Optimal policy for an inventory system with demand dependent on price, time and frequency of advertisement

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Abstract

This paper studies a new lot-size inventory problem for products whose demand pattern is dependent on price, advertising frequency and time. It is considered that the demand rate of an item multiplicatively combines the effects of a power function dependent on the frequency of advertisement and a function dependent on both selling price and time. This last function is additively separable in two power functions, one varies with the selling price and the other depends on the time since the last inventory replenishment. Moreover, it is assumed that the holding cost per unit of item is a non-linear function of time in stock. Shortages are not allowed. The aim consists of determining the frequency of advertisement, the selling price and the length of the stock period to maximize the average profit per unit time. This leads to a mixed integer non-linear inventory problem, which is solved by using an efficient algorithm previously developed. The inventory model considered here extends several inventory models previously proposed in the literature. Some numerical examples are solved to illustrate how the algorithm works to obtain optimal inventory policies. Finally, a sensitivity analysis for the optimal solution with respect to the parameters of the inventory system is presented.

Keywords: inventory; profit maximization; advertisement-dependent demand; time and price-dependent demand; non-linear holding cost

1 Introduction

One of the main objectives in any company or organization that distributes products is to have a sufficient amount of stored items to be able to satisfy the demand of the clients during a reasonable period of time and to sell the items at an acceptable selling price to obtain the maximum profit.

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In general, demand for items is not constant and fluctuates over time due to external causes that occur in the market. Therefore, the control of the inventory of an article is a dynamic activity that requires the continuous revision of the operative methods to reflect the possible changes in the stock level of the products. As a consequence, inventory systems must be adapted to new situations by applying the most efficient inventory policies at any time.

The theory of stock management seeks to maintain adequate levels of inventory to meet customer demand and in turn to obtain the highest possible profit. To carry out this control, it is necessary to develop mathematical models that allow the characteristics of the inventory systems to be represented in order to deduce good properties on the strategies to follow in the stock control. In this way, the optimal inventory policies that should be applied to stock management can be determined.

As in real-life inventory systems, the demand rate often depends on time, so it would be interesting to analyze functions that better represent the demand rate. Thus, Naddor (1966) introduced the power demand pattern as an adequate function to model the customer demand process. This function considers that the demand depends both on the time elapsed since the last replenishment and on the duration of the inventory cycle. After Naddor, other works have appeared in the literature that consider this type of demand. For example, we can cite the articles by Datta and Pal (1988); Singh et al. (2009); Rajeswari and Vanjikkodi (2011); Mishra and Singh (2013); and Mandal and Islam (2015). In all these papers, it is assumed that the length of the inventory cycle is known and fixed. However, Sicilia et al. (2012) and Siclia et al. (2014) developed several inventory systems in which the length of the inventory cycle was unknown and assumed that it was a decision variable of the inventory model. More recently, San-José et al. (2017) studied an inventory system with a power demand pattern and partial backlogging, where the duration of the inventory cycle was a decision variable.

The customer's behavior can also depend on the selling price of the products (see, e.g., Mills, 1959; Karlin and Carr, 1962; Federgruen and Heching, 1999; Petruzzi and Dada, 1999; Smith et al., 2007; and Kabirian, 2012). Usually, if the selling price increases, then demand decreases, and viceversa. A hypothesis frequently used in the literature is that demand depends linearly on the selling price of the article (see, e.g., Roy, 2008; Sundar et al., 2012; Rao et al., 2014; Chaudhary and Sharma, 2015; Alfares and Ghaithan, 2016; and Panda et al., 2017). Chen et al. (2006) extended this type of demand function and assumed a power function of the price in the inventory system with periodic review and finite planning horizon. The above papers assume that demand depends on the selling price, but it does not depend on the time elapsed since the replenishment is received. However, it would be more realistic to consider that the demand function depends on both selling price and time. In the literature, there are several works in which it is considered that the demand rate depends on the time and the selling price. Thus, You (2005) determined the order size and optimal price for a perishable inventory system considering time and price dependent demand. Maihami and Abadi (2012) developed an inventory model for non-instantaneously deteriorating items with price and time dependent demand and permissible delay in payments. Soni (2013) studied an inventory model with an additive demand rate with respect to both the sale price and the stock level. Wu et al. (2014) reviewed and corrected the Soni model. Avinadav et al. (2014)

