Departamento de Física
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Introduction to String Theory

Universidad de La Laguna

Grado en Física

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Agradecimientos

Para comenzar agradecer a mi tutor por todo el tiempo que ha dedicado en este trabajo y toda la ayuda que me ha aportado.

A mis amigos y familiares, en especial a mis padres y mi hermano, por estar presentes ante cualquier circunstancia. A Dani, por darme su apoyo en todo momento y por creer en mí.

A todos los que han compartido conmigo la ilusión que he puesto en este trabajo.

Gracias.

Resumen

La teoría de cuerdas aparece como una propuesta que permite explicar la excistencia de partículas fundamentales como cuerdas idénticas, que se diferencian entre sí según el modo en el que oscilan. Es una teoría de campos que no tiene parámetros libres, solo depende del parámetro que determina la longitud de la cuerda. En este caso se estudiará la teoría de cuerdas bosónica, que no contempla los fermiones.

Se puede derivar a partir del principio de mínima acción. Para la cuerda relativista, la acción depende del área de la hoja del mundo que representa su trayectoria por el espacio-tiempo. Se estudian dos acciones equivalentes: la acción de Nambu-Goto y la de Polyakov. Después de fijar las ligaduras que debe cumplir el sistema, la ecuación de una cuerda relativista queda esencialmente reducida a una ecuación de ondas. Para resolverla se imponen condiciones de contorno correspondientes a una cuerda cerrada o una cuerda abierta con condiciones de Neumann o Dirichlet.

La expresión para la las coordenadas en la hoja del mundo se puede expandir en modos de Fourier, dando lugar a una ecuación que cuenta con una parte traslacional y otra asociada a las vibraciones internas de la cuerda. Para una cuerda abierta los modos solo se propagan en un sentido, mientras que para la cuerda cerrada se deben tener en cuenta ambos. A partir de la definición de los coeficientes de Fourier, se puede determinar el momento lineal, y posteriormente, la masa efectiva.

Para cuantizar las ecuaciones se siguen dos procedimientos. Partiendo de la teoría clásica, se definen los conmutadores a partir de los corchetes de Poisson y los coeficientes pasan a ser operadores. Con este método aparecen estados de norma negativa a los que se les denomina estados fantasma. Al eliminarlos aparecen una serie de restricciones que fijan el número de dimensiones a 26. Por otra parte, haciendo uso de la simetría frente a reparametrizaciones, se puede definir un gauge en el cono de luz que permite cuantizar las cuerdas evitando los estados fantasmas. En cambio, al usar estas nuevas coordenadas, la invariancia de Lorentz no se manifiesta y debe ser impuesta. Al recuperarla se llega nuevamente a las mismas restricciones para la dimensionalidad del espacio-tiempo.

Se puede hacer uso de la fórmula de la masa para obtener el espectro de estados excitados. El estado fundamental, tanto para cuerdas abiertas como cerradas, corresponde con la representación de un taquión. El primer estado excitado para las cuerdas cerradas, corresponde sorprendentemente con la representación del gravitón. Este resultado permite establecer una teoría de gravedad cuántica.

Sin embargo, la teoría de cuerdas bosónica no contiene fermiones en sus representaciones. Para completarla se establecen un conjunto de supersimetrías, que dan lugar a las teorías de supercuerdas. Mediante el uso de un superálgebra, se crea un grupo espacial supersimétrico en el que cada bosón tiene asociado un fermión. Existen 5 tipos de supercuerdas, todas ellas se definen en espacios de 9 dimensiones espaciales y una temporal. Se desarrolla una teoría que engloba todas las posibles teorías de cuerdas, considerando cada una de estas como casos particulares. Es la denominada Teoría-M. Aunque aún es una teoría desconocida, se ha establecido que sus dimensiones aumentan, teniendo ahora 10 espaciales y una temporal.

Abstract

String theory comes as a proposal that allows to explain the excistence of fundamental particles as identical strings, which differ from each other according to the way they oscillate. It is a field theory that has no free parameters, it only depends on the parameter that determines the length of the string. In this case we will study bosonic string theory, which does not consider fermions.

This may be derived from least action principle. For the relativistic string, the action depends on the area of the world-sheet that represents its trajectory through space-time. Two equivalent actions are studied: the Nambu-Goto action and the Polyakov action. After fixing the bindings to be satisfied by the system, the equation of a relativistic string is essentially reduced to a wave equation. To solve it, boundary conditions corresponding to a closed string or an open string with Neumann or Dirichlet conditions are imposed.

The resulting expression for the coordinates on the world sheet can be expanded into Fourier modes, giving rise to an equation that has a translational part and other part associated with the internal vibrations of the string. For an open string the modes only propagate in one direction, while for the closed string both must be taken into account. From the definition of the Fourier coefficients the linear momentum can be determined, and subsequently, the effective mass.

To quantize the equations, two procedures are followed. Starting from the classical theory, the commutators are defined from the Poisson brackets and the coefficients become operators. With this method, negative-norm states are predicted, which are called ghost states. When they are eliminated, a series of restrictions appear that set the number of dimensions to 26. Alternatively, by making use of symmetry against reparametrizations, a gauge can be defined in the light cone that allows quantization of the strings avoiding ghost states. In contrast, when using these new coordinates, the Lorentz invariance does not show manifested and must be imposed. When recovering it, one again arrives at the same restrictions for the dimensionality of space-time.

The mass formula can be used to obtain the spectrum of excited states. The ground state, for both open and closed strings, corresponds to the representation of a tachyon. The first excited state for closed strings, corresponds surprisingly to the representation of the graviton. This result makes it possible to establish a theory of quantum gravity.

However, bosonic string theory does not contain fermions in its representations. To complete the theory, a set of supersymmetries are established, which give rise to superstring theories. By using a superalgebra, a supersymmetric space group is created in which each boson has an associated fermion. There are 5 types of superstrings, all of them are defined in spaces of 9 spatial and one temporal dimension. From them a theory is developed that gathers all the possible string theories, considering each one of them as particular cases. This is the so-called M-theory. Although it is still an unknown theory, it has been proved that its dimensions increase, having now 10 spatial and one temporal.

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Chapter 1
INTRODUCTION

En esta sección se explica la motivación principal de la teoría de cuerdas. Se introduce como teoría unificadora que incluye la gravedad cuántica. Para su derivación se emplea el principio de mínima acción: un principio variacional que permite describir cualquier teoría física a partir de una acción que debe ser estacionaria. También se hace un breve estudio del sistema formado por las cuerdas clásicas.

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1.1 Motivation and justification arguments for string theory

One of the major open problems in physics today is the description of a Grand Unifying Theory that encompasses all the forces and particles present in the universe. As Maxwell did in the 19^{th} century, when he discovered the equivalence between the electric and magnetic fields. Or as happened a hundred years later, when there was established the relationship between weak and electromagnetic interactions. The current aim is to achieve a universal and unifying theory.

The Standard Model is the most widely accepted physical model for describing the fundamental particles and their interactions. It can even be considered the closest to the GUT (Grand Unifying Theory). On the other hand, it has certain gaps or unsolved problems such as: it is not able to explain gravity and it depends on about twenty parameters that cannot be determined.

The initial development of the string theory consists in supposing a system made of indivisible elements. Particles are usually thought of as point elements. However, when thinking of one dimension objects, as strings curved with the shape of a loop, different and nevertheless interesting results are obtained. String theory can be mainly though of as a substitution of point particles by one dimension strings. Unlike the Standard Model, where our four-dimensional space, with three spatial and one temporal dimension is presupposed, in String Theory the dimensionality of the considered space-time is derived. This allows to put an end to the problem of undetermined parameters. For strings, only one parameter results which is associated with the length of the string l_s. There are no other free parameters.

The fundamentals of this theory assume that all known particles constituting the universe are composed of identical strings. Then each particle would correspond to an excited mode of the string. Meaning, that depending on how they vibrate, they will give rise to different particles. It can be thought of as a classical string on an instrument producing music: each mode of vibration gives a different note, just as each mode in string theory represents a fundamental particle.

Depending on the conditions considered different theories can be derived. On the one hand, the theory that emerges from the classical theories can be considered separately. This does not include anticommutation properties, and therefore, fermions cannot appear. This is the *Bosonic String Theory* (that is the one that will be developed in greater depth), where the predicted particles are only bosons. During the derivation it is concluded that these strings live in 26 dimensions. On the other hand, if the bosonic theory is expanded by using super-algebras that include fermions in their representations, both kinds of particles are taken into account. The strings considered now are called *Superstrings*. These live in 10 dimensions (one temporal and 9 spatial) and present a series of supersymmetries.

Supersymmetry is a relation that couples bosons, which are the force-carrying particles, to fermions, which are the matter-carrying particles. Thus, all the fundamental particles are associated in pairs for each excited state, unless symmetry is broken. It can be also imposed to the Standard Model. However, the most significant improvement over the Standard Model is including the particle associated to gravity carriers. In the derivation of string theory, the *graviton* emerges as one of the excited states. From it one can derive a quantum gravity theory and if supersymmetry is considered, supergravity is obtained.

Bosonic strings do not include fermions, so it is not a realistic theory. There are 5 different superstring theories: Type I, Type IIA, Type IIB, Heterotic SO(32) and Heterotic $E_8 \times E_8$. Many interrelations have been observed between them. For this reason, the so-called *M-theory* arises later on. It encompasses all string theories, in a space of 11 dimensions, which are again separated into one temporal and 10 spatial dimensions.

So far, this M-theory remains completely unknown. It can be thought of as a single theory that has different sides. Just as in Einstein's theory the reference systems are presented, the different superstring theories would correspond to the edges of a network of theories aiming to explain particular cases. This kind of duality describes physical systems that apparently are completely different but in fact could be identical.

In addition to the difficulty that working in a space with so many dimensions introduces, the question arises: What happens with these extra dimensions? Why don't we see them in our four-dimensional world? Mainly one could start by arguing that, due to the available technology, it is not possible to observe at such a small scale and check whether it is possible to detect these dimensions. Assuming that both the length of the string and the size of the extra dimensions are of the order of Planck's constant $l_s \sim 10^{-35}\, m$, the minimum observed distance is of the order of $10^{-18}\, m$. Hence, there is a difference of many orders of magnitude.

For this reason, there is no experimental verification of the certainty of this theory to describe reality. If one could actually measure the length of the string or prove the existence of more spatial dimensions, there would be some evidences. So far it is assumed that there are compact dimensions. This consists in supposing that there are objects of dimension D, which, when observed from a much larger scale, appear to be naturally defined in a space of lower dimension (see Figure 1).

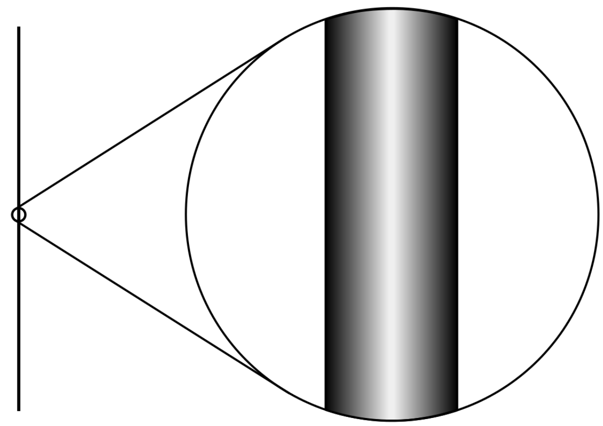

Figure 1: Example of compactification of one circular dimension when observing from a much larger scale.

General physics presents two opposing frameworks that have revolutionized the paradigm followed up to now. These are relativity and quantum physics. In this sense, string theory is an excellent candidate as a unifying theory. With it, quantum gravity appears naturally, filling the gap that was missing to describe the behavior of the universe as a whole.

1.2 The least action as a fundamental principle

The least action principle states that any theory can be described starting from an action. The fundamental equations of physics are obtained by establishing this action as stationary, meaning that a physical system will evolve in time in such a way that the resulting action gives a minimum. If the action for a given mechanical system is denoted by S, the trajectory followed in space and time by the particle under study must correspond to that which gives a variation δS equal to zero.

However, it is more accurate thinking of an extremum value instead of restricting it just to minimum. This assumption comes from general relativity. When considering relativistic particles the trajectories followed are defined by the geodesic, where the critical point could either minimize or a maximize the path, depending on the chosen parametrization.

The least action principle is one of the most important methods because of its generality for the derivation of different theories, models and equations. It can be used in all branches of physics. Each system can be described from a Lagrangian density that gives an action and then some restrictions that are included to satisfy the variational principle. The constraints imposed to make the action stationary give the equations of motion

1.2.1 Action of the relativistic point particle

The motion of a relativistic point particle can be described from the equations of motion derived from the principle of least action. The first objective resides in determining the proper action for the system considered. For the present case, it must satisfy several symmetries, such as Lorentz invariance. The integral cannot depend on the path considered or on the reference frame. Therefore, the only possibility is that the action is a scalar. Intuitively it looks like the interval ds defined between two events in general relativity, should appear since it fulfills all these conditions. However, the dependence can be a proportionality relation. That is why a factor α is included, just to avoid loosing generality and provide the correct dimensions of an action: energy multiplied by time, or similarly ML^2/T.

$$S = -\alpha \int_a^b ds. \tag{1}$$

For a flat spacetime, described by the Minkowski metric, the interval is defined by $ds^2 = -(c\, dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$. Thus, if the particle is considered to be static, its interval should be temporal and all the differentials of the spacial coordinates ($dx^1 = dx,\ dx^2 = dy,\ dx^3 = dz$) equal to zero. This gives the so-called proper time $d\tau$:

$$ds^2 = -c^2\, d\tau^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2,$$

$$d\tau^2 = dt^2 \left(1 - \frac{dx^2 + dy^2 + dz^2}{c^2}\right) \Longrightarrow d\tau = \frac{ds}{c} = dt\sqrt{1 - \frac{v^2}{c^2}}.$$

As for non-moving free particles the proper time is maximum, the minus sign added in equation (1) guarantees that the action will be minimum for that simple case. In order to maintain that statement it can be predicted that α must be a positive constant.

The limits of the integrals are the corresponding positions of the particle at given initial and final times t_1, t_2. In agreement with the least action principle, the action can be written as an integral in time of the form

$$S = \int_{t_1}^{t_2} L\, dt. \tag{2}$$

From this equality and using the previous definition of proper time, it can be observed that the equivalence between these two equations allows changing the integration variable in the equation (1), and expressing it as an integral over time dt. This means:

$$S = -\int_{t_1}^{t_2} \alpha c \sqrt{1 - \frac{v^2}{c^2}}\, dt.$$

Now it can be seen at a glance that the Lagrangian of the system is $L = -\alpha c\sqrt{1 - \frac{v^2}{c^2}}$.

In the non-relativistic limit this should give the same result as the classical Lagrangian for a free particle $L = mv^2/2$. Then the relativistic Lagrangian function is expanded giving the approximate expression

$$L = -\alpha c\sqrt{1-\frac{v^2}{c^2}} \approx -\alpha c + \alpha\frac{v^2}{2c}. \tag{3}$$

The constant terms appearing in the Lagrangian will not be present in the equations of motion. Therefore, to obtain an expression that coincides in the classical limit, the constant term $-\alpha c$ is not taken into account. Consequently in the limit $v \ll c$ this expression coincides with the classical one if the constant α is the mass of the relativistic point particle times the speed of light, i.e.

$$\alpha = mc. \tag{4}$$

That assumption emerges from direct comparison of the second term of equation (3) and the classical expression. What is more, one can observe that for rest particles, the Lagrangian corresponds to minus the rest energy ($E = mc^2$).

