University of La Laguna

Science Faculty

AN INTRODUCTION TO BOSONIC STRING THEORY

Final degree project

By: Himar Bravo Pérez

Supervisor: Jose Mª Gómez Llorente

Abstract

This final degree project is thought to be a theoretical enlargement of the degree knowledge in order to broaden the basic concepts of mathematics and theoretical physics applied to the formulation of String theory.

In this essay, we will make an introduction to the theory of quantum bosonic strings. The ultimate objective of this final project will be to formulate the hypothetical gravity force carrier particle, and how to recover the Einstein´s equations from String theory.

This piece of work will commence by making a classical approach to the relativistic string for the purpose of formulating important results that we will use to quantize it.

We will quantize the theory in three different ways. Firstly, the Canonical Quantization will be applied, secondly, the Lightcone Quantization will be used and, lastly, we will utilize the one needed in order to formulate the general relativity which is The Path Integral approach.

Contents

Introduction

On String theory all the particles emerge as vibrations of a single string after its quantization. Originally, it was developed only for bosons. The integration of fermionic supersymmetric fields gave rise to the superstring theory where there exists a symmetry on the number of bosons and fermions on nature. Therefore, each type of boson particle has a corresponding fermion and vice versa.

There are several versions of Superstring theory: type I, type IIA, type IIB and two Heterotic theories Type HO and Type HE ($SO(32)$ and $E_8 \times E_8$), all of them living in a ten dimensions spacetime. These different theories allow different types of conditions on the strings, as an example, some of them only permit closed strings, while others admit closed and open strings.

It can be proved that the five formulations are key aspects of the same theory, related to each other through dualities. These five theories would be features or points of view of a theory called M-theory.

The strings of these theories can be extended to D-dimensional objects, called D-branes. It can be observed that $D \leq 10$ because it would be senseless to speak of a fifteen dimensional object living in a ten dimensional spacetime.

The string theory has many interesting properties. In the case of bosonic string theory, the number of space-time dimensions is twenty-six but in superstring it is ten, four of them are flat and the others are compactified at really small scales, thus it is difficult to detect them experimentally. This theory has different aspects that make it an excellent candidate as a route to the theory of everything, some of them are the following:

- Every string theory must contain a massless spin two state, whose interaction reduces at low energy to general relativity.
- String theories lead to gauge groups large enough to include the Standard Model.
- String theory does not contain free adjustable parameters, all emerge from the string properties.

These are some of the reasons that make this theory popular and interesting. Even if it is not the theory of everything its study gives us really important tools to understand others fields of physics and other theories.

1 The Action as a Central Point of a Theory

En este primer capítulo se hablará del principio de mínima acción en mecánica clásica y se ejemplificará la formulación de diferentes ecuaciones de la física mediante dicho principio, muchas de ellas serán fundamentales para poder estudiar la física de la cuerda. Enunciaremos la acción para una cuerda y estudiaremos sus propiedades y simetrías.

String theory is built over the least action principle as its starting point. Most of the modern theories use a least action principle to formulate its mechanics, this principle is widely used due to the simplicity of the obtention of the motion equations, conserved quantities and symmetry properties. The Electromagnetism, the Newtonian Dynamics, General and Special Relativity are theories that can be obtained as a minimum of a certain action. The quantum field theory starts taking the wave functions as excitations of classical fields that obey least action principles.

The action is an integral in time of a certain type of function called Lagrangian that depends on dynamical variables such as fields or particles coordinates.

$$S[q_i(t)] = \int_{t_1}^{t_2} dt\, L(q_i(t), \dot{q}_i(t), t).$$

Taking S as the action functional depending on a family of functions, q_i, fixed at the boundary, $\delta q_i(t_1) = \delta q_i(t_2) = 0$. This principle imposes the variation at first order of this quantity to be zero $\delta S = 0$.

$$\delta S = \int_{t_1}^{t_2} dt\, L(q_i(t) + \delta q_i(t), \dot{q}_i(t) + \delta \dot{q}_i(t), t) = 0,$$

this expression is developed in terms of, $\delta q_i(t)$, considering, $\delta \dot{q}_i(t) = \frac{d\delta q_i}{dt}$, we obtain the Euler-Lagrange equations.

$$\delta S = S[q_i + \delta q_i] - S[q_i],$$

$$\delta S = \int_{t_2}^{t_2} dt \left[\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i\right],$$

$$\delta S = \int_{t_1}^{t_2} dt \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\right] \delta q_i + \int_{t_1}^{t_2} dt \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\, \delta q_i\right) = 0 \quad \forall \delta q_i.$$

The second integral is zero as a consequence of the boundary conditions. Therefore, we obtain the Euler-Lagrange equations

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0.$$

As we can see in (Landau L. D. , Lifshitz E. M., 1971); (Lifshitz E. M., V. B. Berestetskii, L.P. Pitaevskii, 1975), we can formulate several actions for different theories, we will see some of them in the following pages.

It is possible to derive the special relativity equations for a free particle from an extremal principle. The extremum condition $\delta S = 0$ must be an invariant under any Lorentz transformation so the quantity, $dt\, L(q_i(t), \dot{q}_i(t), t)$, must be a scalar in this sense. The action is defined here as,

$$S[x(t)] = \int_{t_1}^{t_2} dt\, L(x(t), v(t)) \; = \; -mc^2 \int_{t_1}^{t_2} dt \sqrt{1 - \frac{v^2(t)}{c^2}} \quad ,$$

with $dt\sqrt{1-\frac{v^2(t)}{c^2}} = d\tau = -ds$, here τ is the proper time and ds is the length element of the metric space. The special relativity equations are those that minimize the proper time between two events.

Considering continuous variables such as a family of fields, $\phi^i\,(x^\nu)$, where x^ν is a four-vector in a Minkowski space. The Lagrangian is written as,

$$L = \int d^3x\, \mathcal{L}\left(\phi^i(x^\nu), {\phi^i}_{,\mu}(x^\nu)\right).$$

We have denoted the derivative of the field respect to its coordinates as, $\phi_{,\mu}(x^i) = \frac{\partial \phi}{\partial x^\mu}$. The least action principle is formulated as,

$$\delta S = \int d^4x\, \mathcal{L}\left(\phi^i + \delta\phi^i, {\phi^i}_{,\mu} + \delta{\phi^i}_{,\mu}\right) - \int d^4x\, \mathcal{L}\left(\phi^i, {\phi^i}_{,\mu}\right) = 0.$$

Moreover, the integral is defined in the region bounded by the hypersurfaces with $t = t_1$ and $t = t_2$, setting the boundary conditions, $\delta\phi^i(\boldsymbol{x}, t_1) = \delta\phi^i(\boldsymbol{x}, t_2) = 0$. Plugging this on the variation we obtain

$$\delta S = \int d^4x \left[\frac{\partial\mathcal{L}}{\partial\phi^i} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^i}\right)\right]\delta\phi^i + \int d^4x\, \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^i}\delta\phi^i\right) = 0,$$

$$\int d^4x \left[\frac{\partial\mathcal{L}}{\partial\phi^i} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^i}\right)\right]\delta\phi^i = 0 \quad \forall\; \delta\phi^i,$$

$$\left[\frac{\partial\mathcal{L}}{\partial\phi^i} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^i}\right)\right] = 0.$$

From now on, we will use natural units ($c = \hbar = 1$) in this essay.

From (Lifshitz E. M., V. B. Berestetskii, L.P. Pitaevskii, 1975) and (Landau L. D. , Lifshitz E. M., 1971) we found some wave equations obtained with a least action principle:

1. Schrödinger equation, $i\frac{\partial\psi}{\partial t} = \frac{1}{2m}\nabla^2\psi$, that comes from the Lagrangian density,
$$\mathcal{L} = \frac{1}{2}\left[i\left(\psi^*\psi_{,0} - \psi\psi^*_{,0}\right) + \frac{1}{m}\psi^{*,i}\psi_{,i}\right].$$

This action is invariant under Galileo transformations but it is not a relativistic invariant.

Some examples of invariant wave equations under Lorentz transformations are the Klein Gordon equation and the Dirac equation (both formulated on a Minkowskian space).

2. Klein Gordon equation $(\partial^{\mu}\partial_{\mu} + m^2)\phi = 0$ comes from the Lagrangian density,

$$\mathcal{L} = \partial^{\mu}\phi^{*}\partial_{\mu}\phi - m^2\phi^{*}\phi.$$

3. Dirac field equation $(i\gamma^{\mu}\partial_{\mu} - m)\psi = 0$ comes from the Lagrangian density,

$$\mathcal{L} = \bar{\psi}(i\gamma^{\mu}\partial_{\mu} - m)\psi.$$

Where γ^{μ} are the Dirac gamma matrices and ψ is a four components object called spinor, $\bar{\psi}$ is defined $\bar{\psi} = \psi^{*}\gamma^{0}$. These matrices enable the Dirac equation to be a covariant equation with a first order derivative (thus making it different from the Klein Gordon equation and solving problems of normalization as we see in (Maiani L. ; Benar O., 2016)).

There is not a unique way of quantizing a system. Different approaches can give rise to the same classical state, for example, the canonical quantization and the Feynman path integrals. But the main idea when we quantize a field is to construct the Fock space of the theory. The Fock space has a non-defined number of field excitations. At the covariant approach the states of this space are constructed promoting the fields to creation and destruction operators that fill the vacuum (which has to be previously defined, additional information will be provided on the last chapter) with many excitations of the field modes.

Other examples of the use of the least action principle come from General Relativity. We can generalize the special relativity action to curved spaces to describe the evolution of a free particle on a curved spacetime. This is the starting point of the study of string theory. We can read about this in (Wray, 2011).

Firstly, we define the element of distance on a curved space as,

$$ds^2 = -g_{\mu\nu}(x)dx^{\mu}dx^{\nu},$$

where x^{ν} are the manifold coordinates and $g_{\mu\nu}$ the metric tensor.

The action is, again, defined as the integral with respect to the element of distance times one constant, that has to be the mass to be consistent with the special relativity Lagrangian case.

$$S = -m\int ds = -m\int d\tau\sqrt{-g_{\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau}}.$$

This action is invariant under the parametrization choice on the manifold and represents the path length in the manifold. The equations of motion describe the geodesic on the manifold between two given points.

An equivalent action could be formulated with an additional function $e(\tau)$, that allows us to avoid the square root

$$S' = \frac{1}{2}\int d\tau \left(e(\tau)^{-1} g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} - m^2 e(\tau)\right).$$

To show its equivalence we vary the action respect to $e(\tau)$ and find the stationary result

$$\delta S' = \frac{1}{2}\int d\tau\, \delta\left(e(\tau)^{-1} g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} - m^2 e(\tau)\right)$$

$$= \frac{1}{2}\int d\tau \left(e(\tau)^{-2} g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}\delta e - m^2 \delta e\right)$$

$$= \frac{1}{2}\int d\tau \frac{\delta e}{e^2}\left(g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} - m^2 e^2\right) = 0.$$

We find $e = \sqrt{\frac{-g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}}{m^2}}$, now plugging back in S' we get

$$S' = \frac{1}{2}\int d\tau \left(\left(\frac{-g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}}{m^2}\right)^{-\frac{1}{2}} g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} - m^2\left(\frac{-g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}}{m^2}\right)^{\frac{1}{2}}\right)$$

$$= \frac{1}{2}\int d\tau \left(\left(\frac{-g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}}{m^2}\right)^{-\frac{1}{2}}\left(g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} - m^2\frac{-g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}}{m^2}\right)\right)$$

$$= -m\int d\tau \left(\left(-g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}\right)^{-\frac{1}{2}}\left(-g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}\right)\right)$$

$$= -m\int d\tau \left(\left(-g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}\right)^{\frac{1}{2}}\right) = S.$$

Therefore, it is shown that when the action is minimized regarding the variation of the function, $e(\tau)$, we obtain the action as the length element in the manifold.

The Einstein equations also arise from an extremal principle see (Lifshitz E. M., V. B. Berestetskii, L.P. Pitaevskii, 1975). Starting from the Einstein Hilbert action we can obtain the General relativity equations. This action is written as,

$$S = \int \left(\frac{1}{2\kappa}R + \mathcal{L}_M\right)\sqrt{-\det\left(g_{\alpha\beta}\right)}\, d^4x.$$

Where R is the Ricci scalar, $\kappa = 8\pi G c^{-4}$, with G being the gravitational constant and $\mathcal{L}_M$ the Lagrangian density of the matter fields.

If the action is varied concerning the inverse of the metric, $g^{\mu\nu}$, and we make it zero we obtain,

$$\delta S = \int d^4x \left\{ \frac{1}{2\kappa} \frac{\delta\sqrt{-g}R}{\delta g^{\mu\nu}} + \frac{\delta\sqrt{-g}\mathcal{L}_M}{\delta g^{\mu\nu}} \right\} \delta g^{\mu\nu} =$$

$$\int d^4x \left\{ \frac{1}{2\kappa} R \frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} + \frac{1}{2\kappa}\sqrt{-g} \frac{\delta R}{\delta g^{\mu\nu}} + \frac{\delta\sqrt{-g}\mathcal{L}_M}{\delta g^{\mu\nu}} \right\} \delta g^{\mu\nu} =$$

$$\int d^4x\sqrt{-g} \left\{ \frac{1}{2\kappa} \left(R \frac{1}{\sqrt{-g}} \frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} + \frac{\delta R}{\delta g^{\mu\nu}} \right) + \frac{1}{\sqrt{-g}} \frac{\delta\sqrt{-g}\mathcal{L}_M}{\delta g^{\mu\nu}} \right\} \delta g^{\mu\nu} = 0.$$

(it is assumed that the reader knows the foundations of the functional derivatives)

Hence, we obtain the following equation

$$\frac{1}{2\kappa} \left(R \frac{1}{\sqrt{-g}} \frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} + \frac{\delta R}{\delta g^{\mu\nu}} \right) = -\frac{1}{\sqrt{-g}} \frac{\delta\sqrt{-g}\mathcal{L}_M}{\delta g^{\mu\nu}}.$$

By definition the Energy-momentum tensor with regard to the variation of a metric is $T_{\alpha\beta} \equiv -\frac{1}{\sqrt{-g}} \frac{\delta\sqrt{-g}\mathcal{L}_M}{\delta g^{\mu\nu}}$.

Now, we will show a couple of results before continuing. First,

$$\frac{\delta R}{\delta g^{\mu\nu}} = \frac{\delta g^{\mu\nu} R_{\mu\nu}}{\delta g^{\mu\nu}} =$$

$$\frac{\delta(g^{\mu\nu})R_{\mu\nu}}{\delta g^{\mu\nu}} + \frac{\delta(R_{\mu\nu})g^{\mu\nu}}{\delta g^{\mu\nu}}.$$

Looking at the second term,

$$\delta(R_{\mu\nu})g^{\mu\nu} = (\nabla_\rho \delta\Gamma^\rho_{\nu\mu} - \nabla_\nu \delta\Gamma^\sigma_{\sigma\mu})g^{\mu\nu}.$$

∇_ρ being the covariant derivative. By employing the property $\nabla_\rho g^{\mu\nu} = 0$, we obtain the expression

$$\delta(R_{\mu\nu})g^{\mu\nu} = \nabla_\rho(g^{\mu\nu}\delta\Gamma^\rho_{\nu\mu} - g^{\mu\rho}\delta\Gamma^\sigma_{\sigma\mu}).$$

This is a total derivative that plugged into the action is multiplied by $\sqrt{-g}$, that also has the property $\nabla_\rho\sqrt{-g} = 0$, so we can write this term in the integral as

$$\int d^4x \nabla_\rho\{\sqrt{-g}(g^{\mu\nu}\delta\Gamma^\rho_{\nu\mu} - g^{\mu\rho}\delta\Gamma^\sigma_{\sigma\mu})\}.$$

Employing the Stokes theorem we reduce this expression to boundary terms that vanish because $\delta g^{\mu\nu}$ has to be zero at the boundary. Then,

$$\frac{\delta R}{\delta g^{\mu\nu}} = R_{\mu\nu}.$$

On the other hand, we have,

$$\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} = -\frac{1}{2}\left(\frac{1}{\sqrt{-g}}\right)\frac{\delta g}{\delta g^{\mu\nu}} = -\frac{1}{2}\left(\frac{1}{\sqrt{-g}}\right)\frac{\delta g_{\mu\nu}g^{\mu\nu}g}{\delta g^{\mu\nu}} = -\frac{1}{2}\left(\frac{1}{\sqrt{-g}}\right)\frac{-g_{\mu\nu}\delta g^{\mu\nu}g}{\delta g^{\mu\nu}}$$

$$= -\frac{1}{2}g_{\mu\nu}\sqrt{-g}.$$

(see the page 12 to a more detailed process of the last calculation)

Finally substituting these results we obtain the Einstein equations,

$$\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) = 2\kappa T_{\alpha\beta}$$

To sum up, we have seen how to express the General Relativity equations as a consequence of the least action principle. The string action will emerge as a direct generalization of the relativistic action of a particle moving through a curved spacetime, as we will see in the next section.

