}{k}$. We will denote this polynomial $\sqrt[k]{p}$. If $k = k_1 k_2$ divides $\deg p$, then we have that $\sqrt[k]{p} = \sqrt[k_1]{\sqrt[k_2]{p}}$ (see proposition 3.3 in [PP]) For instance if $p = Y^n - a_1 Y^{n-1} + \dots + a_0$, we have that $\sqrt[n]{p} = Y - \frac{a_1}{n}$. If $q \in A[Y]$ is any monic polynomial of degree $\frac{\deg p}{k}$ we can expand using Euclidean division several times,

$$p = q^k + a_1 q^{k-1} + \dots + a_0,$$

where the polynomials $a_i \in A[Y]$ are of degree $< \frac{\deg p}{k}$. The map $\mathcal{T}_p$ between the set of monic polynomials of degree $\frac{\deg p}{k}$ defined by

$$\mathcal{T}_p(q) = q + \frac{1}{k} a_1,$$

is called the $k$*-Tschirnhausen operator* . It is shown that the $k$-approximate roots can be computed by iterating the $k$-Tschirnhausen operator, i.e,

$$\sqrt[k]{p} = \overbrace{\mathcal{T}_p \circ \dots \mathcal{T}_p}^{\deg p / k}(q),$$

where $q$ is any monic polynomial of degree $\frac{\deg p}{k}$ (see Proposition 6.3 in [PP]).

We are now interested in approximate roots the irreducible quasi-ordinary polynomial $f$.

**Proposition 2.20** *The $e_j$-approximate roots of $f$ are irreducible quasi-ordinary polynomials having order of coincidence $\lambda_{j+1}$ with $f$, for $j = 0, \dots, g$.*

*Proof.* This is trivial for $j = 0$ and it is also trivial for $j = g$, since we have that $\sqrt[1]{f} = f$ satisfies the condition (for $\lambda_{g+1} = \infty$). Suppose the result is true for $1 < j < g$. We show that it is true for $j-1$. We have that $e_{j-1} = n_j e_j$ and $\sqrt[e_{j-1}]{f} = \sqrt[n_j]{\sqrt[e_j]{f}}$. If we set $p = \sqrt[e_j]{f}$, we obtain that

$$\sqrt[e_{j-1}]{f} = \overbrace{\mathcal{T}_p \circ \dots \mathcal{T}_p}^{n_j}(q),$$

where $q$ is any monic polynomial of degree $n_1 \dots n_{j-1}$.

It is sufficient to prove that if $q$ is a $(j-1)$-semi-root, the polynomial $\mathcal{T}_p(q)$ is a $(j-1)$-semi-root. We expand, $p = q^{n_j} + a_1 q^{n_j - 1} + \dots + a_0$, with $a_i \in \mathbb{C}\{X\}[Y]$ polynomials of degree $\deg q$. By the induction hypothesis $p$ is a $j$-semi-root of $f$, This implies (with notations as in the proof of proposition 2.16) that

$$\mathcal{N}(a_1 q^{n_j - 1}(\xi_1, \dots, \xi_d; \xi'_{d+1})) \subset \mathcal{N}(\xi_{d+j+1}).$$

It means that if $u$ is the exponent of any term appearing in the initial polynomial of $a_1(\xi_1, \dots, \xi_d; \xi'_{d+1})$ then $u \geq \gamma_{d+j+1} - (n_j - 1)\gamma_{d+j}$ is $\gneq \gamma_{d+j}$ by lemma 2.11. It follows that $q + \frac{1}{n_j} a_1$ is a $(j-1)$-semi-root of $f$ by proposition 2.14. ◇

## 2.5 Invariance of the semigroup and the inversion formulae

We prove in this section the topological invariance of the semigroup $\Gamma_\zeta$. First, we need the proof of the inversion lemma for quasi-ordinary singularities. This result, allows us to parametrize any quasi-ordinary hypersurface with a normalized quasi-ordinary branch.

### 2.5.1 The Inversion Lemma

**Lemma 2.21** *(see [L1], lemma 2.3, or [Gau], Appendix) Suppose we have a quasi-ordinary projection $(X_1, \dots, X_d, Y) \to (X_1, \dots, X_d)$ from $(\mathbb{C}^{d+1}, 0) \supset (S, 0) \to (\mathbb{C}^d, 0)$ parametrized by a quasi-ordinary branch of the form:*

$$\zeta = X_1^{k/r_1} H(X_1^{1/r_1}, \dots, X_d^{1/r_d}) \text{ with } H(0) \neq 0.$$

*Then, the projection* $(X_1, \ldots, X_d, Y) \rightarrow (X_2, \ldots, X_d, Y)$ *is quasi-ordinary and is parametrized by a quasi-ordinary branch of the form:*

$$\tau = Y^{r_1/k} H'(Y^{1/k}, X_2^{1/r_2}, \ldots, X_d^{1/r_d}) \text{ with } H'(0) \neq 0.$$

*Proof.* If $n$ is the degree of the minimal polynomial $f \in \mathbb{C}\{X_1, \ldots, X_d\}[Y]$ of $\zeta$, then $f(X_1, 0)$ is equal to $X_1^{kn/r_1}\varepsilon_1$, where $kn/r_1$ is a positive integer and the series $\varepsilon_1 \in \mathbb{C}\{X_1\}$ is a unit. By the Weierstrass preparation theorem, there is a polynomial $g \in \mathbb{C}\{Y_1, X_2, \ldots, X_d\}[X_1]$ of degree $kn/r_1$ and a unit $\varepsilon_2 \in \mathbb{C}\{X_1, \ldots, X_d, Y\}$, such that $\varepsilon_2 f = g$. The polynomial $g$ is irreducible, since $g = 0$ defines the germ $(S, 0)$ which is analytically irreducible.

The quasi-ordinary branch $\zeta$ is of the form $\zeta = X_1^{k/r_1} F^k(X_1^{1/r_1}, \ldots, X_d^{1/r_d})$, for a series $F \in \mathbb{C}\{X_1, \ldots, X_d\}$ such that $F^k = H$. The series $W - X_1 F \in \mathbb{C}\{X_1, \ldots, X_d, W\}$ is of order one in $X_1$, hence by the Weierstrass preparation theorem, there exists a unit $\varepsilon_3 \in \mathbb{C}\{X_1, \ldots, X_d, W\}$ such that:

$$\varepsilon_3(W - X_1 F) = X_1 - WG,$$

with $G \in \mathbb{C}\{X_2, \ldots, X_d, W\}$ a unit such that $\varepsilon_3(0, X_2, \ldots, X_d, W) = -WG$.

We substitute $X_i$ by $X_i^{1/r_i}$, for $i = 1, \ldots, d$ and $W$ by $Y^{1/k}$, and we denote by $\mathcal{O}$ the ring $\mathbb{C}\{X_1^{1/r_1}, \ldots, X_d^{1/r_d}, Y^{1/k}\}$.

The series $Y^{1/k} - X_1^{1/r_1} F(X_1^{1/r_1}, \ldots, X_d^{1/r_d})$ and $X^{1/r_1} - Y^{1/k} G(Y^{1/k}, X_2^{1/r_2}, \ldots, X_d^{1/r_d})$ define the same ideal $\mathcal{I}$ in $\mathcal{O}$. Hence we have the isomorphism of $\mathbb{C}$-algebras:

$$\mathbb{C}\{X_1^{1/r_1}, \ldots, X_d^{1/r_d}\} = \mathcal{O}/\mathcal{I} \rightarrow \mathcal{O}/\mathcal{I} = \mathbb{C}\{Y^{1/k}, X_2^{1/r_2}, \ldots, X_d^{1/r_d}\} \tag{2.13}$$

The inverse image of the class of $Y$ under the isomorphism (2.13) is equal to $\zeta$, and the image of the class of $X_1$ is equal to $\tau := Y^{r_1/k} G^{r_1}(Y^{1/k}, X_2^{1/r_2}, \ldots, X_d^{1/r_d})$. Hence, the image by the isomorphism (2.13) of $g(X_1, \ldots, X_d, \zeta)$ is equal to $g(\tau, X_2, \ldots, X_d, Y)$, therefore the series $\tau$ is a root of $g$, since $g = \varepsilon_2 f$, and $\zeta$ is a root of $f$.

The inclusion $\mathbb{C}\{Y, X_2, \ldots, X_d\} \subset \mathbb{C}\{Y^{1/k}, X_2^{1/r_2}, \ldots, X_d^{1/r_d}\}$ induces a finite Galois extension of the respective fields of fractions. Since the polynomial $g$ is irreducible over $\mathbb{C}\{Y, X_2, \ldots, X_d\}$ and it have a root $\tau \in \mathbb{C}\{Y^{1/k}, X_2^{1/r_2}, \ldots, X_d^{1/r_d}\}$, it splits. Hence all the other roots are obtained by the action of the roots of unity on the variables. The discriminant of the polynomial $g$ with respect to $X_1$ is equal to $\Delta_{X_1} g = \prod \frac{\partial g}{\partial X_1}(\tau^{(s)})$, where $\tau^{(s)}$ runs the roots of $g$. Its suffices to show that $\frac{\partial g}{\partial X_1}(\tau)$ is of the form $Y_1^{a_1/k} X_2^{a_2/r_2} \ldots X_d^{a_d/r_d} \varepsilon_4$, for a unit $\varepsilon_4 \in \mathcal{O}/\mathcal{I}$ and non negative integers $a_1, \ldots, a_d$.

When we differentiate, $g(\tau, X_2, \ldots, X_d, Y) = 0$, with respect to the indeterminate $Y^{1/k}$ we get the formula:

$$\frac{\partial g}{\partial X_1}(\tau, X_2, \ldots, X_d, Y)\frac{\partial \tau}{\partial Y^{1/k}} + kY^{k-1/k}\frac{\partial g}{\partial Y}(\tau, X_2, \ldots, X_d, Y) = 0, \tag{2.14}$$

where $\frac{\partial \tau}{\partial Y^{1/k}}$ is of the form $Y^{(r_1-1)/k}\varepsilon_5$, for a unit $\varepsilon_5 \in \mathcal{O}/\mathcal{I}$.

Since $\frac{\partial g}{\partial Y} = f\frac{\partial \varepsilon_2}{\partial Y} + \varepsilon_2\frac{\partial f}{\partial Y}$ and the polynomial $f$ is quasi-ordinary, the inverse image of $\frac{\partial g}{\partial Y}(\tau, X_2, \ldots, X_d, Y)$ by the isomorphism (2.13) is of the form:

$$(f\frac{\partial \varepsilon_2}{\partial Y} + \varepsilon_2\frac{\partial f}{\partial Y})(X_1, \ldots, X_d, \zeta) = (\varepsilon_2\frac{\partial f}{\partial Y})(X_1, \ldots, X_d, \zeta) = X_1^{b_1/r_1}\ldots X_d^{b_d/r_d}\varepsilon_6,$$

for a unit $\varepsilon_6 \in \mathcal{O}/\mathcal{I}$. Hence, $g_Y(\tau, X_2, \ldots, X_d, Y)$ is of the form $Y_1^{b_1/k}X_2^{b_2/r_2}\ldots X_d^{b_d/r_d}\varepsilon_7$, for a unit $\varepsilon_7 \in \mathcal{O}/\mathcal{I}$ and non negative integers $b_1, \ldots, b_d$. By equation (2.14), we have

$$\frac{\partial g}{\partial X_1}(\tau, X_2, \ldots, X_d, Y) = Y^{(k-r_1)/k}\frac{\partial g}{\partial Y}(\tau, X_2, \ldots, X_d, Y)\varepsilon_8,$$

for a unit $\varepsilon_8 \in \mathcal{O}/\mathcal{I}$, and we are done. ◇

### 2.5.2 Invariance of the semigroup

We define the equivalence of parametrizations. The proof of the Inversion Lemma implies an equivalence of parametrizations that we use combined with Gau's results (see [Gau]) to prove that the semigroup $\Gamma_\zeta$ does not depend on the quasi-ordinary branch $\zeta$ parametrizing $S$.

Let $\zeta$ and $\tau$ be two quasi-ordinary branches parametrizing $S$. We say that the parametrizations they define $R \to \mathcal{O}$ and $R \to \mathcal{O}'$ are *equivalent* if there is an isomorphism $\mathcal{O} \to \mathcal{O}'$ such that the following diagram commutes:

$$\begin{array}{ccc} R & \to & \mathcal{O}' \\ \downarrow & \nearrow & \\ \mathcal{O} & & \end{array}$$

**Lemma 2.22** *The parametrizations defined by the quasi-ordinary branches $\zeta$ and $\tau$ in lemma 2.21 are equivalent.*

*Proof.* We notice that the ring $R$ is isomorphic to:

$$\mathbb{C}\{X_1, \ldots, X_d\}[Y]/(f) \cong \mathbb{C}\{X_1, \ldots, X_d, Y\}/(f) \cong \mathbb{C}\{X_1, \ldots, X_d, \zeta\},$$

and we obtain an analogous formula for $\tau$, and $g$ by exchanging $X_1$ and $Y$. The parametrization defined by $\zeta$ (resp. $\tau$) is $R \to \mathcal{O}/\mathcal{I}$ on the left (resp. right) side of the formula (2.13). The isomorphism (2.13) gives the equivalence since it maps $X_1 \mapsto \tau$, $X_i \mapsto X_i$, for $i = 2, \dots, d$ and $\zeta \mapsto Y$. ⋄

**Theorem 2.5** *The semigroup $\Gamma_\zeta$ does not depend on the choice of quasi-ordinary branch $\zeta$ parametrizing the germ $(S,0)$. This semigroup determines and is determined by the embedded topological type of the germ $(S,0)$.*

*Proof.* If $\zeta$ is a quasi-ordinary branch parametrizing $(S,0)$, and its first characteristic exponent is of the form $(\lambda'_{1,1}, 0, \dots, 0)$ with $\lambda'_{1,1} < 1$, we can use lemma 2.21 to obtain a quasi-ordinary branch $\tau$ parametrizing $(S,0)$, whose first characteristic exponent is of the form $(\lambda_{1,1}, 0, \dots, 0)$ with $\lambda_{1,1} > 1$. Otherwise, we set $\zeta = \tau$. In both cases $\tau$ is normalized. By the lemma 2.22, the parametrizations $R \to \mathcal{O}$ and $R \to \mathcal{O}'$ defined by $\tau$ and $\zeta$ respectively are equivalent thus we have $\mathbb{C}[\Gamma_\tau] = \mathrm{gr}_{(T)} R = \mathbb{C}[\Gamma_\zeta]$. It follows that $\Gamma_\tau \cong \Gamma_\zeta$ since the graded isomorphism $\mathrm{gr}_{(T)}\mathcal{O} \to \mathrm{gr}_{(T')}\mathcal{O}'$ is obtained by changing $T$ by $T'$.

Theorem 2.3 gives the equivalence between the data of the characteristic exponents of any normalized quasi-ordinary branch parametrizing $(S,0)$, and the embedded topological type of the germ $(S,0)$. Finally, we use remark 2.12 and formulae (2.6) to recover from the unique minimal set of generators of the semigroup $\Gamma_\zeta$ the characteristic exponents of $\tau$. ⋄

**Definition 2.8** *The semigroup $\Gamma$ associated to a quasi-ordinary hypersurface germ $(S,0)$ is the semigroup $\Gamma_\zeta$ associated to any quasi-ordinary branch $\zeta$ with well ordered variables, parametrizing $(S,0)$.*

### 2.5.3 The inversion formulae

As an application of the Inversion Lemma we give a proof of the corresponding Inversion Formulae. We show the dependence between the characteristic exponents of $\tau$ and $\zeta$ in lemma 2.21. This result is known (see [L1] page 78). The proof given here uses the main ideas of the proof of the inversion formulae for curves in [A2]. Another way to prove this result is to use the invariance of the semigroup and the formulae (2.6) as is done in [PP] in the case of plane branches.

With the same hypothesis and notations of lemma 2.21 we have:

**Lemma 2.23** *If $k = r_1$ then the characteristic exponents of $\tau$ and $\zeta$ coincide.*

*Proof.* Without loss of generality we can suppose that the variables are well ordered. The result follows from the converse in the proof of Theorem 2.5, and from remark 2.12, since the hypothesis means that the quasi-ordinary branches $\tau$ and $\zeta$ are both normalized. ◇

**Proposition 2.24** *Suppose that the rational number $\alpha = k/r_1$ is not an integer. Let $\{\lambda_i\}_{i=1}^g$ be the characteristic exponents of $\zeta$. Denote by $\lambda'_i \in \mathbb{Q}^d$ the vector with coordinates: $\lambda'_{i,1} = \alpha^{-1}\lambda_{i,1} + \alpha^{-1} - 1$ and $\lambda'_{i,q} = \lambda_{i,q}$ for $q = 2, \dots, d$ and $i = 1, \dots, g$. Then the characteristic exponents of $\tau$ are*

$$\begin{cases} \lambda'_1, \dots, \lambda'_g \text{ if } \alpha, \alpha^{-1} \notin \mathbb{Z} \\ \lambda'_2, \dots, \lambda'_g \text{ if } \alpha^{-1} \in \mathbb{Z} \end{cases}$$

*Proof.* We follow the notations in the proof of lemma 2.21. Let $m$ be the l.c.d. of $k$ and $r_1$. We have $k = u'm$ and $r_1 = um$ for positive integers, $u, u'$ with $(u, u') = 1$. We consider $U = X_1^{1/u}$ as an indeterminate. As a consequence of lemma 2.4, the series $\zeta$ defines a quasi-ordinary branch:

$$\zeta_1 = U^{u'} F^{u'm}(U^{1/m}, X_2^{1/r_2}, \dots, X_d^{1/r_d}),$$

whose characteristic exponents $\{\lambda_i(\zeta_1)\}_{i=2}^g$ are obtained from $\{\lambda_i\}_{i=2}^g$ multiplying the first coordinate by $u$.

By lemma 2.8 the series,

$$\zeta_2 = U F^m(U^{1/m}, X_2^{1/r_2}, \dots, X_d^{1/r_d}),$$

is a quasi ordinary branch with characteristic exponents $\{\lambda_i(\zeta_2)\}_{i=2}^g$ with coordinates: $\lambda_{i,1}(\zeta_2) = \lambda_{i,1}(\zeta_1) - u' + 1 = u\lambda_{i,1} - u' + 1$, and, $\lambda_{i,q}(\zeta_2) = \lambda_{i,q}(\zeta_1) = \lambda_{i,q}$ for $q = 2, \dots, d$.

We set $V = Y^{1/u'}$, and define:

$$\tau_1 = V^u G^{um}(V^{1/m}, X_2^{1/r_2}, \dots, X_d^{1/r_d}) \text{ and } \tau_2 = V G^m(V^{1/m}, X_2^{1/r_2}, \dots, X_d^{1/r_d}).$$

In the same manner, we deduce that the characteristic exponents, $\{\lambda_i(\tau)\}_{i=2}^{g'}$ of $\tau$ that are $> \lambda'_1$, are related with the characteristic exponents $\{\lambda_i(\tau_2)\}_{i=2}^{g'}$ of $\tau_2$ by: $\lambda_{i,1}(\tau_2) = u'\lambda_{i,1}(\tau) - u + 1$ and $\lambda_{i,q}(\tau_2) = \lambda_{i,q}(\tau)$, for $q = 2, \dots, d$.