analyzed two inventory models with demand dependent on time and price (one with a multiplicative influence of price and time, and the other with an additive effect). Hossen et al. (2016) analyzed a fuzzy inventory model for deteriorating items, considering that demand is dependent on price and time and assuming an inflation effect. Recently, Herbon and Khmelnitsky (2017) developed an inventory model with an additive demand rate that generalizes the pseudoadditive model suggested in Avinadav et al. (2014). Pervin et al. (2019) developed a multi-item inventory model for deteriorating items where the retailer's demand depends on the stock level and the selling price under a trade-credit policy. Shaikh et al. (2019) developed two inventory models with price- and stock-dependent demand, linearly time-varying carrying cost, and variable unit purchase cost under all-units quantity discount environment. Khan et al. (2019) developed two inventory models (with and without shortages) for deteriorating products, assuming that demand depends linearly on price and the deterioration rate is a time-varying increasing function, which depends on the expiration date.

In classic inventory models, the maintenance cost is considered to be a linear function of time. However, this hypothesis may not be realistic for some articles. Thus, Weiss (1982) analyzed an inventory model with constant demand where the holding cost was a power function of time. He showed that models with non-linear holding cost can be applied to inventory systems in which the value of the product decreases non-linearly with respect to the time spent in the inventory. Later, Ferguson et al. (2007) showed how the deterministic inventory model studied by Weiss was an approximation of the optimal batch size for perishable items. More recently, some authors have considered that the holding cost per unit and per unit of time is a linearly increasing function of time. Pervin et al. (2017, 2018a, 2018b) are some of the papers in this research line.

In the inventory literature, there are several articles on inventory models in which product demand is sensitive to the frequency of advertisement through media (press, radio, television, internet, etc.). Thus, Goyal and Gunasekaran (1995) developed an integrated production-inventory model considering the effect of the price and the advertising frequency on the demand of a deteriorating item. Shah and Pandey (2009) studied the economic replenishment policy in an EOQ model with stock-level and advertisement-frequency dependent demand. Shah et al. (2013) studied an inventory system for non-instantaneous deteriorating items in which the demand rate is a function of the advertisement of an item and the selling price. Rabbani et al. (2015) developed an integrated model for dynamic pricing and inventory control of non-instantaneous deteriorating items. Their demand rate is a function of the selling price, frequency of advertisement and changes in price per time unit. Manna et al. (2017) analyzed an economic production quantity (EPQ) model with imperfect production system and advertisement dependent demand, where the advertisement rate is assumed to be an exponentially increasing function with respect to time, but with a gradually decreasing growth rate. Bhunia et al. (2017) presented a production-inventory model to study the effects of partially integrated production and marketing policy, considering that demand is dependent on the selling price and marketing cost. Chen (2018) developed a production-inventory model of perishable products with pricing and promotion for a single-vendor multi-buyer system comprising one manufacturer and multiple retailers. Rad et al. (2018) studied inventory models of a vendor–buyer supply chain with imperfect products and shortages completely backordered by the buyer, assuming that both the selling price and advertisements influence market demand. Panda et al. (2019) introduced an inventory model for deteriorating items, where demand depends on frequency of advertisement, price and stock under the credit policy approach, considering the frequency of advertising to be constant. Khan et al. (2020) studied two inventory models for a deteriorating product, which has a maximum useful life, assuming that demand depends on the selling price and the frequency of advertisement, while also considering the advance payment.

Table 1 shows a list of papers previously cited that have been developed since 2010, classifying them according to the type of demand, the shortage and the unit cumulative holding cost.

In this paper, we study an inventory model for a single item whose demand depends on marketing strategies, the selling price and time. Thus, we assume that the demand rate is the product of a power function dependent on the advertising frequency and a function dependent on both selling price and time. Moreover, we consider that this last function is additively separable in two power functions, one dependent on the selling price and the other dependent on the time since the last inventory replenishment. As in San-José et al. (2015), we suppose that the holding cost per unit of product has two components: a fixed cost and a variable cost that is a power function of the time that the items spend in stock. The objective is to determine the values of the frequency of advertisement, the selling price and the lot size that maximize the average profit per unit of time. To solve this mixed integer non-linear problem, we have developed an efficient algorithm that finds a maximum. In some steps of the algorithm, we need to solve non-linear equations by using some numerical method of solving equations (for example, the bisection method or the Newton-Raphson method).

This new inventory model can be useful for products whose demand is sensitive to the effect of advertising policies, changes in the selling price and time spent in the inventory. In this sense, we can mention, among others, the following goods: (a) cooked foods, fruit, yoghurts, etc., which have a higher demand at the beginning than at the end of the inventory period; (b) sugar, coffee, oil, salt, water, etc., which may have a lower demand at the beginning of the inventory cycle and this increases as the stock decreases; and (c) kitchen utensils, supplies, consumables, etc., which have an almost constant demand during the inventory cycle. In order to clarify the contribution of our work for the readers, we have shown the main characteristics of the present paper and of previous papers published since 2010 in Table 1.