1.1 The String

The string theory starts by making a natural generalization for the action of a relativistic particle in a curved space to the action that describes the motion of a type of objects called d-branes. The d-branes are d-dimensional hypersurfaces embedded on the space-time background geometry. The 0-brane is a point particle and the 1-brane is a string, which is a surface on the background spacetime.

If we call X^μ the coordinates of the space-time. The string will be a mapping on these coordinates, $X^\mu(\tau,\sigma)$. That conforms the worldsheet, a generalization of the worldline of the relativistic particle. This surface tells us the movement of the string in time as a function that depends on the parameter σ, so we can treat the string as a field (see Fig 1.1). It is generalized to d-branes as $X^\mu(\tau,\sigma_\nu)$ being, τ and σ_ν, the family of parameters of the brane surface in the space-time.

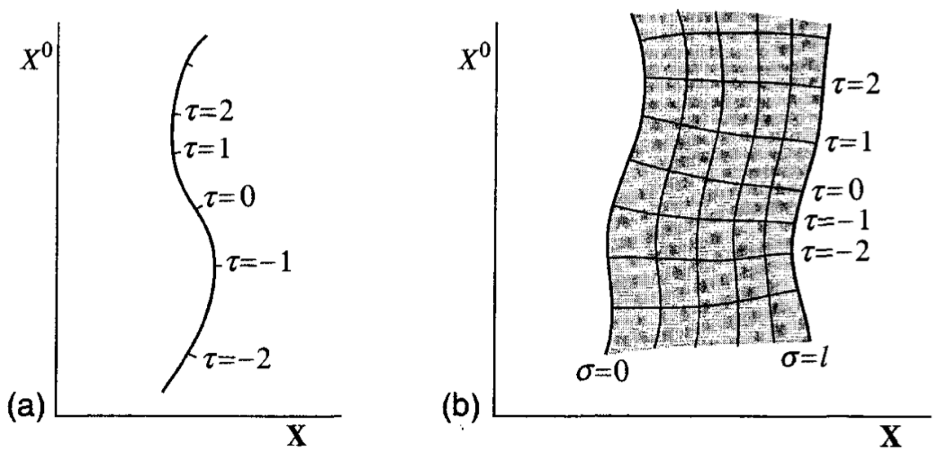


Fig 1.1 In this figure we observe the worldline which describes a particle's motion, and the worldsheet that describes the string's motion.

In order to generalize the worldline concept to the worldvolume of the brane, the relativistic action $S = -m \int ds$, whose argument is the arc element in the manifold. The arc element is now promoted to a d-dimensional hypersurface element on the background manifold $(d\mu_d)$, thus the brane action is expressed as

$$S_d = -T_d \int d\mu_d, \tag{1.1.1}$$

with T_d being the tension of the d-brane and has units of $\frac{[mass]}{[volume]}$. The element $d\mu_d$ is written in terms of the parameters of the brane as

$$d\mu_d = \sqrt{-\det\left(G_{\alpha\beta}(X)\right)}\, d^{p+1}\sigma\,, \quad d^{p+1}\sigma = d\sigma^0 d\sigma^1 \dots d\sigma^p, \quad d\sigma^0 = \tau,$$

where $G_{\alpha\beta}(X)$ is the metric induced into the worldvolume (or worldsheet for p=1). The induced metric can be expressed using the background metric as,

$$G_{\alpha\beta}(X) = \frac{\partial X^\mu}{\partial \sigma^\alpha}\frac{\partial X^\nu}{\partial \sigma^\beta} g_{\mu\nu}(X) \quad \alpha, \beta = 0,1, \dots, p\,.$$

We express the action for a string as,

$$S_{NG} = -T \int d\mu_1\,, \quad p = 1\,.$$

The string $X^\mu(\tau, \sigma)$ is a function of two parameters, one timelike τ, and one spacelike σ. By introducing the notation $X^{\mu\prime} = \frac{\partial X^\mu}{\partial \sigma}$ and $\dot{X}^\mu = \frac{\partial X^\mu}{\partial \tau}$, the metric element, (on a Minkowskian spacetime) $G_{\alpha\beta}(X)$, is written as,

$$G_{\alpha\beta} = \begin{pmatrix} \frac{\partial X^\mu}{\partial \tau}\frac{\partial X^\nu}{\partial \tau}\eta_{\mu\nu}(X) & \frac{\partial X^\mu}{\partial \tau}\frac{\partial X^\nu}{\partial \sigma}\eta_{\mu\nu}(X) \\ \frac{\partial X^\mu}{\partial \sigma}\frac{\partial X^\nu}{\partial \tau}\eta_{\mu\nu}(X) & \frac{\partial X^\mu}{\partial \sigma}\frac{\partial X^\nu}{\partial \sigma^\beta}\eta_{\mu\nu}(X) \end{pmatrix} = \begin{pmatrix} \dot{X}^2 & X'\dot{X} \\ \dot{X}X' & X'^2 \end{pmatrix}$$

Then the action S_{NG} is written as

$$S_{NG} = -T \int \sqrt{(X'\dot{X})(\dot{X}X') - \dot{X}^2 X'^2}\, d\tau d\sigma \tag{1.1.2}$$

This is the Nambu-Goto action, it depends on the area of the worldsheet. It can be shown that the quantity $(X'\dot{X})(\dot{X}X') - \dot{X}^2 X'^2$ is always positive defined as it is shown in (Zwiebach, 2004) just as the area.

If we take a vector on the world surface $v^\mu = \frac{\partial X^\mu}{\partial \tau} + \lambda \frac{\partial X^\mu}{\partial \sigma}$, it can be timelike or spacelike depending on λ, so the quantity v^2 must suffer a change of sign on the values that make $v^2 = 0$,

$$v^2 = 0 \; \rightarrow \; \lambda^2 X'^2 + 2\lambda X'\dot{X} + \dot{X}^2 = 0.$$

It is a polynomial in lambda so if we want this quantity to have two possible signs, this implies that the discriminant has to be positive, so $(X'\dot{X})(\dot{X}X') - \dot{X}^2 X'^2 > 0$. Thus, this proves that the argument of the square root is always positive.

As for the point particle case, it is possible to define an equivalent action by introducing an additional field, $h_{\alpha\beta}(X)$,

$$S_P = -\frac{T}{2}\int \sqrt{-\det(h_{ab})}\, h^{\alpha\beta} \frac{\partial X^{\mu}}{\partial \sigma^{\alpha}} \frac{\partial X^{\nu}}{\partial \sigma^{\beta}} g_{\mu\nu} d\tau d\sigma, \quad \sigma^{\alpha}, \sigma^{\beta} = \sigma, \tau. \tag{1.1.3}$$

This is the Polyakov action, it presents the tangent vectors of the worldsheet out of the square root so this action is better suited for path integral quantization as we will see in the chapter 4.

We make the same method as in the point particle case to see the equivalence of the S_P and S_{NG} actions (see (Wray, 2011)).

We vary S_P with respect to $h^{\alpha\beta}$, and impose the variation to be zero,

$$\frac{\delta S_P}{\delta h^{\alpha\beta}} = 0,$$

$$\delta S_P = \int \frac{\delta S_P}{\delta h^{\alpha\beta}}\, \delta h^{\alpha\beta}.$$

In order to start proving this, we remember that varying any action with respect to a metric we obtain a stress-energy tensor,

$$T_{\alpha\beta} = -\frac{2}{T}\frac{1}{\sqrt{-h}}\frac{\delta S_P}{\delta h^{\alpha\beta}}. \tag{1.1.4}$$

By setting the variation of the action equal to zero we obtain,

$$\delta S_P = \int \frac{\delta S_P}{\delta h^{\alpha\beta}}\, \delta h^{\alpha\beta} = -\frac{T}{2}\int d\sigma d\tau\, T_{\alpha\beta}\sqrt{-h}\; \delta h^{\alpha\beta} = 0.$$

This equation is only possible if $T_{\alpha\beta} = 0$. We observe that the identity, $h = h_{ab}H(a,b)$, (being $H(a,b)$ the adjoint matrix of h_{ab}) is varied with respect to h_{ab} as

$$\delta h = \delta h_{ab} H(a,b) = \delta h_{ab} h^{ab} h.$$

Using,

$$h_{ab}h^{ab} = 1 \rightarrow \delta h_{ab}h^{ab} + h_{ab}\delta h^{ab} = 0,$$

we obtain the expressions

$$\delta h = -h_{ab}\delta h^{ab} h,$$

$$\delta\sqrt{-h} = \frac{-1}{2}\left(\frac{1}{\sqrt{-h}}\right) h_{ab}\delta h^{ab}(-h) = \frac{-1}{2}\sqrt{-h} h_{ab}\delta h^{ab}.$$

To study the form of the stress-energy tensor we vary the Polyakov action with respect to $h^{\alpha\beta}$,

$$\delta S_p = -\frac{T}{2}\int \delta\sqrt{-h}\, h^{\alpha\beta}\frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X^\nu}{\partial\sigma^\beta} g_{\mu\nu} d\tau d\sigma \; -\frac{T}{2}\int \sqrt{-h}\,\delta h^{\alpha\beta}\frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X^\nu}{\partial\sigma^\beta} g_{\mu\nu} d\tau d\sigma,$$

$$\delta S_p = -\frac{T}{2}\int \sqrt{-h}\delta h^{\alpha\beta}\left(\frac{-1}{2} h_{\alpha\beta}h^{dc}\frac{\partial X^\mu}{\partial\sigma^d}\frac{\partial X^\nu}{\partial\sigma^c} g_{\mu\nu} + \frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X^\nu}{\partial\sigma^\beta} g_{\mu\nu}\right) d\tau d\sigma.$$

Then we finally obtain,

$$T_{\alpha\beta} = \frac{-1}{2} h_{\alpha\beta}h^{dc}\frac{\partial X^\mu}{\partial\sigma^d}\frac{\partial X^\nu}{\partial\sigma^c} g_{\mu\nu} + \frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X^\nu}{\partial\sigma^\beta} g_{\mu\nu} = 0\,. \tag{1.1.5}$$

We conclude from the equation (1.1.5) that

$$\frac{1}{2} h_{\alpha\beta}h^{dc}\frac{\partial X^\mu}{\partial\sigma^d}\frac{\partial X^\nu}{\partial\sigma^c} g_{\mu\nu} = \frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X^\nu}{\partial\sigma^\beta} g_{\mu\nu} = G_{\alpha\beta}(X),$$

taking the determinant of this expression,

$$\det(h_{\alpha\beta})\left(\frac{1}{2} h^{dc}\frac{\partial X^\mu}{\partial\sigma^d}\frac{\partial X^\nu}{\partial\sigma^c} g_{\mu\nu}\right)^2 = \det(G_{\alpha\beta}),$$

multiplying by a minus sign and taking the square root we obtain the final conclusion

$$\frac{1}{2}\sqrt{-h}h^{dc}G_{dc} = \sqrt{-G}.$$

This shows that the Polyakov action is classically equivalent to the Nambu-Goto action.

1.2 Symmetries of The String Action.

The symmetries of a theory is a high relevance topic when we start to develop a theory. If we want to start on this matter we have to distinguish two types of symmetries, global and local symmetries (see (Tong, 2012) and (Wray, 2011)).

The global symmetries do not depend on which space-time point are being performed and give rise to global conserved currents and conserved quantities via Noether theorem. One example of global transformation is the Poincare transformations (see on (Maiani L. ; Benar O., 2016) the specific details of this group).

If the strings are embedded in a Minkowskian spacetime, this produces that our strings must respect the same symmetries as this space, in particular the invariance under the Lie group of Poincare transformations, defined as

$$X^{\mu}(\tau,\sigma) = \Lambda^{\mu}_{\nu} X^{\nu}(\tau,\sigma) + b^{\mu},$$

where the Λ^{μ}_{ν} satisfies the equation $\Lambda^{\mu}_{\nu}\eta_{\mu\gamma}\Lambda^{\gamma}_{\sigma} = \eta_{\nu\sigma}$, taking the determinant results in the identity that defines the Lorentz group,

$$-\Lambda^{T}\Lambda = -1 \ \mapsto \ \det\Lambda = \pm 1 .$$

The value of the determinant defines the proper and improper transformations of the Lorentz group that are two not continuously connected parts of the group. The transformation on infinitesimal form is written as

$$X'^{\mu}(\tau,\sigma) = X^{\mu}(\tau,\sigma) + a^{\mu}_{\nu} X^{\nu}(\tau,\sigma) + \epsilon^{\mu},$$

$$X'^{\mu}(\tau,\sigma) - X^{\mu}(\tau,\sigma) = \delta X^{\mu}(\tau,\sigma) = a^{\mu}_{\nu} X^{\nu}(\tau,\sigma) + \epsilon^{\mu},$$

$$\delta h_{\alpha\beta} = 0.$$

Where, a^{μ}_{ν} and ϵ^{μ}, are the infinitesimal generators of the group and if we lower the μ index then, $a_{\mu\nu} = -a_{\nu\mu}$.

The Polyakov action is invariant under this transformation,

$$\delta S_P = -T\int \sqrt{-\det(h_{ab})}\, h^{\alpha\beta} \frac{\partial \delta X^{\mu}}{\partial\sigma^{\alpha}} \frac{\partial X^{\nu}}{\partial\sigma^{\beta}} \eta_{\mu\nu} d\tau d\sigma,$$

$$\delta S_P = -T\int \sqrt{-\det(h_{ab})}\, h^{\alpha\beta} \frac{\partial\left(a^{\mu}_{\varsigma} X^{\varsigma}(\tau,\sigma) + \epsilon^{\mu}\right)}{\partial\sigma^{\alpha}} \frac{\partial X^{\nu}}{\partial\sigma^{\beta}} \eta_{\mu\nu} d\tau d\sigma,$$

$$\delta S_P = -T\int \sqrt{-\det(h_{ab})}\, h^{\alpha\beta}\eta_{\mu\nu} a^{\mu}_{\varsigma} \frac{\partial\left(X^{\varsigma}(\tau,\sigma)\right)}{\partial\sigma^{\alpha}} \frac{\partial X^{\nu}}{\partial\sigma^{\beta}} d\tau d\sigma,$$

$$\delta S_P = -T\int \sqrt{-\det(h_{ab})}\, h^{\alpha\beta} a_{\varsigma\nu} \frac{\partial X^{\varsigma}}{\partial\sigma^{\alpha}} \frac{\partial X^{\nu}}{\partial\sigma^{\beta}} d\tau d\sigma.$$

The term $a_{\varsigma\nu}$ is antisymmetric, so the following product vanishes, $a_{\varsigma\nu}\frac{\partial X^{\varsigma}}{\partial\sigma^{\alpha}}\frac{\partial X^{\nu}}{\partial\sigma^{\beta}} = 0$. This makes the variation to be zero and proves that the Polyakov action is manifestly Poincare invariant.

The Local symmetries are those that do not depend on the points of the manifold where they are being performed. Two crucial examples on our theory are reparameterization invariance and Weyl symmetry.