It is easy to see, from the proof of lemma 2.21, that the quasi-ordinary branches $\zeta_2$ and $\tau_2$ parametrize the same quasi-ordinary hypersurface $(S', 0)$ and they correspond to the quasi-ordinary projections:

$$(U, X_2, \dots, X_d, V) \to (U, X_2, \dots, X_d) \text{ and } (U, X_2, \dots, X_d, V) \to (V, X_2, \dots, X_d)$$

respectively. By the lemma 2.23, the quasi-ordinary branches $\zeta_2$ and $\tau_2$ have the same characteristic exponents. We can deduce from this that $g = g'$, and $\lambda_i(\tau) = \lambda'_i$ for $i = 2, \dots, g$.

If $\alpha^{-1} \in \mathbb{Z}$, the characteristic exponents of $\tau$ are $\{\lambda_i\}_{i=2}^g$ since $u' = 1$ and $\tau = \tau_1$. Since $\alpha \notin \mathbb{Z}$ the first characteristic exponent of $\zeta$ is $\lambda_1 = (\alpha, 0, \dots, 0)$ and if $\lambda'_{1,1} = \alpha^{-1} \notin \mathbb{Z}$, then $\lambda'_1$ is the first characteristic exponent of $\tau$. $\diamond$

## 2.6 The monomial variety

We call the affine complex variety $\mathcal{S}_0 = \mathrm{Spec}\ \mathbb{C}[\Gamma]$, the *monomial variety associated to the germ of quasi-ordinary hypersurface singularity* $(S, 0)$. The one dimensional case, the monomial curve, has been studied by Teissier and Goldin (see [T2], and [G-T]).

We embed the monomial variety in $\mathbb{C}^{d+g}$ and we show that it is an affine (non necessarily normal) toric variety of dimension $d$ defined by $g$ binomials. The germ $(\mathcal{S}_0, 0)$ is a quasi-ordinary singularity, and we determine its singular locus.

The variety $\mathcal{S}_0$ is irreducible since the algebra $\mathbb{C}[\Gamma]$ is a domain. The commutative group $\Gamma + (-\Gamma)$ generated by the semigroup $\Gamma$ is free of rank $d$ hence the homomorphism $\mathbb{C}[\Gamma] \to \mathbb{C}[\Gamma + (-\Gamma)]$ gives the embedding of the torus $(\mathbb{C}^*)^d$ as an open dense subset of $\mathcal{S}_0$. By Prop. 1, Chap. 1 of [KKMS], the affine variety $\mathcal{S}_0$ is a non necessarily normal toric variety.

We recall the embedding of $S \subset \mathbb{C}^{d+g}$ associated to a complete set of semiroots $\xi_1, \dots, \xi_{d+g}$ defined by (2.12). Their initial polynomials are the monomials $T^{\gamma_1}, \dots, T^{\gamma_{d+g}}$ generating the $\mathbb{C}$-algebra $\mathbb{C}[\Gamma]$; therefore they define an embedding of the monomial variety in the affine space $\mathbb{C}^{d+g}$

$$U_1 = T^{\gamma_1} = T_1^n, \quad \dots, \quad U_d = T^{\gamma_d} = T_d^n, \quad \dots, \quad U_{d+1} = T^{\gamma_{d+1}}, \quad \dots, \quad U_{d+g} = T^{\gamma_{d+g}} \tag{2.15}$$

**Proposition 2.25** *The ideal $I(\mathcal{S}_0)$ defining $\mathcal{S}_0 \subset \mathbb{C}^{d+g}$ in the ring $\mathbb{C}[U_1, \dots, U_{d+g}]$ is generated by $g$ binomials, which correspond to the relations*

$$n_i \gamma_{d+i} \in \langle \gamma_1, \dots, \gamma_{d+i-1} \rangle \text{ for } i = 1, \dots, g.$$

*Thus, the monomial variety $\mathcal{S}_0$ is a complete intersection.*

*Proof.* Since the embedding $\mathcal{S}_0 \subset \mathbb{C}^{d+g}$ is defined by the monomials $T^{\gamma_1}, \dots, T^{\gamma_{d+g}}$, the ideal $I(\mathcal{S}_0)$ in $\mathbb{C}[U_1, \dots, U_{d+g}]$ is generated by the binomials, $U^{\omega} - U^{\omega'} \in \mathbb{C}[U_1, \dots, U_{d+g}]$, such that

$$\sum_{j=1}^{d+g} \omega_{d+j} \gamma_j = \sum_{j=1}^{d+g} \omega'_{d+j} \gamma_j \tag{2.16}$$

(see [St], Chapter 4).

Let $n_i \gamma_{d+i} = \sum_{j=1}^{d+i-1} a_j^i \gamma_j$ be any $g$ relations in the semigroup $\Gamma$ (given by lemma 2.11), then the binomials

$$h_1 = U_{d+1}^{n_1} - U_1^{a_1^1} \dots U_d^{a_{d+i-1}^1}, \quad \dots \quad , h_g = U_{d+g}^{n_g} - U_1^{a_1^g} \dots U_{d+g-1}^{a_{d+g-1}^g}, \tag{2.17}$$

belong to the ideal $I(\mathcal{S}_0)$.

We prove that $I(\mathcal{S}_0)$ is contained in the ideal $(h_i)_{i=1}^g$ by showing that any irreducible binomial $B$ in $I(\mathcal{S}_0)$ is zero mod $(h_i)_{i=1}^g$. We replace the binomial $B$ modulo $(h_g)$, by an irreducible binomial of the form $U^\omega - U^{\omega'}$ with $0 \leq \omega_{d+g} < n_g$. This binomial leads to a relation of the form (2.16) which implies that $\omega_{d+g} = 0$, since the integer $n_g$ is the order of the element $\gamma_{d+g}$ in the group $nQ_g / nQ_{g-1}$ (see lemma 2.11). By induction we get a binomial $B'$ in the variables $U_1, \dots, U_d$ such that $B' = B$ modulo the ideal $(h_i)_{i=1}^g$. Since $B'$ belongs to $I(\mathcal{S}_0)$, it leads to a relation of the form (2.16) between the linearly independent vectors $\gamma_1, \dots, \gamma_d$, which implies $B' = 0$. ◇

Since the semigroup $\Gamma$ is contained in a strictly convex cone we can define as in [GP].

**Definition 2.9** *The ring of formal power series with exponents in the semigroup* $\Gamma$ *is* $\mathbb{C}[[\Gamma]] = \{\sum_{u \in \Gamma} c_u T^u / c_u \in \mathbb{C}\}$. *The ring of convergent power series with exponents in the semigroup* $\Gamma$*, is the subring* $\mathbb{C}\{\Gamma\}$ *of* $\mathbb{C}[[\Gamma]]$ *of those series which are absolutely convergent in a neighborhood of* $0 \in \mathcal{S}_0$.

The proof of the following lemma is analogous to the proof of lemma 1.1 in [GP].

**Lemma 2.26** *The ring of germs of holomorphic functions at* $0$ *on the monomial variety* $\mathcal{S}_0$ *is isomorphic to* $\mathbb{C}\{\Gamma\}$. ◇

We deduce from this that the germ $(\mathcal{S}_0, 0)$ is an analytically irreducible complete intersection and its ideal in $\mathbb{C}\{U_1, \dots, U_{d+g}\}$ is the extension of $I(\mathcal{S}_0)$.

**Lemma 2.27** *The morphism* $(\mathbb{C}^d \times \mathbb{C}^g, 0) \supset (\mathcal{S}_0, 0) \to (\mathbb{C}^d, 0)$ *that maps* $(U_1, \dots, U_{d+g}) \mapsto (U_1, \dots, U_d)$ *is quasi-ordinary projection.*

*Proof.* It follows from the form of the generators of the ideal of the monomial variety that this morphism is unramified outside the coordinate hyperplanes of $\mathbb{C}^d$. This morphism is finite since it corresponds algebraically to the quasi-finite, i. e., finite (see [K] Corollary 45.6) homomorphism of analytic algebras $\mathbb{C}\{T^{\gamma_1},\dots,T^{\gamma_d}\}\subset\mathbb{C}\{\Gamma\}$. ◇

We describe the singular locus of the monomial variety $\mathcal{S}_0$.

**Proposition 2.28** *The discriminant locus of the quasi-ordinary projection above is* $\bigcup_{i\in I}\{U_i=0\}\cap\mathcal{S}_0$, *where* $I=\{i/\gamma_{d+g,i}>0\}$. *The singular locus of* $\mathcal{S}_0$ *is the union of* $\mathcal{S}_0\cap\{U_i=0,U_j=0\}$ *for* $1\le i\le j\le d$ *satisfying* $i=j$ *and* $\gamma_{d+g,i}>1$ *or* $i\neq j$ *and* $\gamma_{d+g,i}=\gamma_{d+g,j}=1$.

*Proof.* It follows from lemma 2.11 that $\gamma_{d+g,i}=0$, if and only if for any set of binomials $h_1,\dots,h_g$ defining the ideal of the monomial variety in the proof of proposition 2.25, do not depend on the variable $U_i$ for $i\in\{1,\dots,d\}$. This condition clearly implies that the hyperplane of equation $U_i=0$ is not contained in the discriminant locus of the projection and then $\mathcal{S}_0\cap\{U_i=0\}$ is not contained in the singular locus of $\mathcal{S}_0$.

The singular locus of $\mathcal{S}_0$ is contained in the inverse image of the discriminant of the projection $\mathcal{S}_0\to\mathbb{C}^d$, thus the singular locus of $\mathcal{S}_0$ is contained in the union of coordinate hyperplanes $U_i=0$, for $i\in I$. If $\gamma_{d+g,i}=1$ then we deduce from the formulae (2.6) that $\gamma_{1,i}=\dots=\gamma_{d+g-1,i}=0$, thus the binomials $h_1,\dots,h_{g-1}$ do not depend on the variable $U_i$. We deduce that the exponent $b_i$ of $U_i$ in the binomial $h_g=U_{d+g}^{n_g}-U_1^{b_1}\dots U_{d+g-1}^{b_{d+g-1}}$, is equal to $1$. The Jacobian matrix of $(h_1,\dots,h_g)$ is of rank $g$ on a general point of $\mathcal{S}_0\cap\{U_i=0\}$, hence $\mathcal{S}_0\cap\{U_i=0\}$ is not contained in the singular locus of $\mathcal{S}_0$. In the same way, if $\gamma_{d+g,i}=0$ the intersection $\mathcal{S}_0\cap\{U_i=0,U_j=0\}$ is not contained in the singular locus of $\mathcal{S}_0$. It is easy to see that if $\gamma_{d+g,i}=\gamma_{d+g,j}=1$ for $i\neq j$, the intersection $\mathcal{S}_0\cap\{U_i=0,U_j=0\}$ is contained in the singular locus of $\mathcal{S}_0$, since the Jacobian matrix of $(h_1,\dots,h_g)$ is of rank $<g$ on every point of this intersection. ◇

## 2.7 The normalization of a quasi-ordinary hypersurface

We compute in this section the normalization (see [K] §71) of the quasi-ordinary hypersurface $S$ by showing that it coincides with the normalization of the associated monomial variety $\mathcal{S}_0$.

It is shown by Lipman (see [L4], remark 7.3.2) that a quasi-ordinary hypersurface singularity is *normal* if and only if it is isomorphic to a germ of the form $Y^n-X_1\dots X_c=0$ for some $1\le c\le d$. This equation defines an affine normal toric variety. In general, it is known that the *normalization* of a quasi-ordinary surface singularity is always the germ of an affine toric variety at the *special point* (see [B-P-V], Chapter III, Theorem 5.2). We give in this section an algebraic proof of this fact in any dimension.

We begin by determining the normalization of the monomial variety. The commutative group $\Gamma + (-\Gamma)$ generated by the semigroup $\Gamma$ is free of rank $d$. The saturated semigroup $\tilde{\Gamma}$ of $\Gamma$ in the group $\Gamma + (-\Gamma)$ is the set of elements $u$ of the lattice $\Gamma + (-\Gamma)$ such that $ku \in \Gamma$ for some $k \in \mathbb{Z}_{\geq 0}$. The cone $\sigma$ generated by the elements of the semigroup $\Gamma$ in $\mathbb{R}^d$ is a rational cone for the lattice $\Gamma + (-\Gamma)$ and it is easy to see that $\tilde{\Gamma} = \sigma \cap (\Gamma + (-\Gamma))$. The proof of the following proposition can be found in Chapter 1, Proposition 1, Lemma 1, and Theorem 1 of [KKMS].

**Lemma 2.29** *The integral closure of the ring* $\mathbb{C}[\Gamma]$ *in its field of fractions is equal to the ring* $\mathbb{C}[\tilde{\Gamma}]$. ◇

Recall the definition 2.9 of the ring $\mathbb{C}\{\Gamma\}$. This proposition shows that the normalization of the monomial variety $\mathcal{S}_0$ is the map $\tilde{\mathcal{S}_0} = \text{Spec } \mathbb{C}[\tilde{\Gamma}] \to \mathcal{S}_0$. The inverse image by this map of the point $0 \in \mathcal{S}_0$ is the special point of $\tilde{\mathcal{S}_0}$, which corresponds to the maximal ideal $(T^u / 0 \neq u \in \tilde{\Gamma})$.

**Lemma 2.30** *The integral closure of the ring* $\mathbb{C}\{\Gamma\}$ *in its field of fractions is equal to the ring* $\mathbb{C}\{\tilde{\Gamma}\}$. ◇

We denote by $\text{fr}(A)$ the field of fractions of a domain $A$.

**Lemma 2.31** *The rings* $\mathbb{C}\{\Gamma\}$ *and* $R$ *have the same field of fractions (viewed in the field of fractions of* $\mathbb{C}\{T\}$*).*

*Proof.* By lemma 2.27, $\mathbb{C}\{\Gamma\}$ is a $\mathbb{C}\{T^{\gamma_1}, \dots, T^{\gamma_d}\}$-module of finite type, generated by a finite set of monomials in $\mathbb{C}\{\Gamma\}$ (by Nakayama's Lemma). This shows that

$\mathbb{C}\{\Gamma\} = \mathbb{C}\{T^{\gamma_1}, \dots, T^{\gamma_d}\}[T^{\gamma_{d+1}}, \dots, T^{\gamma_{d+g}}]$ and $\text{fr}(\mathbb{C}\{\Gamma\}) = \text{fr}(\mathbb{C}\{T^{\gamma_1}, \dots, T^{\gamma_d}\})[T^{\gamma_{d+1}}, \dots, T^{\gamma_{d+g}}]$

The field of fractions of $R$ viewed in $\mathbb{C}\{T\}$ by (2.2) is equal to $\text{fr}(\mathbb{C}\{T^{\gamma_1}, \dots, T^{\gamma_d}\})[T^{n\lambda_1}, \dots, T^{n\lambda_g}]$ by lemma 2.5. The result follows from formula (2.6). ◇

**Proposition 2.32** *The integral closure of the ring* $R$ *in its field of fractions is equal to* $\mathbb{C}\{\tilde{\Gamma}\}$.

*Proof.* The ring $\mathbb{C}\{\Gamma\}$ and $R$ are integral over $\mathbb{C}\{T^{\gamma_1}, \dots, T^{\gamma_d}\}$ by (2.2) and lemma 2.27. The integral closures of the rings $\mathbb{C}\{\Gamma\}$ and $R$ in the field

$$K := \text{fr } \mathbb{C}\{\Gamma\} \overset{\text{lemma 2.31}}{=} \text{fr } R$$

coincides with the integral closure of $\mathbb{C}\{T^{\gamma_1}, \dots, T^{\gamma_d}\}$ on $K$ and are equal to $\mathbb{C}\{\tilde{\Gamma}\}$ by lemma 2.30. ◇

# Chapter 3

# Toric embedded resolutions of quasi-ordinary hypersurfaces

In this chapter we use toric methods to construct embedded resolution of an irreducible quasi-ordinary hypersurface singularity $S$ from the characteristic exponents of a quasi-ordinary branch $\zeta$ parametrizing $S$. Both methods make use of the set of generators $\{\xi(T)\}_{j=1}^{d+g}$ of the maximal ideal of $R$. We will give two different resolution procedures.

Firstly, we consider the embedding of $S$ in $\mathbb{C}^{d+1}$ defined by the generators $\xi_1(T), \ldots, \xi_{d+1}(T)$ of the maximal ideal of $R$. We will define an embedded resolution procedure using toroidal maps. Each one of these maps will be quasi-ordinary stable for the hypersurfaces defined by the semiroots at each stage, in particular each one eliminates one characteristic exponent. The strict transform of the hypersurface defined by the first semiroot at each stage will be smooth and will have normal crossings with the critical divisor. We will define from this semiroot canonical coordinates, that provides a quasi-ordinary projection for the strict transform. We will do this locally on each chart, and we show how to define a global map. The composition of these toroidal maps gives an embedded resolution of the hypersurface.

In the second procedure we consider the variety $S$ embedded in $\mathbb{C}^{d+g}$, the embedding is defined by the elements $\xi_1(T), \ldots, \xi_{d+g}(T)$. We define a $d$-parameter deformation $\mathcal{S}$ whose general fiber is isomorphic to $S$ and whose fiber over the origin is isomorphic to $\mathcal{S}_0$, the monomial variety associated to $S$. We show how to construct a toric embedded resolution of the monomial variety. The main result is that this map gives an embedded resolution of $S \subset \mathbb{C}^{d+g}$, and that it provides a simultaneous embedded resolution of the deformation $\mathcal{S}$ and all its fibers. We give an exemple of these methods.

## 3.1 Embedded resolution as a composite of toroidal maps

In this section, we construct a resolution procedure of the surface $S$ embedded in $\mathbb{C}^{d+1}$, that is not unique but it is determined by the characteristic exponents of the quasi-ordinary branch and the generators $\xi_1, \ldots, \xi_{d+g}$ of the maximal ideal of $R$ (of formulae 2.12). This resolution is defined as a composition of toroidal maps. We generalize the well known toroidal resolution of a plane branch, which has been generalized by Lê and Oka to reduced complex analytic curves (see [Le-Ok] and [Ok1]).

### 3.1.1 The toroidal modification

We keep the notations of section 2.14. We consider a fixed set $q_1, \ldots, q_{g-1}$ of semi-roots of $f$. We can suppose that they are polynomials in $\mathbb{C}\{X\}[Y]$ with degrees $\deg q_i = n_0 n_1 \ldots n_i$. They define quasi-ordinary hypersurfaces $S_i \subset \mathbb{C}^{d+1}$, and the restriction of the projection $(X,Y) \mapsto X$ to each hypersurface $S_i$ is a quasi-ordinary projection. We will call the hypersurface $S_i$, the $i^{th}$*-approximate surface of* $S$. The terminology is used to underline the fact that we can take for $q_i$ the characteristic approximate roots (see section 2.4.4). We will suppose that we have good coordinates (i.e., $q_0 := Y$ is a $0$-semi-root of $f$) The Newton polyhedron of a semi-root of $f$ have only one compact edge with a fixed inclination. We study in this section the strict transform of $S$ by a toric modification defined by a regular fan compatible with these Newton polyhedra.