This way one can determine the action of the relativistic point particle with its proper dimensions, again ML^2/T

$$S = -mc\int_a^b ds, \tag{5}$$

and also the Lagrangian function that is finally given by the equation

$$L = -mc^2\sqrt{1-\frac{v^2}{c^2}}. \tag{6}$$

1.2.2 Action for the Schrödinger equation

The Schrödinger equation is a fundamental equation in quantum mechanics that describes the behavior of quantum systems. Being one of the most important equations of modern physics, it can be derived from the least action principle as well, where the action is defined as the integral of the Lagrangian over time. With this another example of the usefulness of this variational principle is provided.

This derivation leads to the general time-dependent equation, which arises from the consideration of the following Lagrangian density $\mathcal{L}$

$$\mathcal{L} = \frac{i\hbar}{2}(\psi^*\dot{\psi} - \dot{\psi}^*\psi) - \frac{\hbar^2}{2m}\nabla\psi^*\cdot\nabla\psi - V(r)\psi^*\psi. \tag{7}$$

Here $\psi(r)$ are complex classical fields that represent the states of a particle in quantum mechanics. The last term on the right hand side is multiplied by $V(r)$, that represents the external potential, and the other two terms are associated with the kinetic energy for the supposed quantum particle of mass m.

The corresponding action, S, is obtained by performing the integration in time of the Lagrangian L, which in turn is found from the integral over the whole volume of the Lagrangian density.

$$L = \int \mathcal{L}\, d\vec{r}\,, \qquad S = \int_{t_i}^{t_f} L\, dt.$$

The Euler-Lagrange equations for this system give the same dynamical result for a field or its complex conjugate. In the present case, the derivatives of the Lagrangian respect to the fields ψ^* and $\partial\psi^*/\partial\vec{r}$ are used. Bearing in mind the form of $\mathcal{L}$ in (7), Euler-Lagrange equations lead to the following equality:

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\psi}^*}\right) + \frac{\partial}{\partial\vec{r}}\left(\frac{\partial\mathcal{L}}{\frac{\partial\psi^*}{\partial\vec{r}}}\right) - \frac{\partial\mathcal{L}}{\partial\psi^*} = 0,$$

$$i\hbar\,\dot{\psi} + \frac{\hbar^2}{2m}\Delta\psi - V(r)\psi = 0.$$

This is the time-dependent Schrödinger equation, which describes the behavior of a particle of mass m in a generic potential $V(r)$.

If instead of using the Euler-Lagrange equations one would develop the action by taking a variation and making it stationary, the same results are obtained. To do so, suppose a variation of the action respect to ψ^*

$$\delta S = \int_{t_i}^{t_f}\int_V \left[\frac{i\hbar}{2}(\delta\psi^*\dot{\psi} - \delta\dot{\psi}^*\psi) - \frac{\hbar^2}{2m}\nabla(\delta\psi^*)\cdot\nabla\psi - V(r)\delta\psi^*\psi\right] d\vec{r}\, dt.$$

Since the variance $\delta\psi^*$ is arbitrary, it is convenient to write the variation of the action as a term, that must be zero, multiplied by $\delta\psi^*$. Therefore, the summands in which instead of $\delta\psi^*$ its derivatives appear, must be integrated by parts. This leads to the emergence of the following boundary terms:

$$\int_{t_i}^{t_f}\int_V \delta\dot{\psi}^*\psi\, d\vec{r}\, dt = \int_V\int_{t_i}^{t_f}\left[\frac{d}{dt}\left(\delta\psi^*\psi\right) - \delta\psi^*\dot{\psi}\right] dt\, d\vec{r}, \tag{8}$$

$$\int_{t_i}^{t_f}\int_V \nabla(\delta\psi^*)\cdot\nabla\psi\, d\vec{r}\, dt = \int_{t_i}^{t_f}\int_V \left[\nabla(\delta\psi^*\cdot\nabla\psi) - \delta\psi^*\cdot\nabla^2\psi\right] d\vec{r}\, dt. \tag{9}$$

where the first term in the first integral can be integrated and evaluated in time, and the first term in the last integral can be transformed into a surface integral using the Stokes theorem. This finally gives

$$\int_V\int_{t_i}^{t_f}\frac{d}{dt}\left(\delta\psi^*\psi\right)\, dt\, d\vec{r} = \int_V \left[\delta\psi^*\psi\right]_{t_i}^{t_f}\, d\vec{r}, \tag{10}$$

$$\int_{t_i}^{t_f}\int_V \nabla(\delta\psi^*\cdot\nabla\psi)\, d\vec{r}\, dt = \int_{t_i}^{t_f}\int_S \delta\psi^*\cdot\nabla\psi\, d\vec{s}\, dt. \tag{11}$$

Since these two terms must also cancel out, they give rise to the initial and boundary conditions that the physical system must satisfy. Thus, the least action principle not only allows to obtain the equations of motion, but also gives the possible boundary conditions, which emerge from the boundary terms coming from the integration by parts.

Notice that Schrödinger equation demand to define proper boundary conditions. The first result shown in equation (10), sets the values for the initial and final times, requiring the variation to be zero. The boundary conditions (11) state that the flow through the surface delimiting the volume under consideration must be zero. This either because the function is zero at the border or because the current across the boundary cancels out.

1.2.3 Other examples of the least action principle

In order to emphasize the versatility of the least action principle, some examples of equations and laws of great importance in physics which can be obtained from an action, are shown below.

The first example are the famous Maxwell equations for the free electromagnetic field. Suppose certain electric field $\vec{E}$ and a magnetic field $\vec{B}$, both defined in a volume V. The whole theory can be derived from a Lagrangian density that leads to the corresponding action

$$S_{EM} = -\frac{1}{8\pi}\int_{t_i}^{t_f}\int_V (\vec{E}^2 - \vec{B}^2) d\vec{r}dt.$$

It can be also convenient to use the tensorial notation, replacing those two fields in the Lagrangian density by the electromagnetic field, denoted as F^{ik}. Therefore, as the action has to be an scalar, it takes the form

$$S_{EM} = -\frac{1}{16\pi c}\int_{t_i}^{t_f}\int_V F_{ik}F^{ik} d\vec{r}dt.$$

On the other hand, it can be also used in quantum mechanics to derive the Dirac equation. This equation describes a fermionic field of spin-1/2 particles. If there are no interactions, the Dirac equation for free fermions emerges from the following Lagrangian density:

$$\mathcal{L} = \psi^{\dagger}\gamma^0(i\hbar c\,\gamma^{\mu}\partial_{\mu} - mc^2)\psi,$$

where ψ is a spinor representing the field of the fermionic particles and γ^{μ} are the so-called Dirac matrices.

Changing the framework under study, one can consider for instance the Einstein equations for general relativity. In the derivation for the relativistic particle it was shown that the action must be a scalar, and it seems to be related with the metric when the whole spacetime is considered and not only a region. Introducing the scalar R that gives the curvature of space, this action is given by

$$S_{GR} = -\frac{c^3}{16\pi G}\int_\Omega R\sqrt{-g}\, d\Omega,$$

being c and G the light speed constant and the gravitational fundamental constant, respectively. Note that in this case the integration is done over a Ω space, that represents all the coordinates, temporal and spatial.

It can be also mentioned that this theory can be derived from this Lagrangian straightfowardly. The problem with general relativity emerges when one tries to quantize it. The classical gravitational field is well defined by the resulting equations, but the quantum field does not seem possible.

1.3 The classical string

Before considering the general case, it can be helpful to study the classical (nonrelativistic) string. Assuming a string of length a with fixed endpoint $(0,0)$ and $(0,a)$ in the (x,y) plane. This makes that only transverse oscillations are allowed, meaning that only the y coordinates of each point of the string may vary in time.

The dynamical configuration of a string is mainly described by the tension T_0. Thinking of a dimensional analysis, this must have units of force, or equally, mass times velocity squared (energy) over length

$$[T_0] = [Force] = \frac{[Energy]}{L} = \frac{M[v^2]}{L}, \tag{12}$$

where the squared brackets are used to denote units. M comes from mass units and L from length units. The mass per unit length is also a known parameter for a string. It is called the mass density and it is usually denoted as μ_0.

Now that the force and the mass parameters are determined, Newton's laws can be applied in order to obtain the equation of motion. Equating the vertical force to the mass times the acceleration, one arrives at the typical wave equation

$$\frac{\partial^2 y}{\partial x^2} - \frac{\mu_0}{T_0}\frac{\partial^2 y}{\partial t^2} = 0. \tag{13}$$

From the dimensions of the tension shown in (12), it can be deduced that the velocity of the string oscillations can be written in terms of the mass density and the tension. Thus, the velocity of the waves taking place on the studied string $v_0 = \sqrt{T_0/\mu_0}$, can be substituted in the previous expression, giving the final wave equation:

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{v_0^2}\frac{\partial^2 y}{\partial t^2} = 0. \tag{14}$$

The equation of motion of the classical string can also be derived from a Lagrangian density. The latter will determine the action to be employed when applying the variational principle. The Lagrangian of a system is typically given by its kinetic and potential energy, denoted by T and V respectively.

$$L = T - V\,; \qquad T = \int_0^a \frac{1}{2}\,(\mu_0\,dx)\left(\frac{\partial y}{\partial t}\right)^2\,, \quad V = \int_0^a \frac{1}{2}\,T_0\,\left(\frac{\partial y}{\partial x}\right)^2\,dx.$$

The integrals are defined along the string, so the total energy is obtained by taking all the infinitesimal segments dx. Note that in the kinetic energy, instead of the total mass m expected for a point particle, there appears the mass density multiplied by the infinitesimal length interval dx. Similarly, the potential energy is determined from the work done over each fragment Δl, that in the matter under consideration is $T_0\,\Delta l$. After taking infinitesimal segments, Δl can be approximated to $\frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2\,dx$.

At last the Lagrangian density $\mathcal{L}$ is defined. It should be integrated to obtain the Lagrangian of the system $L(t)$ and subsequently for the action of the string.

$$\mathcal{L}\left(\frac{\partial y}{\partial t},\frac{\partial y}{\partial x}\right) = \frac{1}{2}\,\mu_0\left(\frac{\partial y}{\partial t}\right)^2 - \frac{1}{2}T_0\left(\frac{\partial y}{\partial x}\right)^2$$
$$L(t) = \int_0^a \mathcal{L}\,dx\,, \quad S = \int_{t_i}^{t_f} L(t)\,dt.$$

Assuming a variation in the action δS and making it vanish, one obtains that the resulting equation of motion is identical to the wave equation (13), as expected. But when integrating by parts, two other terms appear which are evaluated at the boundaries. Since these terms must also cancel out, the boundary conditions and initial conditions are set in such a way that they verify for these terms to be equal to zero [1].

As one must deal with a partial differential equation to solve this kind of problems, those boundary conditions and initial conditions should be set. The most common boundary conditions are Dirichlet and Neumann boundary conditions (see Figure 2). The first are used to describe strings whose ends are fixed (equations (15)), giving the corresponding values at $x = 0$ and $x = a$. Meanwhile the others refer to endpoints that are free to move in y direction. This are obtained evaluating the derivative $\partial y/\partial x$ (equations 16).

$$y(t, x = 0) = y(t, x = a) = 0 \quad \Leftrightarrow \quad \frac{\partial y}{\partial t}(t, x = 0) = \frac{\partial y}{\partial t}(t, x = a) = 0, \tag{15}$$

$$\frac{\partial y}{\partial x}(t, x = 0) = \frac{\partial y}{\partial x}(t, x = a) = 0. \tag{16}$$

Once the proper boundary conditions are set, the general solution for this problem is a superposition of two arbitrary functions travelling in opposite directions

$$y(t, x) = h_+(x - v_0t) + h_-(x + v_0t). \tag{17}$$

From the starting configuration given by the initial conditions, the corresponding value of the function $y(t, x)$ and its time derivative at $t = 0$ is known

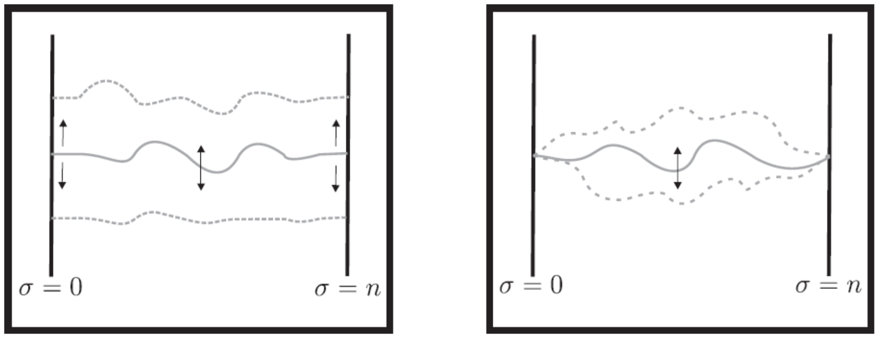

Figure 2: Representation of the motion allowed for the endpoints of a string depending on the boundary conditions imposed. Left: Neumann boundary conditions. Right: Dirichlet boundary conditions.

$$y(0, x) = h_+(x) + h_-(x),$$
$$\frac{\partial y}{\partial t}(0, x) = -v_0 \frac{dh_+}{dx} + v_0 \frac{dh_-}{dx}.$$

These equations allow solving the shape of h_+ and h_- from the fist-order differential equations and using the boundary conditions. When the functions h_+ and h_- are obtained, the full solution of the equation of motion can be calculated from the sum (17).

In the case of ideal oscillations, where each point of the string is in phase and doing a sinusoidal transverse movement, the function $y(t, x)$ would take the form

$$y(t, x) = y(x)\, sin(\omega t + \phi),$$

where ω is the frequency and ϕ the phase constant. This leads to a wave equation were the allowed frequencies can be determined.

Depending on the imposed conditions for the system, one obtains the following nontrivial solutions that depend on an arbitrary constant A_n:

$$\begin{cases} y_n(x) = A_n\, sin\left(\frac{n\pi x}{a}\right), \; n = 1, 2, ... & \text{for Dirichlet BC,} \\ y_n(x) = A_n\, cos\left(\frac{n\pi x}{a}\right), \; n = 0, 1, 2, ... & \text{for Neumann BC.} \end{cases}$$

Substituting that in the resultant wave equation for this problem gives the possible oscillation frequencies.

$$\frac{d^2 y(x)}{dx^2} + \omega^2 \frac{\mu_0}{T_0} y(x) = 0\,, \quad \text{with} \quad \omega_n = \sqrt{\frac{T_0}{\mu_0}} \left(\frac{n\pi}{a}\right).$$

where ω_n gives the shape of the allowed frequencies. In both cases those are the same, except that for Neumann boundary conditions it is admitted the situation with $n = 0$.

Chapter 2

THE RELATIVISTIC STRING

Considerando las condiciones a las que está sujeta una cuerda relativista, se obtiene la acción de Nambu-Goto a partir del área de la hoja del mundo. La acción de Polyakov es equivalente a ésta, por lo que las ecuaciones del movimiento derivadas en ambos casos son idénticas. También deben satisfacer las mismas ligaduras.

Since general relativity it is known what is the path that a particle leaves when traveling through space-time. This corresponds to what is called the *world-line*. It is an intuitive representation of the trajectory of a relativistic point particle when it is moving, where one axis shows the variation in space and the other axis is for time.