Polyakov action is invariant under a change of parameters as $\sigma' = f(\sigma)$ (this transformation on the parameters is also called diffeomorphism). This variable change lets $X^{\mu}(\tau,\sigma)$ invariant, and transforms the metric as a 2-form,

$$X'^{\mu}(\tau,\sigma') = X^{\mu}(\tau,\sigma) \qquad \text{and} \qquad h_{\alpha\beta} = \frac{\partial f^{\delta}}{\partial\sigma^{\alpha}}\frac{\partial f^{\gamma}}{\partial\sigma^{\beta}} h_{\delta\gamma} .$$

Weyl transformations represent changes on the scale of a metric (they are also called conformal transformations and they are basically local changes of scale that keep the angles of the parametrization).

$$h_{\alpha\beta}(\tau,\sigma) \longrightarrow h'_{\alpha\beta}(\tau,\sigma) = e^{2\phi(\sigma)}h_{\alpha\beta}(\tau,\sigma)$$

$$X^{\mu}(\tau,\sigma) \longrightarrow X^{\mu}(\tau,\sigma)$$

The Polyakov action depends on $h_{\alpha\beta}$ in the form $h^{\alpha\beta}\sqrt{-\det(h_{\alpha\beta})}$. The first term, $h^{\alpha\beta}$, transforms under a Weyl transformation as $h'^{\alpha\beta} = e^{-2\phi(\sigma)}h^{\alpha\beta}$, and the second as

$$\sqrt{-\det(h'_{\alpha\beta})} = \sqrt{-e^{2(2\phi(\sigma))}\det(h_{\alpha\beta})} =$$

$$e^{\frac{2(2\phi(\sigma))}{2}}\sqrt{-\det(h_{\alpha\beta})} = e^{2\phi(\sigma)}\sqrt{-\det(h_{\alpha\beta})}.$$

Therefore the product $h^{\alpha\beta}\sqrt{-\det(h_{\alpha\beta})}$ remains symmetric

$$h'^{\alpha\beta}\sqrt{-\det(h'_{\alpha\beta})} = h^{\alpha\beta}\sqrt{-\det(h_{\alpha\beta})}.$$

So the Polyakov action is symmetric under this transformation.

These local symmetries (Gauge symmetries) denote redundancies in the degrees of freedom of our theory, but it is possible to fix this redundancies with additional requirements. This is known as 'gauge fixing'. Gauge fixing can simplify our theory equations, for example, the electromagnetic field is invariant under the gauge group of phase transformations U(1) whose elements are of the form, $e^{i\phi(X)}$. We can fix these gauge by taking the restriction $\partial_{\mu}A^{\mu} = 0$ (where A^{μ} is the gauge field associated to the gauge group U(1)). Thus, we write the Maxwell equations in the compact form

$$\partial_{\mu}\partial^{\mu}A^{\nu} = ej^{\nu}.$$

We will show now that we can fix a gauge to make the metric h_{ab} flat, by using the local symmetries we have just introduced.

The metric $h_{\alpha\beta}$ is a symmetric tensor with three independent components, with two reparameterizations and a Weyl transformation we can make $h_{\alpha\beta}$ Minkowskian. At first, we make a reparameterization $\sigma^{(1)^{\mu}} = f^{(1)^{\mu}}(\sigma^{\alpha})$ to make $h_{01}^{(1)} = 0$; after that, a second reparameterization is performed, $\sigma^{(2)^{\mu}} = f^{(2)^{\mu}}(\sigma^{(1)^{\alpha}})$, which makes $h_{00}^{(2)} = -h_{11}^{(2)}$ and keeps also the previous condition $h_{01}^{(2)} = 0$.

The resultant metric is $h_{\alpha\beta}^{(2)} = \begin{pmatrix} -h_{11}^{(2)} & 0 \\ 0 & h_{11}^{(2)} \end{pmatrix}$. If we define $\xi_{\alpha\beta} \equiv \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ then $h_{\alpha\beta}^{(2)}$ can be written as $h_{\alpha\beta}^{(2)} = h_{11}^{(2)}\xi_{\alpha\beta}$. By employing a Weyl transformation to remove the $h_{11}^{(2)}$ factor, we rewrite the Polyakov action as

$$S_P = -\frac{T}{2}\int \xi^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} d\tau d\sigma \rightarrow$$

$$S_p = \frac{T}{2}\int \left(\dot{X}^2 - X'^2\right) d\tau d\sigma. \qquad (1.2.1)$$

Just assuming a flat Minkowskian background metric we have reduced the difficulty of the problem enormously; we will see why on the next chapter. As a conclusion of this chapter, we have explained the basic aspects in order to be able to understand the classical motion of the string. The next chapter will be focused on the details of the classical motion.

2 Solving Field Equations

En el capítulo anterior enunciamos los detalles necesarios para formular la acción de la cuerda y las simetrías necesarias para simplificarla. Ahora nos centraremos en la resolución de las ecuaciones para dicha acción poniendo atención a las condiciones de contorno y en imponer las restricciones necesarias para su correcto tratamiento.

In the previous section we obtained the action (1.2.1) employing the gauge symmetries of our theory and considering a Minkowskian space-time. The Euler-Lagrange equation of (1.2.1) is the free wave equation.

$$(\partial_t^2 - \partial_\sigma^2)\, X = 0. \tag{2.1.1}$$

But to obtain this result we have to consider firstly certain boundary conditions (all this chapter makes reference to (Tong, 2012) and (Wray, 2011)). We will establish these conditions varying the action (1.2.1) as follows

$$\delta S_p = \frac{T}{2}\int \left(2\dot{X}\delta\dot{X} - 2X'\delta X'\right) d\tau d\sigma =$$

$$T\int (-\partial_t^2 X^\mu + \partial_\sigma^2 X^\mu)\delta X^\mu d\tau d\sigma + T\int d\sigma \dot{X}\delta X^\mu]_{\partial\tau} \tag{2.1.2}$$

$$+\left\{T\int d\tau X'\delta X^\mu]_{\sigma=2\pi} - T\int d\tau X'\delta X^\mu]_{\sigma=0}\right\}.$$

The second term on the right hand side of (2.1.2) is zero because the variation at the boundary of τ is zero, $\delta X^\mu]_{\partial\tau} = 0$.

To remove the last two terms of the right hand side of (2.1.2) we have to stablish constraints at the strings boundaries. Firstly, we have to distinguish between open strings and closed strings. This is a crucial point that differs when we consider different string theories, as we had previously mentioned at the introduction. To make zero the boundary terms corresponding to the sigma coordinate we have three options, one of them is to consider the string to be closed and the other two belong to the opens discusion.

From now on in this essay. The most of the following calculations do not distinguish between closed or open strings, if we do not specify if we are talking about closed or open strings we will refer to closed strings. If the discussion were different for the open strings it will be indicated and explained.

Closed strings are characterized by periodic boundary condition, namely $X^\mu(\tau,\sigma) = X^\mu(\tau,\sigma + 2\pi)$.

The periodic condition implies that $\delta X^\mu(\tau,\sigma) = \delta X^\mu(\tau,\sigma + 2\pi)$ therefore, the boundary terms in (2.1.2) disappear, i.e.

$$T\int d\tau X'\delta X^\mu]_{\sigma=2\pi} - T\int d\tau X'\delta X^\mu]_{\sigma=0} = 0.$$

Besides, there is one thing we still have to consider to use the Polyakov action. We have to satisfy the condition $\frac{\delta S_P}{\delta h^{\alpha\beta}} = 0$. The condition (1.1.5) can be rewritten, by piking our gauge choice ($h_{\alpha\beta} = \xi_{\alpha\beta}$) and a flat Minkowskian space-time as,

$$\begin{cases} T_{00} = T_{11} = \frac{1}{2}\left(\dot{X}^2 + X'^2\right) = 0, \\ T_{10} = T_{01} = \dot{X} \cdot X' = 0. \end{cases} \tag{2.1.3}$$

Recapitulating, our string must obey the three following equations:

$$\begin{cases} (\partial_\tau^2 - \partial_\sigma^2)\, X^\mu = 0, \\ \frac{1}{2}\left(\dot{X}^2 + X'^2\right) = 0, \\ \dot{X} \cdot X' = 0. \end{cases}$$

The last equation of (2.1.3) tells us that the string vibrational part must be orthogonal to the strings time evolution. This is a necessary condition at the quantization to avoid what we will call ghosts states on the next section.

Employing the static gauge, $X^0 \equiv t = R\tau$, then $X^{0'} = 0$, and we use the notation $X^\mu = (t, \vec{x}\,)$. The conditions (2.1.3) are rewritten as,

$$\begin{cases} \dot{\vec{x}} \cdot \vec{x}' = 0, \\ \left(\dot{\vec{x}}^2 + \vec{x}'^2\right) = R^2. \end{cases} \tag{2.1.4}$$

The first condition tells us that the spatial modes of the string must be perpendicular to the string itself so the only allowed oscillations on the string are transverse oscillations. The second condition tells us that if $\dot{\vec{x}} = 0$ the length of the string is:

$$\int d\sigma \sqrt{\left(\frac{d\vec{x}}{d\sigma}\right)^2} = 2\pi R.$$

But it will not stay this way for too long, the string will contract under its own tension while the second equation found in (2.1.4) relates the length of the string with its instantaneous velocity $\dot{\vec{x}}$ determined by (2.1.1)

2.1 Mode Expansion and Noether Theorem

In this section, we will solve the field equation finding also the conserved charges. To solve the field equation we will use the light-cone coordinates on the worldsheet defined as:

$$\sigma^{\pm} = \tau \pm \sigma$$

This transformation affects to the partial derivates in the following way

$$\partial_{\pm} = \frac{1}{2}(\partial_\tau \pm \partial_\sigma).$$

In this coordinate frame the motion equations are rewritten as

$$(\partial_\tau^2 - \partial_\sigma^2)\, X = (\partial_\tau - \partial_\sigma)(\partial_\tau + \partial_\sigma)X = \partial_- \partial_+ X = 0.$$

The (2.1.3) conditions convert to the following restrictions:

On one hand,

$$\frac{1}{2}((\partial_\tau X)^2 + (\partial_\sigma X)^2) = \frac{1}{2}\left((\partial_- X + \partial_+ X)^2 + \left((-\partial_- X + \partial_+ X)\right)^2\right) =$$

$$\frac{1}{2}((\partial_- X)^2 + (\partial_+ X)^2 + 2\partial_- X \partial_+ X + (\partial_- X)^2 + (\partial_+ X)^2 - 2\partial_- X \partial_+ X) =$$

$$(\partial_- X)^2 + (\partial_+ X)^2 = 0.$$

On the other hand,

$$\dot{X} \cdot X' = (\partial_- X + \partial_+ X)(-\partial_- X + \partial_+ X) = (\partial_+ X)^2 - (\partial_- X)^2 = 0.$$

This two restrictions are only matching each other if,

$$\begin{cases} (\partial_- X)^2 = 0 \\ (\partial_+ X)^2 = 0 \end{cases}$$

Therefore, set of conditions that our string have to obey are

$$\begin{cases} (\partial_- X)^2 = 0 \\ (\partial_+ X)^2 = 0 \\ \partial_- \partial_+ X = 0 \end{cases} \tag{2.1.5}$$

We have enough information to introduce the general solution (2.1.5) applying Fourier series. The general solution of the wave equation has the form of a right moving wave plus a left moving wave,

$$X^\mu(\tau, \sigma) = X_R^\mu(\sigma^-) + X_L^\mu(\sigma^+). \tag{2.1.6}$$

Each of them with its corresponding expansion in Fourier modes

$$\begin{cases} X_R^\mu(\sigma^-) = \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n}\, \alpha_n^\mu e^{-in\sigma^-} \\ X_L^\mu(\sigma^+) = \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n}\, \tilde{\alpha}_n^\mu e^{-in\sigma^+} \end{cases} \tag{2.1.7}$$

Of course, the general solution still has to obey the additional constraints and the periodicity condition, $X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + 2\pi)$. Using this, the general solution is written as

$$X^{\mu} = x^{\mu} + \alpha' p^{\mu}\tau + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\left(\tilde{\alpha}_n^{\mu} e^{in\sigma} + \alpha_n^{\mu} e^{-in\sigma}\right)e^{-in\tau}. \quad (2.1.8)$$

With respect to the (2.1.8) expression we have to make some comments:

- Firstly, we can observe that X_R^{μ} and X_L^{μ} do not obey the periodicity conditions by their own but the sum of them does. The reason lay on the linear terms at $\sigma^{\pm}$.
- The factors α' and $\frac{1}{n}$ have been chosen for later convenience (to define the Virasoro algebra from the constraints of (2.1.5)) .
- The coefficients of the Fourier modes require $\alpha_n^{\mu} = (\alpha_{-n}^{\mu})^*$ and $\tilde{\alpha}_n^{\mu} = (\tilde{\alpha}_{-n}^{\mu})^*$, because $X^{\mu}(\tau,\sigma)$ are real fields.
- Lastly, we have to mention that here x^{μ} is the center of mases of the string and the factor p^{μ}, its momentum.

We obtain the string momentum term from the Noether current of the translational invariance with respect to the Poincare group. From the Noether theorem we know that every global symmetry has associated a conserved current and a conserved charge (see (Maiani L. ; Benar O., 2016; Polchinski, 1998) to consult Noether theorem on field theory).

We will now explain how to obtain the conserved quantities such as the linear momentum and angular momentum for field representations of the Poincare group. Then we will apply this to our string discussion.

If we perform an infinitesimal transformation of the Poincare group over the family of fields (that conforms a representation of the Lorentz group) that our theory depends on, $\phi^A(x)$, such, $\phi^A \to \phi^A + \delta\phi^A$, being, $\delta\phi^A = \epsilon_{ij}(M_B^{ij})^A\,\phi^B$, with ϵ_{ij} the infinitesimal parameter of the Lorentz transformation and $(M_B^{ij})^A$, the generators of the transformations from the B component of the representation to the A (all of them antisymmetric in the ij indices). Regarding the translational part the variation would be written as $\delta\phi^A = \epsilon_i(M_B^i)^A\,\phi^B$.