The Newton polyhedron of the minimal polynomial $f$ over $\mathbb{C}\{X\}$ of the quasi-ordinary branch $\zeta$ has only two vertices $(0, \ldots, 0, n)$ and $(n\lambda_{1,1}, \ldots, n\lambda_{1,d}, 0)$. Let us take a regular fan $\Sigma$ compatible with $f$, that is a regular fan compatible with the linear subspace $\ell$ of equation $l(u) = \lambda_{1,1}u_1 + \ldots \lambda_{1,d}u_d - u_{d+1} = 0$ (see Figure 3.1). Denote $\Sigma \cap \ell$ the "sub-fan" of $\Sigma$ given by those cones of $\Sigma$ which are contained in the linear subspace $\ell$. We consider the toric variety $Z(\Sigma \cap \ell)$ as an open set contained in $Z(\Sigma)$. Then we have:

**Lemma 3.1** *The strict transform $\tilde{S}$ of $S$ by $\pi(\Sigma)$ is contained in $Z(\Sigma \cap \ell)$, more precisely the divisors $D(a)$, for $a \in \Sigma^{(1)}$, intersecting $\tilde{S}$ are those defined by vectors $a \in \ell$.*

*Proof:* We study the strict transform $\tilde{f}_\sigma$ of $f$ in the chart defined by the regular cone $\sigma = \langle a^1, \ldots, a^{d+1} \rangle$. Without loss of generality we can suppose that the vectors $a^i$ are in the half-space $\ell^+$ defined by, say $l(u) \geq 0$. This map is written in canonical coordinates as (2.1) (where $Y$ is changed to $X_{d+1}$). Since $f$ is of the form $\prod_{i=1}^{n}(Y - X^{\lambda_1} H_i(X^{1/n}))$ (where the $H_i$ are

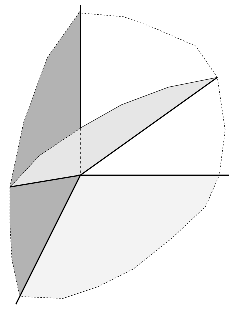

Figure 3.1: The linear subspace $\ell$

units) the transform of $f$ by the map $\pi(\sigma) : Z(\sigma) \to \mathbb{C}^{d+1}$ is of the form:

$$(U_1^{a^1_{d+1}} \cdots U_{d+1}^{a^{d+1}_{d+1}})^n \prod_{i=1}^{n} (1 - U_1^{l(a^1)} \cdots U_{d+1}^{l(a^{d+1})} \tilde{H}_i(U^{1/n})) \text{ with units } \tilde{H}_i \tag{3.1}$$

because $l \geq 0$ on the cone $\sigma$, and this implies that:

$$\inf\{\langle a^i, v' \rangle \,/\, v' \in \mathcal{N}(Y - X^{\lambda_1} H_i(X^{1/n}))\} = a^i_{d+1} \text{ for } i = 1, \ldots, d,$$

hence we factor out from the transform of $Y - X^{\lambda_1} H_i(X^{1/n})$, the transform of $Y$, which is the monomial $U_1^{a^1_{d+1}} \cdots U_{d+1}^{a^{d+1}_{d+1}}$, for $i = 1, \ldots, n$.

We call the strict transform of $f$, on $\pi(\sigma)$ the function $\tilde{f}_\sigma = (U_1^{a^1_{d+1}} \ldots U_{d+1}^{a^{d+1}_{d+1}})^{-n} f \circ \pi(\sigma)$. If we have $a^i \notin \ell$, then the intersection of the zero locus of the strict transform $\tilde{f}_\sigma$ with $U_i = 0$ is empty and $\tilde{f}_\sigma = 0$ is contained in the chart $Z(\sigma')$, where $\sigma'$ is the face of $\sigma$ defined by the $a^j$ which are contained in $\ell$. ◇

For the geometric picture of the strict transform in the case $d = 3$ see figure 3.2, (on the left the case $a^1, a^2 \in \ell$, and on the right the case $a^1 \in \ell$ and $a^2, a^3 \notin \ell$).

**Remark 3.2** *The strict transforms $\tilde{f}_\sigma$, for the cones $\sigma \in (\Sigma)^{(d+1)}$ contained in the half-space $l(u) \geq 0$, glue up to define a function, we call the strict transform of $f$.*

We can do this because they are the strict transforms on charts defined by cones associated to the same vertex of the Newton polyhedron of $f$.

We can restrict ourselves to study the strict transform $\tilde{S}$ in the charts defined by cones $\langle a^1, \ldots, a^{d+1} \rangle$ contained in the half-space $\ell^+ : l(u) \geq 0$ with $a^i \in \ell$, for $i = 1, \ldots d$, because the

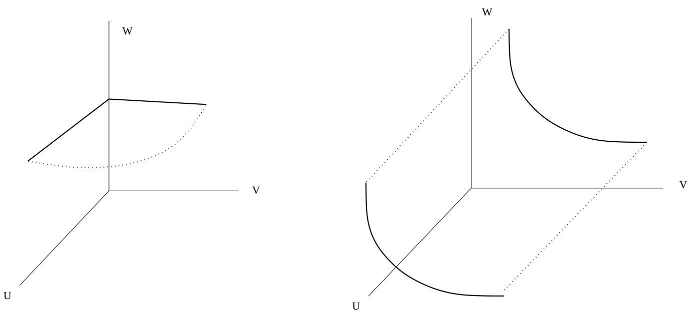


Figure 3.2: The strict transform $\tilde{f}_\sigma = 0$ on different charts.

strict transform on $Z(\sigma)$, for any $\sigma \in \Sigma$, is contained in the chart defined by the cone $\sigma \cap \ell$, which will be contained as an open set of a chart of the desired form. In this case, we have that the strict transform $\tilde{S}$ does not meet the divisor $D(a^{d+1})$.

Recall the integers $e_j$ and $n_i$ from definition 2.2. The symbolic restriction of $f$ to the compact face of its Newton polyhedron is is equal to the product $\prod(Y - \alpha_1^{(i)} X^{\lambda_1})$, for $\alpha_1^{(i)}$ running through the coefficients of $X^{\lambda_1}$ in all the conjugates $\zeta^{(i)}$ of $\zeta$. Since the $\alpha_1^{(i)} X^{\lambda_1}$ are a complete set of conjugates of $\alpha_1 X^{\lambda_1}$, each conjugate being counted $e_1$ times we obtain that:

$$\mathrm{in}(f) = (Y^{n_1} - \alpha_1^{n_1} X^{n_1 \lambda_1})^{e_1}$$

**Lemma 3.3** *We have the following:*

1. *The intersection of the strict transform and the divisors* $\tilde{S} \cap D(a^1) \cap \cdots \cap D(a^d)$ *is a single point* $O_\sigma$ *which has coordinates* $(0, \ldots, 0, \alpha_1^{-n_1})$ *in* $Z(\sigma)$.
2. *Set* $W := 1 - \alpha_1^{n_1} U_{d+1}$, *the germ* $(\tilde{S}, O_\sigma)$ *is defined by a Weierstrass polynomial of degree* $e_1$ *in* $\mathbb{C}\{U_1, \ldots, U_d\}[W]$.

*Proof.* The strict transform of $f$ by the map $\pi(\sigma)$ is of the form:

$$(1 - \alpha_1^{n_1} U_{d+1}^{n_1(l(a^{d+1}))})^{e_1} + \tilde{g}(U)$$

We notice also that $\tilde{g}(0, U_{d+1}) = 0$ since the exponent of any term in $f$ providing a term in $\tilde{g}$ is not in the only compact edge of the Newton polyhedron of $f$.

Since the vectors $a^i$ for $i = 1, \ldots, d$, form part of a basis of $\mathbb{Z}^{d+1}$, the coefficients of the equation of the subspace they span are primitive. This equation is obtained by taking the determinant of the matrix with columns the vectors $a^1, \ldots, a^d$ and the indeterminate vector $u$, i.e.,

$\det \mathrm{Col}[a^1, \ldots, a^d, u] = 0$. By definition of $n_1$, we know also that this equation is $n_1 l(u) = 0$. Since the cone $\sigma$ is regular, we have $|\det \mathrm{Col}[a^1, \ldots, a^d, a^{d+1}]| = 1$, and using that $a^{d+1} \in \ell^+$ we deduce

$$n_1 l(a^{d+1}) = 1. \tag{3.2}$$

Thus the strict transform $\tilde{f}_\sigma$ of $f$ is of the form:

$$\tilde{f}_\sigma = (1 - \alpha_1^{n_1} U_{d+1})^{e_1} + \tilde{g}(U).$$

therefore $\tilde{f}_\sigma = 0$ cuts the curve $\bigcap_{i=1}^d D(a^i)$ at the point given by $U_{d+1} = \alpha_1^{-n_1}$. We define $W := 1 - \alpha_1^{n_1} U_{d+1}$ and then the germ defined by $\tilde{S}$ at the point $O_\sigma$ is defined by $f'_\sigma := \tilde{f}_\sigma(U_1, \ldots, U_d, \alpha_1^{-n_1}(1-W))$. The Weierstrass polynomial corresponding to $f'_\sigma$ is of degree $e_1$ in $W$. ◇

**Remark 3.4** *The series $f'_\sigma$ is a well defined element of $\mathbb{C}\{U_1, \ldots, U_d, W\}$ since we can view it as the strict transform of $f$ by the algebra homomorphism $\mathbb{C}\{X_1, \ldots, X_d, Y\} \to \mathbb{C}\{U_1, \ldots, U_d, W\}$ determined by:*

$$\begin{array}{rcl} X_i & = & \alpha_1^{-n_1 a^i_{d+1}} U_1^{a^i_1} \cdots U_d^{a^i_d} (1-W)^{a^i_{d+1}} \quad \text{for } i = 1, \ldots, d \\ Y & = & \alpha_1^{-n_1 a^{d+1}_{d+1}} U_1^{a^{d+1}_1} \cdots U_d^{a^{d+1}_d} (1-W)^{a^{d+1}_{d+1}} \end{array} \tag{3.3}$$

The homomorphism (3.3) is well defined, if for $1 \leq j \leq d$, we had $a^j_1 = \cdots = a^j_d = 0$, it would imply that $a^j \in \ell$ is the $(d+1)$-canonical basis vector, which is a contradiction.

### 3.1.2 The toroidal embedding defined by the first approximate hypersurface

We show that the morphism $\pi(\Sigma)$ defined in the previous section is an embedded toric resolution of the first approximate hypersurface $S_1$. We compute the intersection of the strict transform of $S_1$, with the critical divisor, we associate to it a toroidal embedding and we describe the associated polyhedral complex. These tools are introduced to facilitate the description of the other toroidal morphisms in section 3.1.4.

If we take a quasi-ordinary branch with only one characteristic exponent, we have $e_1 = 1$ by lemma 2.5. By lemma 3.3, the strict transform $\tilde{S}$ is smooth and transversal to the critical locus of $\pi(\Sigma)$. Moreover, we have the following:

**Lemma 3.5** *If $\zeta$ has only one characteristic exponent $\lambda = (\lambda_1, \ldots, \lambda_d)$ then the toric modification $\pi : Z(\Sigma) \to \mathbb{C}^{d+1}$ induces a toric embedded resolution of the quasi-ordinary hypersurface $S$.*

*Proof.* We only have to show that it induces an isomorphism outside the singular locus of $S$. The intersection of $S$ with $(\mathbb{C}^*)^{d+1}$ is non singular. Theorem 2.4 implies that the intersection $Z_1$ of $S$ with $X_1 = 0$ is not contained in the singular locus if and only if $\lambda_1 = 0$ or $n\lambda_1 = 1$.

In the first case, the minimal polynomial $f$ of $\zeta$ over $\mathbb{C}\{X\}$ is of the form $f = Y^n + \cdots + X_2^{n\lambda_2} \cdots X_d^{n\lambda_d} H(X)$ with $H$ a unit. A "simple" regular fan $\Sigma$ compatible $f = 0$ is a "product" of a $d$-dimensional regular fan contained in $\{0\} \times \mathbb{R}^d$ by the edge defined by the first canonical basis vector. Then the toric modification gives un isomorphism locally on points of the form $(0, x_2, \ldots, x_{d+1})$ with $\prod_{i=2}^d x_i \neq 0$. Otherwise it is a subdivision of a fan $\Sigma'$ of this form. In that case we argue in a similar manner as in the last paragraph of this proof.

In the second case, the minimal polynomial $f$ of $\zeta$ over $\mathbb{C}\{X\}$ is of the form $f = Y^n + \cdots + X_1 X_2^{n\lambda_2} \cdots X_d^{n\lambda_d} H(X)$ with $H$ a unit. We notice that $\partial f / \partial X_1 \neq 0$ if $X_2 \cdots X_d \neq 0$, so we can apply the implicit function theorem locally on these points to obtain a parametrization of $S$ of the form $X_1 = \psi(X_2, \ldots, X_d, Y)$.

We will need to describe some essential vectors (defined in §2) appearing in any regular fan $\Sigma$ compatible with the subspace $\ell$: $\frac{1}{n} u_1 + \lambda_2 u_2 + \cdots + \lambda_d u_d - u_{d+1} = 0$. Clearly, if the intersection of a coordinate subspace with $\ell$ is a 2-dimensional cone $\sigma$, then the essential vectors for $\sigma$ are essential for $\Sigma$ too. We consider the following essential vectors in $\ell$:

$$
\begin{aligned}
a^1 &= (n, 0, \ldots, 0, 1) \\
a^i &= (c_i, 0, \ldots, 0, 1, 0, \ldots, 0, b_i) \quad \text{for } i = 2, \ldots, d,
\end{aligned}
\tag{3.4}
$$

where 1 stands on the $i^{th}$-place in $a^i$ and $c_i, b_i$ are the smallest non negative integers such that $a^i \in \ell$, for $i = 2, \ldots d$. We find that $a^1, \ldots, a^d$ are a basis of the free group $\mathbb{Z}^{d+1} \cap \ell$ since they determine a equation for $\ell$, $\det \operatorname{Col}[a^1, \ldots, a^d, u] = 0$, which has primitive coefficients, namely the equation $u_1 + n\lambda_2 u_2 + \cdots + n\lambda_d u_d - n u_{d+1} = 0$.

We will show first that the intersection $Z_1$ of the hypersurface $S$ with $X_1 = 0$ is not in the discriminant locus of the map $\tilde{S} \to S$ in the case the regular cone $\sigma = \langle a^1, \ldots, a^d \rangle$ is in $\Sigma$.

The map $\pi$ on the chart corresponding to a cone of the form, $\langle a^1, \ldots, a^{d+1} \rangle$ with $l(a^{d+1}) = 1$, is given by formula (2.1). The strict transform $\tilde{S}$ of $S$ on this chart is defined by $U_{d+1} = \phi(U)$, where $\phi(U)$ it is a unit in $\mathbb{C}\{U_1, \ldots, U_d\}$, by lemma 3.3.

The intersection of $S$ with $X_1 = 0$, $X_2 \cdots X_d \neq 0$ is locally isomorphic to the intersection of the strict transform $\tilde{S}$ with $U_1 = 0$, $U_2 \cdots U_{d+1} \neq 0$. We see that this isomorphism can be extended to the strict transforms on a neighborhood of any point of this intersection applying the inverse function theorem to the system obtained from (2.1) by replacing $U_{d+1}$ by $\phi(U)$,

$$\begin{array}{rcl} X_2 & = & U_2\,\phi^{a_2^{d+1}} \\ & \cdots & \\ X_d & = & U_d\,\phi^{a_d^{d+1}} \\ Y & = & U_1 U_2^{b_2}\cdots U_d^{b_d}\,\phi^{a_{d+1}^{d+1}} \end{array}$$

once it is noticed that $S$ is parametrized by $X_1 = \psi(X_2, \ldots, X_d)$. These isomorphisms glue up giving an isomorphism outside the singular locus.

Suppose that the fan $\Sigma$ provides a subdivision $\Xi$ of the essential regular cone $\sigma$, then the toric map $Z(\Xi) \to \mathbb{C}^{d+1}$ is composed with $Z(\Xi) \to Z(\sigma)$. A point $u$ in the intersection in $Z(\sigma)$ of $\tilde{S}$ with the image of a divisor corresponding to a non essential vector added in $\sigma$ have more than one zero in its first $d$ coordinates. This implies that the point $u$ lies over the singular locus of $S$ by theorem 2.4. ◇

We consider the open set $\mathcal{U}_1$ in the variety $\mathcal{X}_1 := Z(\Sigma)$ given by the complementary of the hypersurface $\bigcup_{a\in(\Sigma\bigcap\ell)^{(1)}} D(a) \bigcup \tilde{S}_1$. The components of this hypersurface are clearly non singular and they intersect transversally by lemma 3.5. Thus, we have defined a regular toroidal embedding (see §2), $\mathcal{U}_1 \subset \mathcal{X}_1$, and we say it is *associated to the first approximate hypersurface.*

We have shown that there is a bijection which we denote by $\sigma = \langle a^1, \ldots, a^d\rangle \mapsto O_\sigma$, between the cones in the $d$-skeleton of the fan $\Sigma_1 \cap \ell$ and the $0$-strata of the toroidal embedding $\mathcal{U}_1 \subset \mathcal{X}_1$. By lemma 3.5, we take at the point $O_\sigma$ the *canonical coordinates*, $(U_1, \ldots, U_d, W_1)$, where $U_i = 0$ is the toric equation for the divisor $D(a^i)$, and $W_1 = 0$, is the equation of the strict transform of the $1^{st}$ approximate hypersurface $\tilde{S}_1$.

We now define a fan $\Theta_1$, combinatorially isomorphic to the cone over the dual intersection graph of the components of $\mathcal{X}_1 - \mathcal{U}_1$. Let us fix a vector $c_0 \in \mathbb{Z}^{d+1}$ such that $l(c_0) = 1$. The $(d+1)$-dimensional cones of the fan $\Theta_1$ are spanned by the vector $c_0$ and the $d$-dimensional cones of $\Sigma \cap \ell$. We say that the fan $\Theta_1$ is the $(d+1)$ *dimensional regular fan over* $\Sigma \cap \ell$. We notice that the "$(d+1)$-dimensional regular fan over $\Sigma \cap \ell$" is unique up to *isomorphism of fans* (see [Ew]); that is, if we choose another vector $c_1$ different from $c_0$, we will get an isomorphism of $\mathbb{Z}^d$, which will transform one fan into the other. It is sufficient to define it by $c_1 \mapsto c_0$, and the identity map on $\ell$. (See also the proof of lemma 3.6).

**Lemma 3.6** *The conic polyhedral complex associated to* $\mathcal{U}_1 \subset \mathcal{X}_1$ *is isomorphic to the fan* $\Theta_1$.