The next step would be to do the same procedure but with a one dimensional object. In that case instead of drawing a line, it trivially appears as a two dimensional object. Therefore a string moving in space-time forms the so-called *world-sheet*. In Figure 3 the world-sheet of an open string and a closed string are shown respectively.

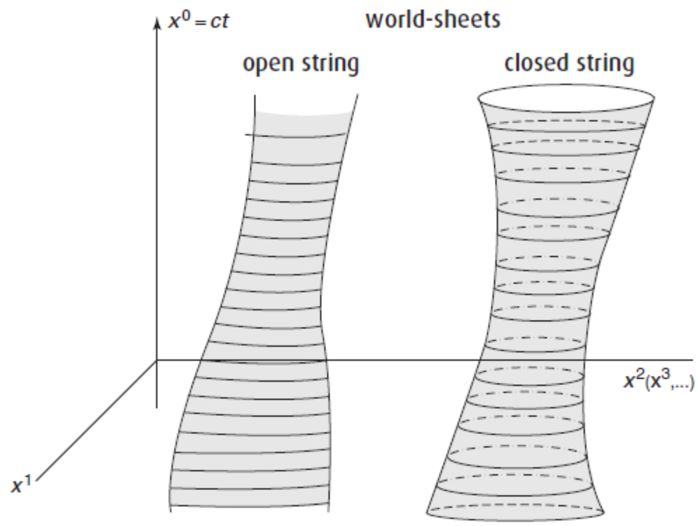

Figure 3: Representation of the world-sheet resultant for open (left) and closed (right) strings when traveling through space-time. All spacial dimensions are included in the horizontal planes where the string is contained at each instant. The temporal dimension is the one represented in the vertical axis. For all fixed times, the world-sheet gives the corresponding position of the string in space.

2.1 Action of a relativistic string

In the following, the behavior of a relativistic string will be studied. For this purpose, the equations of motion must be determined. As in the previous section, they will be obtained from the least action principle (as was done with the relativistic particle) and certain boundary conditions must be set (just like the classical string). In agreement with the procedure followed for the point particle, the action of the relativistic string must be derived first. Therefore, a scalar is also used so that it remains invariant to Lorentz transformations.

For the relativistic point particle the quantity used was the interval that describes its world-line. It seems intuitively reasonable that for a 1-dimensional string this quantity could be the area of the world-sheet created by the string. Through this reasoning one arrives to the Nambu-Goto action (whose name comes from the physicists Yoichiro Nambu and Tetsuo Goto).

Nambu-Goto action:

To describe a two dimensional surface, the coordinates of the string are needed to be parameterized first. This means that each point of the string will be defined by a mapping function that transforms the coordinates from the parameter space into the spacetime. For convenience this parameters are usually denoted as τ and σ, being associated to a temporal and a spacial parameter respectively.

Therefore certain given region from the parameter space (τ, σ) is transformed by the mapping functions $X^\mu(\tau, \sigma)$ (see Figure 4), that are the coordinates of the string

$$X^\mu(\tau, \sigma) = (X^0(\tau, \sigma), X^1(\tau, \sigma), ..., X^d(\tau, \sigma)).$$

Now the two infinitesimal vectors corresponding to a variation in each direction of the parameter space are defined separately:

$$dv_1^\mu = \frac{\partial X^\mu}{\partial \tau} d\tau \, , \qquad dv_2^\mu = \frac{\partial X^\mu}{\partial \sigma} d\sigma.$$

These vectors represent the corresponding quadrilateral surface in the world-sheet. As they together take the generic form of a parallelogram, its area can be calculated by means of the formula:

$$dA = \sqrt{(dv_1 \cdot dv_1)(dv_2 \cdot dv_2) - (dv_1 \cdot dv_2)^2}.$$

Using the scalar product, as well as for the point particle, guarantees that the result is a Lorentz invariant. However, the term in the square root is negative. In order to obtain the action from this quantity, a change in the sign would not affect the result. After this small modification, the definitions of dv_1^μ, dv_2^μ are substituted, and with the relativistic dot product notation, one finally arrives at

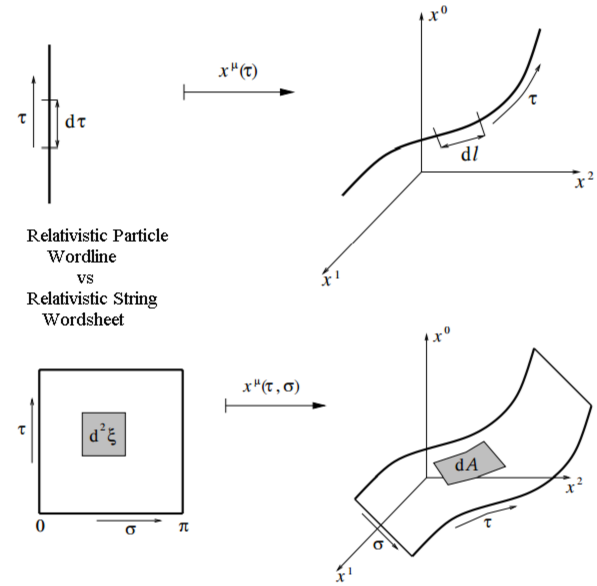

Figure 4: World-line of a relativistic point particle in the parameter space and in the target space. Bellow the world-sheet for an open string is shown, also in the parameter space and in the target space. The change from one system to the other is done mapping the coordinates by expressing them in terms of arbitrary parameters. The way this coordinates transform allows the reparametrization invariance, meaning that they can be written in terms of other parameters and return the same result.

$$A = \int \sqrt{\left(\frac{\partial X}{\partial \tau} \cdot \frac{\partial X}{\partial \sigma}\right)^2 - \left(\frac{\partial X}{\partial \tau}\right)^2 \left(\frac{\partial X}{\partial \sigma}\right)^2} d\tau d\sigma. \tag{18}$$

As it was done with the point particle, there is a constant that should be added, to provide the proper dimensions. In fact, if a dimensional analysis is performed, it can be seen that the action has units of ML^2/T, while those of the area are L^2. The missing constant will have to add units of mass divided by time. This corresponds precisely to the units of tension divided by velocity (12). Because of the relativistic nature of the problem, that velocity seems to correspond to the speed of light. So the action would have as a proportionality constant T_0/c.

Introducing the abbreviated notation for partial derivatives

$$\dot{X}^\mu \equiv \frac{\partial X^\mu}{\partial \tau}, \qquad X^{\mu\prime} \equiv \frac{\partial X^\mu}{\partial \sigma},$$

one reaches the final expression of the Nambu-Goto action:

$$\boxed{S = -\frac{T_0}{c}\int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma \sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2}}. \tag{19}$$

The integral is defined between the limits of an initial and final time corresponding to the values τ_i, τ_f of the temporal parameter, and between $0, \sigma_1$ for the spacial one. If it turns out that σ is periodic, the system corresponds to a closed string.

By direct inspection of equation (5) for a relativistic point particle, one can notice that this action may be defined as a function of the metric. The definition of the relativistic dot product says that for a generic metric the interval can be written as

$$ds^2 = -g_{\mu\nu}(X)dX^\mu dX^\nu. \tag{20}$$

Assume that the coordinates for the one-dimensional world-line are expressed using only one parameter $X(\tau)$. Thereby the action of the point particle is also defined by the following expression

$$S = -mc\int\sqrt{-g_{\mu\nu}\frac{dX^\mu(\tau)}{d\tau}\frac{dX^\nu(\tau)}{d\tau}}d\tau.$$

Returning to the problem at hand, the action of a string will also depend on the metric. In that case, the surface under consideration is a 2-dimensional object, thus two parameters are needed to determine the world-sheet coordinates.

An induced metric is also defined, which will be the one intrinsic to the world-sheet. If flat Minkowski space can be used in this situation, the metric $g_{\mu\nu} = \eta_{\mu\nu}$. The interval in equation (20) can be obtained as a function of the surface parameters by applying the chain rule for the coordinates X^μ, i.e.

$$ds^2 = -\eta_{\mu\nu}dX^\mu dX^\nu = -\eta_{\mu\nu}\frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X^\nu}{\partial\sigma^\beta}d\sigma^\alpha d\sigma^\beta. \tag{21}$$

The generic situation of dimension D, has parameters $\sigma^d = (\sigma^0, \sigma^1, ..., \sigma^{D-1})$. In the proposed scenario, only two parameters are used: $\sigma^0 = \tau, \sigma^1 = \sigma$. This makes that $0, 1$ are the only possible values for the sum over α and β that is implicit in the previous equation.

Here it is defined the induced metric of the surface, that arises from placing the string in a D dimensional space. As σ^d are arbitrary parameters, this equality must remain satisfied and invariant. From this statement one can prove that Nambu-Goto is a *reparametrization invariant* action. So the induced metric ($G_{\alpha\beta}$) will depend on the parameters used

$$G_{\alpha\beta} \equiv \eta_{\mu\nu}\frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X^\nu}{\partial\sigma^\beta} = \frac{\partial X}{\partial\sigma^\alpha}\cdot\frac{\partial X}{\partial\sigma^\beta}. \tag{22}$$

From the definition above, the different terms that result from giving values to $\alpha, \beta = 0, 1$ can be calculated. Matrix representation of this metric reveals its explicit shape

$$
\begin{aligned}
G_{00} &= \left(\frac{\partial X}{\partial \tau}\right)^2 = (\dot{X})^2, \\
G_{11} &= \left(\frac{\partial X}{\partial \sigma}\right)^2 = (X')^2, \\
G_{01} = G_{10} &= \frac{\partial X}{\partial \tau} \cdot \frac{\partial X}{\partial \sigma} = \dot{X} \cdot X', \\
\implies G_{\alpha\beta} &= \begin{pmatrix} \dot{X}^2 & \dot{X} \cdot X' \\ \dot{X} \cdot X' & X'^2 \end{pmatrix}.
\end{aligned} \tag{23}
$$

To complete this derivation, one should note that the determinant of $G_{\alpha\beta}$ is equivalent to the term in the square root of equation (19) for the action. Calling $G = det(G_{\alpha\beta})$, the Nambu-Goto action in its reparametrization invariant form is given by

$$
S = -\frac{T_0}{c} \int \sqrt{-G}\, d\tau d\sigma. \tag{24}
$$

Polyakov action:

In addition to this equation, there exists the Polyakov action. Here the dependence is established using another metric introduced as an auxiliary field: the intrinsic metric $h_{\alpha\beta}(\tau, \sigma)$. This is more general since it has not been reduced to the Minkowskian flat metric.

$$
\boxed{S = -\frac{T_0}{2\,c} \int \sqrt{-h}\, h^{\alpha\beta} \frac{\partial X^\mu}{\partial \sigma^\alpha} \frac{\partial X^\nu}{\partial \sigma^\beta} g_{\mu\nu}\, d\tau\, d\sigma}. \tag{25}
$$

The Polyakov action is a redefinition of the previous one, which, since it does not contain a square root, it is more convenient for certain calculations. Both actions are equivalent and it can be proved by doing small variations of the Polyakov action respect to the field $h_{\alpha\beta}$ (Wray, pp 22-23, [2]). Thus same equations of motion and the same quantization of the strings are obtained in both cases.

2.2 Equation of motion for strings

Once the action is known, the equations of motion can be derived from it. In agreement with the least action principle, if a variation δS is computed, the equations of motion for the relativistic string will emerge by setting $\delta S = 0$.

In this calculations Nambu-Goto action will be used. For simplicity of writing, one begins by explicitly defining the action as a function of a Lagrangian density

$$
\mathcal{L}(\dot{X}^\mu, X^{\mu\prime}) = -\frac{T_0}{c}\sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2},
$$

$$
S = \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma\; \mathcal{L}(\dot{X}^\mu, X^{\mu\prime}).
$$

Once this simple notation has been entered, the variational principle is applied.

$$\delta S = \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma \, \left[\frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} \frac{\partial(\delta X^\mu)}{\partial \tau} + \frac{\partial \mathcal{L}}{\partial X^{\mu\prime}} \frac{\partial(\delta X^\mu)}{\partial \sigma} \right].$$

When the variation is made, the terms $\partial\mathcal{L}/\partial\dot{X}^\mu$ and $\partial\mathcal{L}/\partial X^{\mu\prime}$ arise. These are the momenta of the string and as they are important quantities, they will appear on several occasions. Therefore they are denoted with their own symbols.

$$\mathcal{P}^\tau_\mu \equiv \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = -\frac{T_0}{c} \frac{(\dot{X}\cdot X')X'_\mu - (X')^2 \dot{X}_\mu}{\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2 (X')^2}}, \tag{26}$$

$$\mathcal{P}^\sigma_\mu \equiv \frac{\partial \mathcal{L}}{\partial X^{\mu\prime}} = -\frac{T_0}{c} \frac{(\dot{X}\cdot X')\dot{X}_\mu - (\dot{X})^2 X'_\mu}{\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2 (X')^2}}. \tag{27}$$

This notation is then inserted in the equation for δS. By making some modifications to the derivatives in order to group them in a more convenient way, it results in the following for the term in brackets:

$$\left[\mathcal{P}^\tau_\mu \frac{\partial(\delta X^\mu)}{\partial \tau} + \mathcal{P}^\sigma_\mu \frac{\partial(\delta X^\mu)}{\partial \sigma}\right] = \left[\frac{\partial}{\partial \tau}(\mathcal{P}^\tau_\mu \, \delta X^\mu) + \frac{\partial}{\partial \sigma}(\mathcal{P}^\sigma_\mu \, \delta X^\mu) - \left(\frac{\partial \mathcal{P}^\tau_\mu}{\partial \tau} + \frac{\partial \mathcal{P}^\sigma_\mu}{\partial \sigma}\right) \delta X^\mu \right].$$

The first two terms in the right-hand side are full derivatives, so after doing the integral by parts they will depend on the initial and boundary conditions. Cancelling the boundary terms requires a different study about these constraints (as was done for (11) and (10) in the derivation of Schrödinger equation). However, the last term must go to zero always, for every arbitrary variation carried out. No other possibility is left but to make equal zero the term in parenthesis, yielding the *equations of motion* for the relativistic string

$$\delta S = \int_0^{\sigma_1} d\sigma \, \left[\mathcal{P}^\tau_\mu \, \delta X^\mu\right]_{\tau_i}^{\tau_f} + \int_{\tau_i}^{\tau_f} d\tau \, \left[\mathcal{P}^\sigma_\mu \, \delta X^\mu\right]_0^{\sigma_1} - \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma \, \left[\left(\frac{\partial \mathcal{P}^\tau_\mu}{\partial \tau} + \frac{\partial \mathcal{P}^\sigma_\mu}{\partial \sigma}\right) \delta X^\mu\right]$$

$$\Longrightarrow \boxed{\frac{\partial \mathcal{P}^\tau_\mu}{\partial \tau} + \frac{\partial \mathcal{P}^\sigma_\mu}{\partial \sigma} = 0}. \tag{28}$$

As mentioned above, the first two boundary integrals appearing at the right hand side of the equality must go to zero as well. For the first one to vanish, one can simply consider the initial and final states such that their variation is zero $\delta X^\mu(\tau_f, \sigma) = \delta X^\mu(\tau_i, \sigma) = 0$. The second integral is related with the endpoints of the string, and $2D$ boundary conditions are required. Following the same procedure as for the classic string, Dirichlet boundary conditions are set for the spatial terms

$$\frac{\partial X^\mu}{\partial \tau}(\tau, \sigma_*) = 0 \,, \qquad \mu \neq 0 \,, \qquad \sigma_* = 0, \, \sigma_1;$$

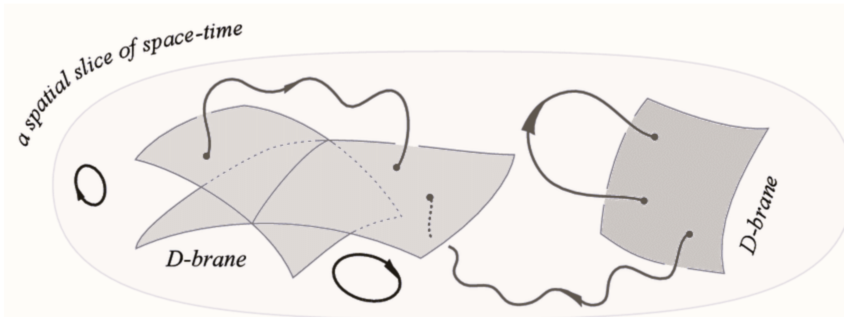

Figure 5: Possible boundary conditions for the relativistic string: the closed strings that has no endpoints and open strings fixed to D-branes. The open strings can have both endpoints fixed to the same brane, or start in a brane and end in a different one.

where $\mu = 0$ must be excluded because time varies with τ. Therefore, a condition can be set to force $\mathcal{P}_0^\sigma(\tau, \sigma_*)$ to vanish as well. These are the free endpoints condition and can be associated with Neumann boundary condition

$$\mathcal{P}_0^\sigma(\tau, \sigma_*) = 0 \ , \qquad \sigma_* = 0, \, \sigma_1 .$$

Dirichlet boundary conditions arise from assuming fixed endpoints in the string. In the relativistic string the objects to which the string is fixed are called *D-branes*. The dimensionality of these branes depend on the freedom the endpoints have to move. It can be a point, what means a 0 dimension brane (D0-brane), a line (D1-Brane), a plane (D2-brane) or a hiper surface (Dp-brane). For instance, if the endpoints have free boundary to move means that there is a D-brane filling the space. Figure 5 shows some examples of how strings that have boundary conditions associated with branes are distributed.