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi^A}\delta\phi^A + \frac{\partial\mathcal{L}}{\partial\partial_\nu\phi^A}\partial_\nu\delta\phi^A = \left(\frac{\partial\mathcal{L}}{\partial\phi^A} + \partial_\nu\frac{\partial\mathcal{L}}{\partial\partial_\nu\phi^A}\right)\delta\phi^A + \partial_\nu\left(\frac{\partial\mathcal{L}}{\partial\partial_\nu\phi^A}\delta\phi^A\right) = 0.$$

The first term of the last step is zero, because it is the Euler-Lagrange equation. The second term defines an integral that has to be zero to make $\delta S = 0$,

$$\int dx^D\,\partial_\nu\left(\frac{\partial\mathcal{L}}{\partial\partial_\nu\phi^A}\delta\phi^A\right) = \int dx^D\,\partial_\nu\left(\frac{\partial\mathcal{L}}{\partial\partial_\nu\phi^A}\epsilon_{ij}(M_B^{ij})^A\,\phi^B\right)$$

$$= \epsilon_{ij}\int dx^D\,\partial_\nu\left(\frac{\partial\mathcal{L}}{\partial\partial_\nu\phi^A}(M_B^{ij})^A\,\phi^B\right) = \epsilon_{ij}\int dx^D\,\partial_\nu(J^{ij})^\nu = 0.$$

So if we want this to be satisfied for every, ϵ_{ij}, this requires that,

$$\int dx^D\,\partial_0(J^{ij})^0 + \partial_\nu(J^{ij})^\nu = \frac{dQ^{ij}}{d\tau} + \int dx^{D-1}\,\partial_\nu(J^{ij})^\nu = 0 \quad (2.1.9)$$

For the Lorentz transformations of the Poincare group for our string coordinates, $\delta\phi^A = \epsilon_{ij}(M_B^{ij})^A\,\phi^B \rightarrow \delta X^\mu = a_\nu^\mu X^\nu(\tau,\sigma)$,

$$\int d\sigma^2\,\partial_\nu\left(\frac{\partial\mathcal{L}}{\partial\partial_\alpha X^\mu}\delta X^\mu\right) = \int d\sigma^2\,\partial_\alpha\left(\frac{\partial\mathcal{L}}{\partial\partial_\alpha X^\mu}a_\nu^\mu X^\nu(\tau,\sigma)\right)$$

$$= a_{\mu\nu}\int d\sigma^2\,\partial_\alpha\left(\eta^{\mu\rho}\frac{\partial\mathcal{L}}{\partial\partial_\alpha X^\rho}X^\nu(\tau,\sigma)\right),$$

as $a_{\mu\nu}$ is antisymmetric the associated current must be also antisymmetric so

$$J_\alpha^{\mu\nu} = \frac{1}{2}\left[\eta^{\mu\rho}\frac{\partial\mathcal{L}}{\partial\partial_\alpha X^\rho}X^\nu(\tau,\sigma) - \eta^{\mu\rho}\frac{\partial\mathcal{L}}{\partial\partial_\alpha X^\rho}X^\nu(\tau,\sigma)\right], \tag{2.1.10}$$

for the for the Lorentz transformations the generators of the transformation are the Q^{ij},

$$L^{\mu\nu} = \int d\sigma\,J_0^{\mu\nu} = \int d\sigma\frac{1}{2}\left[\eta^{\mu\rho}\frac{\partial\mathcal{L}}{\partial\partial_0 X^\rho}X^\nu(\tau,\sigma) - \eta^{\mu\rho}\frac{\partial\mathcal{L}}{\partial\partial_0 X^\rho}X^\nu(\tau,\sigma)\right]. \tag{2.1.11}$$

As we have gauge fixed the action to be

$$S_P = -\frac{T}{2}\int \xi^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu\eta_{\mu\nu}d\tau d\sigma.$$

Then, $\mathcal{L} = -\frac{T}{2}\xi^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu\eta_{\mu\nu}$, and we obtain,

$$\frac{\partial\mathcal{L}}{\partial\partial_\alpha X^\mu} = -\frac{T}{2}\xi^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu\eta_{\mu\nu} = -\frac{T}{2}\xi^{\alpha\beta}\partial_\beta X^\nu\eta_{\mu\nu} - \frac{T}{2}\xi^{\beta\alpha}\partial_\beta X^\mu\eta_{\mu\nu} = -T\partial^\alpha X_\mu,$$

$$L^{\mu\nu} = \int d\sigma\frac{T}{2}\left[\eta^{\mu\rho}\partial^0 X_\rho X^\nu(\tau,\sigma) - \eta^{\nu\rho}\partial^0 X_\rho X^\mu(\tau,\sigma)\right],$$

Therefore, we obtain the expression we were looking for the angular momentum

$$L^{\mu\nu} = \int d\sigma\frac{T}{2}\left[\partial^0(X^\mu)X^\nu - \partial^0(X^\nu)X^\mu\right]. \tag{2.1.12}$$

For the translational part of the Poincare group we obtain $X^\mu(\tau,\sigma) \rightarrow X^\mu(\tau,\sigma) + b^\mu$ so

$$\int d\sigma^2\,\partial_\nu\left(\frac{\partial\mathcal{L}}{\partial\partial_\alpha X^\mu}\delta X^\mu\right) = \int d\sigma^2\,\partial_\alpha\left(\frac{\partial\mathcal{L}}{\partial\partial_\alpha X^\mu}b^\mu\right) = b^\mu\int d\sigma^2\,\partial_\alpha\left(\frac{\partial\mathcal{L}}{\partial\partial_\alpha X^\mu}\right).$$

The conserved current of the μ component on the α coordinate corresponds to the energy-momentum tensor and is written as

$$(j_\mu)^\alpha = -T\partial^\alpha X_\mu \tag{2.1.13}$$

Then, the conserved charge of this tensor is the linear momentum

$$P^\nu = \int_0^{2\pi} d\sigma\, \eta^{\nu\mu}(j_\mu)^0 = \int_0^{2\pi} d\sigma\, T\partial^0 X^\mu \tag{2.1.14}$$

This integral eliminates the $e^{\pm in\sigma}$ factor of the modes so the result is

$$P^\nu = 2\pi T\alpha' p^\nu$$

So here making $\alpha' = \frac{1}{2\pi T}$ we obtain that the p^ν factor in the mode expansion is in fact the total momentum of the string.

2.2 Constraints Applied to The String

In the light con gauge, the constraints from (2.1.5) are (we will denote α_n^μ just as α_n and the scalar product in the Minkowski space as $\alpha_m \cdot \alpha_n$)

$$(\partial_- X)^2 = (\partial_- X_R)^2 = \left(\frac{\alpha'}{2}p^\mu + \sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0} \alpha_n^\mu e^{-in\sigma^-}\right)^2 = 0.$$

If we define $\alpha_0^\mu \equiv \sqrt{\frac{\alpha'}{2}}p^\mu$ we can write it in a compact form

$$\frac{\alpha'}{2}\left(\sum_{n,m} \alpha_m \cdot \alpha_n e^{-i(n+m)\sigma^-}\right) = \frac{\alpha'}{2}\left(\sum_{m,p} \alpha_m \cdot \alpha_{p-m} e^{-i(p)\sigma^-}\right) = 0.$$

By defining the Virasoro generators $L_p = \frac{1}{2}\sum_m \alpha_m \cdot \alpha_{p-m}$ and $\tilde{L}_p = \frac{1}{2}\sum_m \tilde{\alpha}_m \cdot \tilde{\alpha}_{p-m}$,

we write $(\partial_- X)^2 = 0$ as

$$\alpha'\left(\sum_p L_p e^{-i(p)\sigma^-}\right) = 0.$$

And by using a similar develop $(\partial_+ X)^2 = 0$ as

$$\alpha' \left(\sum_p \tilde{L}_p e^{-i(p)\sigma^+} \right) = 0$$

This is only possible if $L_p = \tilde{L}_p = 0$ for all p in Z.

One of this constraints has a special property. The L_0 and $\tilde{L}_0$ contain a term proportional to $p_\mu p^\mu$ that is the square of the mass at rest of the particle, $p_\mu p^\mu = -M^2$. Then the mass of the particle can be written as

$$M^2 = \frac{4}{\alpha'} \sum_{n>0} \alpha_n \cdot \alpha_{-n} = \frac{4}{\alpha'} \sum_{n>0} \tilde{\alpha}_n \cdot \tilde{\alpha}_{-n}.$$

These two terms must be equal to each other. This necessary condition is known as level matching and it is the only constraint between the left and right modes for the closed strings.

Here, changing the discussion to the open strings, the parameter sigma goes from zero to π, $\sigma \in [0, \pi]$, then the boundary term that has to vanish is

$$T \int d\tau X' \delta X^\mu]_{\sigma=\pi} - T \int d\tau X' \delta X^\mu]_{\sigma=0} = 0$$

For the open strings, this requires that $X' \delta X^\mu = 0$ on the endpoints of the string. We can impose two types of boundary conditions to achieve this.

- Newman boundary conditions:

$$\begin{cases} X^{\mu\prime}(\tau, \sigma = 0) = 0 \\ X^{\mu\prime}(\tau, \sigma = \pi) = 0 \end{cases} \tag{2.2.2}$$

  These conditions impose $\alpha_n^\mu = \tilde{\alpha}_n^\mu$ which makes zero the momentum at the boundaries. The string endpoints are not fixed but their derivatives vanish at the boundary. The endpoints can move freely but they still have to obey the eq. (2.1.3). If we set again the static gauge we obtain the equations (2.1.4) with the condition $X^{0\prime} = 0 \to \vec{x}' = 0$ so we are left with $\left(\dot{\vec{x}}^2\right) = R^2 \to \left|\frac{d\vec{x}}{d\tau}\right| = 1$, so the endpoints move at the speed of light.

- Dirichlet boundary conditions:

$$\begin{cases} \delta X^\mu(\tau, \sigma = 0) = 0 \\ \delta X^\mu(\tau, \sigma = \pi) = 0 \end{cases}$$

  Then

$$\begin{cases} X^\mu(\tau, \sigma = 0) = X_0^\mu \\ X^\mu(\tau, \sigma = 2\pi) = X_\pi^\mu \end{cases} \tag{2.2.3}$$

This condition imposes the constraint that in the mode expansion $\alpha_n^\mu = -\tilde{\alpha}_n^\mu$. This boundary conditions are a little bit odd. This condition makes the reader to ask himself how to do physics with them, if their boundaries do not move. Which is the mining of a fixed boundary at X^0 and if the string is fixed to an instant which is the meaning of τ.

To solve this let us imagine we had mixed conditions,

$$X^{\mu\prime} = 0 \quad \text{for} \quad \mu = 0, \dots, p$$

$$\delta X^\mu = 0 \quad \text{for} \quad \mu = p+1, \dots, D-1$$

This force the endpoints to lie in a $p+1$ dimensional hypersurface so the Lorentz group SO$(1, D-1)$ is now broken into SO$(1,p)\times$SO$(D-p-1)$. This object is called Dp-brane where the D comes from Dirichlet and p is the number of spatial dimentions. This surface has to be introduced as a new dynamical object on the theoretical frame, but this is not the topic of this essay.

Concerning the mode expansion of open strings,

$$\begin{cases} X_R^\mu(\sigma^-) = \frac{1}{2}x^\mu + \alpha' p^\mu \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}\, \alpha_n^\mu e^{-in\sigma^-} \\ X_L^\mu(\sigma^+) = \frac{1}{2}x^\mu + \alpha' p^\mu \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}\, \tilde{\alpha}_n^\mu e^{-in\sigma^+} \end{cases} \tag{2.2.4}$$

We notice that we have $\alpha' p^\mu$ instead of $\frac{1}{2}\alpha' p^\mu$. This occurs due to the fact that the conserved charge P^ν must remain the same for open and closed strings; then as we have to integrate over σ from 0 to π, instead of 0 to 2π, we put this factor of two in the linear term

$$P^\nu = \int_0^\pi d\sigma\, \eta^{\nu\mu} (j_\mu)^0 = \int_0^\pi d\sigma \frac{1}{2\pi\alpha'} \partial^0 X^\mu,$$

$$P^\nu = \frac{1}{2\pi\alpha'} \int_0^\pi d\sigma 2\alpha' p^\nu.$$

So as the string momentum must remain unchanged the linear term must change by this factor 2 (The tension remains the same for open and closed strings $\frac{1}{2\pi\alpha'} = T$). This has consequences in the mass formula. Let us see closely the gauge constraints,

$$(\partial_- X)^2 = (\partial_- X_R)^2 = \left(\alpha' p^\mu + \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \alpha_n^\mu e^{-in\sigma^-} \right)^2 = 0.$$

If we define $\sqrt{\frac{\alpha'}{2}} \alpha_0^\mu \equiv \alpha' p^\mu$ the condition L_0=0 can be written as

$$L_0 = \frac{1}{2} \sum_m \alpha_m \cdot \alpha_{-m} = \frac{1}{2} \alpha_0^2 + \sum_{m>0} \alpha_m \cdot \alpha_{-m} = \frac{1}{2} 2\alpha' p_\mu p^\mu + \sum_{m>0} \alpha_m \cdot \alpha_{-m} = 0$$

Then we finally obtain the mass formula for open strings

$$M^2 = \frac{1}{\alpha'} \sum_{m>0} \alpha_m \cdot \alpha_{-m} \qquad (2.2.5)$$

We observe a difference of a 4 factor with the closed string mass. The previous constraints on the modes of the Dirichlet and Newman conditions make that if we start with $(\partial_+ X)^2 = 0$ we obtain the same result.

3 The Quantum String

En este capítulo nos centraremos en la cuantización de la cuerda mostrando las dificultades que esta conlleva, así como su resolución tomando diferentes caminos y como esto conlleva restricciones sobre el número de dimensiones en los que vive la teoría.

We have observed in the previous chapters that the bosonic string theory is a gauge theory. If we are working with the canonical formalism that we will use to quantize the string, there are different ways to proceed. We have here two choices.

We could firstly impose the canonical quantization rules and, starting from them, to impose the gauge constraints as operator equations. As a equivalent way, we could first impose the gauge conditions to simplify the classical equations and then quantize the system. These two methods should agree if we do them correctly.

On this chapter we will follow (Green, Schwatrz, & Witten, 1987) and (Tong, 2012).

3.1 Canonical Quantization

The canonical quantization procedure changes the canonical variables of the theory by operators in a Hilbert space and the Poisson brackets by commutation relations between them.

If we want to first quantize a single string we have to quantize the field that defines the worldsheet.

We impose equal-time commutator relations on the field (taking into account that the conjugated momentum $\Pi_\nu(\tau,\sigma') = \frac{1}{2\pi\alpha\prime}\dot{X}_\nu$)

$$[X^\mu(\tau,\sigma), \Pi_\nu(\tau,\sigma')] = i\delta(\sigma-\sigma')\delta^\mu_\nu,$$
$$[X^\mu(\tau,\sigma), X^\nu(\tau,\sigma)] = 0, \quad [\Pi_\mu(\tau,\sigma'), \Pi_\nu(\tau,\sigma')] = 0. \quad (3.1.1)$$

This is traduced to commutator relations in terms of the mode expansion operators $\hat{x}^\mu$, $\hat{p}_\mu$, $\hat{\alpha}_n{}^\mu$, $\hat{\tilde{\alpha}}_n{}^\mu$ as

$$[\hat{x}^\mu, \hat{p}_\nu] = i\delta^\mu_\nu, \ \ [\hat{\alpha}_n{}^\mu, \hat{\alpha}_m{}^\nu] = n\eta^{\mu\nu}\delta_{n+m,0}, \ \ [\hat{\tilde{\alpha}}_n{}^\mu, \hat{\tilde{\alpha}}_m{}^\nu] = n\eta^{\mu\nu}\delta_{n+m,0}, \quad (3.1.2)$$

with all others zero (the hat on the operators represents that we are talking about operators, not variables but, later on, in some cases, we will leave this notation without giving rise to confusion). If we define $\hat{\alpha}_n{}^\mu = \sqrt{n}\hat{a}_n{}^\mu$, the $\hat{a}_n{}^\mu$ operator relations are those of the harmonic oscillator except for the 0^{th} component where we have a negative sign due to the signature of the metric.

As in quantum field theory we must construct the Fock space of our field theory, starting from the vacuum state, that here, is not a space-time vacuum as its analog in field theory, but a vacuum on the worldsheet. This vacuum (the ground state) is a state which goes to zero under the action of all the mode operators of the string except $\hat{x}^{\mu}$ and $\hat{p}_{\mu}$. It is defined as $|0,p\rangle$ where $\hat{p}_{\mu}|0,p\rangle = p_{\mu}|0,p\rangle$,

$$\hat{\alpha}_n{}^{\mu}|0,p\rangle = \hat{\tilde{\alpha}}_n{}^{\mu}|0,p\rangle = 0,$$

for $n > 0$.

The physical states are built with the rising operators $\hat{a}_n{}^{\mu\dagger}$ such as,

$$|\phi\rangle = \hat{a}_{n_1}{}^{\mu_1\dagger}\hat{a}_{n_2}{}^{\mu_2\dagger}\hat{a}_{n_3}{}^{\mu_3\dagger} \ldots \hat{a}_{n_i}{}^{\mu_i\dagger}|0,p\rangle.$$

The problem comes from the negative norm states built with the 0^{th} component of the modes as we have already mentioned,

$$\left\langle 0 \middle| \hat{a}_n{}^0 \hat{a}_n{}^{0\dagger} \middle| 0 \right\rangle = \left\langle 0 \middle| \hat{a}_n{}^{0\dagger} \hat{a}_n{}^0 \middle| 0 \right\rangle + (-1)\langle 0|0\rangle = (-1)\langle 0|0\rangle.$$

These negative norm states are called 'ghosts' and we have to fix our theory to eliminate them. We will see that this constricts our bosonic string theory to live in a 26-dimensional space-time, one temporal and 25 spatial others.

As we have said at the beginning of this chapter there are two ways of quantizing a gauge theory with the canonical formalism, we can gauge fix before quantizing or impose the gauge fixing as operator equations after that. Following the second path, we will impose the gauge restrictions as operator equations over the physical states $|\phi\rangle$.