*Proof.* Take a cone $\sigma = \langle a^1, \ldots, a^d\rangle$ in $(\Sigma \cap \ell)^d$ and the unique $0$-stratum $O_\sigma$. The canonical coordinates $\{U, \ldots, U_d, W_1\}$ on $O_\sigma$ give a basis of $M^{O_\sigma}$ and also the generators of the regular

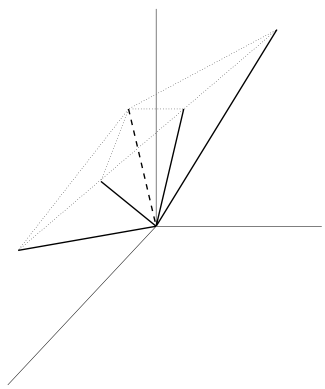

Figure 3.3: The fan $\Theta_1$

cone of effective divisors on $M_{\mathbb{R}}^{O_\sigma}$. We identify its dual basis with $\{a^1, \ldots, a^d, c_0\}$. This identification provides the isomorphism between the rational cones $\rho^{O_\sigma}$ and $\langle a^1, \ldots, a^d, c_0 \rangle$ which extends to an isomorphism of the complex polyhedral complex associated to $\mathcal{U}_1 \subset \mathcal{X}_1$ onto the fan $\Theta_1$. ◇

**Lemma 3.7** *With respect to the canonical coordinates $U_1, \ldots, U_d, , W_1$ corresponding to a 0-stratum $O_\sigma$, for $\sigma$ running through in the $d$-skeleton of the fan $\Theta_1 \cap \ell$, the coefficients of the terms of the strict transform $\tilde{f}_\sigma$ viewed as a power series in the ring $\mathbb{C}\{U_1, \ldots, U_d, W_1\}$ are independent of the cone $\sigma$, the corresponding exponents define the coordinates of vectors in $\mathbb{Z}^{d+1}$ with respect to the basis formed by the minimal set of generators of the semigroup $S_\sigma$ which are independent of the cone $\sigma$.*

*Proof.* Let us take adjacent cones $\sigma = \langle a^1, \ldots, a^d \rangle$ and $\bar{\sigma} = \langle a^1, \ldots, a^{d-1}, \bar{b} \rangle$. Then we have that $b = \sum_{i=1}^{d-1} s_i a^i - \bar{b}$ for some $s_i \in \mathbb{Z}_{\geq 0}$. Then the canonical coordinates $U_1, \ldots, U_d, W_1$ on $O_\sigma$ and $\bar{U}_1, \ldots, \bar{U}_d, \bar{W}_1$ on $O_{\bar{\sigma}}$ are related on an open set by

$$\begin{array}{rcll} \bar{U}_i &=& U_i U_d^{s_i} & \text{for } i = 1, \ldots, d-1 \\ \bar{U}_d &=& U_d^{-1} & \\ \bar{W}_1 &=& W_1 & \end{array} \tag{3.5}$$

This is because the strict transform of the first approximate polynomial $q_1$ is a well defined function equal to $W_1$ and to $\bar{W}_1$ on respective charts, by remark 3.2 . For the same reason, the strict transform of $f$ is a well defined function $\tilde{f}$. Then the result follows. ◇

### 3.1.3 Quasi-ordinary character of the strict transform

We show in this section the "stability" of quasi-ordinary singularities with respect to the suitable toric morphisms of the previous section.

**Proposition 3.8** *For each cone $\sigma = \langle a^1, \dots, a^d \rangle$ in the $d$-skeleton of the fan $\Theta_1 \cap \ell$, we take canonical coordinates $(U_1 \dots, U_d, W_1)$ at the $0$-strata $O_\sigma$. Then, the projection of germs $(\tilde{S}, O_\sigma) \to (\mathbb{C}^d, 0)$ defined by $(U, W_1) \mapsto U$ is a quasi-ordinary projection of degree $e_1$. The germ $(\tilde{S}, O_\sigma)$ is analytically irreducible and is parametrized by a quasi-ordinary branch which has $g-1$ characteristic monomials in $U$ with exponents $\{\tilde{\lambda}_i(\sigma)\}_{i=2}^g$ which are obtained from $\{\lambda_i - \lambda_1\}_{i=2}^g$ by multiplication by the matrix:*

$$A(\sigma) = \begin{pmatrix} a_1^1 & a_2^1 & \dots & a_d^1 \\ a_1^2 & a_2^2 & \dots & a_d^2 \\ \dots & \dots & \dots & \dots \\ a_1^d & a_2^d & \dots & a_d^d \end{pmatrix} \tag{3.6}$$

*Proof.* By lemma 3.3 it is sufficient to study the strict transform of $(S, 0)$ at the charts $Z(\sigma)$ for $\sigma$ in the $d$-skeleton of the fan $\Theta_1 \cap \ell$. We begin with the analysis of the lattice $Q_\ell$ dual to the lattice $\oplus_{i=1}^d a^i \mathbb{Z}$ of integral vectors in the linear subspace $\ell$. We denote by $Q'_0$ (resp. $Q'_1$) the lattices $Q_0 \oplus e_{d+1}\mathbb{Z} = \mathbb{Z}^{d+1}$ (resp. $Q_1 \oplus e_{d+1}\mathbb{Z}$) where $Q_0$ is the lattice $\mathbb{Z}^d \times \{0\}$, $e_1, \dots, e_{d+1}$ denotes the canonical basis of $\mathbb{Z}^{d+1}$ and $Q_1$ is the first characteristic lattice.

The lattice isomorphism $\phi : Q'_0 \to Q_1$ that maps $e_{d+1} \mapsto \lambda_1$ and fixes the elements of $Q_0$ has kernel equal to $m(e_{d+1} - \lambda_1)\mathbb{Z}$ where $m = |(Q_0 + \lambda_1 \mathbb{Z})/Q_0|$. By definition we obtain that $m = n_1$. If we replace $Q_0$ and $Q'_0$ by $Q_1$ and $Q'_1$ respectively the integer $m' := n'_1$ obtained is equal to one.

We denote by $u_1, \dots, u_d$ the dual basis of $a^1, \dots, a^d$ in $Q_\ell$. We obtain isomorphisms:

$$\sigma^\vee \cap Q'_0 \cong n_1(e_{d+1} - \lambda_1)\mathbb{Z} \oplus (u_1 \mathbb{Z}_{\geq 0} \oplus \dots \oplus u_d \mathbb{Z}_{\geq 0}), \tag{3.7}$$

$$\sigma^\vee \cap Q'_1 \cong (e_{d+1} - \lambda_1)\mathbb{Z} \oplus (u_1 \mathbb{Z}_{\geq 0} \oplus \dots \oplus u_d \mathbb{Z}_{\geq 0}). \tag{3.8}$$

If $\mathcal{S}$ is a semigroup we denote by $\mathbb{C}[\mathcal{S}]$ the algebra of the semigroup $\mathcal{S}$ with coefficients on $\mathbb{C}$. We have a commutative diagram :

$$\begin{array}{ccc} \mathbb{C}[\bigoplus_{j=1}^d u_j \mathbb{Z}_{\geq 0}] & \longrightarrow & \mathbb{C}[\sigma^\vee \cap Q'_1] \\ \mathrm{id} \uparrow & & p_{n_1} \uparrow \\ \mathbb{C}[\bigoplus_{j=1}^d u_j \mathbb{Z}_{\geq 0}] & \longrightarrow & \mathbb{C}[\sigma^\vee \cap Q'_0] \end{array} \tag{3.9}$$

where the bottom row corresponds to the linear projection $Z(\sigma) \to \mathbb{C}^d$ with kernel the orbit $\mathbf{O}_\sigma$. Recall that the coordinate ring of the orbit $\mathbf{O}_\sigma$ is equal to $\mathbb{C}[n_1(e_{d+1} - \lambda_1)\mathbb{Z}$ and with the natural coordinates of $Z(\sigma)$ (which are provided by the right-hand of formula 3.7) the projection is given by $(U_1, \ldots, U_d, U_{d+1}) \mapsto (U_1, \ldots, U_d)$. An analogous statement holds for the top row.

Notice that the inclusion $p_{n_1}$ corresponds geometrically to a unramified covering of degree $n_1$ by 3.7 and 3.8. Therefore it is sufficient to prove that the strict transform of $(S, 0)$ in $\text{Spec}\mathbb{C}[\sigma^\vee \cap Q'_1]$ is quasi-ordinary with respect to the projection defined by $\mathbb{C}[\bigoplus_{j=1}^d u_j \mathbb{Z}_{\geq 0}] \to \mathbb{C}[\sigma^\vee \cap Q'_1]$ and then to deduce that the strict transform of the first approximate hypersurface is a $0^{th}$-approximate hypersurface of the strict transform of $(S, 0)$.

We denote by $Q_{0,\geq 0}$ and $Q'_{0,\geq 0}$ (resp. by $Q_{1,\geq 0}$ and $Q'_{1,\geq 0}$) the semigroups $Q_0 \cap \mathbb{R}^d_{\geq 0}$ and $Q'_0 \cap \mathbb{R}^{d+1}_{\geq 0}$ (resp. $Q_1 \cap \mathbb{R}^d_{\geq 0}$ and $Q'_1 \cap \mathbb{R}^{d+1}_{\geq 0}$). We have a commutative diagramm:

$$\begin{array}{ccc} \mathbb{C}[Q'_{1,\geq 0}] & \longrightarrow & \mathbb{C}[\sigma^\vee \cap Q'_1] \\ j_{n_1} \uparrow & & p_{n_1} \uparrow \\ \mathbb{C}[Q'_{0,\geq 0}] & \longrightarrow & \mathbb{C}[\sigma^\vee \cap Q'_0] \end{array} \tag{3.10}$$

We have that $\mathbb{C}[Q'_{0,\geq 0}] = \mathbb{C}[Q_{0,\geq 0}][Y]$ and the quasi-ordinary polynomial $f$ belongs to $\mathbb{C}\{Q_{0,\geq 0}\}[Y]$. We denote by $F$ its image in the ring $\mathbb{C}\{Q_{1,\geq 0}\}[Y]$. The polynomial $F$ is toric quasi-ordinary in the sense of chapter 1. It factors as $F = F_1 \ldots F_{n_1}$ where each $F_i = \prod_{k=1}^{e_1}(Y - \zeta_i^{(k)})$ where the product collects the roots of $f$ which have the same coefficient $\alpha_1^{(k)} := c_i$ of the term $X^{\lambda_1}$, for $i = 1, \ldots, n_1$.

By lemma 3.3 the strict transform of the factor $F_i = 0$ in $\text{Spec}\mathbb{C}[\sigma^\vee \cap Q'_1]$ is defined by $X^{-e_1\lambda_1} F_i$. It follows also that the point of intersection with the orbit $U_1 = \cdots = U_d = 0$ with the strict transform of $F_i$ and $F_j$ for $i \neq j$ are distinct points since $c_i \neq c_j$ and that $n'_1 = 1$.

For simplicity we study $F_1$. Let $\zeta$ denote a root of $F_1$. We decompose the sum $YX^{-\lambda_1} - X^{-\lambda_1}\zeta$ as

$$YX^{-\lambda_1} - X^{-\lambda_1}\zeta = YX^{-\lambda_1} - c_i + \cdots + X^{-\lambda_1}(p_2 + \cdots + p_g).$$

The strict transform of the approximate hypersurface of $f$ gives an approximate hypersuface for the strict transform of $F_1$ which is smooth and defined by $W'_1 = X^{-\lambda_1}(Y - p_1) = X^{-\lambda_1}Y - c_i + \ldots$.

The terms of the series $X^{-\lambda_1}(p_2 + \cdots + p_g)$ can be viewed in the ring $\mathbb{C}[\bigoplus_{j=1}^d \frac{u_j}{e_1}\mathbb{Z}_{\geq 0}]$ by $X^\gamma \mapsto U^{A(\sigma)\gamma}$. We apply then lemma 2.4 to conclude that it defines a quasi-ordinary branch with characteristic exponents $\{A(\sigma)(\lambda_i - \lambda_1)_{i=2}^g\}$. $\diamond$

**Remark 3.9** *We can apply this proposition also for the strict transforms $\tilde{S}_i$ of the approximate hypersurfaces of $S_i$ for $i = 2, \ldots, g$.*

The coordinates $(U_1, \ldots, U_d, W_1)$ are good for $(\tilde{S}_i, O_\sigma)$ for $i = 2, \ldots, g$.

### 3.1.4 The resolution algorithm

We have seen that taking a suitable toric modification with respect to good coordinates decreases the "local complexity" of an embedded quasi-ordinary singularity of hypersurface. We continue this procedure to define an embedded resolution.

First, we built the regular toroidal embedding $(\mathcal{U}_1, \mathcal{X}_1)$ associated to the first quasi-ordinary hypersurface. Then, with respect to the canonical coordinates at each $0$-stratum, we have a quasi-ordinary projection of the strict transform, in good coordinates. And the germ defined by the strict transform is parametrized by a quasi-ordinary branch with $g-1$ characteristic monomials. Hence, the conditions stated at the beginning of this section are satisfied locally.

We are going to iterate this process by taking the modification compatible with the Newton polyhedron of the strict transform $\tilde{f}$ with respect to the canonical coordinates defined at each $0$-strata. To this globally, we will use the proper map defined by a regular subdivision of the complex polyhedral complex associated to the toroidal embedding $(\mathcal{U}_1, \mathcal{X}_1)$ that we define as follows.

By proposition 3.8, the canonical coordinates $U_1, \ldots, U_d, W_1$ associated to a $0$-stratum $O_\sigma$ are good. This means that the Newton polyhedron of $\tilde{f}_\sigma$ viewed as an element of $\mathbb{C}\{U_1, \ldots, U_d, W_1\}$ has only one compact face, which is an edge with vertices $(0, \ldots, 0, e_1)$ and $(e_1 \tilde{\lambda}_2(\sigma), 0)$, where $\tilde{\lambda}_2(\sigma) \in \mathbb{Q}^d_{\geq 0}$ is the image of the second characteristic exponent $\lambda_2$ by the affine map (3.6) corresponding to $\sigma$ in proposition 3.8, and the integer $e_1$ is defined in formula (2.3). The linear subspace of $(\mathbb{R}^{d+1})^*$, orthogonal to this compact face induces a subdivision of the regular cone $\sigma$ in the fan $\Theta_1$ (this works exactly as at the beginning of this section by replacing the cone $(\mathbb{R}^{d+1})^*_{\geq 0}$ by $\sigma$). By lemma 3.7, this linear subspace does not depend on the cone $\sigma$, hence we obtain a linear subspace $\ell_2$ that subdivides all the $(d+1)$-dimensional cones in the fan $\Theta_1$.

The toroidal modification $\pi_2 : \mathcal{X}_2 \to \mathcal{X}_1$ associated to the subdivision $\Sigma_2$ is a proper map satisfying the conditions we need (see remark 2.1). We can study the strict transform of $\tilde{S}$ by $\pi_2$ on the charts defined by cones contained in the half-space $l_2 \geq 0$ by lemma 3.1. Using remark 3.2, we define the strict transforms $\bar{f}_i$ by $\pi_2$ of the functions $\tilde{f}_i$, these being the strict transforms of the semi-roots $q_i$ by $\pi_1$, for $i = 2, \ldots, g$.

It follows from proposition 3.8 and lemma 3.5 that the hypersurface in $\mathcal{X}_2$ defined by the strict transform of $\tilde{f}_2$ by $\pi_2$ is smooth and transversal to the stratification of the critical divisor of $\pi_2$. Since it coincides with the strict transform of the second approximate hypersurface by

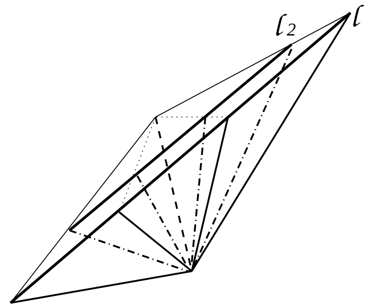


Figure 3.4: The subdivision of $\Theta_1$ defined by $\ell_2$

the modification $\pi_1 \circ \pi_2$, a fortiori we obtain that this hypersurface is also transversal to an stratification of the critical divisor of $\pi_1 \circ \pi_2$.

Thus, we can define in a similar way a new toroidal embedding $\mathcal{U}_2 \subset \mathcal{X}_2$, which has by lemma 3.6 an associated convex polyhedral complex $\Theta_2$ isomorphic to the $(d+1)$-regular fan over $\Sigma_2 \cap \ell_2$. Also by proposition 3.8 the strict transform of the hypersurface $S$ by $\pi_1 \circ \pi_2$ defines germs of quasi-ordinary singularity at the $0$-orbits of the embedding $\mathcal{U}_2 \subset \mathcal{X}_2$. The canonical coordinates provide a quasi-ordinary projection in good coordinates, parametrized by a quasi-ordinary branch with $g-2$ characteristic monomials at each of these points.

We can iterate to get in $g$ steps:

1. A proper morphism $\pi_k : \mathcal{X}_k \to \mathcal{X}_{k-1}$, between non singular complex varieties of dimension $d+1$, for $k = 1, \dots, g$.

2. A regular toroidal embedding $(\mathcal{X}_k, \mathcal{U}_k)$ corresponding to the strict transform of the $k^{th}$-approximate surface with associated polyhedral complex the fan $\Theta_k$, for $k = 1, \dots, g$,

3. A regular subdivision $\Sigma_k$ of $\Theta_k$ compatible with the linear subspace $\ell_k$ orthogonal to the compact edge of the Newton polyhedron of the strict transform of $f$ by the morphism $\pi_1 \circ \dots \circ \pi_k$, with respect to canonical coordinates, for $k = 1, \dots, g-1$.

4. The morphism $\pi_{k+1} : \mathcal{X}_{k+1} \to \mathcal{X}_k$, is defined by the subdivision $\Sigma_k$, for $k = 1, \dots, g-1$.

5. The strict transform of $f$ by $\pi_1 \circ \dots \circ \pi_k$ has only quasi-ordinary singularities at the $0$-strata of the embedding $(\mathcal{X}_k, \mathcal{U}_k)$, which are parametrized by a quasi-ordinary branch with $g-k$ characteristic monomials, for $k = 1, \dots, g$.

Hence the strict transform of $S$ by the composite $\pi = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_g$ is non singular, and by the lemma 3.5 it will be transverse to a stratification of the critical locus of $\pi_g$, (a fortiori a stratification of the critical locus of $\pi$). So we have an embedded "toroidal"pseudo-resolution of the hypersurface $S$.

**Theorem 3.1** *The proper morphism $\pi : \mathcal{X}_g \to \mathbb{C}^{d+1}$ defined above is an embedded resolution of the quasi-ordinary hypersurface germ $(S, 0)$.*

*Proof.* We verify that this procedure gives us a resolution of the quasi-ordinary hypersurface $S$. We know that the singular locus of $S$ is contained in its intersection $Z_i$ with the coordinate planes $X_i = 0$, for $i = 1, \dots, d$. We have to verify only that $\pi_i$ is an isomorphism outside the singular locus of the strict transform of $S$ by $\pi_1 \circ \pi_2 \circ \cdots \circ \pi_{i-1}$. We have shown in lemma 3.5 the case, $g = 1$, so we suppose that $g > 1$. We show first that $\pi_1$ is an isomorphism outside of the singular locus of $S$. We only have to study the case where $Z_i$ is not a component of the singular locus of $S$.

If $\lambda_{1,i} = 0$, then we have that the fan $\Sigma_1 \cap \ell$ is a subdivision of a regular fan $\Xi$, such that all its $(d+1)$-dimensional cones contain the vector $e_i$. Clearly, this condition implies that the map $\pi(\Xi)$ is not ramified over $X_i = 0$. Then the map $\pi_1$ on the open set $Z(\Sigma_1 \cap \ell)$ is equal to a composite of toric maps corresponding to these subdivisions. We can argue as in the end of the proof of lemma 3.5 to show that $\pi_1$ is an isomorphism over generic points of $Z_i$. Also, the equality $\lambda_{1,i} = 0$ implies that the canonical basis vector $e_i$ defines an edge of the fan $\Sigma_1 \cap \ell$. If we take the cone $\sigma = \langle a^1, \dots, a^d \rangle$ of the fan $\Sigma_1 \cap \ell$ with $a^i = e_i$, then the $i$-coordinate of the first characteristic exponent of the quasi-ordinary branch parametrizing the strict transform at the $0$-stratum $O_\sigma$ to is equal to $\lambda_{2,i}$, by proposition 3.8.