Chapter 3

SYMMETRIES OF THE BOSONIC STRING

Las acciones de la cuerda relativista presentan simetrías globales y locales. En esta sección se demuestra que la acción de Polyakov cumple las simetrías correspondiantes a las transformaciones del grupo de Poincaré, la invariancia frente a reparametrizaciones y la simetría frente a un cambio de escala local (simetría de Weyl). Al aplicarlas se obtienen restricciones adicionales que se deben verificar junto con la ecuación de onda.

According to the actions that determine the world-sheet of a bosonic string, there are some interesting transformations that maintain these actions invariant. These can be divided in two kinds: global symmetries and local symmetries. Both will be studied in this section, giving rise to new restrictions that must be met.

The global symmetries are those that do not depend on the coordinates of the spacetime where the transformation is applied. These are satisfied in all the spacetime considered, and their parameters do not depend on the coordinates. For the case of local symmetries, the changes that affect in different regions of spacetime are involved, so they will depend on where the transformation is performed.

3.1 Global symmetries

The response of the whole spacetime to these global transformations must include the ones corresponding to the background. Taking the Minkowski space, that must comply with the Lorentz invariance, the first global symmetry that our system presents is against the so-called Poincaré transformations.

Poincaré's group consists of an extension of the Lorentz transformations with the group of spatial translations and rotations of the Minkowski space. The generic form of the Poincaré transformations is as follows

$$\begin{cases} \delta X^\mu(\tau,\sigma) = a^\mu_\nu \, X^\nu(\tau,\sigma) + b^\mu, \\ \delta h_{\alpha\beta}(\tau,\sigma) = 0. \end{cases}$$

This emerges from an infinitesimal Lorentz transformation

$$X'^\mu = \Lambda^\mu_\nu \, X^\nu \ , \qquad \text{with} \quad \Lambda^\mu_\nu = \delta^\mu_\nu + a^\mu_\nu.$$

Thus, the variation under Lorentz transformation is given by $\delta X^\mu = a^\mu_\nu X^\nu$. But Lorentz invariance should be imposed here, meaning that for any spacetime interval $\delta(\eta_{\mu\nu} X^\mu X^\nu) = 0$. Using the property of a symmetric $\eta_{\mu\nu}$, one can deduce that this condition is satisfied if a^μ_ν is antisymmetric

$$\delta(\eta_{\mu\nu}X^\mu X^\nu) = 2\,\eta_{\mu\nu}\,\delta(X^\mu)X^\nu = 2\,\eta_{\mu\nu}(a^\mu_\rho X^\rho)X^\nu = 2\,a_{\rho\nu}X^\rho X^\nu = 0.$$

Looking at the resulting expression it can be seen that the only way to obtain zero is that the terms on which the sum is made cancel out. Which yields the general solution $a_{\rho\nu} = -a_{\nu\rho}$.

Knowing the meaning of these transformations, the main interest is to show that actions are invariant against them. For this purpose, the Polyakov action (25) will be used in the following development. In addition, the expressions can be simplified by setting $c = 1$. By doing this, the writting gets simpler as constant c will not appear anymore. But the general units of the problem are changed. Under this consideration it is said that normalized units are employed, and they will be used hereafter.

A variation performed in this action according to the Poincaré transformations, keeps the metric $h_{\alpha\beta}(\tau,\sigma)$ defined in the world-sheet unchanged. However the two derivatives $\partial_\alpha X^\mu$, $\partial_\beta X^\nu$ present variations that will be considered separately and summed. If the indices are exchanged in that sum, one reaches the following variation of the action

$$\delta S = -T_0 \int d\tau d\sigma\,\sqrt{-h}\,h^{\alpha\beta}\,\partial_\alpha(\delta X^\mu)\,\partial_\beta X^\nu g_{\mu\nu}.$$

But the variation of X^μ was already defined by $a^\mu_{\ \rho}X^\rho + b^\mu$. None of the components of a^μ_ν and b^μ depend on the spacetime coordinates, so their derivatives are zero, making possible to simplify the previous expression

$$\begin{aligned}
\delta S &= -T_0 \int d\tau d\sigma\,\sqrt{-h}\,h^{\alpha\beta}\,\partial_\alpha(a^\mu_{\ \rho}X^\rho + b^\mu)\,\partial_\beta X^\nu\,g_{\mu\nu} \\
&= -T_0 \int d\tau d\sigma\,\sqrt{-h}\,h^{\alpha\beta}\,a^\mu_{\ \rho}\,\partial_\alpha X^\rho\,\partial_\beta X^\nu\,g_{\mu\nu} \\
&= -T_0 \int d\tau d\sigma\,\sqrt{-h}\,a_{\nu\rho}(h^{\alpha\beta}\,\partial_\alpha X^\rho\,\partial_\beta X^\nu),
\end{aligned}$$

where the last equality is obtained using the general metric tensor $g_{\mu\nu}$ to lower the indices of $a^\mu_{\ \rho}$. At first glance it may not appear to be zero, but it is. It has been shown above that $a_{\nu\rho}$ (with both indices lowered) is antisymmetric, and the term in parentheses is symmetric, since it depends only on the intrinsic metric and the derivatives of the string coordinates. The product of a symmetric term and an antisymmetric term enables us to conclude that the variation is zero $\delta S = 0$. At this point it is finally proved that the Polyakov action is invariant against Poincaré transformations.

3.2 Local symmetries

Bosonic strings also present local symmetries. Those are transformations that depend on the metric and on the coordinates of the world-sheet.

3.2.1 Reparametrization invariance

This is a property that depends on the chosen parameters of the world-sheet. As mentioned in the derivation of the Nambu-Goto action, reparametrization invariance must be satisfied. Analogously for Polyakov action, due to the fact that both actions are equivalent.

Simply looking at the equations that give the actions (19), (25), the components of the string coordinates X^μ are transformed as scalars for different parameters $\sigma \to \sigma' = f(\sigma)$. Besides, in the Polyakov action, the metric $h^{\alpha\beta}$ varies as a second order tensor.

$$X^\mu(\tau,\sigma) = X'^\mu(\tau,\sigma')\ , \quad h_{\alpha\beta}(\tau,\sigma) = \frac{\partial f^\gamma}{\partial\sigma^\alpha}\frac{\partial f^\lambda}{\partial\sigma^\beta}\, h'_{\gamma\lambda}(\tau,\sigma').$$

This kind of symmetries shows a redundancy in this theory. Meaning that bosonic string theory actually has fewer degrees of freedom than initially expected.

3.2.2 Weyl symmetry

In addition to the aforementioned symmetries, the Polyakov action fulfills one more symmetry. Since it is defined by the intrinsic metric of the world-sheet, if a transformation were to change the scale, this metric must remain invariant. These are the so-called Weyl transformations:

$$h_{\alpha\beta}(\tau,\sigma) \to h'_{\alpha\beta}(\tau,\sigma) = e^{2\phi(\sigma)} h_{\alpha\beta}(\tau,\sigma). \tag{29}$$

Note that this transformations are local because the function $\phi(\sigma)$ depends exclusively on the world-sheet parameters, that are defined locally.

A change in the scale leaves the variation of the coordinates $X^\mu(\tau,\sigma)$ equal to zero, i.e. $\delta X^\mu(\tau,\sigma) = 0$. However, one must check whether the other quantities appearing in the action are invariant as well. Beginning with the term of the determinant

$$\sqrt{-h'} = \sqrt{-det(h'_{\alpha\beta})} = e^{2\phi(\sigma)}\sqrt{-det(h_{\alpha\beta})} = e^{2\phi(\sigma)}\sqrt{-h}. \tag{30}$$

Doing a Taylor expansion of the exponential function and considering an infinitesimal variation, all the terms of higher order in ϕ can be neglected, leaving only the linear ones. Thus, for the metric with both upper indices

$$h'^{\alpha\beta} = e^{-2\phi(\sigma)} h^{\alpha\beta} = (1 - 2\phi + \mathcal{O}(2))h^{\alpha\beta} \approx (1-2\phi)h^{\alpha\beta}\ , \quad \delta h^{\alpha\beta} = -2\phi\, h^{\alpha\beta}.$$

Now these transformed terms are multiplied and substituted using the first equality of the previous equation, which gives

$$\sqrt{-h'}h'^{\alpha\beta} = e^{2\phi(\sigma)}\sqrt{-h}e^{-2\phi(\sigma)}h^{\alpha\beta} = \sqrt{-h}h^{\alpha\beta}. \tag{31}$$

This equivalence confirms the Weyl symmetry of the Polyakov action under these transformations, i.e. independently on the scaled metric that is being used, the action S does not change.

3.3 String equation and its constraints

Knowing these properties, the gauge symmetry of this theory can be used to simplify the equations of motion. Since the actions are equivalent, their corresponding equations of motions must be equivalent as well. For the Polyakov action, it is explicitly written as

$$\frac{\partial}{\partial\sigma^\alpha}\left(\sqrt{-h}\,h^{\alpha\beta}\,\frac{\partial X^\mu}{\partial\sigma^\beta}\right)=\partial_\alpha\left(\sqrt{-h}\,h^{\alpha\beta}\,\partial_\beta X^\mu\right)=0.$$

A locally flat metric can be chosen for this purpose. Using invariance against reparametrizations, a so-called conformal gauge metric is strategically selected. If $\phi(\tau,\sigma)$ is a function of the world-sheet, the form of this metric will be

$$h_{\alpha\beta}=e^{2\phi}\eta_{\alpha\beta},$$

where, of course, one can select the condition of $\phi=0$ and reach the Minkowski flat metric. With that choice, the Polyakov action (25) simplifies significantly

$$S=-\frac{T_0}{2}\int d^2\sigma\,\partial_\alpha X\cdot\partial^\alpha X,$$

and the equation of motion for coordinates X^μ reduces to the free wave equation, in equivalence to the classical string equation (14)

$$\partial_\alpha\partial^\alpha X^\mu=0. \tag{32}$$

Note that in this expression there is an implicit sum over α, and the sacalar product is actually the relativistic dot product.

This equation now is satisfied whether the metric is set to be flat. However it may be generalized for an arbitrary $h_{\alpha\beta}$. To ensure so, the action must not change when the metric varies; i.e. the derivative of the Polyakov action respect to $h_{\alpha\beta}$ must be zero. But this derivative is a very important quantity, as it defines the *stress-energy* tensor $T_{\alpha\beta}$. Applying certain normalization conditions, just for convenience, the stress-energy tensor is expressed as

$$T_{\alpha\beta}=-\frac{2}{T_0}\frac{1}{\sqrt{-h}}\frac{\partial S}{\partial h^{\alpha\beta}}. \tag{33}$$

Now a variation of the action under a generic transformation of the metric is calculated. Restricting it only to Weyl transformations (30), the resulting variation is

$$\delta S=-\frac{T_0}{2}\int d\tau d\sigma\,\sqrt{-h}\,\delta h^{\alpha\beta}\,T_{\alpha\beta}=-\frac{T_0}{2}\int d\tau d\sigma\,\sqrt{-h}\,(-2\phi)\,h^{\alpha\beta}\,T_{\alpha\beta}.$$

From this result, for the least action principle to be satisfied, the only possibility (since $\sqrt{-h}$ and ϕ can take any arbitrary value) is provided by the condition

$$h^{\alpha\beta}T_{\alpha\beta}=0. \tag{34}$$

Returning to the previous assumption in which a flat metric is used, this equality leads to a traceless stress-energy tensor. Considering a matrix corresponding to a two-dimensional space, and knowing that the metric $h_{\alpha\beta} = \eta_{\alpha\beta}$ is symmetric, this result may be expressed explicitly as

$$h_{\alpha\beta}(X) = \eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \Longrightarrow \begin{cases} T_{01} = T_{10} = \dot{X} \cdot X' = 0, \\ \\ T_{00} = T_{11} = \frac{1}{2}(\dot{X}^2 + X'^2) = 0, \end{cases}$$

$$\boxed{\dot{X} \cdot X' = 0 \,, \quad (\dot{X}^2 + X'^2) = 0}\,. \tag{35}$$

The assumption of a flat space can only be done locally. In general, this condition cannot be extended, although these constraints must be satisfied for all the local regions considered independently of the boundary conditions of the system. This means that they are needed to fulfill the least action principle and to continue with the development of the theory.

To summarize, it can be seen that the motion of a relativistic string is just equivalent to solving a wave problem with given boundary conditions. Even now the three possibilities supposed for the endpoints (closed string, open string with Dirichtlet boundary conditions or open string with Neumann boundary conditions) led to the same constraints. By joining the wave equation (32) in its expanded form and the constraints to be fulfilled (35) compacted in one expression, the problem is finally reduced to the equations

$$\left(\partial_\tau^2 - \partial_\sigma^2\right)\, X^\mu(\tau,\sigma) = 0 \,, \quad \text{for} \quad \left(\dot{X} \pm X'\right)^2 = 0. \tag{36}$$

Chapter 4

MODE EXPANSION

Se resuelve la ecuación de ondas a partir de una solución general para las coordenadas en la hoja del mundo. Realizando una expansión en modos de Fourier el movimiento de la cuerda se descompone en un término traslacional y otro para las oscilaciones internas. Se determinan nuevos coeficientes que se relacionan con el momento lineal y permiten obtener la relación para la masa asociada a cada modo.
...