The rest of the paper is organized as follows. Section 2 introduces the properties of the inventory system and shows the notation that will be used throughout the paper. Section 3 presents the formulation of the mathematical model and finds the profit function per unit time to be maximized. In Section 4, an algorithm to determine an optimal inventory policy is developed. Section 5 presents some numerical examples to illustrate the theoretical results and the application of the algorithm. Moreover, a numerical sensitivity analysis for the optimal values of the decision variables (frequency of advertisement, selling price and length of inventory cycle), the lot size and the maximum profit with respect to the parameters of the inventory system is given. Finally, the conclusions are drawn and future research lines are addressed in Section 6.

Table 1. Summary of literature published since 2010

Authors	Demand rate					Shortage	Unit cumulative holding cost	
	Constant	Stock-dependent	Time-dependent	Price-dependent	Advertisement -dependent		Fixed	Time-dependent
Alfares and Ghaithan (2016)				Linear				Quadratic
Avinadav et al. (2014)			General	General				Linear
Bhunia et al. (2017)				Linear	Yes	PBO		Linear
Chaudhary and Sharma (2015)				Linear		PBO		Linear
Chen (2018)				Mixture		PBO		Linear
Herbon and Khmelnitsky (2017)			General	General				Linear
Hossen et al. (2016)			Linear	Linear		PBO		Fuzzy
Kabirian (2012)				General				Linear
Khan et al. (2019)				Linear		PBO		Quadratic
Khan et al. (2020)				Linear	Yes	PBO		Quadratic
Maihami and Abadi (2012)			Exponential	Linear		PBO		Linear
Mandal and Islam (2015)			Power pattern			PBO		Fuzzy
Manna et al. (2017)			Mixture		Yes			Linear
Mishra and Singh (2013)			Power pattern			PBO		Linear
Panda et al. (2017)				Linear				Linear
Panda et al. (2019)		Yes		Linear	Yes	PBO		Linear
Pervin et al. (2017)		Yes				PBO		Quadratic
Pervin et al. (2018a)		Yes				PBO		Quadratic
Pervin et al. (2018b)			Linear			PBO		Quadratic
Pervin et al. (2019)		Yes		Exponential		FBO		Linear
Rabbani et al. (2015)			Changes in price	Mixture	Yes			General
Rad et al. (2018)				Iso-elastic	Yes	FBO		Linear
Rao et al. (2014)				Linear		FBO		Linear
Rajeswari and Vanjikkodi (2011)			Power pattern			PBO		Linear
San-José et al. (2015)	Yes					PBO	Yes	Power
San-José et al. (2017)			Power pattern			PBO		Linear
Shah et al. (2013)				Iso-elastic	Yes			General
Shaikh et al. (2019)		Yes		Linear		PBO		Quadratic
Sicilia et al. (2012)			Power pattern			FBO		Linear
Sicilia et al. (2014)			Power pattern					Linear
Soni (2013)		Yes		General				Linear
Sundar et al. (2012)				Linear				Linear
Wu et al. (2014)		Yes		General				Linear
This paper			Power pattern	Polynomial	Yes		Yes	Power

PBO: partial backordering; FBO: full backordering

2 Hypothesis and notation

The notation used throughout the paper is presented in Table 2.

The mathematical model for the inventory system studied here is based on the following hypotheses:

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Table 2. Notation

K	Cost of placing an order (> 0)
c	Unit purchasing cost (> 0)
h_0	Fixed accommodating cost per stored unit (≥ 0)
h	Scale parameter for the holding cost (> 0)
θ	Elasticity of the holding cost (≥ 1)
$H(t)$	Cumulative holding cost for t units of time, that is, $H(t) = h_0 + ht^\theta$
p	Unit selling price ($p \geq c$, *decision variable*)
p_m	Maximum unit price for sale
A	Frequency of advertisement per cycle (≥ 0, *decision variable*)
v	Cost for each advertisement
T	Inventory cycle length (≥ 0, *decision variable*)
$D(A, p, t)$	Demand per unit of time at time t when the frequency of advertisement is A and the selling price is p, with $0 < t < T$
η	Power of the advertising frequency in the demand rate
α	Scale parameter of the part of the price-dependent demand
β	Sensitivity parameter of the demand with respect to price
γ	Exponent of the selling price in the demand rate
λ	Scale parameter of the part of the time-dependent demand
δ	Index of the power demand pattern (> 0)
$I(A, p, t)$	Inventory level at time t when the frequency of advertisement is A and the selling price is p, with $0 \leq t < T$
Q	Lot size (> 0)
$B(A, p, T)$	Average profit per unit of time