Now we continue the same analysis for the modification $\pi_2 : \mathcal{X}_2 \to \mathcal{X}_1$ associated to the strict transform of $S$ by $\pi_1$, and we study it locally as before. We follow the algorithm, and we arrive at two different cases:

1. All $\lambda_{j,i}$ are $0$ for $j = 1, \dots, g$, (equisingular case). Then, the composition $\pi = \pi_1 \circ \cdots \circ \pi_g$ is an isomorphism over generic points of $Z_i = 0$.

2. We have $\lambda_{1,i} = \cdots = \lambda_{g-1,i} = 0$, and $\lambda_{g,i} = (n_g)^{-1}$, (with respect to a sequence of cones in $\Theta_k \cap \ell_k$, for $k = 0, 1, \dots, g-1$). Then the strict transform of $Z_i$ by $\pi_1 \circ \cdots \circ \pi_{g-1}$ is not contained in the singular locus of of the strict transform of $S$, and then we can apply lemma 3.5 and the previous discussion locally at each $0$-stratum to obtain the result. $\diamond$

## 3.2 Deformation of the quasi-ordinary hypersurface and simultaneous resolution

In this section we generalize the work of Goldin and Teissier from plane branches to quasi-ordinary hypersurface singularities (see [G-T]). We construct a $d$-parameter specialization $\mathcal{S}$ of the variety $S$ into the monomial variety $\mathcal{S}_0$ defined in the previous chapter. The total deformation space $\mathcal{S}$ is itself a quasi-ordinary singularity (non hypersurface in general). We describe the toric morphisms providing a toric embedded resolution of singularities of $\mathcal{S}_0$. We will use a simultaneous parametrization of $\mathcal{S}$ to show that all the fibers are quasi-ordinary singularities "with semigroup $\Gamma$". We describe a set of equations of $\mathcal{S}$ and their Newton polyhedra from a fixed set of equations of the monomial variety using some properties of flatness.

Finally, we show that the toric morphism which resolves the monomial variety provides a toric embedded resolution of the variety $S$ in this particular embedding, and furthermore that it induces an embedded resolution of the deformation $\mathcal{S}$ and all the members of the family.

### 3.2.1 Toric resolution of the monomial variety

We show that the non normal toric variety $\mathcal{S}_0$ admits an embedded toric resolution, constructed from a fixed set of binomials equations $h_1 = 0, \cdots, h_g = 0$ defined by proposition 2.25.

Denote by $h$ the polynomial $h_1 \dots h_g$. The polyhedron $\mathcal{N}(h)$ has a unique $g$-dimensional compact face $\mathcal{E}$ since it is equal to the Minkowski sum $\sum_{j=1}^{g} \mathcal{N}(h_j)$, and by proposition 2.25 and lemma 2.11 these polyhedra have affinely independent edges $\mathcal{E}_i$.

The cone $\sigma(\mathcal{E})$ of vectors in $(\mathbb{R}^{d+g})^*_{\geq 0}$ associated to the compact face $\mathcal{E}$ of $\mathcal{N}(h)$ is of dimension $d$ and generates a linear subspace $\ell$, which does not depend on the choice of the equations $h_1, \dots, h_g$ of the monomial variety in lemma 2.25. It is the vector subspace orthogonal to the kernel of the homomorphism $\mathbb{R}^{d+g} \to \mathbb{R}^d$, that maps the canonical basis vector $e_i$ to $\bar{\lambda}_i$. If the binomial $h_i$ is of the form $h_i = U^{v_i} - U^{w_i}$, any vector $a$ belongs to the subspace $\ell$ if and only if $\langle a, v_i - w_i \rangle = 0$ for $i = 1, \dots, g$ (see Proposition 2.14). We deduce from this that, $\sigma(\mathcal{E}) = \ell \cap (\mathbb{R}^{d+g})^*_{\geq 0}$. We call $\ell$ the linear subspace associated to the monomial variety.

Let $\Sigma$ be a regular fan supported on $\mathbb{R}^{d+g}_{\geq 0}$ compatible with each polyhedron $\mathcal{N}(h_j)$. The toric morphism $\pi(\Sigma) : Z(\Sigma) \to \mathbb{C}^{d+g}$ is described in the affine chart corresponding to a cone $\sigma = \langle a^1, \dots, a^{d+g} \rangle$ by formula (2.1), where the variable $X$ is changed to $U$, the variable $U$ is changed to $Y$ and the integer $d+1$ is changed to $d+g$.

**Lemma 3.10** *Let $a$ be a primitive vector defining an edge of the fan $\Sigma$. The divisor $D(a)$ intersects the strict transform $\tilde{\mathcal{S}}_0$ of the monomial variety if and only if the vector $a$ belongs to the linear subspace $\ell$.*

*Proof.* Lets consider any chart $Z(\sigma)$ intersecting the divisor $D(a)$. The cone $\sigma$ is then defined by the vectors $a = a^1, a^2, \dots, a^{d+g}$. In this coordinates the divisor $D(a)$ is defined by $Y_1 = 0$. The strict transform of $h_i$ by the morphism $\pi(\sigma)$ is of the form $\widetilde{h_i} = \pm(1 - Y^{\alpha^i})$, where the coordinate $\alpha^i_j = \langle a_j, v_i - w_i \rangle$ is a non negative integer. Then, $\widetilde{h_i} = 0$ intersects $D(a^1)$ if and only if $\alpha^i_1 = \langle a^1, v_i - w_i \rangle = 0$. ◇

Teissier shows that any irreducible binomial variety admits a toric resolution, ([T3] prop. 6.3). We give a proof of this fact in a particular case.

**Proposition 3.11** *The morphism $\pi(\Sigma) : Z(\Sigma) \to \mathbb{C}^{d+g}$ is a toric embedded resolution of the monomial variety.*

*Proof.* The set of binomials $h_1, \dots, h_g$ are non degenerate, i.e., the Jacobian matrix of $h_1, \dots, h_g$ is of maximal rank $g$ on any point of the torus $(\mathbb{C}^*)^{d+g}$. By corollary 5.1 of [G-T], the morphism $\pi(\Sigma)$ is a toric pseudo-resolution.

We show that the restriction $\tilde{\mathcal{S}}_0 \to \mathcal{S}_0$ of $\pi(\Sigma)$ to the strict transform, is an isomorphism outside the singular locus of $\mathcal{S}_0$.

The binomial $h_g$ is of the form $U_{d+g}^{n_g} - U_1^{b_1} \dots U_{d+g-1}^{b_{d+g-1}}$ and defines a quasi-ordinary hypersurface $H$ with only one characteristic exponent in good coordinates. Since the fan $\Sigma$ is compatible with $\mathcal{N}(h_g)$, the morphism $\pi(\Sigma)$ defines an isomorphism $\tilde{H} \to H$ outside the singular locus of $H$ by lemma 3.5. We can verify using proposition 2.28 and theorem 2.4, that the smooth points of the monomial variety $\mathcal{S}_0$ are contained in the smooth part of the hypersurface $H$ and the result follows. ◇

### 3.2.2 Deformation of the quasi-ordinary singularity

We define a $(V_1, \dots, V_d)$-parameter deformation $\mathcal{S}$ of the variety $S$. The deformation is a quasi-ordinary singularity and we describe its singular locus. The deformation is defined by a simultaneous parametrization which is a homomorphism compatible suitable filtrations. We describe a set of equations of the deformation from a set of equations of the monomial variety and we give some information about their Newton polyhedra. We do this by showing that the algebra $\mathcal{R}$ of the deformation is a flat $\mathbb{C}\{V\}$-algebra. We give a reminder of results about flatness in commutative algebra at the end of the chapter.

We introduce some notations:

Consider any ring $A$ and $0 \neq \phi \in C$ of formal power series in $d$ variables $T = (T_1, \dots, T_d)$ with coefficients in $A$, i.e., $C = A[[T]]$. The *Newton initial polynomial* $\mathrm{in}_{\mathcal{N}}(\phi)$ is the sum of the terms of $\phi$ having exponents lying on the compact faces of the Newton polyhedron of $\phi$ (this polynomial appears also on [Ko]).

We denote by $\mathcal{N}_c$ the union of compact faces of a polyhedron $\mathcal{N}$. The set of points *below the compact part* $\mathcal{N}_c$ of the polyhedron $\mathcal{N} \in \mathcal{J}$ is

$$B(\mathcal{N}) := \{u \in \mathbb{R}^d_{\geq 0} /\ \text{there exists } u' \in \mathcal{N}_c \text{ such that } u \leq u' \text{ coordinate wise }\}$$

The set of points *strictly below* the compact part of the polyhedron $\mathcal{N}$ is

$$B^-(\mathcal{N}) := \{u \in \mathbb{R}^d_{\geq 0} /\ \text{there exists } u' \in \mathcal{N}_c \text{ such that } u \lneq u' (\text{where } \lneq \text{ means } \neq \text{ and } \leq\ )\ \}.$$

The *support* $\mathrm{supp}(\phi)$ of a series $\phi \in A[[T]]$ is the set of exponents of terms with non zero coefficient. For any Newton polyhedron $\mathcal{N}$ we define the ideals of $C$:

$$\mathfrak{F}(C)_{\mathcal{N}} = \{\phi \in C \,/\, \mathrm{supp}(\phi) \cap B^-(\mathcal{N}) = \emptyset\}, \qquad \mathfrak{F}(C)^+_{\mathcal{N}} = \{\phi \in C \,/\, \mathrm{supp}(\phi) \cap B(\mathcal{N}) = \emptyset\}.$$

Let $\mu_1, \dots, \mu_g$ be any non zero integral vectors in $\mathbb{R}^d_{\geq 0}$. We denote by $B$ the ring $A[[U_1, \dots, U_g]]$ We give *weight* $\mu_i$ to the variable $U_i$, for $i = 1, \dots, g$ and define the *weight of term* $U^\alpha$, appearing in a formal power series $f = \sum c_\alpha U^\alpha$, is the vector $\alpha_1\mu_1 + \cdots + \alpha_g\mu_g$. The weighted support, $\mathrm{supp}_w(f)$, of $f$ is the set of weights of terms appearing in $f$. We say that a polynomial in $A[U]$ is *weighted homogeneous* of degree $\gamma \in \mathbb{Z}^d_{\geq 0}$ if all its terms have weight $\gamma$. We define analogously the ideals:

$$\mathfrak{F}_w(B)_{\mathcal{N}} = \{\phi \in B \,/\, \mathrm{supp}_w(\phi) \cap B^-(\mathcal{N}) = \emptyset\}, \qquad \mathfrak{F}_w(B)^+_{\mathcal{N}} = \{\phi \in B \,/\, \mathrm{supp}_w(\phi) \cap B(\mathcal{N}) = \emptyset\}$$

Given any irrational vector $\omega \in \mathbb{R}^d_{>0}$ we can consider on the ring $C$ the filtration defined by the associated valuation (see [GP]) and on the ring $B$ the weighted filtration defined by analogously. The associated graded rings are $A[T]$ and $A[U]$ respectively with the grading defined by $\omega$ (resp. and $\omega$ and the weights). Any a non zero homomorphism of rings $\psi : B \to C$ such that

$$\mathrm{supp}(\psi(h)) \subset \mathrm{supp}_w(h) + \mathbb{R}^d_{\geq 0},\ \forall h \in A[[U]] \tag{3.11}$$

is compatible with the filtrations defined by $\omega$. In this case $\psi$ induces a homomorphism in the graded rings $\mathrm{gr}_w A[U] \to \mathrm{gr} A[T]$ with the grading defined by $\omega$.

We are going to define a $d$-parameter family of germs of quasi-ordinary singularities embedded in $\mathbb{C}^{d+g}$ whose general member is isomorphic to the germ $(S,0)$ and which has the monomial variety $(X,0)$ as a special member.

The total space $\mathcal{S} \subset (\mathbb{C}^{d+g} \times \mathbb{C}^d, 0)$ of this family is defined by the kernel of the homomorphism of $\mathbb{C}\{V\}$-algebras that maps $U_i$ to $\xi_i(VT)V^{-\gamma_i}$ for $i = 1, \ldots, d+g$, i.e., the homomorphism $\psi : \mathbb{C}\{V, U\} \to \mathbb{C}\{V, T\}$ defined by

$$U_{d+i} = \xi_{d+i}(VT)V^{-\gamma_{d+i}} \text{ for } i = 1, \ldots, d+g. \tag{3.12}$$

(where we denote the set of variables $T_1, \ldots, T_d$ by $T$, $U_1, \ldots, U_{d+g}$ by $U$, $V_1, \ldots, V_d$ by $V$, and $V_1T_1, \ldots, V_dT_d$ by $VT$).

**Lemma 3.12** *The* $\mathbb{C}\{V\}$*-module* $\mathbb{C}\{V,T\}/Im(\psi)$ *is flat* [1].

*Proof.* The homomorphism $\psi$ is a quasi-finite local homomorphism of analytic algebras since $\psi((U,V))$ defines an ideal containing a power of the maximal ideal of $\mathbb{C}\{V,T\}$. Hence, $\mathbb{C}\{V,T\}$ is a $\mathrm{Im}(\psi)$-module of finite type (see [K], Corollary 45.6).

We denote by $A$ the ring $\mathbb{C}[[V]]$, and by $B$ the ring $\mathbb{C}[[V,U]]$. The homomorphism $\psi$ extends to an homomorphism of $A$-modules $\hat{\psi} : B \to C$, where $C$ is the completion of the $\mathrm{Im}(\psi)$-module $\mathbb{C}\{V,T\}$ with respect to the ideal $\psi((V,U))$. It is easy to see that $C$ coincides, as a module over the ring $\mathrm{Im}(\hat{\psi})$, with $\mathbb{C}[[T,V]]$. Since the rings $\mathbb{C}\{V\}$, $\mathbb{C}\{V,U\}$ are Noetherian and the $\mathrm{Im}(\psi)$-module, $\mathbb{C}\{V,T\}/\mathrm{Im}(\psi)$, is of finite type, we deduce from assertion *3* in lemma 3.26 that the $\mathbb{C}\{V\}$-module, $\mathbb{C}\{V,T\}/\mathrm{Im}(\psi)$ is flat if and only if the $A$-module $C/\mathrm{Im}(\hat{\psi})$ is flat.

We consider the ring $B$ as the ring of formal power series in the variables $U$ with coefficients in the domain $A$. We give weight $\bar{\lambda}_i$ to the variables $U_i$ for $i = 1, \ldots, d+g$. We consider $C$ as a ring of formal power series in the variables $T$ with coefficients on the domain $A$. For any $f \in B$ we have $\mathrm{supp}(\psi(f)) \subset \mathrm{supp}_w(f) + \mathbb{R}^d_{\geq 0}$ thus the homomorphism $\hat{\psi}$ is compatible the filtrations defined by any fixed irrational vector $\omega \in \mathbb{R}^d_{\geq 0}$ by (3.11). The graded homomorphism of $A$-algebras $\mathrm{gr}(\hat{\psi}) : A[U] \to A[T]$ induced by $\hat{\psi}$ applies $U_i \mapsto T^{\gamma_i}$, since this is the corresponding $\omega$-initial polynomial of $\hat{\psi}(U_i)$, for $i = 1, \ldots, d+g$. If we forget the gradings this ring homomorphism is independent of $\omega$.

[1] See section 3.4 for definitions and results of commutative algebra.

By assertion *1* in lemma 3.26, the $A$-module $C/\mathrm{Im}(\hat{\psi})$ is flat is and only if for any ideal $I$ of $A$ we have $\hat{\psi}(B) \cap IC \subset I\hat{\psi}(B)$.

We show that for any $g \in \hat{\psi}(B) \cap IC$, there is an element $l \in IB$ such that

$$\psi(g) = \psi(l) \text{ and } \mathrm{supp}_w(\mathrm{in}(l)) = \mathrm{supp}(\mathrm{in}(g)). \tag{3.13}$$

The hypothesis on the element $g$ implies that the Newton initial polynomial, $\mathrm{in}(g)$, belongs to the ideal $IA[T]$ and also to the image of the homomorphism $\mathrm{gr}(\hat{\psi})$. Since the image of the restriction of the homomorphism $\mathrm{gr}(\hat{\psi})$ to $\mathbb{C}[U]$ is contained in $\mathbb{C}[T]$, the $A$-module $A[T]/\mathrm{Im}\ (\mathrm{gr}(\hat{\psi}))$ is free, and thus flat. By lemma 3.26, this implies that

$$I \cdot \mathrm{Im}\ (\mathrm{gr}(\hat{\psi})) = \mathrm{Im}\ (\mathrm{gr}(\hat{\psi})) \cap IA[T].$$

Therefore, there exists a polynomial $l_0 \in IA[U]$ such that $\mathrm{gr}(\psi)(l_0) = \mathrm{in}(g)$. Since the homomorphism $\mathrm{gr}(\hat{\psi})$ is defined by monomials we can find $l_0$ such that $\mathrm{supp}_w \mathrm{in}(l_0) = \mathrm{supp}(\mathrm{in}(g))$.

If $g - \psi(l_0) = 0$ we are done, otherwise we replace $g$ by $g_1 := g - \psi(l_0)$ and notice that it belongs to the ideal $\mathfrak{F}(C)^+_{\mathcal{N}(g)}$. By induction, we define a sequence $l_j$ of polynomials in the ideal $IA[U] \bigcap \mathfrak{F}_w(B)^+_{\mathcal{N}(l_{j-1})}$ for all $j \geq 0$, such that if $g_j = g - \sum_{i=0}^{j-1} \hat{\psi}(l_i)$ we have $\mathrm{gr}(\hat{\psi})(l_{j+1}) = \mathrm{in}(g_j)$ and $\mathrm{supp}_w(\mathrm{in}(l_{j+1})) = \mathrm{supp}(\mathrm{in}(g_j))$.

It is easy to see that the element $l = \sum_{j \geq 0} l_j$ belongs to the $(U)$-adic completion of $IB$, which is $IB$ (since $B$ is Noetherian and complete). Also, we have $g = \sum_{j \geq 0} \hat{\psi}(l_j) = \hat{\psi}(l)$ since $C$ is separated, hence the element $g$ belongs to $I\psi(B)$ and we have $\mathrm{supp}_w \mathrm{in}(l) = \mathrm{supp}(\mathrm{in}(g))$.
$\diamond$

Any Newton polyhedron $\mathcal{N}$ in $\mathbb{R}^{d+g}$ defines two ideals $\mathfrak{B} := \mathfrak{F}_w(B) \cap \mathbb{C}\{V, U\}$ and $\mathfrak{C} := \mathfrak{F}(C) \cap \mathbb{C}\{V, T\}$ of the rings $\mathbb{C}\{V, U\}$ and $\mathbb{C}\{V, T\}$ respectively. These ideals are generated by a finite set of monomials in $U$ and $T$ respectively, $\mathfrak{B}$ and $\mathfrak{C}$ are flat $\mathbb{C}\{V\}$-modules by example 3.2. We deduce using that $\psi$ is quasi-finite and that $\mathrm{Im}(\psi)$ is a Noetherian ring, that $\mathfrak{C}/\psi(\mathfrak{B})$ is of finite type over $\mathrm{Im}(\psi)$.