The space of all physical states is a subspace of the total Fock space. In the classical model the physical states are those that obey the gauge fixing conditions.

$$L_p = \tfrac{1}{2}\textstyle\sum_m \alpha_m \cdot \alpha_{p-m} = \tilde{L}_n = \tilde{L}_p = \tfrac{1}{2}\textstyle\sum_m \tilde{\alpha}_m \cdot \tilde{\alpha}_{p-m} = 0 \ \textit{for all n in Z}\ .$$

As it has been already mentioned the α_m variables are promoted to operators, so we have operator order ambiguities when we try to write the gauge conditions. For p different from zero α_m commutes with α_{p-m} so the only ambiguity we are left with, appears in the case $p = 0$.

We set the correct order to be the normal ordering

$$\hat{L}_0 = \frac{1}{2}\hat{\alpha}_0^2 + \sum_{m>0} :\hat{\alpha}_{-m} \cdot \hat{\alpha}_m:$$

The normal ordering, just like it is done to exclude infinites on field theories, obeys the following rule

$$:\hat{\alpha}_{-m} \cdot \hat{\alpha}_m: = \hat{\alpha}_{-m} \cdot \hat{\alpha}_m \ , \qquad :\hat{\alpha}_m \cdot \hat{\alpha}_{-m}: = \hat{\alpha}_{-m} \cdot \hat{\alpha}_m$$

We define

$$:\hat{\alpha}_m\cdot\hat{\alpha}_n: \;=\begin{cases}\hat{\alpha}_m\cdot\hat{\alpha}_n\,, & n>m\\ \hat{\alpha}_n\cdot\hat{\alpha}_m\,, & m>n\end{cases}$$

The condition of the vanishing L_0 that defines the allowed movements of the classical string traduces in the quantum theory as the requirement that any physical state must be zero under the action of the $\hat{L}_0$ operator. The normal order condition we have imposed makes us introduce a constant in this condition to correct possible wrong physical states. So, $L_0 = 0$, is now:

$$(\hat{L}_0 - a)\,|\phi\rangle = (\hat{\tilde{L}}_0 - a)\,|\phi\rangle = 0 \qquad (3.1.\underline{3})$$

This equation will give us the mass operator M of the string states,

$$(\hat{L}_0 - a)\,|\phi\rangle = (\frac{1}{2}\hat{\alpha}_0^2 + \sum_{m>0} :\hat{\alpha}_{-m}\cdot\hat{\alpha}_m: - a)|\phi\rangle = 0,$$

$$(\frac{\alpha'}{4}p^2 + \sum_{m>0} :\hat{\alpha}_{-m}\cdot\hat{\alpha}_m: - a)|\phi\rangle = 0,$$

$$M^2 = \frac{4}{\alpha'}\left(\sum_{m>0} :\hat{\alpha}_{-m}\cdot\hat{\alpha}_m: - a\right) = \frac{4}{\alpha'}(N-a) = \frac{4}{\alpha'}(\tilde{N}-a). \qquad (3.1.\underline{4})$$

We define $\sum_{m>0} :\hat{\alpha}_{-m}\cdot\hat{\alpha}_m:$ as the number operator.

The number operator, N, is written as $N = \sum_{m>0} m{:}\,\hat{\alpha}_{-m}\cdot\hat{\alpha}_m{:}$. We notice that from (3.1.4) we obtain N=$\tilde{N}$; this is the level matching condition for the oscillator excitation states. The eigenvalues of the number operator give the values of the permitted masses, $M^2 = \frac{4}{\alpha'}(n-a)$ for closed strings and $M^2 = \frac{1}{\alpha'}(n-a)$, for the open ones. We will only discuss closed ones, but it is interesting to mention that the photon arises as an open string state.

1. The ground state mass obeys the following formula; $M^2 = \frac{4}{\alpha'}(-a)$, this corresponds to a tachyon, a particle with imaginary mass. This is a problem of the bosonic string theory that is solved in superstring theory using the GSO conditions, see (Green, Schwatrz, & Witten, 1987).
2. The first excited state is a massless state $M^2 = \frac{4}{\alpha'}(1-a) = 0$ ($a = 1$ as we will soon see) this corresponds with the graviton on the closed string discussion. We will see it later, on the last chapter.

If we want to have a spectrum free of ghosts we need to impose restrictions over the variables a and D (the number of dimensions of the space-time). But before introducing this topic, firstly the study of the Virasoro algebra will be developed.

3.2 Virasoro Algebra

The classical Virasoro algebra comes defined by the Poisson brackets of the generators $\{L_n\}$

$$\{L_n, L_m\} = (n-m)L_{n+m}. \tag{3.2.1}$$

When we introduce the canonical quantization and the normal ordering these commutation relations change a little bit. If $n+m \neq 0$ these commutator relations remain unchanged because the mode operators commute, but if $n+m=0$ the commutation relations change. As the order ambiguities on these relations introduced by the normal ordering will only involve a number, we are guaranteed to have

$$[\hat{L}_m, \hat{L}_n] = (m-n)\hat{L}_{m+n} + A(m)\delta_{m+n,0}. \tag{3.2.2}$$

This is known as the central extension of the Virasoro Algebra and the additional term is called the anomaly term in that algebra. We have some trivial relations of this factor as $A(m) = -A(-m)$ and $A(0) = 0$, hence it is enough to find $A(m)$ for positive m.

To find the form of $A(m)$ we will use the Jacobi identity that is satisfied by the generators of any Lie algebra,

$$\left[\hat{L}_k, [\hat{L}_m, \hat{L}_n]\right] + \left[\hat{L}_m, [\hat{L}_n, \hat{L}_k]\right] + \left[\hat{L}_n, [\hat{L}_k, \hat{L}_m]\right] = 0.$$

For the choice: $k+n+m=0$, we have

$$(m-n)A(k) + (n-k)A(m) + (k-m)A(n) = 0.$$

Setting $k=1$ and $m=n+1$ it gives

$$A(n+1) = \frac{nA(n) - A(1)}{n-1}$$

This enables us to obtain all the $A(m)$ in terms of $A(1)$ and $A(2)$, that here are two unknown coefficients. The general solution that obey these relations is

$$A(m) = c_3 m^3 + c_1 m \tag{3.2.3}$$

With c_3, c_1 as constants. To obtain these constants we have to be careful with the choice of the state and the m selection. We will use the commutator of $\hat{L}_2 = \frac{1}{2}\sum_n \alpha_{2-n}\alpha_n$ and $\hat{L}_{-2} = \frac{1}{2}\sum_n \alpha_n\alpha_{n-2}$,

$$A(2) = \langle 0,0|[\hat{L}_2, \hat{L}_{-2}]|0,0\rangle = \langle 0,0|\hat{L}_2\hat{L}_{-2}|0,0\rangle = \frac{1}{4}\langle 0,0|\alpha_1 \cdot \alpha_1\alpha_{-1} \cdot \alpha_{-1}|0,0\rangle$$

using $[\hat{\alpha}_n{}^{\mu}, \hat{\alpha}_m{}^{\nu}] = n\eta^{\mu\nu}\delta_{n+m,0} \rightarrow \alpha_1^{\nu}\alpha_{-1}^{\mu} = \eta^{\nu\mu} + \alpha_{-1}^{\mu}\alpha_1^{\nu}$,

$$\langle 0,0|\hat{L}_2\hat{L}_{-2}|0,0\rangle = \frac{1}{4}\langle 0,0|\alpha_1 \cdot \alpha_{-1}\alpha_1 \cdot \alpha_{-1}|0,0\rangle + \frac{1}{4}\left\langle 0,0\middle|\eta^{\nu'\mu'}\eta_{\mu'\nu}\alpha_1^{\nu}\eta_{\mu\nu'}\alpha_{-1}^{\mu}\middle|0,0\right\rangle =$$

$$\frac{1}{4}\left\langle 0,0\middle|\eta_{\mu\nu}\alpha_1^{\nu}\alpha_{-1}^{\nu'}\eta_{\mu'\nu'}\alpha_1^{\mu}\alpha_{-1}^{\mu'}\middle|0,0\right\rangle + \frac{1}{4}\left\langle 0,0\middle|\eta_{\mu\nu}\alpha_1^{\nu}\alpha_{-1}^{\mu}\middle|0,0\right\rangle =$$

$$\frac{1}{4}\left\langle 0,0\middle|\eta_{\mu\nu}\eta^{\nu\nu'}\eta_{\mu'\nu'}\alpha_1^{\mu}\alpha_{-1}^{\mu'}\middle|0,0\right\rangle + \frac{1}{4}\left\langle 0,0\middle|\eta_{\mu\nu}\alpha_1^{\nu}\alpha_{-1}^{\mu}\middle|0,0\right\rangle =$$

$$\frac{1}{4}\langle 0,0|\eta_{\mu\nu}\alpha_1^\mu\alpha_{-1}^\nu|0,0\rangle + \frac{1}{4}\langle 0,0|\eta_{\mu\nu}\alpha_1^\nu\alpha_{-1}^\mu|0,0\rangle =$$

$$\frac{1}{2}\eta_{\mu\nu}\eta^{\nu\mu}\langle 0,0|\alpha_1^\mu\alpha_{-1}^\nu|0,0\rangle = \frac{1}{2}\eta_{\mu\nu}\eta^{\nu\mu} = \frac{1}{2}D.$$

$$\frac{D}{2} = A(2) = c_3 8 + c_1 2.$$

As we also know that $\langle 0,0|[\hat{L}_1,\hat{L}_{-1}]|0,0\rangle = 0$, because for the $m = 1$ p^μ annihilates the zero-momentum ground state $0 = A(1) = c_3 + c_1 \rightarrow c_1 = -c_3$. We finally have

$$A(m) = \frac{D}{12}(m^3 - m).$$

Eventually we obtain the searched term,

$$[\hat{L}_m,\hat{L}_n] = (m-n)\hat{L}_{m+n} + \frac{D}{12}(m^3 - m)\delta_{m+n,0}. \quad (3.2.4)$$

This condition will allow us to impose the correct gauge restrictions as operator equations. Let us consider a physical state $|\phi\rangle$. If we promote just the gauge restrictions to operators as $\hat{L}_m|\phi\rangle = 0$ for all $m \neq 0$ then

$$\langle\phi|[\hat{L}_m,\hat{L}_n]|\phi\rangle = \left\langle\phi\middle|(m-n)\hat{L}_{m+n} + \frac{D}{12}(m^3 - m)\delta_{m+n,0}\middle|\phi\right\rangle.$$

If $m + n \neq 0$ then $\langle\phi|\hat{L}_{m+n}|\phi\rangle = 0$ so we only can consider $m = 0,1,-1$. In fact $\{\hat{L}_1,\hat{L}_0,\hat{L}_{-1}\}$ constitute a closed subalgebra of the Virasoro algebra.

Instead of doing this, we will only impose, $\hat{L}_m|\phi\rangle = 0$, for all $m > 0$.

This way, the set of physical states are then characterized by the conditions

$$(\hat{L}_m - a\delta_{m,0})\,|\phi\rangle \quad \text{for all } m \geq 0. \quad (3.2.5)$$

We notice that $L_m \approx p \cdot \alpha_m$ plus additional terms; so if the rest of the terms were absent, as p has to be timelike, α_m is necessarily spacelike. So we are making something good in the construction on our physical states, we are eliminating the timelike components. In the next section, we will see how the ghost states are eliminated from the theory.

3.3 Eliminating Ghosts

The negative norm states vary when we change the values of the variables a and D. The proof of the no-ghosts theorem sets that for the values of the undetermined constants a and D, we have to impose $a = 1$ and $D = 26$. This way the string has only transverse oscillator excitations. In this section, we will not prove this result but we will give some clues about how that happens.

At first, we will make some definitions:

1. A state, $|\phi\rangle$, is called physical if it satisfies the following conditions

$$(\hat{L}_0 - a)|\phi\rangle = 0, \qquad \hat{L}_{m>0}|\phi\rangle = 0 \, . \tag{3.3.1}$$

2. A state, $|\psi\rangle$, is called spurious if it is orthogonal to all physical states and if it satisfy the mass shell condition.

A state $|\psi\rangle$ orthogonal to all physical states by definition can be written as

$$|\psi\rangle = \sum \hat{L}_{-m}|\chi_m\rangle, \quad (m > 0). \tag{3.3.2}$$

$$\langle\phi|\psi\rangle = \sum \langle\phi|\hat{L}_{-m}|\chi_m\rangle = \sum \langle\chi_m|\hat{L}_m|\phi\rangle^* = 0, \quad (m > 0).$$

if $|\psi\rangle$ satisfies the mass shell condition then $|\chi_m\rangle$ states satisfy a modified mass shell condition

$$(\hat{L}_0 - a)|\psi\rangle = 0,$$

$$(\hat{L}_0 - a)\sum \hat{L}_{-m}|\chi_m\rangle = 0,$$

$$\sum (\hat{L}_0 - a)\hat{L}_{-m}|\chi_m\rangle = 0,$$

$$\sum \{\hat{L}_{-m}(\hat{L}_0 - a) + m\hat{L}_{-m}\}|\chi_m\rangle = 0,$$

$$\sum \hat{L}_{-m}(\hat{L}_0 - a + m)|\chi_m\rangle = 0.$$

In other words, if $(\hat{L}_0 - a)|\psi\rangle = 0$ this implies $(\hat{L}_0 - a + m)|\chi_m\rangle = 0$ for all $m > 0$. These are modified mass shell conditions that define the $|\chi_m\rangle$ states

$$(\hat{L}_0 - a + m)|\chi_m\rangle = 0 \text{ for all } m > 0. \tag{3.3.3}$$

By a similar argument we can say that $|\chi_m\rangle$ are eigenstates of the $\hat{L}_0$ operator with eigenvalues $m - a$ and since $\hat{L}_0\hat{L}_{-n}|\chi_m\rangle = (m - a + n)\hat{L}_{-n}|\chi_m\rangle$ we can say that $\hat{L}_{-n}|\chi_m\rangle = |\chi_{m+n}\rangle$.

Any operator $\hat{L}_n$ for $n > 2$ can be written using $[\hat{L}_1, \hat{L}_{n-1}] = (n-2)\hat{L}_n$ in a iterative way starting from $\hat{L}_2$. Thus, we can construct any $\hat{L}_m$ from $\hat{L}_1$ and $\hat{L}_2$, then the conditions $\hat{L}_{m>0}|\phi\rangle = 0$ are simplified to $\hat{L}_{m=1,2}|\phi\rangle = 0$ and the spurious state can be truncated to orther two,

$$|\psi\rangle = \sum \hat{L}_{-m}|\chi_m\rangle, \quad (m = 1,2). \tag{3.3.4}$$

The spurious physical states are states orthogonal to themselves, therefore, they have zero norm $\langle\psi|\psi\rangle = 0$.

The zero norm states have crucial conditions on their construction that will allow us to figure out how the parameters a and D give rise to a theory free of ghosts.

The first condition we will impose to build the zero normed state is, $\hat{L}_1\hat{L}_{-1}|\chi_1\rangle = 0$, in other words $\hat{L}_1$ eliminates the first order spurious state,

$$\hat{L}_1\hat{L}_{-1}|\chi_1\rangle = 0 \rightarrow \hat{L}_1\hat{L}_{-1}|\chi_1\rangle = 2\hat{L}_0|\chi_1\rangle = 2(1-a)|\chi_1\rangle = 0 \rightarrow a = 1.$$

After obtaining the first condition to construct these states the second arises logically as

$$\hat{L}_1(\hat{L}_{-2} + \gamma\hat{L}_{-1}\hat{L}_{-1})|\chi_2\rangle = 0,$$

$$\hat{L}_2(\hat{L}_{-2} + \gamma\hat{L}_{-1}\hat{L}_{-1})|\chi_2\rangle = 0,$$

where the constant γ has been introduced to ensure that $(\hat{L}_{-2} + \gamma\hat{L}_{-1}\hat{L}_{-1})|\chi_2\rangle$ has zero norm. From, $\hat{L}_1(\hat{L}_{-2} + \gamma\hat{L}_{-1}\hat{L}_{-1})|\chi_2\rangle = 0$, we obtain $\gamma = \frac{3}{2}$.