Once the constraints required for a relativistic string have been established, it enables further development and resolution of the equations that govern its nature. Following the same procedure as with the classical string, the wave equation (32) can be solved now. For this, a generic solution for two arbitrary functions f^μ, g^μ is proposed. Analogously, one is travelling to the right and the other to the left; therefore the resulting function takes the form

$$X^\mu = \frac{1}{2}\left(f^\mu(\tau+\sigma) + g^\mu(\tau-\sigma)\right).$$

Suppose an open string moving in a space-filling D-brane as a general situation. The boundary conditions are imposed in such a way that the free endpoints must satisfy Neumann conditions. Therefore at $\sigma = 0$ it leads to

$$\frac{\partial X^\mu}{\partial \sigma}(\tau, 0) = \frac{1}{2}\left(f^{\mu\prime}(\tau) - g^{\mu\prime}(\tau)\right) = 0.$$

So the derivatives of f^μ and g^μ are equal, giving that the functions can only differ in a constant. But the constant can be included if a redefinition of f^μ is done, then

$$X^\mu = \frac{1}{2}\left(f^\mu(\tau+\sigma) + f^\mu(\tau-\sigma)\right). \tag{37}$$

If the same condition is placed on the other boundary $\sigma = \pi$, it is clear that the derivative $f^{\mu\prime}$ is periodic, with natural period 2π. As a generic periodic function, it can be expanded in a Fourier series and then integrated

$$f^{\mu\prime}(u) = f_1^\mu + \sum_{n=1}^{\infty}\left[a_n^\mu\, cos(nu) + b_n^\mu\, sin(nu)\right],$$

$$f^\mu(u) = f_0^\mu + f_1^\mu\, u + \sum_{n=1}^{\infty}\left[A_n^\mu\, cos(nu) + B_n^\mu\, sin(nu)\right];$$

where in the second expression the integration constant was absorbed by the new coefficients of the expansion and f_0^μ is added.

Substituting this result in the proposed solution gives a proper expression for the fields X^μ. Using some trigonometric relations, the arguments can be separated in terms of τ or σ for more simplicity, giving rise to the expanded function

$$X^\mu(\tau,\sigma) = f_0^\mu + f_1^\mu\,\tau + \sum_{n=1}^{\infty}\left[A_n^\mu\, cos(n\tau) + B_n^\mu\, sin(n\tau)\right] cos(n\sigma).$$

This can be also written in terms of complex exponential functions. Before doing so, new constants will be defined to change them for those appearing in this equation.

So far, all equations depend on one parameter: T_0. Since the beginning of string theory the slope parameter α', started to be used. It is obtained from the proportionality relation between the angular momentum of the string and its energy squared. It is defined by

$$\alpha' = \frac{1}{2\pi\, T_0\, \hbar c}\,, \quad \text{or} \quad \alpha' = \frac{1}{2\pi\, T_0} \quad \text{in natural units } (c = \hbar = 1),$$

and it is proportional to the string tension.

By thinking of the units, one can easily see that the slope parameter should have dimensions of length squared, when natural units are used. That gives a hint about its relation with the string length l_s. Therefore it can be also defined in an alternative way

$$l_s = \hbar c\,\sqrt{\alpha'} \qquad \text{or} \qquad l_s = \sqrt{\alpha'}.$$

The constraints obtained from the stress-energy tensor (35) allows to simplify enormously the expressions that define the momentum density of the string (26) and (27). If this parameter is also inserted into the mentioned equations, they will reduce respectively to

$$\mathcal{P}^{\tau\mu} = \frac{1}{2\pi\,\alpha'}\dot{X}^\mu\,, \quad \mathcal{P}^{\sigma\mu} = -\frac{1}{2\pi\,\alpha'}X^{\mu\prime}. \tag{38}$$

These quantities are densities defined respect to the parameters. When an integration is carried out for $P^{\tau\mu}$, using the previous definition of X^μ and the periodicity condition in σ, the result gives the total momentum

$$p^\mu = \int_0^\pi \mathcal{P}^{\tau\mu}\, d\sigma = \frac{1}{2\pi\,\alpha'}\,\pi f_1^\mu \implies f_1^\mu = 2\alpha' p^\mu.$$

Finally, the name of the constant f_0^μ is changed for a more intuitive one: $f_0^\mu = x_0^\mu$. Now the parameters have been redefined and the resulting expansion gives:

$$X^\mu(\tau,\sigma) = x_0^\mu + 2\alpha' p^\mu\,\tau - i\sqrt{2\alpha'}\sum_{n=1}^{\infty}\left(a_n^{\mu *}\, e^{in\tau} - a_n^\mu\, e^{-in\tau}\right)\frac{cos(n\sigma)}{\sqrt{n}}.$$

At this point, the physical meaning of this equation can be appreciated. The fields $X^\mu(\tau,\sigma)$, which define the coordinates of the string, are dictated by the sum of two basic motions. On the left hand side, there is a constant and a term associated with the momentum. This matches the movement of the string as a whole, i.e. the translation motion.

On the other hand, the last term on the right represents the oscillations. In fact, if all the coefficients a_n^μ (coming from the Fourier series) were cancelled, the equation turns out to be the equation of motion of a point particle located at the center. This expansion introduces the internal modes, that represent a significant difference between the string and the particle theories.

Moreover, other coefficients can be used to define this function as a sum over the positive and negative integers of a single exponential. These are

$$\alpha_0^\mu = \sqrt{2\alpha'}\, p^\mu, \tag{39}$$

$$\alpha_n^\mu = a_n^\mu \sqrt{n}\,, \quad \alpha_{-n}^\mu = a_n^{\mu *} \sqrt{n} \qquad \text{for} \quad n \geq 1; \tag{40}$$

where $\alpha_{-n}^\mu = (\alpha_n^\mu)^*$ is an important property of these coefficients.

In order to maintain the form of the equation as a sum of a translational motion and the oscillations, it is convenient to keep outside the summation the term for $n = 0$ when introducing these new coefficients. Thus this leads to:

$$X^\mu(\tau, \sigma) = x_0^\mu + \sqrt{2\alpha'}\, \alpha_0^\mu\, \tau + i\sqrt{2\alpha'} \sum_{n \neq 0} \left(\frac{1}{n}\, \alpha_n^\mu\, e^{in\tau} \right) cos(n\sigma). \tag{41}$$

4.1 Light-cone coordinates

The equation obtained above fulfills the wave equation, but the constraints must still be imposed. For that purpose the *world-sheet light-cone coordinates* are introduced. As in general relativity, these are defined as a linear combination of the parameters τ, σ, meaning a change in the coordinates used in the world-sheet

$$\sigma^\pm = \tau \pm \sigma \Longrightarrow \begin{cases} \tau = \frac{1}{2}(\sigma^+ + \sigma^-), \\ \sigma = \frac{1}{2}(\sigma^+ - \sigma^-). \end{cases}$$

If closed strings are considered now, the previous generic solution proposed for X^μ (see equation (37)) corresponds to the sum of the right moving function that depends on σ^- and the left moving function with argument σ^+, i.e. $X^\mu = X_R^\mu(\sigma^-) + X_L^\mu(\sigma^+)$. Giving their own two solutions after computing the Fourier series

$$X_R^\mu(\sigma^-) = \frac{1}{2} x_0^\mu + \frac{1}{2}\alpha'\, p^\mu\, \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu\, e^{-in\sigma^-},$$

$$X_L^\mu(\sigma^+) = \frac{1}{2} x_0^\mu + \frac{1}{2}\alpha'\, p^\mu\, \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \tilde{\alpha}_n^\mu\, e^{-in\sigma^+}.$$

Note that here two different coefficients are defined, one for each direction of propagation.

With the definition of these new parametrization, the wave equation and the constraints (36) are respectively written in the following way

$$\partial_+\partial_- X^\mu = 0,$$

$$(\partial_+ X^\mu)^2 = (\partial_- X^\mu)^2 = 0.$$

Reviewing the constraints, further conditions that must be fulfilled by the coefficients of the Fourier modes and the momenta p^μ will be obtained. One can take now the derivatives to analyze them, starting with $\partial_- X^\mu$

$$\partial_- X^\mu = \partial_- X^\mu_R = \frac{1}{2}\alpha' p^\mu + \sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \alpha^\mu_n \, e^{-in\,\sigma^-} = \sqrt{\frac{\alpha'}{2}} \sum_{n=0}^{\infty} \alpha^\mu_n \, e^{-in\,\sigma^-},$$

$$(\partial_- X^\mu)^2 = \frac{\alpha'}{2} \sum_{n,k} \alpha_n \cdot \alpha_k \, e^{-i(n+k)\,\sigma^-} = \frac{\alpha'}{2} \sum_{n,m} \alpha_n \cdot \alpha_{m-n} \, e^{-im\,\sigma^-}.$$

In the first relation, to obtain the second equality it is assumed that p^μ and α' are related with α_0 by

$$\alpha^\mu_0 = \sqrt{\frac{\alpha'}{2}} p^\mu. \tag{42}$$

Meanwhile for the second equation what proceeds is to rename the sum of modes. The quadratic combination of oscillators is a remarkable operator that gives rise to the so-called *Virasoro modes*

$$L_m = \frac{1}{2} \sum_{n\in\mathbb{Z}} \alpha_n \cdot \alpha_{m-n}. \tag{43}$$

The same steps can be followed with the derivative with respect to σ^+. The results are equivalent, giving the constants

$$\tilde{\alpha}^\mu_0 = \sqrt{\frac{\alpha'}{2}} p^\mu\,, \quad \tilde{L}_m = \frac{1}{2} \sum_{n\in\mathbb{Z}} \tilde{\alpha}_n \cdot \tilde{\alpha}_{m-n}. \tag{44}$$

Note that clearly the zero-order coefficients coincide in both representations $\alpha^\mu_0 = \tilde{\alpha}^\mu_0$.

Then the constraints of the stress-energy tensor in this basis are written in terms of the Virasoro modes as

$$\begin{cases} T_{--} = (\partial_- X^\mu)^2 = \alpha' \sum_m L_m \, e^{-im\,\sigma^-} = 0, \\[2ex] T_{++} = (\partial_+ X^\mu)^2 = \alpha' \sum_m \tilde{L}_m \, e^{-im\,\sigma^+} = 0, \\[2ex] T_{+-} = T_{-+} = 0. \end{cases}$$

This explicitly shows that the constraints that must be satisfied imply having an infinite number of null coefficients $L_m = \tilde{L}_m = 0$, for $m \in \mathbb{Z}$.

4.2 Mass formula for a bosonic string

From the definition of L_m and $\tilde{L}_m$ one can note that the corresponding values for $m = 0$ include the momentum squared. This is an special product, because from relativity it is known that the rest mass can be calculated from it

$$M^2 = -p^\mu p_\mu.$$

The mass of the particle arises from the appearance of the four-momentum in the mode expansion. The linear momentum is a constant of motion, so the mass of the particle cannot change. Due to the bindings of the system, the momentum of the string depends on the oscillations, and therefore the mass as well. The state of the string and its mass are clearly related. If the string does not interact, the oscillation mode stays the same, therefore its momentum and mass will remain constant.

Depending on whether one considers open or closed strings, two moving modes are considered or only one. For closed string left moving and right moving modes contribute; meanwhile for the open strings, a mode propagating in one direction reaches the endpoint and is reflected. Thus after all, both modes are essentially the same in that case. This arises from the fact that for open strings, σ is defined with values between 0 and π; meanwhile for closed strings there are no endpoints so σ is periodic with values between 0 and 2π. This gives different definitions on the coefficients α_0^μ for open and closed strings.

The constraints obtained in the previous section must be also satisfied for L_0 and $\tilde{L}_0$. But the definition of the Virasoro operators shows that these quantities depend on α_0^μ and $\tilde{\alpha}_0^\mu$ respectively. Equally, they depend on the linear momenta of the string, i.e. p^μ. Then the effective mass of a string could be given in terms of L_0 and $\tilde{L}_0$.

For an open string, in the mode expansion there is only one mode, because left and right moving modes are forced to combine into standing waves. The corresponding mass formula is then

$$M^2 = \frac{1}{\alpha'} \sum_{n=1}^{\infty} \alpha_n \cdot \alpha_{-n}. \tag{45}$$

However, for the closed string there can be left-moving and right-moving modes, leading to the following

$$M^2 = \frac{2}{\alpha'} \sum_{n=1}^{\infty} (\alpha_n \cdot \alpha_{-n} + \tilde{\alpha}_n \cdot \tilde{\alpha}_{-n}). \tag{46}$$

These are the mass-shell relations of the string. From these expressions the corresponding effective mass related to each state of the string can be calculated.

Chapter 5

QUANTIZATION OF THE STRINGS

Mediante dos procedimientos distintos se llega a la cuantización de las cuerdas. La cuantización canónica surge de la transformación directa de la teoría clásica, pero predice estados de norma negativa que deben ser eliminados. Para las coordenadas en el cono de luz se pierde la simetría de Lorentz, por lo que debe ser impuesta después de cuantizar. En ambos casos se establecen las ligaduras correspondientes a la dimensionalidad del espacio-tiempo para las cuerdas bosónicas.

In order to continue with the development of this theory, the conformal fields X^μ need to be quantized. There are several alternatives, but here two of them are going to be performed: canonical quantization and light-cone quantization. Both should give the same outcome. However, weak arguments are used. The rigorous way to prove that the results are true is by means of conformal field theory [2, 3, 4]. This derivation is rigorous and verifies that the conclusions reached are correct.

5.1 Canonical quantization

Canonical quantization is the one that emerges from transforming the classical theory directly to quantum. It is reached by changing the Poisson brackets from the Hamiltonian formalism into commutators. In the same way, the fields that describe the coordinates and the momentum will be treated as operators. As they are functions that depend on the modes coming from the Fourier expansion, the coefficients of this expansion must become operators as well.

The canonical quantization should be started by promoting the fields X^μ and canonical momenta $\mathcal{P}_\mu = 1/(2\pi\alpha')\dot{X}^\mu$ to operators. These must obey the following commutation relations

$$[X^\mu(\tau,\sigma),\, \mathcal{P}_\nu(\tau,\sigma')] = i\delta(\sigma-\sigma')\delta^\mu_\nu,$$
$$[X^\mu(\tau,\sigma),\, X^\nu(\tau,\sigma')] = [\mathcal{P}_\mu(\tau,\sigma),\, \mathcal{P}_\nu(\tau,\sigma')] = 0,$$

being these the results obtained from transforming the Poisson brackets into commutators.