**Remark 3.13** *For any Newton polyhedron $\mathcal{N}$ in $\mathbb{R}^{d+g}$, the $\mathbb{C}\{V\}$-module $\mathfrak{C}/\psi(\mathfrak{B})$ is flat.*

To see this, we argue as the proof lemma 3.12. The completion $\hat{\mathfrak{B}}$ of $\mathfrak{B}$ with respect to the $(U)$-adic topology is equal to $\mathfrak{F}_w(B)$, and the completion $\hat{\mathfrak{C}}$ of the $\mathrm{Im}\ (\psi)$-module $\mathfrak{C}$ with respect to the ideal $\psi((V, U))$ is equal to $\mathfrak{F}(C)$. The completion $\hat{\mathfrak{C}}$ is a flat $A$-module since it is generated by monomials in $T$. By assertion *3* of lemma 3.26 and example 3.2, it is enough to show that $\hat{\mathfrak{C}}/\hat{\psi}(\hat{\mathfrak{B}})$ is a flat $A$-module. This in turn follows analogously from assertion (3.13).

**Proposition 3.14** *Given any binomials $h_1, \dots, h_g$, generating the ideal of the monomial variety in the proof of proposition 2.25, there exist $H_1, \dots, H_g \in \mathbb{C}\{V, U\}$ generating the kernel of $\psi$ such that*

(a) $H_i = h_i - G_i$, *with $G_i \in (V)\mathbb{C}\{V, U\}$*

(b) *The weighted Newton initial polynomial of $H_i$ is equal to $h_i$.*

*Proof.* The binomial $h_i$ is weighted homogeneous of degree $n_i\gamma_{d+i}$, for $i = 1, \dots, g$. It follows from the definition that the image $\psi(h_i)$ belongs to the ideal $(V)\mathfrak{C}^+_{\mathcal{N}(h_i)}$. By remark 3.13 and assertion *2* of lemma 3.26 there is an element $G \in (V)\mathfrak{B}_{\mathcal{N}(\psi(h_i))}$ such that $\psi(h_i) = \psi(G_i)$, for $i = 1, \dots, d$. Then the series $H_i := h_i - G_i$ belong to the kernel of the homomorphism $\psi$ and they verify the conditions (a) and (b) above.

Fix any irrational vector $\omega \in \mathbb{R}^d_{>0}$, We can use Proposition 12 N$^o$ 9, §2, Chapitre III, of [Bbk] to show that the kernel of $\hat{\psi}$ is generated by the elements $H_i = h_i - G_i$, for $i = 1, \dots, g$, since their initial polynomials generate the graded ideal

$$\mathrm{gr}_w(\mathrm{Ker}(\hat{\psi})) = \mathrm{Ker}(\mathrm{gr}(\hat{\psi})).$$

for the grading defined by any fixed irrational vector $\omega \in \mathbb{R}^d_{>0}$. Then, the ideal $\mathrm{Ker}(\psi) = \mathbb{C}\{V, U\} \cap (H_1, \dots, H_g)\mathbb{C}[[V, U]]$ is generated by $(H_1, \dots, H_g)$, by assertion *4* in lemma 3.26, since $\mathbb{C}[[V, U]]$ is a faithfully flat $\mathbb{C}\{V, U\}$-module (see example 3.1, number 3). ⋄

We denote by $\mathcal{S}$ a representative of the total space of the deformation $\mathcal{S} \subset (\mathbb{C}^{d+g} \times \mathbb{C}^d, 0)$ of the quasi-ordinary singularity. We denote by $\mathcal{S}_v$ for $v \in \mathbb{C}^d$, the members of this family, i.e., the fibers of the projection $q : \mathcal{S} \to \mathbb{C}^d$ defined by $(U, V) \mapsto V$.

The homomorphism $\psi$ defines also a *simultaneous parametrization* of all the fibers. We mean by this that the fiber $\mathcal{S}_v$ is defined by the kernel of the homomorphism

$$\psi_v : \mathbb{C}\{U\} \to \mathbb{C}\{T\} \text{ by } U_i \mapsto \xi_i(vT)v^{-\gamma_i}.$$

In particular we find the fiber $\mathcal{S}_0$ is equal to the monomial variety, and the fiber corresponding to $v = (1, \dots, 1)$ gives the embedding of the quasi-ordinary singularity $S$ in $\mathbb{C}^{d+g}$ defined by formulae (2.12).

**Remark 3.15** *If we replace $V$ by $v \in \mathbb{C}^d$, in the equations $H_i = 0$ of the above proposition we get series $H_{v,i} \in \mathbb{C}\{U\}$ which generate the defining ideal of the fiber $\mathcal{S}_v$.*

It is clear from the definitions that $H_{v,i}$ are well defined and belong to the kernel of $\psi_v$. This homomorphism is compatible with the filtrations defined by any fixed irrational vector $\omega \in \mathbb{R}^d_{>0}$

on $\mathbb{C}\{U\}$ and $\mathbb{C}\{V\}$ respectively. We can argue as in the proof of the previous proposition to show that $H_{v,i}$ generate the kernel of $\psi_v$.

**Proposition 3.16** *For any $v \in (\mathbb{C}^*)^d$, the ideal of the fiber $\mathcal{S}_v$ is generated by elements of the form:*

$$\begin{array}{l} h_i + c_i(v)U_{d+i+1} + r_i(U) = 0, \quad \text{with } c_i(v) \in \mathbb{C}^*, \text{ for } i = 1, \dots, g-1, \\ h_g + r_g(U), \end{array} \tag{3.14}$$

*where $r_i \in \mathbb{C}\{U_1, \dots, U_{d+i}\}$ and its terms have weight $>$ than $n_i\gamma_i$.*

*Proof.* The power series $H_{v,1}, \dots, H_{v,g}$ generate the defining ideal $I$ of the fiber $\mathcal{S}_v$ in $\mathbb{C}\{U\}$. The analytic algebra $R$ of the hypersurface $\mathcal{S}_v$ is generated by $U_1, \dots, U_{d+1} \mod I$. This implies that we have relations in the ring $\mathbb{C}\{U\}$ of the form:

$$U_{d+j+1} = P_j + \alpha_{j,1}H_{v,1} + \dots + \alpha_{j,1}H_{v,g}$$

where $P_j$ is a power series in the variables $U_1, \dots, U_{d+1}$ and $\alpha_{j,1}$ are power series in the variables $U_1, \dots, U_{d+g}$, for $j = 1, \dots, g-1$.

The relation obtained for $j = 1$ implies that $U_{d+2}$ appears linearly on one of the summands of the right-hand side. It cannot appear neither in $P_1$ nor in $\alpha_{2,1}H_{v,2} + \dots + \alpha_{j,1}H_{v,g}$ since the weight $\gamma_{d+2}$ of $U_{d+2}$ is strictly smaller than the possible weights of the terms appearing in $H_{v,2}, \dots, H_{v,g}$ by Proposition 3.14 and Lemma 2.11. Hence it must appear in $H_{v,1}$ since this series is not a unit. By the Weierstrass Preparation Theorem, we obtain a non unit power series $\phi_1 \in \mathbb{C}\{U_1, \dots, U_{d+1}, U_{d+3}, \dots, U_{d+g}\}$ such that $H_{v,1}(U_1, \dots, U_{d+1}, \phi_1, U_{d+3}, \dots, U_{d+g}) = 0$. When we substitute $U_{d+2}$ by $\phi_1$ in the relation obtained for $j = 2$, we obtain by the same argument that $U_{d+3}$ appears in $H_2$. It follows by induction that $U_{d+j+1}$ appears in the series $H_{v,j}$ with non zero coefficient for $j = 1, \dots, g-1$. We can replace $H_{v,j}$ by the associated Weierstrass polynomial in $U_{d+j+1}$ for $j = 1, \dots, g-1$. We obtain generators of the form 3.16, since multiplication by a unit in $\mathbb{C}\{U\}$ does not modify the weighted Newton polyhedron. ◇

**Remark 3.17** *We recover an equation for $S \subset \mathbb{C}^{d+1}$ by eliminating $U_{d+2}, \dots, U_{d+g}$ from the first $g-1$ equations of $\mathcal{S}_v$ for $v = (1, \dots, 1)$.*

By eliminating, $U_{d+2}, \dots, U_{d+g}$ from the first $g-1$ generators of the fiber, we get a polynomial of degree $n$ in $U_{d+1}$ and coefficients in $\mathbb{C}\{U_1, \dots, U_d\}$, that gives the equation of the embedding $S \subset \mathbb{C}^{d+1}$ defined by $(\xi_1, \dots, \xi_{d+1})$.

Consider the morphism of germs defining the parametrization (3.12) above.

$$\psi^* : (\mathbb{C}^d \times \mathbb{C}^d, 0) \to (\mathcal{S}, 0) \subset (\mathbb{C}^{d+g} \times \mathbb{C}^d, 0)$$
$$(t, v) \mapsto (u, v) \text{ with } u = (t^{\gamma_1}, \dots, t^{\gamma_d}, \xi_{d+1}(vt)v^{-\gamma_{d+1}}, \dots, \xi_{d+g}(vt)v^{-\gamma_{d+g}})$$

We have an action of the torus $(\mathbb{C}^*)^d$, on the product $\mathbb{C}^d \times \mathbb{C}^d$, given by

$$w \cdot (t, v) = (w^{-1} \cdot t, w \cdot v) = (w_1^{-1} t_1, \dots, w_d^{-1} t_d; w_1 v_1, \dots, w_d v_d).$$

We define an action of $(\mathbb{C}^*)^d$ on $\mathbb{C}^{d+g} \times \mathbb{C}^d$, by $w \cdot (u, v) = (w \cdot u, w \cdot v)$ where

$$w \cdot u := (w^{-\gamma_1} u_1, \dots, w^{-\gamma_{d+g}} u_{d+g}) \text{ and } w \cdot v := (w_1 v_1, \dots, w_d v_d).$$

It follows from the definitions that, the parametrization above is an *equivariant morphism* for these actions, i.e., we have

$$\psi^*(w \cdot (t, v)) = w \cdot \psi^*(t, v), \text{ for all } w;$$

this implies that the action of $(\mathbb{C}^*)^d$ on $\mathbb{C}^{d+g} \times \mathbb{C}^d$ induces an action on $\mathcal{S}$.

**Lemma 3.18** *The map*

$$\begin{array}{rcl} S \times (\mathbb{C}^*)^d & \to & q^{-1}((\mathbb{C}^*)^d) \subset \mathcal{S} \\ ((u, (1, \dots, 1)), v) & \mapsto & (v \cdot u, v) \end{array}$$

*is an isomorphism, hence the fibers $\mathcal{S}_v$, for $v \in (\mathbb{C}^*)^d$ are isomorphic to the quasi-ordinary singularity $S$.*

*Proof.* The inverse of the morphism above is the map defined by:

$$\begin{array}{rcl} q^{-1}((\mathbb{C}^*)^d) & \to & S \times (\mathbb{C}^*)^d \\ (u, v) & \mapsto & (v^{-1} \cdot (u, v), v) \end{array}$$

These morphism are defined from the action of $(\mathbb{C}^*)^d$ and they induce an fiber-wise isomorphism $\mathcal{S}_v \cong S \times \{v\}$, for $v \in (\mathbb{C}^*)^d$. ◇

**Lemma 3.19** *The deformation space $\mathcal{S}$ and all the fibers $\mathcal{S}_v$ are quasi-ordinary singularities. The map*

$$\begin{array}{rcl} \mathbb{C}^{d+g} \times \mathbb{C}^d \supset (\mathcal{S}, 0) & \to & (\mathbb{C}^d \times \mathbb{C}^d, 0) \\ (u_1, \dots, u_{d+g}, v) & \mapsto & (u_1, \dots, u_d, v) \end{array}$$

*is a quasi-ordinary projection of degree* $n$ *and the same holds for its restrictions to the fibers* $\mathcal{S}_v$ *for all* $v \in \mathbb{C}^d$*. The equation of the discriminant locus of the quasi-ordinary projection of* $\mathcal{S}$ *(resp. of the fiber* $\mathcal{S}_v$ *for all* $v \in \mathbb{C}^d$*) is equal to* $\prod_{i \in I} U_i = 0$ *(where the set* $I \subset \{1, \dots, d\}$ *is defined in proposition 2.28). Moreover, for any* $v \in \mathbb{C}^d$ *and* $i, j \in I$*, the following assertions are equivalent:*

1. $\mathcal{S} \cap \{U_i = 0, U_j = 0\}_{i,j \in I} \subset$ *Sing* $(\mathcal{S})$.
2. $\mathcal{S}_v \cap \{U_i = 0, U_j = 0\}_{i,j \in I} \subset$ *Sing* $(\mathcal{S}_v)$.

*Proof.* On the open set $q^{-1}((\mathbb{C}^*)^d)$ of $\mathcal{S}$, the projection above is isomorphic to the morphism $S \times (\mathbb{C}^*)^d \mapsto \mathbb{C}^d \times (\mathbb{C}^*)^d$, that maps $(u_1, \dots, u_{d+g}, v) \mapsto (u_1, \dots, u_d, v)$ by lemma 3.18. By the definition of the embedding $S \subset \mathbb{C}^{d+g}$ (see formula (2.12)), this projection is quasi-ordinary of degree $n$ and it is not ramified on this open set outside the hypersurface $\prod_{i \in I} U_i = 0$ (where $I$ is defined in proposition 2.28).

We have shown that the projection is not ramified, outside the hypersurface $V_1 \dots V_d \prod_{i \in I} U_i = 0$. We show that $V_i = 0$ is not contained in the discriminant locus. The monomial variety corresponds to the fiber $\mathcal{S}_0$ and the restriction of this projection to it is a quasi-ordinary projection of degree $n$, unramified outside $\prod_{i \in I} U_i = 0$ by lemma 2.27 and proposition 2.28. This is an open condition and the result follows since this projection is equivariant for the action of $(\mathbb{C}^*)^d$ defined above. Hence the projection above is quasi-ordinary for $\mathcal{S}$ and all the fibers $\mathcal{S}_v$.

Theorem 2.4, describes exactly the condition for the subset $q^{-1}((\mathbb{C}^*)^d) \cap \{U_i = 0, U_j = 0\}_{i,j \in I}$, to be contained in the singular locus of $q^{-1}((\mathbb{C}^*)^d)$ (and then in the singular locus of $\mathcal{S}$). By proposition 2.28, this condition holds if and only if $\mathcal{S}_0 \cap \{U_i = 0, U_j = 0\}_{i,j \in I}$, is contained in the singular locus of the monomial variety. The equations of the fiber $\mathcal{S}_v$ are obtained by specialization of the equations $H_i = 0$ of $\mathcal{S}$ by remark 3.15. This implies that the falsehood of assertion *2* for the fiber $\mathcal{S}_v$ implies the falsehood of assertion *1*. Conversely, suppose for some $i, j \in I$, that the assertion *1* is false, then the statement *2* is false for some $v \in (\mathbb{C}^*)^d$ and then for $v = 0$ by theorem 2.4 and proposition 2.28. We deduce from the equations of the fibers that the assertion *2* is false for $v$ in a neighborhood of the origin. Finally, the falsehood of statement *2* is preserved by the action of $(\mathbb{C}^*)^d$ and the result follows. ⋄

The proposition 3.14, contains some interesting information about the Newton polyhedra of the generators $H_i$ of the kernel of $\psi$, this time considered as powers series in the variables $U_1, \dots U_{d+g}$ with coefficients in $\mathbb{C}\{V\}$. Hence these Newton polyhedra are contained in $\mathbb{R}^{d+g}_{\geq 0}$.

To express this information clearly we are going to attach to the generators $\gamma_1, \ldots, \gamma_{d+g}$ of the semigroup $\Gamma$ a set of polyhedra $\mathcal{P}_1, \ldots, \mathcal{P}_g$ in $\mathbb{R}^{d+g}_{\geq 0}$, with vertices in $\mathbb{Q}^{d+g}$.

$$\mathcal{P}_i := \{u \in \mathbb{R}^{d+g}_{\geq 0} \ / \ u_1\gamma_1 + \cdots + u_{d+g}\gamma_{d+g} \geq n_i \gamma_{d+i}\}$$

The polyhedron $\mathcal{P}_i$ has a unique compact face,

$$\mathcal{F}_i := \{u \in \mathbb{R}^{d+g}_{\geq 0} \ / \ u_1\gamma_1 + \cdots + u_{d+g}\gamma_{d+g} = n_i \gamma_{d+i}\}$$

and using *3* in lemma 2.11, this face is equal to

$$\{u \in \mathbb{R}^{d+g}_{\geq 0} \ / u_{d+i+1} = 0, \ldots, u_{d+g} = 0, \ u_1\gamma_1 + \cdots + u_{d+i}\gamma_{d+i} = n_i \gamma_{d+i}\}$$

hence is clearly of dimension $i$, and orthogonal to the linear subspace $\ell$ associated to the monomial variety (defined in §2.6), for $i = 1, \ldots, g$. The Minkowski sum $\mathcal{P} := \sum \mathcal{P}_i$ has then a unique compact part $\mathcal{F} := \sum \mathcal{F}_i$, which is orthogonal to the linear subspace $\ell$, and thus is a face of dimension $g$.

Proposition 3.14 implies the inclusion:

$$\mathcal{N}(H_i) \subset \mathcal{P}_i, \text{ for } i = 1, \ldots, g. \tag{3.15}$$

The face of the polyhedron $\mathcal{N}(H_i)$ that is contained on the compact face $\mathcal{F}_i$ of $\mathcal{P}_i$ is the compact edge $\mathcal{E}_i$ of the polyhedron $\mathcal{N}(h_i)$, for $i = 1, \ldots, g$.

**Lemma 3.20** *The face of the polyhedron* $\mathcal{N}(H_i)$ *associated to any non zero vector* $a \in \ell \cap (\mathbb{R}^{d+g})^*_{\geq 0}$ *is equal to* $\mathcal{E}_i$, *for* $i = 1, \ldots, g$. *The Minkowski sum* $\mathcal{E} = \sum \mathcal{E}_i$ *is a face of the polyhedron* $\mathcal{N}(H_1 \cdots H_g)$ *and the cone of vectors in* $(\mathbb{R}^{d+g})^*_{\geq 0}$ *associated to this face is* $(\mathbb{R}^{d+g})^*_{\geq 0} \cap \ell$.

*Proof.* If $0 \neq a \in \ell \cap (\mathbb{R}^{d+g})^*_{\geq 0}$, we have that

$$\min_{u \in \mathcal{N}(H_i)} \langle a, u \rangle \geq \min_{u \in \mathcal{P}_i} \langle a, u \rangle.$$

On the left hand side of the formula above, this minimum value is attained at the face $\mathcal{F}_i$ of $\mathcal{P}_i$. By the inclusion (3.15), this minimum coincide on the right hand side with the value on the face $\mathcal{E}_i$ of $\mathcal{N}(H_i)$. This implies that the cone $(\mathbb{R}^{d+g})^*_{\geq 0} \cap \ell$, is contained in the cone associated to the face $\mathcal{E}$ of the polyhedron $\mathcal{N}(H)$ where $H = H_1 \cdots H_g$. We have seen in section at the beginning of this section that this face is of dimension $g$, hence the associated cone in $(\mathbb{R}^{d+g})^*_{\geq 0}$ is of dimension $d$ and it must coincide with the cone $(\mathbb{R}^{d+g})^*_{\geq 0} \cap \ell$. ◇

**Lemma 3.21** *Let $\Sigma$ be a regular fan compatible with the polyhedra $\mathcal{N}(h_i)$, for $i = 1, \ldots, g$. Then there exist a regular fan $\Sigma'$ refining $\Sigma$ such that:*

1. *The fans $\Sigma \cap \ell$ and $\Sigma' \cap \ell$ coincide.*
2. *If $\sigma = \langle a^1, \ldots, a^{d+g} \rangle \in \Sigma'$ with $a^1, \ldots, a^d \in \ell$, all the vectors $a^{d+j}$ for $j = 1, \ldots, g$, determine the same face of the polyhedron $\mathcal{N}(H_i)$, and this face is a vertex of the compact edge $\mathcal{E}_i$, for $i = 1, \ldots, g$.*

*Proof.* The polyhedra $\mathcal{N}(H)$ defines a subdivision $\Sigma_1$ of $(\mathbb{R}^{d+g})^*_{\geq 0}$. By lemma 3.20, the cone associated to the face $\mathcal{E}$ is $(\mathbb{R}^{d+g})^*_{\geq 0} \cap \ell$, and we saw that this cone is subdivided by the fan $\Sigma$. Hence the "intersection of the fans" $\Sigma$ and $\Sigma_1$ does not modify the cones of $\Sigma \cap \ell$. By theorem 2.1, we can refine this intersection into a regular fan $\Sigma_2$ without modifying the cones in $\Sigma \cap \ell$, which are already regular.