From the $\hat{L}_2(\hat{L}_{-2} + \gamma\hat{L}_{-1}\hat{L}_{-1})|\chi_2\rangle = 0$, condition we will obtain the anomalous term of the Virasoro algebra due to the commutator $[\hat{L}_2, \hat{L}_{-2}]$; then we will establish the dimension, $D = 26$.

This increment of zero norm spurious terms for the values $D = 26$ and $a = 1$, is the clue we were looking for. The boundary between the negative norm and positive norm are the zero norm physical states. The critical dimension that sets the limit to the emergence of negative norm states is $D = 26$ where we finally obtain a theory free of ghosts. In the super string theory the number of dimensions is 10.

3.4 Lightcone Gauge

We have already introduced the lightcone coordinates on the worldsheet. The use of these coordinates on the background Minkowskian spacetime will allow us to quantize only the transverse oscillators, which will give us the positive normed Hilbert space we were looking for. But in the process, we will lose the Lorentz invariance, that we will recover setting, again, $D = 26$ and $a = 1$.

We implement the lightcone coordinates as

$$X^{\pm} = \sqrt{\frac{1}{2}}(X^0 \pm X^{D-1}), \quad X^1, \dots, X^{D-2} = X^1, \dots, X^{D-2}$$

Here we lose the Lorentz invariance manifestly because we pick a preferential direction on the coordinates to make the transformation.

Some properties of the lightcone coordinates are; $A_+ = -A^-,\ A_- = -A^+,\ A_i = A^i$ (i refers to the $X^1, \dots, X^{D-2}$ coordinates). This way the scalar product is written as

$$A \cdot B = -A^+B^- - B^+A^- + \sum_{i=1}^{D-2} B^iA^i.$$

On the solutions we have a remaining diffeomorphism symmetry that permits us to make the change $\sigma^{\pm} \to \sigma'^{\pm} = \zeta^{\pm}(\sigma^{\pm})$.

We can, therefore, define $\tau' = \frac{1}{2}\left(\zeta^{+}(\sigma^{+}) + \zeta^{-}(\sigma^{-})\right)$, but we will use it to relate τ' to one of the coordinates, in this case we choose X^{+}.

$$\tau' = \frac{1}{2}\left(\zeta^{+}(\sigma^{+}) + \zeta^{-}(\sigma^{-})\right) = \frac{X_R^{+}(\sigma^{-}) + X_L^{+}(\sigma^{+})}{\alpha' p^{+}} + x^{+} \to$$
$$X^{+} = x^{+} + \alpha' p^{+} \frac{1}{2}\left(\sigma'^{+} + \sigma'^{-}\right). \tag{3.4.1}$$

The form of the X^{-} coordinate comes from the restrictions (2.1.5), each of them will give us the $X_R^{-}(\sigma^{-})$ and the $X_L^{-}(\sigma^{+})$ respectively.

From the first condition we obtain that,

$$(\partial_{-} X)^2 = (\partial_{-} X_R)^2 = -2\partial_{-} X_R^{+} \partial_{-} X_R^{-} + \sum_{i=1}^{D-2} \partial_{-} X_R^{i} \partial_{-} X_R^{i} = 0 \to \partial_{-} X_R^{-} = \frac{1}{\alpha' p^{+}} \sum_{i=1}^{D-2} \partial_{-} X_R^{i} \partial_{-} X_R^{i}.$$

The same happens to the other equation,

$$\partial_{+} X_L^{-} = \frac{1}{\alpha' p^{+}} \sum_{i=1}^{D-2} \partial_{+} X_L^{i} \partial_{+} X_L^{i}.$$

Using these two results the usual mode expansion for the string is written as

$$\begin{cases} X_R^{-}(\sigma^{-}) = \frac{1}{2} x^{-} + \frac{1}{2} \alpha' p^{-} \sigma^{-} + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \alpha_n^{-} e^{-in\sigma^{-}}, \\ X_L^{-}(\sigma^{+}) = \frac{1}{2} x^{-} + \frac{1}{2} \alpha' p^{-} \sigma^{+} + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \tilde{\alpha}_n^{-} e^{-in\sigma^{+}}. \end{cases} \tag{3.4.2}$$

But now, the modes α_n^{-} have the form

$$\alpha_n^{-} = \sqrt{\frac{1}{2\alpha'}} \frac{1}{p^{+}} \sum_{m=-\infty}^{+\infty} \sum_{i=1}^{D-2} \alpha_{n-m}^{i} \alpha_m^{i} \ , \quad \tilde{\alpha}_n^{-} = \sqrt{\frac{1}{2\alpha'}} \frac{1}{p^{+}} \sum_{m=-\infty}^{+\infty} \sum_{i=1}^{D-2} \tilde{\alpha}_{n-m}^{-} \tilde{\alpha}_m^{-}, \tag{3.4.3}$$

where

$$p^- = \sqrt{\frac{2}{\alpha'}}\alpha_0^- = \frac{1}{\alpha' p^+}\sum_{m=-\infty}^{+\infty}\sum_{i=1}^{D-2}\alpha_{n-m}^i\alpha_m^i$$

$$p^- = \frac{1}{\alpha' p^+}\sum_{i=1}^{D-2}(\frac{1}{2}\alpha' p^i p^i + 2\sum_{m=1}^{+\infty}\alpha_{-m}^i\alpha_m^i) \tag{3.4.4}$$

The mass shell condition is then written as

$$M^2 = -p^2 = 2p^+p^- - \sum_{i=1}^{D-2}p^i p^i = \frac{4}{\alpha'}\sum_{i=1}^{D-2}\sum_{m=1}^{+\infty}\alpha_{-m}^i\alpha_m^i \tag{3.4.5}$$

We sum up over the transverse oscillators, this removes the problematic modes. The quantum excitations of the string will be those of the transverse oscillators. The x^+ can be absorbed on a shift on τ and the p^- comes determined by other variables. p^- can be thought as the light cone Hamiltonian, which generates translations on x^+ , $\{x^+, p^-\} = 1$, so as x^+ shifts time this is equivalent to τ translations. In the next section, we will see how this coordinate choice permits the quantization of the string.

3.5 Lightcone Gauge Quantization

The quantization consists on promoting the physical degrees of freedom to operators of the Fock space, in a similar way as we have already done. The non-zero equal time commutation relations are:

$$\begin{gathered}[x^i, p^j] = i\delta^{ij}, \ \ [x^-, p^+] = -i, \\ [\alpha_n^i, \alpha_m^j] = [\tilde{\alpha}_n^i, \tilde{\alpha}_m^j] = n\delta^{ij}\delta_{m+n,0}, \quad i,j = 1,2 \dots D-2.\end{gathered} \tag{3.5.1}$$

To promote α_n^- to an operator we impose normal ordering,

$$\alpha_n^- = \sqrt{\frac{1}{2\alpha'}}\frac{1}{p^+}\sum_{m=-\infty}^{+\infty}\sum_{i=1}^{D-2} :\alpha_{n-m}^i\alpha_m^i: - a\delta_{n,0}. \tag{3.5.2}$$

In the last section, we discussed the relation between x^+ and p^-. When we promote these variables to operators we obtain $[x^+, p^-] = -i$. This result is similar to $[t, H] = -i$ which is correct in a formal level.

It could be said that, as this is a gauge choice of a Lorentz invariant theory, it is also implicitly Lorentz invariant, but when we change to the quantum frame we usually lose classical symmetries.

In order to restore the Lorentz invariance, we will study the Lorentz generators of the Worldsheet that we have already obtained in (2.1.12) and (2.1.14).

When we proceed to do these calculations substituting X^μ, we observe how the generators are now expressed as

$$J^{\mu\nu} = l^{\mu\nu} + E^{\mu\nu},$$

$$l^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu, \tag{3.5.3}$$

$$E^{\mu\nu} = -i\sum_{n=1}^{\infty}\frac{1}{n}\left(\alpha_{-n}^{\mu}\alpha_{n}^{\nu} - \alpha_{-n}^{\nu}\alpha_{n}^{\mu}\right).$$

These operators must generate the Lorentz algebra. Most of the commutators of these quantities give the correct result for any number of dimensions, but the commutation relations of J^{i-} must be treated carefully, in particular $[J^{i-}, J^{j-}]$, which must be zero to obtain the Lorentz invariance. $[J^{i-}, J^{j-}]$ have terms quartic or quadratic in oscillators, the quartic terms cancel, just like in the classical case. So $[J^{i-}, J^{j-}]$ must have the following form:

$$\left[J^{i-}, J^{j-}\right] = -\frac{1}{(p^+)^2}\sum_{n=1}^{\infty}\Delta(n)\left(\alpha_{-n}^{\mu}\alpha_{n}^{\nu} - \alpha_{-n}^{\nu}\alpha_{n}^{\mu}\right).$$

From a long calculation we obtain the result expressed below:

$$\Delta(n) = n\left(\frac{26-D}{12}\right) + \frac{1}{n}\left(\frac{D-26}{12} + 2(1-a)\right)$$

If we require Lorentz invariance then $\Delta(n) = 0$ for any n, this is only possible if $D = 26$ and $a = 1$ as expected.

In the next section, we will see the path integral quantization. This quantization method is the one which will make possible the treatment of interactions between strings and it will carry us to general relativity.

4. Path Integral approach

En este capítulo hablaremos de la formulación de la mecánica cuántica empleando integrales de caminos, dado que esta es la única manera de introducir interacciones en el modelo cuántico de la cuerda.

4.1 Path Integral

In this chapter we will mainly follow (Polchinski, 1998). The path integral arises from the next idea about the propagators, $\langle q_f, T | q_i, 0 \rangle$ (this represents the probability amplitude of transition between two position states, one at time 0 and another one at time T), of a space of states corresponding to a dynamical variable q. The propagator is defined as:

$$\psi(T, q) = \int dq' \langle q, T | q', 0 \rangle \, \psi(0, q').$$

The idea is to create intermediate divisions in time, from the initial to the final instants, $t_m = m\epsilon, \ \epsilon = \frac{T}{N}$, and then, to introduce the complete set of states on each division, and then make the number of divisions $N \to \infty$.

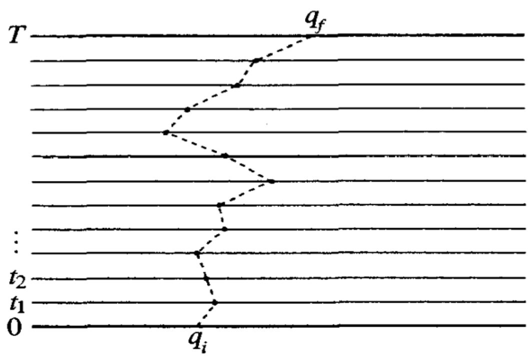


Fig 4.1

The propagator will take the form of a sum over paths, summing up phase factors for each path. We proceed with a finite number of time intervals and after the calculation we go to the continuum.

$$\langle q_f, T | q_i, 0 \rangle = \int dq_{N-1} \dots dq_1 \prod_{m=0}^{N-1} \langle q_{m+1}, t_{m+1} | q_m, t_m \rangle =$$

The generator of the temporal evolution is the Hamiltonian operator that has the form $\hat{H}(\hat{p}, \hat{q}) = T(\hat{p}) + V(\hat{q})$.

$$\int dq_{N-1} \dots dq_1 \prod_{m=0}^{N-1} \left\langle q_{m+1} \right| \exp\left(-\frac{i\hat{H}\epsilon}{\hbar}\right) \left| q_m, t_m \right\rangle =$$

$$\int dq_{N-1} \dots dq_1 dp_{N-1} \dots dp_1 \prod_{m=0}^{N-1} \langle q_{m+1} | p_m \rangle \left\langle p_m \right| \exp\left(-\frac{i\hat{H}\epsilon}{\hbar}\right) \left| q_m, t_m \right\rangle =$$

$$\int dq_{N-1} \dots dq_1 \frac{dp_{N-1}}{2\pi} \dots \frac{dp_1}{2\pi} \exp\left\{-\frac{i}{\hbar}\sum_{m=0}^{N-1} \epsilon \left[H(p_m, q_m) - \frac{p_m(q_{m+1} - q_m)}{\epsilon}\right]\right\} =$$

(As N goes to infinity we go into the continuum, we introduce the notation $\int dq_{N-1} \dots dq_1 \frac{dp_{N-1}}{2\pi} \dots \frac{dp_1}{2\pi} = \int [dqdp] = \int DqDp$)

$$\int [dqdp] \exp\left\{\frac{i}{\hbar}\int_0^T dt(p\dot{q} - H(p,q))\right\} = \int [dq] \exp\left\{\frac{i}{\hbar}\int_0^T dtL(q,\dot{q})\right\} = \int Dq\, e^{\frac{iS}{\hbar}},$$

$$\langle q_f, T | q_i, 0\rangle = \int Dq\, e^{\frac{iS}{\hbar}}. \tag{4.1.1}$$

The propagator is the sum over all possible paths of the variable q weighted by a factor, $e^{\frac{iS}{\hbar}}$.

When we go to high scales where $\hbar$ is close to zero compared with S the integral is dominated by the stationary phase conditions, $\frac{\delta S}{\delta q} = 0$, that corresponds to the classical solution. The rest of the terms are cancelled by their close variations, but near the classical path the sum is constructive, thus, only the trajectories close to the classicals have a considerable apportion to the propagator.

By a similar procedure we obtain that the expected value of q at time t is expressed as $\langle q(t)\rangle = \int Dq\, q(t)e^{\frac{iS}{\hbar}}$ and similarly $\langle T[q(t')q(t)]\rangle = \int Dqq(t')\, q(t)e^{\frac{iS}{\hbar}}$ where T means we are considering a time ordered product (see (Polchinski, 1998)) (the order of the time-dependent operators set the sooner at the right side, $T[q(t')q(t)] = q(t')q(t),\ if, t' > t$, and the other case if , $t > t'$).

4.2 Functional Quantization on Field Theory

In this section we will present in a condensed way some important results of the path integral formalism in quantum field theory. The objective is to formulate the generating functional, that is the object that contains all the information about any process or expected value of the field. It will permit also to introduce the interactions in the field formalism giving us the Feynman diagrams and probabilities of transitions between field excitations (see (Peskin & Schroeder, 1995), (Weimberg, 1995)).

Now, we are working in the second quantization context, where we have that the action we are considering depends on a generic field, f, not on a variable, q, so the action functional will take the form $S[f]$. Instead of considering all possible paths we study all possible field configurations.

In general, to solve the path integral over a function $f(X)$ we take the argument we used to introduce the formula (4.1.1) in reverse, we discretize the continuum, not only in time but using all the field coordinates. When we compute $\int [df(X)]\, e^{iS[f]}$, we are making an integration over $f\left(X^{(i)}\right)$ for every point on a lattice of spacetime points $X^{(i)}$. The expression $[df(X)]$ will take the form $[df(X)] = df\left(X^{(1)}\right)df\left(X^{(2)}\right)df\left(X^{(3)}\right) \dots$

We are now enouncing the integral for a certain type of actions corresponding to fields that obey a generic field equation $\Delta f(X) = 0$, where Δ is a differential operator (with similar operator properties as the Klein-Gordon operator $[\partial^{\mu}\,\partial_{\mu}+m^2]$). The field action and corresponding path integral are written as

$$S[f]=\frac{1}{2}\int dX\,\{f(X)\Delta f(X)+f(X)J(X)\},$$

$$\int[df(X)]\,e^{iS[f,J]}=\int[df(X)]e^{i\frac{1}{2}\int dX\,\{f(X)\Delta f(X)+f(X)J(X)\}}.$$

The path integrals that can be expressed in a similar way are called 'gaussians'. We solve them performing a change of variables, $f'(X)=f(X)-i\int dY\Delta^{-1}(X,Y)J(Y)$. The element $\Delta^{-1}(X,Y)$ is the green function of the Δ operator:

$$\frac{1}{2}\int dX\,\{f(X)\Delta f(X)+f(X)J(X)\}=\frac{1}{2}\int dX\,f'(X)\Delta f'(X)-\frac{1}{2}\int dXdYJ(X)\Delta^{-1}(X,Y)J(Y).$$

The Jacobian factor of this change of variables is one because the transformation is a translation.

$$\int[df(X)]e^{i\frac{1}{2}\int dX\,\{f(X)\Delta f(X)+f(X)J(X)\}}=Ne^{-\frac{1}{2}\int dXdYJ(X)\Delta^{-1}(X,Y)J(Y)},\tag{4.1.2}$$

with, $N=\int[df(X)]e^{i\frac{1}{2}\int dX\,\{f(X)\Delta f(X)\}}$. The right hand side of the equation (4.1.2) shows that the sources are independent of the field integral over $f'(X)$; then the source term goes out of the path integral.