The coefficients of the expansion will become operators analogously. So the only non-zero commutation relations obtained for the modes α, $\tilde{\alpha}$ and the constants x^μ, p^μ are the following

$$[\hat{x}^\mu, \hat{p}_\nu] = i\delta^\mu_\nu,$$

$$[\hat{\alpha}_n^\mu,\, \hat{\alpha}_m^\nu] = [\hat{\tilde{\alpha}}_n^\mu,\, \hat{\tilde{\alpha}}_m^\nu] = n\eta^{\mu\nu}\,\delta_{n+m,\,0}.$$

The relations for modes $\hat{\alpha}_m^\mu$ and $\hat{\tilde{\alpha}}_m^\mu$ are reminiscent of the commutation relations for the creation and annihilation operators of the quantum harmonic oscillator. In quantum mechanics the second quantization introduces a simple notation just by using these two operators. With the same purpose, having operators that share exactly the same structure as creation and annihilation operators, gives a familiar view of string theory in terms of second quantization. In order to use operators constructed in such a way that they satisfy the creation/annihilation relations, the operators $\hat{a}_m^\mu$ and $\hat{a}_m^{\mu\dagger}$ are defined

$$\hat{a}_m^\mu = \frac{\alpha_m^\mu}{\sqrt{m}}\,, \qquad \hat{a}_m^{\mu\dagger} = \frac{\alpha_{-m}^{\mu\dagger}}{\sqrt{m}}; \tag{47}$$

$$[\hat{a}_m^\mu,\, \hat{a}_m^{\nu\dagger}] = [\hat{\tilde{a}}_m^\mu,\, \hat{\tilde{a}}_m^{\nu\dagger}] = \eta^{\mu\nu}\delta_{m,n} \qquad \text{for} \quad m, n > 0. \tag{48}$$

Now, the ground state $|0\rangle$ can be described as the state obtained under the action of all the lowering operators, $\hat{a}_m^\mu$

$$\hat{a}_m^\mu|0\rangle = 0 \qquad \text{for} \quad m > 0.$$

Equally, any generic physical state $|\phi\rangle$ can be also defined as the state resulting after the action of raising operators, $\hat{a}_m^{\mu\dagger}$, over the ground state

$$|\phi\rangle = \hat{a}_{m_1}^{\mu_1\dagger}\,\hat{a}_{m_2}^{\mu_2\dagger}\ldots\hat{a}_{m_n}^{\mu_n\dagger}|0; k^\mu\rangle.$$

This method is the one followed to construct the Fock space: adding creation operators to the ground state, i.e., a representation of the new particles created from the vacuum initial state. Each state of the Fock space corresponds to a different excitation of the string. These physical states will also be eigenstates of the momentum operator. Therefore their corresponding eigenvalues in the momentum space, introduce new quantum numbers

$$\hat{p}^\mu|\phi\rangle = k^\mu|\phi\rangle.$$

Another aspect one should notice, is that there are some exceptional cases in the definition of the commutators for the creation/annihilation operators (48), where the commutator is negative. This occurs when $\mu = \nu = 0$ due to the nature of the metric, as Minkowskian flat space is being considered

$$\left[\hat{a}_m^0,\, \hat{a}_n^{0\dagger}\right] = \eta^{00}\delta_{m,n} = -\delta_{m,n}.$$

Under these conditions, the predicted states lead to results that make no physical sense. Considering a state $|\psi\rangle = \hat{a}_m^{0\dagger}|0; k^\mu\rangle$, for $m > 0$, its norm can be calculated by

$$\||\psi\rangle\|^2 = \langle 0|\hat{a}_m^0\hat{a}_m^{0\dagger}|0\rangle = \langle 0|\left[\hat{a}_m^0,\, \hat{a}_m^{0\dagger}\right]|0\rangle = -\langle 0|0\rangle,$$

where the condition $\hat{a}_m^\mu|0\rangle = 0$ was used for the ground state.

The result shows that there can be states with negative norm: if $\langle 0|0\rangle$ is defined to be positive, all the states having that form must have negative norm; on the contrary, if $\langle 0|0\rangle$ is negative, there will be other states whose norm is less than zero. States with negative norm are meaningless results when thinking of a physical interpretation. Since the norm of a state is related to a probability, its values must belong to the interval $[0,1]$ to have any real interpretation.

The reasoning followed for the canonical quantization predicts these states with no physical sense, that are called *ghost states*. To continue with the development of the theory, these resulting unphysical states must be removed. However, avoiding them leads to new restrictions on the number of dimensions of the background space-time in which our theory is being defined.

The definition of the physical states must take into account the constraints. In the classical theory the constraints where written as $L_m = 0$ for all m, which arise from the vanishing of stress-energy tensor. This generators will also become operators as they are defined in terms of the α_m modes. When doing the quantization, normal order of the operators is chosen as a convention coming from Quantum Field theory. It is denoted by (:) and represents the following:

$$: \alpha_i \cdot \alpha_j := \begin{cases} \alpha_i \cdot \alpha_j & \text{for} \quad i \leq j, \\ \alpha_j \cdot \alpha_i & \text{for} \quad i > j, \end{cases}$$

which says that the operator with lower index is placed on the left of the operator with higher index. When the two operators do not commute, the normal ordering may add additional constants to the quantum expressions that must be determined using other physical arguments.

According to the form of the L_m generators given by (43), they can be transformed using modes operators that emerge after quantization, but setting the normal order. Leading to the following operators that define the Virasoro algebra

$$\hat{L}_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} : \hat{\alpha}_{m-n} \cdot \hat{\alpha}_n : . \tag{49}$$

With this description the operators are well defined (no additional constants are required from the commutation relations), except the $m = 0$ case. In the equation given for $\hat{L}_0$ there is still ambiguity, even after setting the normal order

$$\hat{L}_0 = \frac{1}{2} \sum_{n=-\infty}^{\infty} : \hat{\alpha}_{-n} \cdot \hat{\alpha}_n := \frac{1}{2}\hat{\alpha}_0^2 + \sum_{n=1}^{\infty} \hat{\alpha}_{-n} \cdot \hat{\alpha}_n.$$

The ambiguity is shown in the definition of $\hat{L}_0$ because $\hat{\alpha}_n$ y $\hat{\alpha}_{-n}$ do not commute and it is not clear which order is the correct one. Also, since they do not commute the result will not be the same, but differ by a commutation constant. Then one of the orders is assumed, such that if the arrangement of the operators is switched, the extra constant which arises from the commutation rules appears. These constant is added to ensure that all possible ordered representations leads to the same result and its value should be calculated.

Depending on the value given to this constant, that will be called a, different results are obtained. Apparently each chosen value of the commutation constant give rise to a different theory. Since these operators can be used in the notation of numerous equations and definitions, the value of a will affect all of these functions.

The commutation relations for operators $\hat{L}_m$ are therefore obtained from the corresponding relations for $\hat{\alpha}_n$, and the result can be written by

$$[\hat{L}_m, \hat{L}_n] = (m-n)\hat{L}_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m,-n}, \tag{50}$$

where c is a constant called the central charge, which appears, as the constant a, from the ambiguities in the operator ordering. In bosonic string theory, it is equal to the space-time dimension. The resulting algebra is the well known quantum Virasoro algebra. Also note that for $m = -1, 0, 1$ the term containing c vanishes, what defines a subalgebra containing the elements $\{\hat{L}_{-1}, \hat{L}_0, \hat{L}_1\}$ that fulfill the relations

$$[\hat{L}_m, \hat{L}_n] = (m-n)\hat{L}_{m+n}. \tag{51}$$

5.1.1 Constraints and physical states

The classical constraints implies that $L_m = \tilde{L}_m = 0$ for all m, and consequently $L_0 = 0$. But in the quantized theory, it cannot be established for an operator $\hat{L}_n = 0$ or $\hat{L}_n|\phi\rangle = 0$ for all physical states $|\phi\rangle$. Instead, this constraint is introduced by calculating the expected value between two physical states and making that quantity zero, which is sufficient condition to obtain

$$\langle\phi'|\hat{L}_n|\phi\rangle = 0 \quad \Longrightarrow \hat{L}_n|\phi\rangle = 0 \quad \text{for} \quad n > 0. \tag{52}$$

If one focuses in the $n = 0$ constraints, the ambiguity present on these terms due to normal ordering, introduces constants that yield different constraint equations. For some constant a, the equality reached for an open string is

$$(\hat{L}_0 - a)|\phi\rangle = 0. \tag{53}$$

Meanwhile, for a closed string with right moving and left moving modes, represented respectively by the corresponding operators $\hat{L}_0$ and $\hat{\tilde{L}}_0$, the relations are

$$(\hat{L}_0 - a)|\phi\rangle = 0 \ , \quad (\hat{\tilde{L}}_0 - a)|\phi\rangle = 0. \tag{54}$$

These are the so-called mass-shell condition for open and closed string. In agreement with the previous section, the normal ordering also have a correction for the mass formula. First one can define the number operator $\hat{N}$ taking into account the order of operators

$$\hat{N} = \sum_{n=1}^{\infty} : \hat{\alpha}_{-n} \cdot \hat{\alpha}_n := \sum_{n=1}^{\infty} n : \hat{a}_n^{\dagger} \cdot \hat{a}_n : . \tag{55}$$

The resulting mass formulas describing the states of open and closed strings are given, respectively, by

$$\alpha' M^2 = \frac{1}{\alpha'} \sum_{n=1}^{\infty} : \hat{\alpha}_{-n} \cdot \hat{\alpha}_n : -a = \hat{N} - a, \tag{56}$$

$$\frac{4}{\alpha'} M^2 = \sum_{n=1}^{\infty} : \hat{\alpha}_{-n} \cdot \hat{\alpha}_n : -a = \sum_{n=1}^{\infty} : \hat{\hat{\alpha}}_{-n} \cdot \hat{\hat{\alpha}}_n : -a \quad \text{or} \quad \hat{N} - a = \hat{\hat{N}} - a, \tag{57}$$

where the second equation is for the closed string, and can be expressed using $\hat{N}$ as number operator for the right movers and $\hat{\hat{N}}$ as the number operator for the left movers.

One interesting relation between the number operators is achieved by doing the subtraction of the two conditions for each moving operator (54):

$$(\hat{L}_0 - a - \hat{\hat{L}}_0 + a)|\phi\rangle = (\hat{L}_0 - \hat{\hat{L}}_0)|\phi\rangle = 0.$$

Analyzing the definition of the operators $\hat{N}$ and $\hat{L}_0$, from the above equality the following is obtained

$$\hat{N} = \hat{\hat{N}}. \tag{58}$$

One observes a relation of equivalence between the left moving and right moving modes, known as *level matching condition* of the bosonic string. It implies that the direction in which modes propagate in a closed string is not relevant. The same excited states can be obtained by modes traveling to the left and to the right, as long as the value of the number operator is the same.

From the mass formula and the number operator, the mass spectrum of the string may be constructed. Giving different values to n returns the corresponding excited states, which together form the desired spectrum:

$$\begin{aligned}
&\alpha' M^2 = -a && \text{for} \quad n = 0 \quad \text{(ground state)},\\
&\alpha' M^2 = -a + 1 && \text{for} \quad n = 1 \quad \text{(first excited state state)},\\
&\alpha' M^2 = -a + 2 && \text{for} \quad n = 2 \quad \text{(second excited state)},\\
&\quad\vdots
\end{aligned}$$

5.1.2 Spurious States

The states that satisfy the mass-shell condition (i.e. equation (53)) and that are orthogonal to all the other physical states are called *spurious states*. They can be though of as vacuum states. A general spurious state can be written as a linear combination of states using as coefficients the generator operators for the Virasoro algebra

$$|\psi\rangle = \sum_{n=1}^{\infty} \hat{L}_{-n}|\chi_n\rangle, \tag{59}$$

where $|\chi_n\rangle$ is a state that also verifies the modified mass-shell condition. Thus the two conditions to have spurious states are: the orthogonality and that they are eigenstates of $\hat{L}_0$. This type of states are not very common in physics, so they are one of the characteristic phenomena of strings.

The equation (53) can be applied multiplying the spurious state $|\psi\rangle$ by the term $(\hat{L}_0 - a)$. After some calculations, using the previous definition of $|\psi\rangle$ and the commutation relations for operators $\hat{L}_m$, in particular for $[\hat{L}_0, \hat{L}_{-n}] = n\,\hat{L}_{-n}$, one arrives to

$$\begin{aligned}
(\hat{L}_0 - a)|\psi\rangle &= \hat{L}_0 \left(\sum_{n=1}^{\infty} \hat{L}_{-n}|\chi_n\rangle\right) - a\left(\sum_{n=1}^{\infty} \hat{L}_{-n}|\chi_n\rangle\right) = \\
&= \sum_{n=1}^{\infty} \left([\hat{L}_0, \hat{L}_{-n}] + \hat{L}_{-n}\hat{L}_0\right)|\chi_n\rangle - \sum_{n=1}^{\infty} a\,\hat{L}_{-n}|\chi_n\rangle = \\
&= \sum_{n=1}^{\infty} \left(\hat{L}_{-n}n + \hat{L}_{-n}\hat{L}_0 - \hat{L}_{-n}a\right)|\chi_n\rangle = 0 \\
&\Longrightarrow \left(\hat{L}_0 - a + n\right)|\chi_n\rangle = 0.
\end{aligned}$$

The verification of the last equality allows determining the value of the constant a for certain state with fixed n. This requirement is one of the new constraints needed to eliminate non-physical states.

Being orthogonal to all physical states is one of the most interesting aspects of these states. That is because they do not contribute when they are combined to any other physical state. Meaning that if a spurious state is added to any physical state, the result is the same, due to the orthogonality.

In order to prove that any spurious state $|\psi\rangle$ is orthogonal to all physical states, suppose any physical state $|\phi\rangle$. Making use of the fact that $\hat{L}_{-n}^{\dagger} = \hat{L}_n$ and the restrictions (52) for $n > 0$, when doing the scalar product it is shown that

$$\langle\phi|\psi\rangle = \sum_{n=1}^{\infty}\langle\phi|\hat{L}_{-n}|\chi_n\rangle = \sum_{n=1}^{\infty}\left(\langle\chi_n|\hat{L}_n|\phi\rangle\right)^* = 0.$$

Moreover, as spurious states are orthogonal to all physical states, if a spurious state $|\psi\rangle$ is also assumed to be physical state, it must be orthogonal to itself. Concluding that it should be a zero-norm state

$$\||\psi\rangle\|^2 = \langle\psi|\psi\rangle = 0.$$

When dealing with the problem of eliminating the ghosts, the study of spurious states is of great interest. Negative norm states are also orthogonal to the positive norm states, but they are not physical and must be removed. In this sense, one can consider them also as spurious states. If one is able to make all spurious states physical, then all spurious states will have zero norm and the negative-norm states would have hopefully disappeared.

This rough idea makes it possible to arrive at the optimal situation in which ghost states are eliminated. However, it can also be proved mathematically by following a more rigorous reasoning.

5.1.3 Removing ghost states

To remove the so-called ghost states, some constant values need to be fixed. Analyzing the spurious states allows to establish a and c. For simplicity, it seems logical to start with the lower level states. Hence, level 1 spurious states are the ones used in the beginning because it is the simplest expression that can be used to find a. For that purpose it is defined: $|\psi\rangle = \hat{L}_{-1}|\chi_1\rangle$.

For $|\psi\rangle$ to be a physical state, and making use of the conditions involved by $|\chi_1\rangle$, is it imposed:

$$(\hat{L}_0 - a + 1)|\chi_1\rangle = 0\,, \qquad \hat{L}_{m>0}|\chi_1\rangle = 0 \Longrightarrow \hat{L}_1|\chi_1\rangle = 0, \tag{60}$$

$$(\hat{L}_0 - a)|\psi\rangle = 0\,, \qquad \hat{L}_{m>0}|\psi\rangle = 0 \Longrightarrow \hat{L}_1|\psi\rangle = 0; \tag{61}$$

where the last conditions are the particular cases for $m = 1$. By developing it through the use of the commutators and their corresponding relationships, one arrives at

$$\hat{L}_1|\psi\rangle = \hat{L}_1(\hat{L}_{-1}|\chi_1\rangle) = [\hat{L}_1, \hat{L}_{-1}]|\chi_1\rangle = 2\hat{L}_0|\chi_1\rangle = 2(a-1)|\chi_1\rangle = 0.$$

So finally, to have that last term equals zero, the only solution is $a = 1$. This restriction represents the boundary between the spaces constituted by the spurious and ghost states.

Recalling the origin of constant a, this assumption establishes the commutation relations between operators $\hat{\alpha}_{-n}$ and $\hat{\alpha}_n$. The given solution states that the correction arising from the normal ordering is, indeed, a constant $a = 1$. Now the ambiguity problem in $\hat{L}_0$ has come to an end.