Take a cone $\sigma = \langle a^1, \ldots, a^{d+g} \rangle \in \Sigma_2$ with $a^1, \ldots, a^d \in \ell$. By construction, the cone $\sigma$ defines a vertex of the polyhedron $\mathcal{N}(H)$, hence it defines a vertex of the polyhedron $\mathcal{N}(H_i)$, which belongs to the face $\mathcal{E}_i$ defined by the face $\sigma \cap \ell$ of $\sigma$. This vertex is defined by any vector in the interior of $\sigma$. Then, we subdivide the cone $\sigma$ without modifying the face $\sigma \cap \ell$, to obtain a subdivision of $\sigma$ such that all the vectors $a \notin \ell$, defining the edges of $(d+g)$-cones containing the cone $\sigma \cap \ell$, lie in the interior of $\sigma$. For this purpose, we can use for instance the stellar subdivisions (see [Ew]). This method provides a regular fan $\Sigma'$ as required. ◇

### 3.2.3 Simultaneous resolution

We prove the main result of the section. The embedded resolution of the monomial variety provides a simultaneous embedded resolution of the deformation $\mathcal{S}$ and all its fibers. In particular an embedded resolution of the embedding of $S$ in $\mathbb{C}^{d+g}$.

Let $\Sigma$ be a regular fan compatible with a fixed set of equations $h_1, \ldots, h_g$ of the monomial variety (given by proposition 2.25). We denote by $\mathcal{Y}$ the strict transform of $\mathcal{S}$ by the proper toric morphism

$$\Pi := \pi(\Sigma) \times Id : Z(\Sigma) \times \mathbb{C}^d \to \mathbb{C}^{d+g} \times \mathbb{C}^d,$$

by $\pi : \mathcal{Y} \to \mathcal{S}$ its restriction to $\mathcal{Y}$, and by $\Delta_\Pi$ the reduced discriminant locus of the toric morphism $\Pi$.

**Lemma 3.22** *The set $\pi^{-1}(\Delta_\Pi \cap \mathcal{S})$ is connected.*

*Proof.* The restriction $\pi : \mathcal{Y} \to \mathcal{S}$ of the morphism $\Pi$ is a proper modification. By the Main Theorem of Zariski [2], the fiber $\pi^{-1}(0)$ is connected since $\mathcal{S}$ is analytically irreducible at the origin. The set $\mathcal{S} \cap \Delta_\Pi$ is connected and is naturally stratified by the stratification induced by the coordinate hyperplanes. The strata of this stratification are orbits of the action of $(\mathbb{C}^*)^d$ on $\mathcal{S}$, this implies that for each stratum, the number of *branches of* $\mathcal{S}$ *at any point*[3] of the stratum is constant, hence the branches of $\mathcal{S}$ on any point of a stratum are also the branches of $\mathcal{S}$ at all the points of the stratum. If $\mathcal{Z}$ is branch of $S$ at a given stratum $E$, the restriction of the morphism $\pi$ to the strict transform $\tilde{\mathcal{Z}}$ of the branch $\mathcal{Z}$, $\pi_{\mathcal{Z}} : \tilde{\mathcal{Z}} \to \mathcal{Z}$, is a proper modification. By the Main Theorem of Zariski applied on all the points of the stratum, the set $\pi_{\mathcal{Z}}^{-1}(E)$ is connected and its closure intersects $\pi^{-1}(0)$ since $\pi$ is proper. We have that $\pi^{-1}(E) = \bigcup_{\mathcal{Z}'} \pi_{\mathcal{Z}'}^{-1}(E)$, for $\mathcal{Z}'$ running through the branches of $\mathcal{S}$ at the strata $E$. Hence, we deduce that the set $\pi^{-1}(\mathcal{S} \cap \Delta_\Pi)$ is connected. ◇

We denote by $\mathcal{Y}_v$ the fibers of the restriction to $\mathcal{Y}$ of the map $q \circ \Pi$, for $v \in \mathbb{C}^d$, and by $\tilde{\mathcal{S}}_v$ the strict transform of the fiber $\mathcal{S}_v$ by $\Pi$, which is clearly contained in $\mathcal{Y}_v$, for $v \in \mathbb{C}^d$.

**Lemma 3.23** *Suppose that the fan $\Sigma$ satisfies the conditions in lemma 3.21. Let $a \in \Sigma^{(1)}$ define a critical divisor $D(a) \times \mathbb{C}^d$ , then its intersection with the strict transform $\mathcal{Y}$ is empty if $a \notin \ell$. Moreover, let $\sigma = \langle a^1, \dots, a^{d+g} \rangle$ a cone in $\Sigma$, with $a^1, \dots, a^d \in \ell$. On the chart $Z(\sigma) \times \mathbb{C}^d$ we have:*

1. *The strict transform $\mathcal{Y}$ and the fibers $\mathcal{Y}_v$ are smooth.*
2. *The critical divisors $\{D(a^1) \times \mathbb{C}^d, \dots, D(a^d) \times \mathbb{C}^d\}$ have transversal intersections with the the strict transform $\mathcal{Y}$ and with all its fibers $\mathcal{Y}_v$.*

*Proof.* We prove first the assertion on the chart $Z(\sigma) \times \mathbb{C}^d$. By lemma 3.20, the face of the polyhedron $\mathcal{N}(H_i)$ defined by the cone $\tau := \langle a^1, \dots, a^d \rangle$ is equal to the compact edge $\mathcal{E}_i$ of the polyhedron $\mathcal{N}(h_i)$. Since $\tau \subset \sigma$, the face of the polyhedron $\mathcal{N}(H_i)$ defined by $\sigma$ is a vertex which corresponds to a term of $h_i$ (for $i = 1, \dots, g$). Since the fan $\Sigma$ verifies the assertion of lemma 3.21, this vertex is also equal to the face defined by any of the vectors $a^{d+1}, \dots, a^{d+g}$.

The morphism $\pi(\sigma) \times Id$ is described by by formula (2.1), where the variable $X$ is changed by $U$, the variable $U$ is changed by $Y$ and the integer $d+1$ is changed by $d+g$; and the variable $V$ is left unchanged. The normal crossing divisor $\sum_{i=1}^{d+g} D(a^i) \times \mathbb{C}^d$ defines a stratification of the

[2] We recall the topological form of the Main Theorem of Zariski (see 16. Theorem Chapter III, Sec C, of [G-R], and the homonymous section in [Mu].

[3] i.e. a connected component of $\mathcal{U} - \mathrm{Sing}(\mathcal{S})$ where $\mathcal{U}$ is a "small enough" neighborhood of the point considered.

chart $Z(\sigma) \times \mathbb{C}^d$. If $\emptyset \neq J \subset \{1, \ldots, d+g\}$, we associate to $J$ the stratum defined by $Y_i = 0$ if $i \in J$, and $Y_j \neq 0$ if $j \notin J$. We must show that the intersection of $\mathcal{Y}$ with the stratum defined by $J$ is smooth and transverse. This also implies that $\mathcal{Y}$ is smooth on this chart, since the singular locus of $\mathcal{S}$ is contained in the discriminant locus of the map $\Pi$.

The strict transform $\mathcal{Y}$ is given on the chart $Z(\sigma) \times Id$ by equations of the form:

$$\tilde{H}_i = 1 - Y_{d+1}^{\alpha^i_{d+1}} \cdots Y_{d+g}^{\alpha^i_{d+g}} + \tilde{G}_i(V, Y) = 0 \text{ , for } i = 1 \ldots g. \tag{3.16}$$

The first part, $1 - Y_{d+1}^{\alpha^i_{d+1}} \cdots Y_{d+g}^{\alpha^i_{d+g}}$ is equal to the strict transform of the equation of the monomial variety, and the series $\tilde{G}_i(V, Y)$ satisfies that $\tilde{G}_i(0, Y) = 0$, since it comes from the terms in the summand $G_i$ of $H_i$.

To compute the intersection with $D(a^{d+j}) \times \mathbb{C}^d$, we set $Y_{d+j} = 0$ on the series $\tilde{H}_i$. It is easy to see that the equations we obtain are equal to the strict transform of the symbolic restriction of $H_i$ to the face defined by $a^{d+j}$, for $i = 1, \ldots, d$. Since the fan $\Sigma$ satisfies the assertion of lemma 3.21, this initial polynomial is a monomial, hence its strict transform is equal to $1$. We have shown that:

$$\mathcal{Y} \cap (D(a^{d+j}) \times \mathbb{C}^d) = \emptyset, \text{ in the chart } Z(\sigma) \times \mathbb{C}^d. \tag{3.17}$$

We check the condition of non singularity and of transversality with the intersections of the stratum corresponding to subsets $\emptyset \neq J = \{i_1, \ldots i_s\} \subset \{1, \ldots, d\}$. If $w$ is a generic vector of a non zero face $\langle a^{i_1}, \ldots, a^{i_s} \rangle$ of $\tau$, the $w$-initial polynomial of $H_i$ is equal to the $w$-initial polynomial of $h_i$. Since the polynomials $h_1, \ldots h_g$ are clearly non degenerate, we can argue as in the proof of Theorem 5.1 of [G-T], to show that the intersection of $\mathcal{Y}$ with the strata defined by $J$ is smooth and transversal. This is also valid for all the fibers $\mathcal{Y}_v$, because their equations are obtained by specialization of (3.16) on this chart. Notice also that $\mathcal{Y}_v$ coincides on this chart with the strict transform of the fiber $\mathcal{S}_v$ by remark 3.15.

We show finally, the first assertion of the lemma: If $a \notin \ell$ is a primitive vector in the 1-skeleton of $\Sigma$ defining the critical divisor $D(a) \times \mathbb{C}^d$, then $D = D(a) \times \mathbb{C}^d \cap \mathcal{Y}$ is empty. The open set

$$\mathcal{W} = \bigcup_{\dim \sigma \cap \ell = d} Z(\sigma) \times \mathbb{C}^d$$

cuts $\pi^{-1}(\Delta_\Pi \cap \mathcal{S})$ in a non empty set by lemma 3.10. Since $\pi^{-1}(\mathcal{S} \cap \Delta_\Pi)$ is a connected set (by lemma 3.22) which contains $D$ and we have that $\mathcal{W} \cap D = \emptyset$ by (3.17), we deduce that $D$ is empty. $\diamond$

We show the main result of this section which is a generalization of Corollary, Theorem 6.1 in [G-T].

**Theorem 3.2** *The morphism* $\Pi$ *induces a toric embedded resolution of singularities of the* $\tilde{\mathcal{S}}$ *and all fibers* $\tilde{\mathcal{S}}_v$ *simultaneously.*

*Proof.* First, we show that the morphism $\Pi$ is a embedded toric pseudo-resolution of $\mathcal{Y}$. There is a regular subdivision $\Sigma'$ of the fan $\Sigma$ verifying the conditions of lemma 3.21. This subdivision induces a proper toric morphism

$$\Pi' : Z(\Sigma') \times \mathbb{C}^d \to Z(\Sigma) \times \mathbb{C}^d.$$

The composition $\Pi' \circ \Pi$ is equal to the toric morphism $\Pi'' : Z(\Sigma') \times \mathbb{C}^d \to \mathbb{C}^{d+g} \times \mathbb{C}^d$ induced by the subdivision $\Sigma'$ of $\mathbb{R}^{d+g}_{\geq 0}$. It verifies the hypotheses of the lemma 3.23, hence the strict transform $\mathcal{Y}'$ of $\mathcal{S}$ by $\Pi''$ does not intersect the divisors $D(a) \times \mathbb{C}^d$ if $a \notin \ell$. Moreover, if $a \in \ell$ defines the critical divisor $D(a) \times \mathbb{C}^d$ of $\Pi''$, we deduce from the fact that $a$ defines an edge of $\Sigma$, that $D(a) \times \mathbb{C}^d$ is not a critical divisor for the morphism $\Pi'$. These conditions imply that $\Pi'$ is an isomorphism between two neighborhoods of $\mathcal{Y}'$ and $\mathcal{Y}$. Hence, if $\Pi''$ is a toric pseudo resolution of $\mathcal{Y}'$ then the morphism $\Pi$ is a toric pseudo resolution of $\mathcal{Y}$, and the same thing holds for all the fibers.

We still have to show that $\Pi$ is a toric resolution. We have described the singular locus of $\mathcal{S}$ and of the monomial variety in propositions 2.28 and 3.19. Suppose that $\{U_j = 0\} \cap \mathcal{S}$ is not in the singular locus of $\mathcal{S}$. We have by proposition 3.11 that the restriction of the morphism $\Pi$ to $\mathcal{Y}_0 \to \mathcal{S}_0$ is a toric resolution. Hence this morphism is an isomorphism over a generic point $(u, 0)$ of the intersection $\{U_j = 0\} \cap \mathcal{S}_0$. Since $\mathcal{Y}$ is smooth and the morphism $\Pi$ is of the form $\Pi = \pi(\Sigma) \times Id$, we get that $\Pi$ is an isomorphism over the intersection of $\mathcal{S}$ with a poly-disk centered at the point $(u, 0)$. These isomorphisms glue up, and glue up with the ones obtained for all the intersections $\{U_s = 0\} \cap \mathcal{S} \nsubseteq$ Sing $\mathcal{S}$. We can use the equivariant action of $(\mathbb{C}^*)^d$ on $\mathcal{S}$ to extend this isomorphism to an isomorphism

$$\mathcal{Y} - \Pi^{-1}(\mathrm{Sing}(\mathcal{S})) \to \mathcal{S} - \mathrm{Sing}(\mathcal{S}).$$

Then it is easy to see, using proposition 3.19, we have also a fiber-wise embedded toric resolution.

$$\mathcal{Y}_v - \Pi^{-1}(\mathrm{Sing}(\mathcal{S}_v)) \to \mathcal{S}_v - \ \mathrm{Sing}\ \mathcal{S}_v$$

for all $v \in \mathbb{C}^d$. ◇

## 3.3 An example

In this section we give examples of embedded resolution of a germ of quasi-ordinary surface in $\mathbb{C}^3$ parametrized by a quasi-ordinary branch with two characteristic exponents.

### 3.3.1 Resolution of a surface embedded in $\mathbb{C}^3$

**1.** The fractional polynomial

$$\zeta = X_1^{3/2} + X_1^2 X_2^{1/2}$$

defines a quasi-ordinary branch by lemma 2.4. It has two characteristic exponents $\lambda_1 = (\frac{3}{2}, 0)$ and $\lambda_2 = (2, \frac{1}{2})$. We have that $n_1 = n_2 = 2$. We compute the minimal polynomial $f$ of $\zeta$ from the conjugates $\pm X^{3/2} \pm X^2 Y^{1/2}$ of $\zeta$ and we obtain.

$$f = (Y^2 - X_1^3)^2 + (-2X_1^7 X_2 - 2X_1^4 X_2 Y^2).$$

We have that $f$ is a quasi-ordinary polynomial in *good coordinates* defining a quasi-ordinary surface $S$. Its Newton polyhedron has only one compact edge $e$ and the symbolic restriction of $f$ to this edge is $f_{|e} = (Y^2 - X_1^3)^2$. On the other hand, it is easy to see that the polynomial $q_1 := Y^2 - X_1^3$ is a 1-semi-root of $f$.

The linear subspace $\ell_1$ orthogonal to this compact edge is the subspace generated by the vectors $a^1 = (2, 0, 3)$ and $a^2 = (0, 1, 0)$ of $(\mathbb{R}^3)^*$. They form a regular cone. We can take any regular subdivision $\Sigma$ of $(\mathbb{R}^3)^*_{\geq 0}$ compatible with $\ell : 3u_1 - 2u_3 = 0$, and thus containing the cone $\langle a^1, a^2 \rangle$. This means that there is vector $c \in \ell^+$ such that the cone $\sigma = \langle a^1, a^2, c \rangle$ belongs to $\Sigma_1$, for instance $c = (1, 0, 1)$. We can take as $\Sigma_1$ the fan of 1-skeleton:

$$c^1 := (1,0,0), c^2 := c, c^3 := a^1, c^4 := (1,0,2), c^5 := (0,0,1), c^6 := (0,1,0)$$

A transversal section of the fan $\Sigma_1$ is represented by Figure 3.5

By lemma 3.1, the strict transform of $S$ by $\pi(\Sigma)$ is contained in the chart $Z(\sigma)$. The morphism $\pi(\sigma) : Z(\sigma) \to \mathbb{C}^3$ is given by:

$$\begin{aligned} X_1 &= U_1^2 U_3^1 \\ X_2 &= U_2 \\ Y &= U_1^3 U_3 \end{aligned}$$

The transform of $f$ is the polynomial:

$$U_1^{12} U_3^4 \left((1 - U_3)^2 + -2U_1^2 U_2 U_3^3 + -2U_1^2 U_2 U_3^2\right)$$

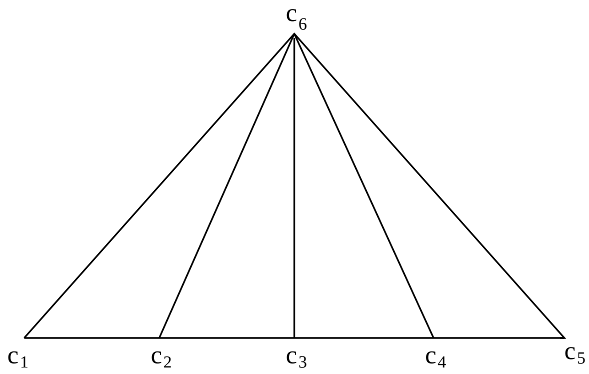


Figure 3.5: A transversal section of the fan $\Sigma_1$

and the strict transform is:

$$\tilde{f} = (1 - U_3)^2 + -2U_1^2 U_2 U_3^3 + -2U_1^2 U_2 U_3^2$$

The strict transform of the 1-semi-root $q_1$ is $W := 1 - U_3$. Taking $W$ as a coordinate means to substitute $U_3$ by $1 - W$ in $\tilde{f}$. We obtain:

$$F := W^2 + -2U_1^2 U_2 (1 - W)^3 - 2U_1^2 U_2 (1 - W)^2$$

The polynomial $F$ has a Newton polyhedron with only one compact edge $e_2$; the symbolic restriction of $F$ to this edge is the polynomial $F_{|e_2} = W^2 - 2U_1^2 U_2$. If we look as $F$ as a polynomial in $\mathbb{C}[U_1, U_2][W]$, the polynomial $t^2 - 2$ associated to the edge $e_2$ (see section 1.3) has no multiple root. It follows from the proof of Théorème 1.4, that the Weierstrass polynomial of $F$ is a quasi-ordinary polynomial.