We pass again to the $Df(X)$ notation. The generating functional will permit us to generate green functions $\Delta^{-1}(X,Y)$. It is written as $\mathcal{Z}[J(X)]$, for a field source $J(X)$ introduced for later convenience.

$$\mathcal{Z}[J(X)]\equiv\frac{\int D[\phi(X)]e^{-i\frac{1}{2}\int\phi\Delta\phi+i\int\phi J(X)}}{\int D[\phi(X)]\,e^{-i\frac{1}{2}\int\phi\Delta\phi}}=e^{-\frac{1}{2}\int dXdYJ(X)\Delta^{-1}(X,Y)J(Y)}.\tag{4.1.3}$$

This expression will remove the N factor, giving us the Green functions or correlation function for the field at n points as it follows

$$G_0^{(n)}(X_1,\dots,X_n)=\langle T\{\phi(X_1)\dots\phi(X_n)\}\rangle=\frac{1}{i^n}\frac{1}{\mathcal{Z}[0]}\frac{\delta\mathcal{Z}[J(X)]}{\delta J(X_1)\dots\delta J(X_n)}\Bigg]_{J=0}\tag{4.1.4}$$

The two points correlation function is the propagator

$$\langle\phi(x_1)\phi(x_2)\rangle=\Delta^{-1}(x_1,x_2)=\frac{\int D[\phi]\,\phi(X)\phi(Y)e^{-\frac{1}{2}\int\phi\Delta\phi}}{\int D[\phi]\,e^{-\frac{1}{2}\int\phi\Delta\phi}}$$

The equation (4.1.4) is the base of the Wicks theorem on path formulation of quantum field theory. By using (4.1.4), we obtain the correlation function as a sum over configurations of propagators between four points as shown in Fig: 4.2. Let us see an example by computing the four-point correlation function

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \Delta^{-1}(x_1,x_2)\Delta^{-1}(x_3,x_4) + \ \Delta^{-1}(x_1,x_3)\Delta^{-1}(x_4,x_4) +$$

$$\Delta^{-1}(x_1,x_4)\Delta^{-1}(x_2,x_3).$$

We say that we develop this expression over what is called contractions and its expressed in terms of Feynman diagrams will be the one that is shown in Fig: 4.2.

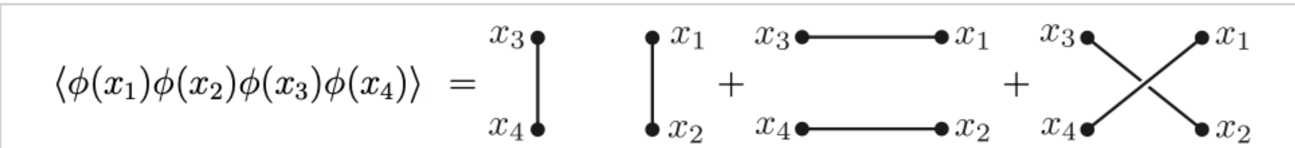


Fig: 4.2

We can plug the interactions on the theory by using $\mathcal{L}[\phi] = \mathcal{L}_{free}[\phi] + \mathcal{L}_{interactive}[\phi]$, (see (Weimberg, 1995)). Then we write, $S = S_0 + S_i$; this way, we can expand the exponential of $S_i[\phi]$ as, $e^{iS} = e^{iS_0}\sum_{N=0}^{\infty}\frac{i^N}{N!}(S_i)^N$.

This result, plugged into the path integral, is commonly used at a perturbative level giving the interaction corrections to the free theory. The terms of the sum $\sum_{N=0}^{\infty}\frac{i^N}{N!}(S_i)^N$ are the successive vertex operators that plug the corrections to the propagator of the free theory as follows

$$\Delta^{-1}_{interactive}(X,Y) = \frac{\sum_{N=0}^{\infty}\int D[\phi]\frac{i^N}{N!}(S_i)^N\phi(X)\phi(Y)e^{-\frac{1}{2}\int \phi\Delta\phi}}{\sum_{N=0}^{\infty}\int D[\phi]\frac{i^N}{N!}(S_i)^N e^{-\frac{1}{2}\int \phi\Delta\phi}}. \qquad (4.1.5)$$

From the numerator of (4.1.5) we obtain the corresponding Feynman diagrams as the successive vertex corrections. The vertex operators are formulated in terms of the fields, (an example could be $S_i = -\frac{\lambda}{4!}\int \phi^4$) in such a way that when we perform the contraction we observe the apparition of terms such as $\Delta^{-1}(z,z)\Delta^{-1}(z,z)\Delta^{-1}(X,Y)$. Here the propagators $\Delta^{-1}(z,z)$ on the expansion do not give any information and correspond to what is called vacuum diagrams. The denominator of (4.1.5) removes the contribution of the vacuum diagrams, so we are only left with the connected contributions part that is the one we see in the right hand side of Fig: 4.3

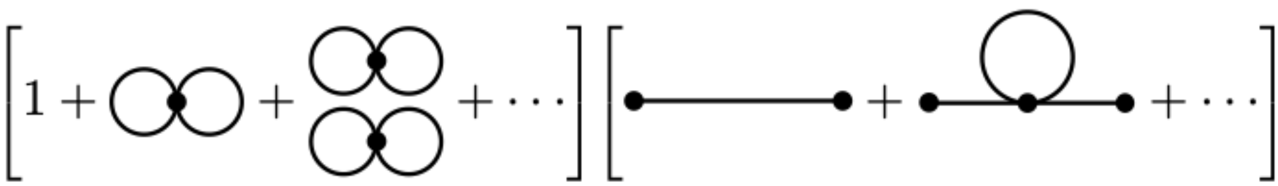


Fig: 4.3 (Feynman diagrams of the self interactive field, $S_i = -\frac{\lambda}{4!}\int \phi^4$. If other fields are involved we have to plug other field interactions and give a different type of diagrams)

The resulting propagator is a sum over loops for this type of vertex.

4.3 Path Integral on String Theory

In this sub-chapter, one key frame will be introduced, the path integral formalism applied to string theory (See (Polchinski, 1998)) (we have to mention that the discussion is not a completely analog to the field theory, it is more complicated). As we have seen, the string is a field of the worldsheet coordinates, we can use this to introduce the interaction terms on the theory.

The natural approach for the Polyakov path integral could be formulated as $\int [dhdX] \exp(-S) \equiv Z$ (It is written in the Euclidean formalism, where we make a Wick rotation replacing t with $-iu$ and changing the metric by the Euclidean metric) but this is not truthful. The diffeomorphism-conformal invariance makes many of the configurations equivalent between them, and we would make an overcounting on the number of configurations. The problem is equivalent to the quantization of the electromagnetic fields in the Yang-Mills theory (see (Peskin & Schroeder, 1995)). The right way would be to count each physical configuration only once, or to divide the expression by the resulting contribution of this overcounting, that we will express as the 'volume' of the diffeomorphism-Weyl transformation local group $Vol_{Diff-Weyl}$,

$$Z \equiv \int \frac{[dhdX]}{Vol_{Diff-Weyl}} \exp(-S).$$

We will fix this counting on each gauge equivalence class using the Faddeev-Popov method (see (Polchinski, 1998)). The metric has to obey $h^{\zeta}_{\alpha\beta}(\tau,\sigma) = e^{2\phi(\sigma)} \frac{\partial f^{\delta}}{\partial \sigma^{\alpha}} \frac{\partial f^{\gamma}}{\partial \sigma^{\beta}} h_{\delta\gamma}$. We use this to define the Faddeev-Popov determinant, $\Delta_{FP}(h)$.

$$1 = \Delta_{FP}(h) \int [d\zeta] \delta(h - \tilde{h}^{\zeta}). \qquad (4.1.6)$$

We will call, $\tilde{h}$, the fiducial metric. The generating functional will depend on this metric as it will be observed. The $\Delta_{FP}(g)$ will remove the volume of configurations due to the diff-Weyl symmetry. Essentially this will permit us to integrate over the equivalent classes of diff-Weyl connected metrics, instead of all the configurations of the metric, the Faddeev-Popov determinant is a Jacobian in this sense, then,

$$Z[\tilde{h}] \equiv \int \frac{[dhdXd\zeta]}{Vol_{Diff-Weyl}} \delta(h - \tilde{h}^{\zeta}) \Delta_{FP}(\tilde{h}) \exp(-S) =$$

$$\int \frac{[dXd\zeta]}{Vol_{Diff-Weyl}} \Delta_{FP}(\tilde{h}) \exp(-S).$$

We still have to integrate all over the equivalence class of the metrics connected to the fiducial metric by a diff-Weyl transformation, corresponding in the integral by $[d\zeta]$, but none of the terms on the integral changes with these transformations, it is a symmetry on the action so the integral on $[d\zeta]$ just produce the 'volume' of the diffeomorphism-Weyl transformation local group and cancels it, letting the generating functional as,

$$Z[\tilde{h}] = \int [dX]\Delta_{FP}(\tilde{h}) \exp(-S[\tilde{h}, X]) \tag{4.1.7}$$

The reader can consult (Polchinski, 1998) to see how to obtain the form of $\Delta_{FP}(\tilde{h})$ as a path integral over Grassmann or anticommuting fields,

$$\Delta_{FP}(\tilde{h}) = \int [dbdc] \exp(-S_g).$$

Where S_g is the ghost action and b, c are the Grassmann fields, also called ghosts fields. This procedure removes the ghosts states from the counting. For the open strings we also need a term for the string boundaries in the action but as our discussion takes only closed ones we will not mention it.

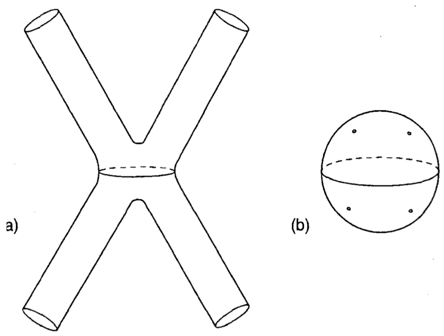
The reason behind the popularity of this procedure is that it allows us to introduce the interactions on the string worldsheet. We want to build the vertex operators for the string. As the string theory is a field theory, the vertex operators are worldsheet operators that represent an emission or absorption of a physical string mode.

When we talk about open strings, the vertex operators must act on the boundary of the worldsheet, when we refer to closed strings the operator must act on the interior of the worldsheet. In the case of closed strings that we are dealing with, we have to sum up over all possible particle emission points on the worldsheet (just like it is done in quantum field theory in the spacetime), so we must integrate the operators over the worldsheet coordinates as, $g_s \int V_\phi \, d\sigma^2$, ($g_s$ is the string coupling constant) the label ϕ specify the state that is being absorbed or emitted. If the emitted particle has momentum k the vertex operator should contain a factor $e^{ik\cdot x}$, working on a string this factor has to be generalized to $e^{ik\cdot X}$.

We have defined the state of the string as $|\phi\rangle = \prod_i \hat{\alpha}_{n_i}{}^{\mu_i \dagger} \prod_j \hat{\tilde{\alpha}}_{m_j}{}^{\nu_j \dagger} \, |0, k\rangle$. To build the vertex operators corresponding to a given state we have to know how to produce it (see (Polchinski, 1998)). The Conformal field theory (CFT) is the work frame that enables us to achieve this. Through the use of CFT we obtain the correspondence $\alpha_{-n}{}^{\mu} \to \partial^n X^\mu$. If a state is built by acting $\hat{\alpha}_{n_i}{}^{\mu_i \dagger}$ n times on the ground state, we will have to plug the n^{th} power of $\partial^n X^\mu$ into the vertex operator (see (Polchinski, 1998)). This gives us an intuition of how to construct the vertex operator of the Graviton state $s_{\mu\nu}\alpha_{-1}^{\mu}\tilde{\alpha}_{-1}^{\nu}|0, k\rangle$ ($s_{\mu\nu}$ is a symmetric factor that contain its polarization) as

$$V = -4\pi g_c s_{\mu\nu} \int h^{\frac{1}{2}} :\partial_\alpha X^\mu \partial^\alpha X^\nu e^{ikX}: d\tau d\sigma \tag{4.1.8}$$

The most we can do by introducing vertex operators on the string discussion is to compute the S-matrix (matrix of transition amplitudes between asymptotically free states) for strings. By taking the sources as initial string states the resulting discussion is equivalent to find the topology of a compact form with as many holes on the surface as external legs on the S-matrix (See Fig 4.4) . Therefore, the discussion about the interactions can be focused on finding the different topologies of this compact forms (Fig 4.5) that would correspond to the right part terms in (Fig 4.3) as Feynman diagrams. If we compute a loop in the closed string it will be a torus.



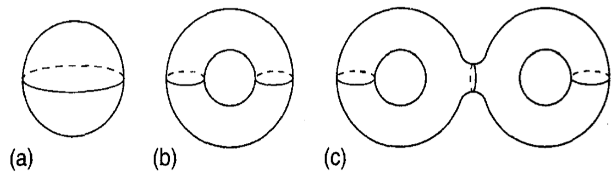


Fig: 4.5

Fig: 4.4

The topologies of (Fig: 4.5) are Riemann surfaces of n handles, we call them Riemann surfaces of genus n.

Summing up over topologies we generalize the sum over loops of the Feynman diagrams, with the difference that at each loop we have all the possible types particles. Eventually, we can write the S-matrix elements for n external legs on j state as

$$S_{J_1 J_2 \dots J_n}(k_1, \dots, k_n) = \sum_{\substack{all\ compact \\ topologies}} \int [dXdh] \exp\left(-S_X - S_g\right) \prod_{i=1}^{n} \int d^2\sigma_i \, h^{\frac{1}{2}}(\sigma_i) V_{j_i}(k_i, \sigma_i) \qquad (4.1.9)$$

Notice that in the expression (4.1.9) we do not divide by $\langle 1 \rangle$ to cancel divergencies as we did in (4.1.5). In string theory, this is a much more delicate issue, see (Polchinski, 1998).

In the next section, we will see a mechanism to formulate the general relativity starting from the string theory.

5 The Graviton and General Relativity

En este capítulo se expondrán los argumentos que dan lugar a las ecuaciones de la relatividad general a partir de la teoría de cuerdas. Para ello emplearemos la integral de camino que hemos previamente introducido.

The first excited state of a closed string is obtained by acting with the creation operator α_{-1}^{i} on the ground state. The level matching condition imposes that we also have to use $\tilde{\alpha}_{-1}^{i}$ so the first excited state is written as,

$$\alpha_{-1}^{i}\tilde{\alpha}_{-1}^{j}|0,\phi\rangle.$$

Each of this α_{-1}^{i} operator transform as a $SO(D-2)$ representation but we want our final states to transform as a full Poincare group, $SO(1,D-1)$, representation. It is not possible to fit a vector with $(D-2)^2$ states, as the graviton, in a representation of the $SO(D-1)$ group, but it is not all lost here. If the state is massless the Poincare group is not expressed in the same way, let us see why.

If we consult the Wigner's classification of the Poincare group representations we observe that massless particles have different representations of their little group (the group that lets the temporal momentum magnitude (p_0) invariant, identified with the spatial rotations) (see (Maiani L. ; Benar O., 2016)). The massive particles little group is $SO(D-1)$, but when we consider massless particles the little group is $SO(D-2)$, this is due to the fact that p_0 must come from the momentum on a defined direction on space so this one must be unchanged too under the little group. In other words, massless particles must have fewer states than massive ones, the massless particles are representations of $SO(D-2)$ while massive ones are representations of $SO(D-1)$.