In order to determine the value of the central charge c, i.e., the dimension of the background space; it is needed to work with two level spurious state. Since any operator $\hat{L}_{-n}$

with $n \leq 1$ can be written in terms of $\hat{L}_{-1}$ and $\hat{L}_{-2}$ as a linear combination (Wray [2], pp 66-67), general spurious states can be also described in terms of those two operators. But to ensure that the resulting norm is zero, it is also needed a constant, denoted as γ. Then the spurious state is given by

$$|\psi\rangle = (\hat{L}_{-2} + \gamma \hat{L}_{-1}\hat{L}_{-1})|\chi_2\rangle.$$

As in the previous assumption, the spurious state $|\psi\rangle$ must satisfy the same conditions (61), and $|\chi_2\rangle$ has the following relations

$$(\hat{L}_0 - a + 2)|\chi_2\rangle = 0\,, \qquad \hat{L}_{m>0}|\chi_2\rangle = 0$$

Again, following the same procedure based on the particular constraint for $m = 1$:

$$\begin{aligned}
\hat{L}_1|\psi\rangle &= \hat{L}_1(\hat{L}_{-2} + \gamma \hat{L}_{-1}\hat{L}_{-1})|\chi_2\rangle = \left([\hat{L}_1, \hat{L}_{-2} + \gamma \hat{L}_{-1}\hat{L}_{-1}]\right)|\chi_2\rangle = \\
&= \left(3\hat{L}_{-1} + 2\gamma \hat{L}_0\hat{L}_{-1} + 2\gamma \hat{L}_{-1}\hat{L}_0\right)|\chi_2\rangle = \left(3\hat{L}_{-1} + 4\gamma \hat{L}_{-1}\hat{L}_0 + 2\gamma \hat{L}_{-1}\right)|\chi_2\rangle = \\
&= \left(3\hat{L}_{-1} - 4\gamma \hat{L}_{-1} + 2\gamma \hat{L}_{-1}\right)|\chi_2\rangle = (3 - 2\gamma)\hat{L}_{-1}|\chi_2\rangle = 0.
\end{aligned}$$

Note that here the commutation relations (51) were used in the first step and then for $[\hat{L}_{-1}, \hat{L}_0]$. Likewise from the above constraints, some results such as $\hat{L}_1|\chi_2\rangle = 0$ and $\hat{L}_0|\chi_2\rangle = (a-2)||\chi_2\rangle$ were also employed, knowing now that $a = 1$, and therefore $\hat{L}_0|\chi_2\rangle = -|\chi_2\rangle$.

From the last equality one obtains the value of γ, being necessarily $\gamma = 3/2$. Thus, for a two level spurious state, that is as well a physical state, this value must be imposed in order to maintain the zero-norm condition.

However, the aim of this calculation is to fix c. For the two level spurious states, there were two undefined constants, so other equation will be needed. As already discussed, the subalgebra formed by the operators $\{\hat{L}_{-1}, \hat{L}_0, \hat{L}_1\}$ does not have the central charge constant in its commutation relations. A greater value of m is required to obtain an equation of similar form to (51), where c appears. Following an analogous procedure to the previous ones, in this instance the constraint with $m = 2$ will be used, where in the commutation relation c will show off

$$\begin{aligned}
\hat{L}_2|\psi\rangle &= \hat{L}_2(\hat{L}_{-2} + \frac{3}{2}\hat{L}_{-1}\hat{L}_{-1})|\chi_2\rangle = \left([\hat{L}_2, \hat{L}_{-2} + \frac{3}{2}\hat{L}_{-1}\hat{L}_{-1}]\right)|\chi_2\rangle = \\
&= \left(13\hat{L}_0 + 9\hat{L}_{-1}\hat{L}_1 + \frac{c}{2}\right)|\chi_2\rangle = \left(-13 + \frac{c}{2}\right)|\chi_2\rangle = 0 \\
&\Longrightarrow \boxed{c = 26}.
\end{aligned}$$

Finally, it is concluded that if the state $|\psi\rangle$ is to be spurious and physical simultaneously, it must be defined in a space with $c = 26$. Then fixing the value of this constant eliminates the non-physical states of the boundary, and hence it is argued that they are hopefully removed.

This constant is equal to the dimensionality of the space in which the strings live. Meaning that the background space-time where the bosonic theory is defined has 26 dimensions, where one must be temporal and the remaining 25 are spacial. There can be bosonic string theories with other values for this constants, like $a \leq 1$ and $c \leq 25$. These are the so-called non-critial theories. On the other hand, if it is required to remove the ghost states with negative norm, then restricting the values of these constants to $a = 1$, $\gamma = 3/2$ and $c = 26$ is needed.

5.2 Light-cone quantization

In the previous quantization process, Lorentz invariance was first imposed. Then quantization was proceeded directly from the classical expression, which gave rise to ghost states of negative norm, that had to be eliminated. In the present derivation of the quantized string, a new gauge defined on the light-cone coordinates will be used. Taking advantage of the symmetry under reparameterization, it has been proved that a change in the spacetime light-cone coordinates of the form $\sigma^\pm \to \tilde{\sigma}^\pm = \xi^\pm(\sigma^\pm)$, will not change the problem. Notice that in previous sections it have been used the light-cone coordinates of the world-sheet described by the string, while now the coordinates to be worked on are the light-cone coordinates of the background spacetime itself.

Choosing the right parameterization is the way to avoid the occurrence of negative norm states. As usual, the coordinates of the light-cone are defined as a linear combination of the time coordinate with a spatial coordinate. The selection of this transverse coordinate is arbitrary and in this model the $D-1$ coordinate is picked. Therefore, the light-cone coordinates X^+, X^-, along with the other $D-2$ spatial coordinates compound the set that describes the spacetime $\{X^-, X^+, X^i\}_{i=1}^{D-2}$

$$X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^{D-1}).$$

In the resulting system, vector operations like the dot product or the raising and lowering indices follow the rules given by

$$A \cdot B = -A^+B^- - A^-B^+ + \sum_{i=1}^{D-2} A^i B^i, \tag{62}$$

$$A_+ = -A^-, \tag{63}$$

$$A_- = -A^+, \tag{64}$$

$$A_i = A^i. \tag{65}$$

Since the two light-cone coordinates are treated differently from the rest, there is no Lorentz invariance manifest. The symmetry of the group that owns the corresponding symmetries of the Lorentz group is $SO(1, D-1)$, which was the group initially considered. But taking the light-cone coordinates causes the dimensions that remain invariant to be reduced. Now the representations becomes those of the group $SO(D-2)$.

The residual gauge symmetry allows to reparametrize τ and σ, as they are functions of σ^+ and σ^-. According to the above mentioned change in these coordinates, their new definition have the generic form

$$\tau \to \tilde{\tau} = \frac{1}{2}(\tilde{\sigma}^+ + \tilde{\sigma}^-) = \frac{1}{2}(\xi^+(\sigma^+) + \xi^-(\sigma^-)),$$
$$\sigma \to \tilde{\sigma} = \frac{1}{2}(\tilde{\sigma}^+ - \tilde{\sigma}^-) = \frac{1}{2}(\xi^+(\sigma^+) - \xi^-(\sigma^-)).$$

The given function for $\tilde{\tau}$ has the same form as the proposed solutions to the wave equation. Furthermore, for the Minkowski metric in D dimensions, the space-time coordinates $X^\mu(\tau,\sigma)$ must also satisfy the wave equation. In terms of the light-cone coordinates, this equation is written as follows

$$\partial_+\partial_-\tilde{\tau} = 0 \qquad \text{and} \qquad \partial_+\partial_- X^\mu(\tau,\sigma) = 0.$$

This implies that from the residual gauge freedom, a reparametrization relating $\tilde{\tau}$ to $X^\mu(\tau,\sigma)$ can be chosen, such that the bosonic theory is considerably simplified:

$$X^+ = x^+ + l_s^2 p^+ \tilde{\tau}. \tag{66}$$

This is the *lightcone gauge*, for some arbitrary constant x^+. Since $\tilde{\tau}$ fulfills the wave equation, writing the coordinates in terms of parameter $\tilde{\tau}$ is a convenient way to enforce that the equations of motion for the strings are satisfied. Taking into account that the wave equation and its restrictions fulfill the properties of linearity, the defined gauge represents X^+ as linear function of $\tilde{\tau}$.

The effects of setting this gauge can be observed comparing equation (66) with the mode expansion of X^+ for an open string

$$X^+(\tau,\sigma) = x^+ + l_s^2 p^+ \tau + \sum_{n\neq 0} \frac{1}{n}\alpha_n^+ e^{-in\tau} cos(n\sigma).$$

The contribution of all α_n^+ modes has been removed for $n \neq 0$. Therefore, this gauge reduces the number of constraints, since in the previous study infinite L_n modes were required to be equal to zero. Analogously, for the closed string, the result would be similar but with both, left moving and right moving modes equal to zero, i.e. $\alpha_n^+ = (\alpha_n^+)^\dagger = 0$.

Then for the expansion of X^- the procedure is non-trivial. Suppose an open string with Neumann boundary conditions. Then applying the Virasoro constraints in the mode expansion gives

$$X^-(\tau,\sigma) = x^- + l_s^2 p^- \tau + \sum_{n\neq 0} \frac{1}{n}\alpha_n^- e^{-in\tau} cos(n\sigma)$$

with

$$\alpha_n^- = \frac{1}{p^+ l_s}\left(\frac{1}{2}\sum_{i=1}^{D-2}\sum_{m=-\infty}^{\infty} :\alpha_{n-m}^i\alpha_m^i: -a\,\delta_{n,0}\right).$$

Hence, only the zero modes are relevant for X^- and X^+, making possible to describe the bosonic strings just using transverse oscillators. As the string is not necessarily living in the plane formed by $X^0 - X^{D-1}$, these are not literally transversal modes, but they are referred this way because they correspond to the components that are not contained in the light-cone.

With the recent definitions, the mass formula in light-cone coordinates is rewritten using the corresponding dot product rule (62). Therefore it is given by

$$M^2 = -p^\mu p_\mu = 2p^+p_- - \sum_{i=1}^{D-2} p^i p^i.$$

The previous definition of α_n^- for an open string, can be used to determine the momentum associated to the light-cone coordinates, i.e. the product p^+p_-. One can see that for $n = 0$

$$\alpha_0^- = \frac{1}{p^+ l_s}\left(\frac{1}{2}\sum_{i=1}^{D-2}\sum_{m=-\infty}^{\infty} :\alpha_{-m}^i\alpha_m^i: -a\right) = \frac{1}{p^+ l_s}\left[\frac{1}{2}(\alpha_0^i)^2 + N - a\right]. \tag{67}$$

But it follows from its definition that $\alpha_0^- \equiv p^- l_s$, and similarly $\alpha_0^i \equiv p^i l_s$. Plugging this into the last expression and moving to one side all the terms containing momentum components, the mass-shell relation for an open string in the light-cone gauge emerges as

$$l_s^2 p^+ p^- = \frac{1}{2}(p^i)^2 l_s^2 + (N-a) \Longrightarrow 2p^+p^- - (p^i)^2 = -p^\mu p_\mu = \frac{2}{l_s^2}(N-a),$$

$$M^2 = \frac{2}{l_s^2}(N-a). \tag{68}$$

These are the so-called level matching conditions of the string, and they are functions of the transverse oscillators, as can be seen from the definition of $\hat{N}$ in equation (55). Actually the general solution can be described by $2(D-2)$ transverse modes α_n^i, and for the closed string $\tilde{\alpha}_n^i$ too.

The excitation of the string can be predicted from the above equation. As mentioned, when using the light-cone gauge it only depends on the transversal modes α_n^i for $i = 1, ..., D-2$. The 00 term is therefore excluded, which implies that the negative sign of the commutation relations will not contribute. Consequently, the light-cone quantization does not include the negative-norm states. Now the theory does not predict non-physical states, but one cannot forget about the Lorentz symmetry. Manifest invariance is lost, so it must be re-imposed by hand once the ghost states have disappeared.

5.2.1 Recovering Lorentz invariance

By expressing the light-cone coordinates as a linear combination of a spatial and a temporal coordinate, the Lorentz symmetry can be lost. It is not guaranteed that the system described by the light-cone gauge remains invariant to these transformations. Fortunately, it can be recovered if explicitly required.

Computing the first excited states allows determining the values of the constants a and c, required for this theory to have physical meaning. The first excited state comes from setting $N = 1$ and can be obtained by acting with the raising operator over the ground state, i.e. $\alpha^i_{-1}|0;k^\mu\rangle$. Now Lorentz invariance has to be imposed. But the physical states that are invariant under Lorentz transformations form a representation of $SO(D-1)$ for massive states and $SO(D-2)$ for massless states. The considered first excited state belongs to a representation of the group $SO(D-2)$ in the transverse space. From this condition, it is deduced that α^i_{-1} must correspond to a state with zero mass. Acting on the mass operator, or equivalently on its square, it must provide 0 as eigenvalue. Therefore,

$$M^2\ \left(\alpha^i_{-1}|0;k^\mu\rangle\right) = \frac{2}{l_s^2}(1-a)\left(\alpha^i_{-1}|0;k^\mu\rangle\right) \quad \Longrightarrow \quad \boxed{a=1}.$$

Thus, to require Lorentz invariance the first excited state should be massless, with an eigenvalue equal zero. This only occurs for fixed $a = 1$. Once this value is obtained, the central charge c can be calculated, and it also determines the dimensionality of the spacetime D.

To find the number of dimensions, first assume that the ambiguity in the ordering of α^i_n, α^i_{-n} operators was not detected. If the number operator $\hat{N}$ is redefined keeping the symmetrized ordering, that comes from the classical theory, it should satisfy

$$\frac{1}{2}\sum_{n\neq 0}\alpha^i_{-n}\alpha^i_n = \frac{1}{2}\sum_{n<0}\alpha^i_{-n}\alpha^i_n + \frac{1}{2}\sum_{n>0}\alpha^i_{-n}\alpha^i_n,$$

where the sum over $i = 1, ..., D-2$ is implicit. If the normal order is now imposed, the resulting expression must be the same as before, i.e. the extra constant must be equal to a.

Setting the normal order means placing the annihilation operators on the right side. These are α^i_n for $n > 0$, so in the sum over $n < 0$ it is needed to change acting order of these operators. To do that, the commutation relations $[\alpha^i_n, \alpha^i_{-n}] = n\,\delta_{ij}$ are used, giving

$$\frac{1}{2}\sum_{i=1}^{D-2}\sum_{n\neq 0}\alpha^i_{-n}\alpha^i_n = \frac{1}{2}\sum_{i=1}^{D-2}\sum_{n>0} :\alpha^i_{-n}\alpha^i_n: + \frac{D-2}{2}\sum_{n=1}^{\infty} n. \tag{69}$$

The last summation on the right hand side diverges as it is a sum over infinite terms. To solve this problem, a regularization procedure is followed, which consists of using the $\zeta(s)$ Riemann function

$$\zeta(s) = \sum_{m=1}^{\infty}\frac{1}{m^s}\,, \quad \text{with} \quad s\in\mathbb{C} \quad \text{and} \quad \mathbb{R}e(s) > 1,$$

that is equal to the summation mentioned before, if one takes $s = -1$.

It turns out that the Riemann ζ-function converges when $\mathbb{R}e(s) > 1$. But using analytic continuation, this function can be defined for all possible arguments, giving finite results (except for $s = 1$). The particular solution for $s = -1$ is $\zeta(-1) = -1/12$. That is quite surprising, because it suggests a solution for an infinite sum of natural numbers, which is not only less than one, but also negative.