The linear subspace $\ell_2$ orthogonal to the compact edge $e_2$ is the subspace generated by the vectors $b^1 = (1,0,1)$ and $b^2 = (0,2,1)$ of $(\mathbb{R}^3)^*$. They form a regular cone. We can take any regular subdivision $\Sigma_2$ of $(\mathbb{R}^3)^*_{\geq 0}$ compatible with $\ell_2 : 2u_1 + u_2 - 2u_3 = 0$, and thus containing the cone $\langle b^1, b^2 \rangle$. This means that there is vector $d \in \ell^+$ such that the cone $\sigma = \langle b^1, b^2 d \rangle$ belongs to $\Sigma_2$, for instance $d = (0,1,0)$. We can take as $\Sigma_2$ the regular fan supported on $\mathbb{R}^3_{\geq 0}$ having 1-skeleton the vectors:

$$d^1 := (1,0,0),\ d^2 := (0,1,0),\ d^3 := b^2,\ d^4 := b_1,\ d^5 := (0,1,1),\ d^6 := (0,0,1)$$

By lemma 3.1, the strict transform of $\tilde{S}$ by $\pi(\Sigma)$ is contained in the chart $Z(\sigma)$. The

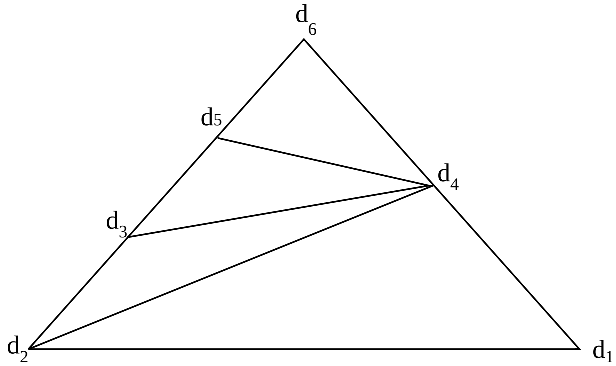


Figure 3.6: A transversal section of the fan $\Sigma_2$

morphism $\pi(\sigma) : Z(\sigma) \to \mathbb{C}^3$ is given by:

$$\begin{array}{rcl} U_1 & = & V_1 \\ U_2 & = & V_2^2 V_3 \\ W & = & V_1 V_2 \end{array}$$

The transform of $F$ is:

$$V_1^2 V_2^2 (1 - 2V_3(1 - V_1\, V_2)^3 - 2V_3(1 - V_1\, V_2)^2),$$

and the strict transform is

$$1 - 2V_3(1 - V_1\, V_2)^3 - 2V_3(1 - V_1\, V_2)^2.$$

The intersection with the strict transform with $V_1 = V_2 = 0$ defines the point $V_3 = 1/4$. We have clearly that the strict transform is smooth and transversal to the divisors $D(b^1) : V_1 = 0$ and $D(b^2) : V_2 = 0$.

**Remark 3.24** *In this case, the cones* $\ell_1 \cap (\mathbb{R}^3)^*_{\geq 0}$ *are regular. This is not true in general.*

The dual intersection graph of the critical divisors of the embedded toroidal resolution can be drawn from figures 3.5 and 3.6. The circles represent the critical divisors, and the squares represent the surfaces corresponding to the generators of the semigroup. The last square $D(d_6)$ represents also the strict transform of the hypersurface. Lines and triangles between circles or squares corresponds to the intersection of the corresponding two surfaces (curves) or three surfaces (points).

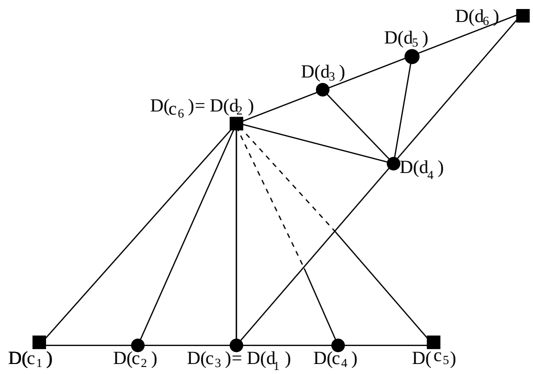


Figure 3.7: Dual intersection graph of the critical divisors of the embedded toroidal resolution

### 3.3.2 Resolution via deformation on the monomial variety

We continue with the example **1** above. We compute the semigroup $\Gamma$ from the formulae 2.6. We obtain:

$$\gamma_1 = (2,0), \quad \gamma_2 = (0,2), \quad \gamma_3 = (3,0), \quad \gamma_4 = (7,1).$$

The quasi-ordinary branch defines the parametrization

$$R = \mathbb{C}\{X_1, X_2\}[\zeta] \to \mathbb{C}\{T_1, T_2\}$$

by

$$X_i \mapsto \xi_i := T_i^2 \text{ and } \zeta \mapsto \xi_3 := T_1^3 + T_1^4 T_2.$$

The class in $R$ of the semi-root, and $q_1 = \frac{1}{2}(Y^2 - X^3)$ is $\xi_4 = T_1^7 T_2 + T_1^8 T_2^2$. Thus, the embedding of the deformation $\mathcal{S}$ in $\mathbb{C}^4 \times \mathbb{C}^2$ is defined by:

$$\begin{array}{rcl} U_1 & = & \xi_1(VT)V^{-\gamma_1} = T_1^2 \\ U_2 & = & \xi_2(VT)V^{-\gamma_2} = T_2^2 \\ U_3 & = & \xi_3(VT)V^{-\gamma_3} = T_1^3(1 + V_1 V_2 T_1 T_2) \\ U_4 & = & \xi_4(VT)V^{-\gamma_4} = T_1^7 T_2(1 + \frac{1}{2} V_1 V_2 T_1 T_2) \end{array} \tag{3.18}$$

Remark that replacing the right hand-side by their initial polynomials or setting $V_1 = V_2 = 0$ leads to the same result, the equations of the embedding of the monomial variety $\mathcal{S}_0$.

We compute, by hand, the following relations between the generators of the semigroup $\Gamma$.

$$\left\{ \begin{array}{rcl} 2\gamma_3 & = & 3\gamma_1 \\ 2\gamma_4 & = & 4\gamma_1 + \gamma_2 + 2\gamma_3 \end{array} \right.$$

This relations give a set of generators for the ideal of the monomial variety $\mathcal{S}_{\prime}$.

$$h_1 := U_3^2 - U_1^3, \quad h_2 := U_4^2 - U_1^4 U_2 U_3^2.$$

We can use the main ideas of the proof of proposition 3.14 to produce an algorithm to lift a set of equations of the monomial variety to a set of equations of the deformation. The formula 3.18 defines an homomorphism $\psi : \mathbb{C}\{U, V\} \to \mathbb{C}\{V, T\}$ of $\mathbb{C}\{V\}$-algebras. Then we have:

$$\begin{array}{rcl} \psi(h_1) & = & 2\, V_1\, V_2\, T_1^7\, T_2 + V_1^2\, V_2^2\, T_1^8\, T_2^2 = \psi(2 V_1\, V_2\, U_4) \\ \psi(h_2) & = & -V_1\, V_2\, T_1^{15}\, T_2^3 - \frac{3}{4}\, V_1^2\, V_2^2\, T_1^{16}\, T_2^4 \end{array}$$

The idea is to find a polynomial $r_i$ in $\mathbb{C}[[U, V]]$ such that $\mathrm{in}\psi(h_i) = \mathrm{in}\psi(r_i)$; then we replace $h_i$ by $h_i - r_i$ and we iterate this procedure to produce the terms of series $H_1$ and $H_2$ that generate the kernel of $\psi$. For instance, we have $\mathrm{in}\psi(h_2) = V_1\, V_2\, T_1^{15}\, T_2^3 = \mathrm{in}(\psi(-V_1\, V_2\, U_1^4\, U_2\, U_4)$ and then

$$\psi(h_2 + V_1\, V_2\, U_1^4\, U_2\, U_4) = -\frac{1}{4}\, V_1^2\, V_2^2\, T_1^{16}\, T_2^4 = \psi(-\frac{1}{4}\, V_1^2\, V_2^2\, U_1^8\, U_2^2).$$

We obtain from this the following generators of the ideal of the deformation (i.e. the kernel of the homomorphism $\psi$).

$$\begin{array}{rcl} H_1 & = & U_3^2 - U_1^3 - 2 V_1\, V_2\, U_4 \\ H_2 & = & U_4^2 - U_1^4 U_2 U_3^2 + V_1\, V_2\, U_1^4\, U_2 U_4 + \frac{1}{4}\, V_1^2\, V_2^2\, U_1^8\, U_2^2. \end{array}$$

and we verify that they satisfy the conditions of proposition 3.14.

**Remark 3.25** *A multiple of the original equation of the quasi-ordinary surface is recovered when we substitute $V_1$ and $V_2$ by $1$, and we eliminate $U_4$ ($U_4 = \frac{1}{2} h_1$).*

We describe now a regular fan $\Sigma$ supported on $\mathbb{R}^4_{\geq 0}$ compatible with the equations $h_1 = h_2 = 0$ of the monomial variety. This is a fan compatible with the linear subspaces:

$$\ell_1 : -2u_3 + 3u_1 = 0, \quad \ell_2 : -2u_4 + 4u_1 + u_2 + 2u_3.$$

In particular the fan $\Sigma$ must be compatible also with the intersection $\ell_1 \cap \ell_2$, which is the linear subspace generated by the vectors $a^1 := (0, 2, 0, 1)$ and $a^2 := (2, 0, 3, 7)$. These two vectors define a regular cone contained in the fan $\Sigma$. This does not happen in general.

The intersection of $\ell_1$ with $\mathbb{R}^4_{\geq 0}$ is the regular cone generated by the vectors:

$$(2, 0, 3, 0), \quad (0, 1, 0, 0), \quad (0, 0, 0, 1)$$

With respect to this basis the equation of the plane $\ell_1 \cap \ell_2$ is $14 s_1 + s_2 - 2 s_3 = 0$. This equation allows to choose one vector in each half-space $(\ell_1 \cap \ell_2)^{\pm}$ of $\lambda_1$, such that with $a^1, a^2$ define a regular basis of $\ell_1 \cap \mathbb{Z}^3$. We can take for instance $c^1 := (0,3,0,1)$ and $c^2 = (0,1,0,1)$.

The intersection of $\ell_2$ with $\mathbb{R}^4_{\geq 0}$ is the regular cone generated by the vectors:

$$(0,0,1,1), \quad (1,0,0,2), \quad (0,2,0,1)$$

With respect to this basis the equation of the plane $\ell_1 \cap \ell_2$ is $2t_1 - 3t_2 = 0$. This equation allows us to choose one vector in each half-space $(\ell_1 \cap \ell_2)^{\pm}$ of $\ell_2$, such that with $a^1, a^2$ define a regular base of $\ell_2 \cap \mathbb{Z}^3$. We can take for instance $d^1 := (1,0,2,4)$ and $d^2 = (1,0,1,3)$.

Then we verify easily that the cones $\sigma_{ij} := \langle a^1, a^2, c^i, d^j \rangle$ are regular for $i, j = 1, 2$ and that they are part of a regular fan $\Sigma$ compatible with the equations of monomial variety. This is a rather simple case, since the cones $\ell_i \cap \mathbb{R}^4_{\geq 0}$ for $i = 1, 2$ and $\ell_1 \cap \ell_2 \cap \mathbb{R}^4_{\geq 0}$ are regular in general we may try this method by finding first regular subdivisions of this cones using the algorithms from [Ag] and [Bo-Go].

The strict transform of the quasi-ordinary surface embedded in $\mathbb{C}^4$ (obtained by substituting $V_1$ and $V_2$ by 1 in formulae 3.18) by the toric morphism $\pi(\Sigma)$ is covered by the charts $Z(\sigma_{ij})$. For example, the morphism $\pi(\Sigma)$ is defined on the chart $Z(\sigma_{11})$ by:

$$\begin{array}{rcl} U_1 & = & W_2^2 W_4 \\ U_2 & = & W_1^2 W_3^3 \\ U_3 & = & W_2^3 W_4^2 \\ U_4 & = & W^1 W_2^7 W_3 W_4^4 \end{array}$$

This strict transform is defined by the strict transforms of the polynomials: $H_1(U,1) = U_3^2 - U_1^3 - 2U_4$ and $H_2(U,1) = U_4^2 - U_1^4 U_2 U_3^2 + U_1^4 U_2 U_4 + \frac{1}{4} U_1^8 U_2^2$.

The transform of $H_1(U,1)$ is

$$W_2^6 W_4^3 (W_4 - 1 - 2 W_1 W_2 W_3 W_4),$$

and the transform of $H_2(U,1)$ is

$$W_1^2 W_2^{14} W_3^2 W_4^8 \left(1 - W_3 + W_2 W_3^2 + \frac{1}{4} W_1^2 W_2^2 W_3^4\right)$$

The strict transforms are left inside the parentheses. We remark immediately that it is smooth and transversal to $W_1 = 0$ and $W_2 = 0$ and that the same holds for the strict transform of the monomial variety.

## 3.4 Appendix: Some lemmas on flatness

We introduce some results of commutative algebra, from [Bbk].

Let $A$ be a commutative ring, $E$ an $A$-module. Consider the sequences of $A$-modules:

$$0 \to G \to H \to F \to 0 \tag{3.19}$$

and

$$0 \to G \otimes E \to H \otimes E \to F \otimes E \to 0 \tag{3.20}$$

The $A$-module $E$ is *flat* (resp. *faithfully flat*) if we have that the sequence (3.20) is exact if (resp. if and only if) the sequence (3.19) is exact.

**Exemple 3.1**

1. Any free $A$ module is faithfully flat.

2. If $I$ is an ideal of a Noetherian ring $A$, then the $I$-adic completion of $A$ is a flat $A$-module, (see Théorème 3, N$^o$ 4, §3, Chapitre III of [Bbk]). This implies that if $A$ is a Noetherian ring, the ring of formal power series $B = A[[X]]$ (with $X = (X_1, \dots, X_s)$) is flat over $A$. Firstly, $B$ is flat as a $A[X]$-module since it is the $(X)$-completion of the Noetherian ring $A[X]$. Secondly the polynomial ring $A[X]$ is flat over $A$ since it is free as $A$-module. Finally, the result follows from transitivity property of flatness, (see Corollaire 3, N$^o$ 8, §2, Chapitre III of [Bbk]).

3. The ring $\mathbb{C}[[X]]$ is a faithfully flat $\mathbb{C}\{X\}$-module. The inclusion $\mathbb{C}\{X\} \hookrightarrow \mathbb{C}[[X]]$ is a local homomorphism, continuous for the $(X)$-adic topologies, which extends to the identity homomorphism between the respective completions. Then, the result follows from Proposition 10, N$^o$ 5, §3, Chapitre III of [Bbk].

**Lemma 3.26**

1. *Suppose that $0 \to M' \to M \to M'' \to 0$ is an exact sequence of $A$-modules and that $M''$ is flat. Then $M$ is flat if and only if $M'$ is flat.*

2. *Suppose that $M$ is a flat $A$-module, and $M'$ is a submodule of $M$. Then $M/M'$ is flat if and only if for any ideal $I$ of $A$ we have $M' \cap IM \subset IM'$.*

3. *Let $A$ and $B$ be two Noetherian rings, $\phi : A \to B$ be a ring homomorphism. Let $I$ be an ideal of $A$, $J$ and ideal of $B$, such that $IB \subset J$, and $J$ is contained in the radical of the ring $B$. Denote by $\hat{A}$ the separated $I$-adic completion of the ring $A$, and by $\hat{B}$ the separated $J$-adic completion of the ring $B$. The homomorphism $\phi$ is continuous for these topologies and defines an homomorphism $\hat{\phi} : \hat{A} \to \hat{B}$. Let $M$ be a $B$ module of finite type, and $\hat{M}$ its $J$-adic completion. The following statements are equivalent:*

   (a) *$M$ is a flat $A$-module.*
   (b) *$\hat{M}$ is a flat $A$-module.*
   (c) *$\hat{M}$ is a flat $\hat{A}$-module.*

4. *Let $\phi : A \to B$ be a ring homomorphism. The following statements are equivalent:*

   (a) *$B$ is a faithfully flat $A$-module.*
   (b) *$B$ is a flat $A$-module, and for any ideal $\mathfrak{a}$ of $A$ we have that $\phi^{-1}(\mathfrak{a}B) = \mathfrak{a}$.*
   (c) *$\phi$ is injective and $B/\phi(A)$ is a flat $A$-module.*

*Proof.* For the first assertion see Proposition 5, N$^o$ 5, §2, Chapitre I of [Bbk]. The second assertion follows from Corollaire, Proposition 7, N$^o$ 6, §2, Chapitre I of [Bbk]. The third one is the Proposition 4, N$^o$ 4, §5, Chapitre III of [Bbk]. The last assertion is stated in Proposition 9, N$^o$ 5, §3, Chapitre I of [Bbk]. ◇

**Exemple 3.2** *If $\mathfrak{a}$ is an ideal of $\mathbb{C}[U]$ contained in the ideal $(U)$, then the $\mathbb{C}\{V\}$-modules $\mathfrak{a}\mathbb{C}\{V,U\}$ and $\mathbb{C}\{V,U\}/\mathfrak{a}\mathbb{C}\{V,U\}$ are flat.*

Firstly, by lemma 3.26, the $\mathbb{C}\{V\}$ module $\mathbb{C}\{V,U\}/\mathfrak{a}\mathbb{C}\{V,U\}$ is flat if and only if the $\mathbb{C}[[V]]$ module $\mathbb{C}[[V,U]]/\mathfrak{a}\mathbb{C}[[V,U]]$ is flat. The ring $\mathbb{C}[[V,U]]/\mathfrak{a}\mathbb{C}[[V,U]]$ is not equal to zero since we have that $\mathfrak{a} \subset (U)$. This ring is the completion of the Noetherian ring $\mathbb{C}[V,U]/\mathfrak{a}\mathbb{C}[V,U]$ with respect to the $(V,U)$-adic topology, hence its is a flat $\mathbb{C}[V,U]/\mathfrak{a}\mathbb{C}[V,U]$ module. Finally, the ring $\mathbb{C}[V,U]/\mathfrak{a}\mathbb{C}[V,U]$ is a flat $\mathbb{C}[V]$-module, since its free over $\mathbb{C}[V]$ because the ideal $\mathfrak{a}$ is contained in $\mathbb{C}[U]$. This shows for $\mathfrak{a} = 0$, that $\mathbb{C}\{V,U\}$ is a flat $\mathbb{C}\{V\}$ module. For a general ideal $\mathfrak{a}$ we have a exact sequence of $\mathbb{C}\{V\}$-modules, with the two latter modules flat:

$$0 \to \mathfrak{a}\mathbb{C}\{V,U\} \to \mathbb{C}\{V,U\} \to \mathbb{C}\{V,U\}/\mathfrak{a}\mathbb{C}\{V,U\} \to 0$$

We deduce from this sequence and lemma 3.26 that $\mathfrak{a}\mathbb{C}\{V,U\}$ is a flat $\mathbb{C}\{V\}$-module.

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