If we consider that the first excited state is massless, then it fits in a representation of $SO(D-2)$ (It is interesting to mention that this is only possible if $D=26$ and the state corresponds to a 24×24 representation).

The quantum state $\alpha_{-1}^{i}\tilde{\alpha}_{-1}^{j}|0,\phi\rangle$ could be identified with the quanta of a spin two field corresponding with a 2-form with the three irreducible representations of 24×24, symmetric, antisymmetric and trace: $G_{\mu\nu}(X)$ (The graviton, corresponding with the symmetric part), $B_{\mu\nu}(X)$(The antisymmetric part) and $\Phi(X)$ (The Dilaton, that corresponds with the trace).

5.1 Non-linear sigma model, the string in a curved spacetime

As we have already discussed, if we want to generalize the string motion in a curved spacetime, the Polyakov action takes the form,

$$S_\sigma = -\frac{T}{2}\int h^{\frac{1}{2}}\ h^{\alpha\beta}g_{\mu\nu}(X)\frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X^\nu}{\partial\sigma^\beta}d\tau d\sigma. \quad (5.1.1)$$

We can expand $g_{\mu\nu}(X)$ over the flat metric with a perturbation as $g_{\mu\nu}(X)=\eta_{\mu\nu}+\chi_{\mu\nu}(X)$ (we call the action S_σ instead S_p because of historical reasons the actions of this form are called as non-linear sigma models).

If we expand the metric perturbation inside the exponential of the path integral we obtain,

$$\exp(-S_\sigma) = \exp(-S_p)\exp\left(-\frac{T}{2}\int h^{\frac{1}{2}}\ h^{\alpha\beta}\chi_{\mu\nu}(X)\frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X^\nu}{\partial\sigma^\beta}d\tau d\sigma\right)$$
$$= \exp(-S_p)\left\{1-\frac{T}{2}\int h^{\frac{1}{2}}\ h^{\alpha\beta}\chi_{\mu\nu}(X)\frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X^\nu}{\partial\sigma^\beta}d\tau d\sigma+\cdots\right\}. \tag{5.1.2}$$

The successive terms on the brackets of the last part of (5.1.2) are similar to the vertex operators on the path integral of the graviton state with, $\chi_{\mu\nu}(X) = -4\pi g_c s_{\mu\nu}e^{ikX}$.

Using that argument in reverse, we consider that the string interacts with a undefined number of gravitons in a macroscopic scale, in such a way that they conform a coherent state. Then the vertex operators can be exponentiated again.

$$\left\{1-\frac{T}{2}\int h^{\frac{1}{2}}\ h^{\alpha\beta}\left(-4\pi g_c s_{\mu\nu}e^{ikX}\right)\frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X^\nu}{\partial\sigma^\beta}d\tau d\sigma+\cdots\right\} =$$
$$\exp\left(-\frac{T}{2}\int h^{\frac{1}{2}}\ h^{\alpha\beta}(-4\pi g_c s_{\mu\nu}e^{ikX})\frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X^\nu}{\partial\sigma^\beta}d\tau d\sigma\right).$$

Looking at the form of the generating functional we obtain

$$Z = \int [dXdh]\exp(-S_p)\exp\left(-\frac{T}{2}\int h^{\frac{1}{2}}\ h^{\alpha\beta}\chi_{\mu\nu}(X)\frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X^\nu}{\partial\sigma^\beta}d\tau d\sigma\right),$$
$$Z = \int [dXdh]\exp\left(-\frac{T}{2}\int h^{\frac{1}{2}}\ h^{\alpha\beta}(\eta_{\mu\nu}+\chi_{\mu\nu}(X))\frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X^\nu}{\partial\sigma^\beta}d\tau d\sigma\right).$$

This is completely analog to the string in a curved spacetime but now the curved metric is a consequence of the continuous interchange of graviton states in the string motion.

At this point we introduce the concept of effective action; if it is possible to change the action to add quantum mechanical corrections to the classical part we say we construct an effective action. In this case the interaction of any string with a coherent background of graviton states introduces a quantum correction. This suggest to define an effective action of the form

$$S_\sigma = \frac{T}{2}\int h^{\frac{1}{2}}\ h^{\alpha\beta}G_{\mu\nu}(X)\frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X^\nu}{\partial\sigma^\beta}d\tau d\sigma.$$

We have summed up, $\left(\eta_{\mu\nu}+\chi_{\mu\nu}(X)\right) = G_{\mu\nu}(X)$. Here, $G_{\mu\nu}(X)$ is not a variable of the path integral but the gravitons background contribution to the action, it is not affected by, $[dXdh]$, then it can be considered as a metric. If we consider a coherent background of gravitons we have to include the rest of irreducible representations of the first excited state of the string, so we conclude that our effective action must have the following form:

$$S_\sigma = -\frac{1}{4\pi\alpha'}\int h^{\frac{1}{2}}\left[h^{\alpha\beta}G_{\mu\nu}(X)\frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X^\nu}{\partial\sigma^\beta}+i\epsilon^{\alpha\beta}B_{\mu\nu}(X)\frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X^\nu}{\partial\sigma^\beta}+\alpha' R\Phi(X)\right]d\tau d\sigma. \tag{5.1.3}$$

The constants $i\epsilon^{\alpha\beta}$ and R are obtained from the vertex operator terms of the corresponding representations, this terms obey $h^{1/2}\, i\epsilon^{\alpha\beta} = 1$ and R is the Ricci scalar on the worldsheet.

Now, this effective action must respect the previous local symmetries that are essential to build the string states consistently. We notice that these terms in the action break the Weyl invariance. To keep Weyl invariance we have to impose constraints over (5.1.3).

Through the change of variables, $X^{\mu}(\sigma) = x^{\mu} + \sqrt{\alpha'}Y^{\mu}(\sigma)$, the x^{μ} is the classical solution that corresponds with the vacuum expectation value of our field $\langle X^{\mu}\rangle = x^{\mu}$ (there exists a fundamental difference on the vacuum state of our theory and those of the standard model fields, one is a field of the worldsheet coordinates and the others are fields over the spacetime). In field theory, the vacuum expectation value is the one that minimizes the effective potential, so the quantization is performed over the, Y^{μ}, variables that obey, $\langle Y^{\mu}\rangle = 0$,so, $Y^{\mu}(\sigma)$, is a dimensionless function that will give us the fluctuations over the classical solution (in quantum field theory the Higgs mechanism is an example of this process, where the form of the potential is changed by giving a non-null vacuum expectation value for the Higgs boson that recovers the masses of the particles in the model). The graviton contribution is expanded over the classical solution on the worldsheet as follows:

$$G_{\mu\nu}(X)\frac{\partial X^{\mu}}{\partial\sigma^{\alpha}}\frac{\partial X^{\nu}}{\partial\sigma^{\beta}} = \\ = \alpha'\left[G_{\mu\nu}(x) + G_{\mu\nu,w}(x)Y^{w}(\sigma) + \frac{\alpha'}{2}G_{\mu\nu,wv}(x)Y^{w}(\sigma)Y^{v}(\sigma) + \cdots\right]\frac{\partial Y^{\mu}}{\partial\sigma^{\alpha}}\frac{\partial Y^{\nu}}{\partial\sigma^{\beta}}. \tag{5.1.4}$$

This way, each of the G derivatives in (5.1.4) are coupling constants of the interacting Y fields. We have written the quantum string motion on a curved spacetime as an interactive quantum two-dimensional field theory. It can be shown (see on (Polchinski, 1998) or (Green, Schwatrz, & Witten, 1987)) that to make the effective action Weyl invariant we require the energy-momentum tensor to be traceless $T_{a}^{a} = 0$. This condition implies that the Couplings beta functions must go to zero: $\beta_{\mu\nu}(G) = \beta_{\mu\nu}(B) = \beta(\Phi) = 0$. (the beta function regularization is a method in field theory to avoid divergencies by reformulating the field magnitudes to cancel this divergences in the integral, we add counterterms. The difference with respect to other quantum gravity theories is that we only need a finite number of them, this makes this field theory a renormalizable theory. The objects that study how the field magnitudes as coupling constants change with the energy scale μ are called β-functions). We will not specify how we obtain the specific form of the beta functions so the reader can consult the bibliography. The beta functions are

$$\begin{cases} \beta_{\mu\nu}(G) = \alpha' R_{\mu\nu} + 2\alpha'\nabla_{\mu}\nabla_{\nu}\Phi - \frac{\alpha'}{4}H_{\mu\lambda k}H_{\nu}^{\lambda k} + o(\alpha'^2) = 0, \\ \beta_{\mu\nu}(B) = -\frac{\alpha'}{2}\nabla^{\lambda}H_{\lambda\mu\nu} + \alpha'\nabla^{\lambda}\Phi H_{\lambda\mu\nu} + o(\alpha'^2) = 0, \\ \beta(\Phi) = -\frac{\alpha'}{2}\nabla^{2}\Phi + \alpha'\nabla_{\nu}\Phi\nabla^{\nu}\Phi - \frac{\alpha'}{24}H_{\mu\lambda k}H^{\mu\lambda k} + o(\alpha'^2) = 0, \end{cases} \tag{5.1.5}$$

where $H_{\mu\lambda k} = \partial_\mu B_{\lambda k} + \partial_\lambda B_{k\mu} + \partial_k B_{\mu\lambda}$ and $R_{\mu\nu}$ the corresponding Ricci tensor of the $G_{\mu\nu}$ metric.

If the radius of curvature R_c is small compared to the string scale then we can develop the theory perturbatively ignoring the internal degrees of freedom of the string and cutting the energy scale at the α'^2 term. In (5.1.4) the addition of the successive terms will add the corrections of high energy to the general relativity. The equations (5.1.5) have to be taken as the motion equations over G, B and Φ that now acquire a dynamical character in the classical way. Therefore, the (5.1.5) equations can be considered as coming from the following low energy effective action,

$$S = \frac{1}{2\kappa_0^2}\int d^{26}x(-G)^{1/2}e^{-2\Phi}\left[R + 4(\partial\Phi)^2 - \frac{1}{12}H_{\mu\lambda k}H^{\mu\lambda k} + o(\alpha'^2)\right]. \quad (5.1.6)$$

The first term in the bracket is very familiar to the Einstein-Hilbert action except for the $e^{-2\Phi}$ factor. It is possible to make a change of variables to go to what is called as Einstein frame.

The change starts by redefining $\Phi = \tilde{\Phi} + \Phi_0$, where $\tilde{\Phi}$ has a vanishing expectation value, $\tilde{G}_{\mu\nu} = e^{-\frac{4(\tilde{\Phi})}{D-2}}G_{\mu\nu}$. This way we have to redefine R as $\tilde{R} = e^{\frac{4(\tilde{\Phi})}{D-2}}[R - 2(D-1)\nabla^2 e^{-\frac{2(\tilde{\Phi})}{D-2}} - (D-2)(D-1)\partial_\mu e^{-\frac{2(\tilde{\Phi})}{D-2}}\partial^\mu e^{-\frac{2(\tilde{\Phi}+\Phi_0)}{D-2}}]$. Hence, the action becomes

$$S = \frac{1}{2\kappa_0^2}\int d^{26}x\left(-\tilde{G}\right)^{1/2}\left[\tilde{R} + \frac{4}{D-2}\left(\partial\tilde{\Phi}\right)^2 - \frac{1}{12}e^{-\frac{8(\tilde{\Phi}+\Phi_0)}{D-2}}H_{\mu\lambda k}H^{\mu\lambda k} + o(\alpha'^2)\right]. \quad (5.1.7)$$

The action (5.1.7) recovers the Einstein-Hilbert action except for the lack of mater fields. In the context of Superstring theory massive fields are introduced in the equations completing the analogy. The other aspect that is unsatisfactory of (5.1.7) is the possibility of defining different metrics, this is achievable because we have a scalar massless field Φ. The presence of this field would change the way we rule distances and would break the equivalence principle. Superstring theory gives also a way to make the Dilaton massive, making the Dilaton forces short range, and letting only two long range interactions to rule the space at long distances, the Graviton, and the antisymmetric part. This way the equivalence principle would be recovered at large scales, and with it, the general relativity.

As a summary of this chapter, we will go through all the different aspects that have been treated to clarify the process. Firstly, the Polyakov path integral in a curved background can be reconstructed by introducing the interaction of the string with a coherent state of gravitons in the path integral. This way, we can introduce an effective theory that incorporates this corrections to the string motion. We have to include also the rest of the representations of the string first excited state, the antisymmetric and the Dilaton. From this point we develop the new effective action as an interactive field theory. Also, new action breaks the Weyl invariance so we have to observe under which circumstances it is recovered at quantum level. The conclusion is that it is required the coupling constants beta functions to be zero to keep this symmetry. Taking only low energy apportions to the beta components we obtain three equations that can be taken as coming from a low energy effective action. Using a change of parameters we can obtain the Einstein Hilbert action with massless extra fields.

Conclusions

The string theory is, perhaps, the most ambitious theory of the history of physics. High relevance results have been explained with this theory such as the loss of information on the blackholes. Maybe the most attractive fact about this theory is that everything in universe is explained with vibrating segments on the spacetime, every interaction and every particle arise from a vibrating string.

Firstly, It has been enounced the least action principle applied to general relativity, field theory, special relativity, and strings moving through the spacetime in different ways. We have studied the string's global and local symmetries to simplify the equations of motion, studying its specific treatment as a gauge theory.

After that, we obtained the different solution of the motion equations discerning between different boundary conditions and we have used the Noether Theorem to obtain the conserved currents and charges of a field theory applied to the string. We also formulated the conditions of the equivalence among the Nambu-Goto action and the Polyakov action in form of the Virasoro conditions and obtained the mass formula for closed and open strings.

Later, we talked about the different ways of quantizing a gauge theory to remove the string's ghosts states. We studied in the process the covariant quantization and imposing the gauge restrictions in two different ways: the covariant approach and the lightcone quantization, mentioning their advantages and disadvantages, we found in both cases that the number of dimensions have to be twenty-six. We obtained in the process the form of the states of the quantized string, as the graviton, by using the central extension of the Virasoro Algebra.

Lastly, we studied the path integral approach applied to String Theory. Permitting us to introduce the interactions in the formalism. We employed this tool to formulate the general relativity as a low energy approach of an effective theory, built by considering the interaction of the string with a coherent state of gravitons.

Bibliography

Green, M. B., Schwatrz, J. H., & Witten, E. (1987). *Superstring Theory.* Cambridge: Cambridge University Press.

Landau L. D. , Lifshitz E. M. (1971). *Classical theory of fields.* Oxford: Pergamon Press.

Lifshitz E. M., V. B. Berestetskii, L.P. Pitaevskii. (1975). *Teoría cuántica relativista.* Barcelona: Reverté.

Maiani L. ; Benar O. (2016). *Relativistic Quantum Mechanics, An Introduction to Relativistic Quantum Fields.* CRC Press.

Peskin, M., & Schroeder, D. (1995). *Quantum Field Theory.* Addison-Wesley Pub. Co.

Polchinski, J. (1998). *String Theory.* Cambridge: Cambridge University Press.

Tong, D. (2012, February). *University of Cambridge* . Retrieved from David Tong: Lectures on String Theory: http://www.damtp.cam.ac.uk/user/tong/string.html

Weimberg, S. (1995). *The Quantum Theory of Fields.* Cambridge: Cambridge University Press.

Wray, K. (2011, May 6). Retrieved from An Introduction to String Theory: https://math.berkeley.edu/~kwray/papers/string_theory.pdf

Zwiebach, B. (2004). *A First Course in String Theory.* Cambridge: Cambridge University Press.

Images taken from:

- Fig 4.2, Fig 4.3 at pages 11 and 15 at: https://www.physics.umd.edu/courses/Phys851/Luty/notes/diagrams.pdf
- Fig 1.1, 4.1, 4.4, Fig 4.5 at: String Theory by Joseph Polchinski at pages 10, 330, 98 and 100

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