Placing that result in equation (69) gives an alternative expression after the normal ordering of the operators, that should be equal to the one obtained when the ordering constant a was used

$$\frac{1}{2}\sum_{i=1}^{D-2}\sum_{n>0} : \alpha_{-n}^{i}\alpha_{n}^{i} : -\frac{D-2}{24} = \frac{1}{2}\sum_{i=1}^{D-2}\sum_{n>0} : \alpha_{-n}^{i}\alpha_{n}^{i} : -a.$$

Since the ordering constant was calculated above, it is assumed this value is known and right, i.e. $a = 1$. Therefore the dimensionality of the background spacetime is finally fixed

$$\frac{D-2}{24} = a \Longrightarrow \boxed{D = 26}.$$

The light-cone quantization allows to derive the theory avoiding predictions of non physical states, such as ghost states. However, as the gauge used breaks the common structure of Lorentz transformations, invariance is not guaranteed and must be required subsequently. It is then when the restrictions are obtained: bosonic string theory satisfies the Lorentz invariance if $a = 1$ and if the space is defined to have dimension $D = 26$.

After performing those two processes of quantization, it is seen that effectively, both lead to the same constraints for the background space-time. Starting with the classical theory that includes the manifestly Lorentz symmetries, seems to be an easy and intuitive method to obtain quantized strings. However it gets complicated when the ghost states with negative norm are introduced. Eliminating them is a tedious procedure, but gives the values of the constants. From another point of view, a gauge in the coordinates can be established to get rid of the ghosts, following the light-cone quantization. This way, the non-physical states are not predicted. Avoiding this problem also carries the lost of Lorentz invariance. When guaranteed again, the same values for the same constants reappear. Therefore, both paths lead to equal parameters of bosonic string theory, these being $a = 1$ and the space-time dimension $D = 26$.

Chapter 6

SPECTRUM OF EXCITED STATES

Empleando la fórmula de la masa obtenida en secciones anteriores, se construye el espectro de masas para cuerdas abiertas y cerradas. Cada estado excitado corresponde con la represtentación de una partícula. Aparece el taquión como estado fundamental y el gravitón como primer estado excitado de las cuerdas cerradas.

After doing the quantization, the conditions that must be fulfilled for Lorentz invariance and symmetries of the strings, fix the values of the constants that describe the bosonic string theory. The level matching condition and the mass formula were rewritten in terms of operators. Then the mass spectrum of the string may be analyzed. The excited states are calculated by raising the value of the number operator N, and also $\tilde{N}$ under the consideration of closed strings. Due to the contribution of two directions of propagation for the modes of the closed string, and only one for the open string, the mass formulas are different for those kinds of strings and their analysis will be done separately.

6.1 Open string

As it was said from the begining, the string theory has the extraordinary characteristic that it has only one parameter to determine: the length of the string, l_s. Open strings have the constant α' that implicitly includes this parameter, since it is defined as $\alpha' = l_s^2/2$. From equation (68) in the canonical quantization, or analogously, (56) in the light-cone quantization; the mass of the resulting particle can be obtained for each excited state. This values are expressed just in terms of the slope parameter.

To built the mass spectrum it seems appropriate to start for the ground state, that should be the one associated to $N = 0$. Substituting all these values in the mentioned equations, the bosonic vector representation for the ground state is $|0; k^\mu\rangle$. From it one obtains a mass squared negative

$$M^2 = -\frac{1}{\alpha'}.$$

This results gives a particle with imaginary mass. Although apparently meaningless, it is a representation corresponding to particles that appear in special relativity, and whose main quality is that they admit velocities higher than the speed of light. This particle is a *tachyon*.

Having an imaginary mass is equivalent to having an imaginary energy in the ground state. A complex energy state is interpreted physically as an unstable state, and therefore it will decay. From this result it follows that this theory is not adequate, since it should be possible to begin from stable states.

For the first excited level, the physical states are those having $N = 1$. These are obtained adding a creation operator to the ground state $\alpha^i_{-1}|0; k^\mu\rangle$ and gives a massless boson. As they are defined as a vector, their representation is associated to spin-1 particle, corresponding then with a photon.

Then the next excited states are obtained simultaneously, but instead of a vector, higher order tensors arise since there are more possibilities of reaching the excited state. For example, for $N = 2$, the states are $\alpha^i_{-1}\alpha^j_{-1}|0; k^\mu\rangle$ or $\alpha^i_{-2}|0; k^\mu\rangle$, giving a total number of 324 states. The representation is a second-rank tensor, that consists of a spin-2 particle. Using again the mass formula (68), the resulting $M^2 = 1/\alpha'$ shows that this is the first massive particle state in the mass spectrum.

6.2 Closed strings

Next, the same study will be performed with the closed strings. Now the mass formulas are the ones appearing in (68) and (57), where left moving and right moving modes are being considered. The equivalence between the excitation of both moving modes, was shown above, i.e. (58). The excited states are reached by forming a composition of both: the right-moving sector and the left moving sector

$$M^2 = \frac{4}{\alpha'}(N-1) = \frac{4}{\alpha'}(\bar{N}-1).$$

Again at the beginning the first value is $N = 0$, corresponding to the ground state. Notice that the closed string have a different definition for the slope parameter $\alpha' = l_s^2$, but it still depending on the length of the string l_s. Therefore the squared mass gives another value, $M^2 = -4/\alpha'$, that is also negative. Hence, the ground state for the closed string is a tachyon as well, and it is also an unstable state represented by a particle field that decays.

Unlike the previous case, two operators are already present in the first excited state. These correspond to each of the propagation directions, as previously mentioned. In agreement with the level matching condition, when one creation operator α^i_{-1} acts over the ground state, an operator $\tilde{\alpha}^j_{-1}$ is also needed. Then there are $(D-2)^2$ first excited states of the form $\tilde{\alpha}^j_{-1}\alpha^i_{-1}|0; k^\mu\rangle$. As it was seen in the derivation, the space-time constraints were fixed under the assumption of a massless first excited state, satisfying the mass formula $M^2 = 4/\alpha'(N-1) = 0$ for $N = 1$. Therefore, these states have zero mass and are given by

$$|\Omega^{ij}\rangle = \alpha^i_{-1}\tilde{\alpha}^j_{-1}|0; k^\mu\rangle,$$

which is the result of a tensor product between the two massless vectors, corresponding to a spin-2 particle.

The tensor $|\Omega^{ij}\rangle$ can be splitted into a symmetric and antisymmetric part. The symmetric part is more interesting, which in turn can have two components: $\delta_{ij}\,|\Omega^{ij}\rangle$ and one with zero trace. The trace term is just a scalar that gives rise to the so-called *dilaton*. The traceless part has no mass and spin 2, but in this case is a field representation of the group $SO(24)$. These characteristics are the ones expected for the *graviton*.

The graviton is the particle defining the gravity field and it determines the metric of spacetime. It can be transformed into an operator and introduce the associated creation $a^\dagger_{\mu\nu}$ and annihilation operators $a_{\mu\nu}$ to obtain a quantized theory. Under the proper symmetry conditions for the metric, the graviton field can be quantized, finally giving rise to a theory of quantum gravity. This fact is indeed one of the most remarkable results that emerges from string theory.

There are other excited states that can be calculated by the tensor product of the vectors resulting for each excited state. Again for $N = \tilde{N} = 2$, there are two options of acting creation operators that must be included. Then one must consider the right moving and left moving sectors

$$(\alpha^i_{-1}\alpha^j_{-1} + \alpha^i_{-2}) \otimes (\tilde{\alpha}^i_{-1}\tilde{\alpha}^j_{-1} + \tilde{\alpha}^i_{-2})|0; k^\mu\rangle,$$

all these states being massive, in agreement with the resulting mass equation $M^2 = 4/\alpha'$.

6.3 Open and closed strings in a same theory

It should be pointed out that the symmetries required by the string framework forces to fix a space in which the constants are $a = 1$ and $D = 26$. These results are obtained equally well whether open or closed strings are assumed. In other words, open and closed strings are part of the same theory.

Some theories may only include the boundary conditions corresponding to the closed string. It is simply to assume that it has no endpoints. In contrast, all theories that include open strings must necessarily include closed strings as well. The reason for this is that when considering interactions, an open string can join its extremes until it becomes closed.

In bosonic string theory both open and closed strings are included. Generally superstring theories also study the two types, but there are certainly some theories that only take into account closed strings.

Chapter 7

CONCLUSION AND FUNDAMENTAL RESULTS

Analizando las predicciones que hace la teoría de cuerdas bosónicas, se deduce que hay redundancias en su definición y da lugar a un estado fundamental inestable. En cambio, la aparición del gravitón es uno de los resultados más impactantes. Tampoco contempla la existencia de fermiones, por tanto la teoría debe ser completada.

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So far, the consideration of one dimensional body instead of a point particle, has led to this whole theory. Relativistic string action, based on its world-sheet, gave an equation of motion and later some constraints that must be also imposed. At the end those equations reduce to the wave equation, meaning that they behave as classical strings. Thus, the coordinates used to describe the motion of the string were expanded in Fourier modes. The resulting expression for the world-sheet coordinates have two components: one associated to the motion of the center of mass (although there is no mass distribution) and one for the vibrational modes.

This is followed by the quantization, that is when complications appear. On the one side, ghost states can be predicted, but they are eliminated by restringing the dimension of the background space. On the other side, reparametrization invariance is employed to define a gauge in the light-cone that allows an alternative way to quantize the strings. Again, imposing Lorentz symmetries gives the same restricting results as before.

During these processes some equations for the states of the string are obtained. Considering different excited states, the mass spectrum is constructed, where each state has a representation that can be associated with a particle. In this theory, only with bosons. The most impressive result is the spontaneous appearance of the graviton, the particle describing the gravitational field at quantum levels. This is a result that has never been achieved before. The idea of having a quantum gravity theory causes a shocking impression of string theory.

The graviton comes from the representation of the first excited state for a closed string. It emerges as a zero-trace second rank tensor, together with the dilaton, that is a scalar field, and with an antisymmetric tensor. The correspondence between this state of the closed string and the graviton is established when one observes that it is a massless state and has spin 2.

However, the bosonic string theory presents some failures that also show off when doing the mass spectrum, like the appearance of the tachyon as the ground state. Having an unstable ground state shows that there are other problems in the definition of the theory. Despite this, the principal shortcomings is the absence of fermions. The only way of including fermions is to expand the theory, by using new symmetries. These are the so-called supersymmetries and their representation needs a superalgebra that collects the bosonic string theory and those supersymmetries.

Chapter 8

FURTHER STEPS TO SUPERSTRINGS

Al introducir supersimetrías la dimensionalidad se reduce a 9 dimensiones espaciales y una temporal. Además el taquión se elimina del espectro. Ahora se consideran tanto fermiones como bosones que aparecen formando superparejas. Hay cinco tipos de supercuerdas. La teoría que engloba todas las anteriores es aún desconocida y se denomina Teoría-M.

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The mass spectrum predicts several kinds of particles when computing their excited states. But fermions do not appear there. Without fermions, there is no way to describe ordinary matter. The need of including those particle fields in this theory was what caused the introduction of supersymmetries. These are symmetries that associate a fermion to each boson [5].

What is done here is to extend the working space to a 2D supersymmetric space, where there are the scalar fields $X^\mu(\tau, \sigma)$ of the world sheet for bosons and their corresponding supersymmetric partners $\psi^\mu(\tau, \sigma)$ for fermions. There are defined creation operators for bosons and for fermions, separately. Each boson is associated with a fermion of the corresponding excited state forming a superpartner. The pairs are established between particles that differ 1/2 in their spin. Using a superalgebra it is possible to combine both particle representations [10]. However, since bosons commute and fermions anticommute, they can be combined if they are considered as the even and odd elements, respectively, of a new supergroup.

The addition of these supersymmetries not only imply the inclusion of fermions, it also makes that for those theories where supersymmetry is present in the world-sheet and also in the spacetime spectrum, the tachyon disappears. Thus, two unlikely results of the bosonic theory have been eliminated. Moreover, after setting this symmetry, the constraint in the number of dimensions changes, and now it is reduced to 10 (1 temporal and 9 spatial).

In contrast to bosonic strings, depending on the considerations for the allowed boundary conditions and moving modes, the theory gives rise to five different string theories that are supersymmetric in spacetime [4, 5, 12, 15]:

- *Type I:* includes open and closed strings moving in a 10-dimensional space. These have one supersymmetry generator where bosons and fermions are defined. The representations of Type I superstrings correspond to those of the gauge group $SO(32)$.
- *Type IIA:* it has right-moving and left-moving closed strings where those modes are chosen to have opposite chirality. There could be stable Dp-branes for p even. The superstrings live in a 10 dimensional space with two supersymmetry generators. The resulting spectrum includes the graviton and their superpartners, which are a gravitino

and two fermions, all of them with opposite chirality. Thus, supergravity appears with non-chiralty property. This theory predicts fermions, but it does not predict force carriers.

- *Type IIB:* it is similar to Type IIA, but now left and right moving modes are defined with the same chirality. These have stable Dp-branes with p odd. Also the supergravity now is chiral, what emerges from the fact that the superpartners, gravitino and fermions, have the same chirality.

- *Heterotic $SO(32)$:* heterotic strings only consider closed strings, combining right-moving modes for fermions in 10 dimensions and left-moving modes for bosons in 26 dimensions. Therefore, there are 16 extra dimensions for the bosonic sector, which is a possible assumption since the different modes do not interact. However the extra dimensions must be compactified, leading to a gauge group corresponding to SO(32), for the present case.

- *Heterotic $E_8 \times E_8$:* they are identical to the previous ones, except that now the gauge group used is $E_8 \times E_8$. In both heterotic strings the graviton field appears with its superpartners and other gauge generators in a space with one supersymmetry. Neither of these two types contemplates branes.

Since 5 theories appear, one might think that the unicity has been lost, but later research work has shown that they are related to each other [16]. There are dualities that relate theories belonging to different limits, such that all theories are related. Then, it is deduced that all these theories are manifestations of the same theory.

The *M-theory* is proposed as a string and branes theory. It also includes a theory of supergravity in 11 dimensions. The theories listed above are interpreted as different visions that are included in the *landscape* of the M-theory. However, it is a theory that has not been fully determined, although some aspects of it are known.

So far, only free strings have been considered. To introduce interactions no additional nonlinear terms should be added, because this would not be consistent with the symmetries. On the contrary, all the information needed is contained in the world sheet described by the Polyakov action. A perturbative theory is developed, where the interactions are included through the Feyman path integrals. This means doing a sum over all possible topologies that the world sheet can have. In comparison with the particle interactions described by quantum field theory, Feynman diagrams for strings form smooth surfaces. They have no vertices.

Figure 6 shows the representation of an interaction between two strings. If one looks locally at each diagram, it is seen how every region appears to be representing a freely propagating string. Looking globally is when the representation of the interaction is understood.

There are evidences that string theory has numerous applications for the resolution of problems in different areas of physics. It has proved to be of great utility especially for the

Figure 6: Representation of the interaction between two strings as a sum of all possible topologies of the world-sheet.

development of models. For example, the use of string theory made it possible to calculate the entropy of black holes [1, 13]. These result comes from the supergravity theories arising from superstrings. In addition, a proposed solution for understanding dark matter has also been arrived at from strings [12, 14].

Due to string theory a new mathematical framework has been developed and it has been useful in other areas of physics. Dualities relate apparently different systems, allowing them to be solved from different points of view. For example, in condensed matter, dualities and other methods obtained from string theory have been used to solve problems [17].

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