1. The inventory refers to a single product.
2. The inventory is reviewed continuously and the planning horizon is infinite.
3. The inventory replenishment is instantaneous.
4. Shortages are not allowed.
5. The replenishing quantity or lot size is constant, but unknown.
6. The cost K of placing an order is constant and independent of the lot size.
7. The price c of acquisition of a unit of the product is a known constant.
8. The demand rate $D(A, p, t)$ is a function of the frequency of advertisement, the selling price and time. Thus, it is assumed that $D(A, p, t) = (A+1)^\eta \left[\alpha - \beta p^\gamma + \lambda\delta \left(\frac{t}{T}\right)^{\delta-1}\right]$, with $\alpha, \beta, \lambda, \delta > 0$, $\gamma \geq 1$, $0 < \eta < 1$ and $c \leq p \leq p_m$, where p_m is the maximum selling price, that is,
$$p_m = (\alpha/\beta)^{1/\gamma}.$$

Hence, the demand rate combines the effects of the unit selling price, the power demand pattern and the frequency of advertisement.

Given the advertising frequency A and the unit selling price p, the demand rate $D(A, p, t)$ describes the way by which demanded quantities are taken out of the inventory. There exist several real-life products that can follow this type of demand pattern. Thus, demand for cooked products, such as sweets, breads, cakes, etc., is higher at the beginning of the inventory cycle because customers prefer goods that have just been made. Also, other products, such as fish, fruit, yoghurts, etc., have greater demand at the beginning than at the end of the inventory cycle, because these products have an expiry date and, in general, people prefer to buy these products when they are recently put up for sale and their expiration date is far away. These situations are considered in the demand rate function, assuming a demand pattern index $n > 1$. There exist other products where demand at the beginning of the inventory period is lower than at the end of the period. Thus, household goods such as sugar, milk, coffee, oil, etc., have major demand when the amount in the inventory decreases significantly, because if people detect that these products are becoming scarce, then the demand for them grows considerably because they are basic products in daily use. In this case, the fluctuation of demand can be modeled considering a demand pattern index $n < 1$. Lastly, other products have a uniform demand rate along the inventory cycle. For instance, electrical goods, supplies, furniture, kitchen utensils and appliances, etc., have a more or less constant demand during the replenishment cycle. This situation can be modeled by using a demand pattern index $n = 1$ in the function $D(A, p, t)$. Therefore, the function $D(A, p, t)$ allows to describe the behavior of customer demand for a wide variety of products.

9. The cumulative cost of holding per unit of product up to time t is a non-linear function of storage time. It is assumed, as in San-José et al. (2015), that $H(t) = h_0 + ht^\theta$, with $h_0 \geq 0$, $h > 0$ and $\theta \geq 1$. Thus, h_0 represents a fixed accommodating cost per stored unit (independent of time), h is the scaling factor and θ is the shape parameter.
10. As no shortages are allowed, the lot size Q must be equal to the total demand during the inventory cycle, that is, $Q = \int_0^T D(A, p, t)dt$.

3 The mathematical model

It is considered that an order of Q units is received at time $t = 0$, so that at the beginning of the inventory cycle there are Q items in the inventory. During the period $(0, T)$, the inventory level decreases due to demand. Therefore, for all $t \in [0, T)$, the inventory level at time t is given by

$$I(A, p, t) = Q - \int_0^t D(A, p, x)dx = \int_t^T D(A, p, x)dx = \left\{(\alpha - \beta p^\gamma)(T - t) + \lambda T \left[1 - \left(\frac{t}{T}\right)^\delta\right]\right\} (A + 1)^\eta . \quad (1)$$

The lot size Q is equal to $I(A, p, 0)$. Thus, we have

$$Q = (\alpha - \beta p^\gamma + \lambda) T (A + 1)^\eta . \quad (2)$$

Taking into account the previous hypotheses, the income and costs in each cycle are calculated:

- Revenue: $pQ = pT(\alpha - \beta p^{\gamma} + \lambda)\,(A+1)^{\eta}$
- Purchasing cost: $cQ = cT(\alpha - \beta p^{\gamma} + \lambda)\,(A+1)^{\eta}$
- Ordering cost: K
- Advertisement cost: vA
- Holding cost: $\int_0^T H(t)D(A,p,t)dt = h_0(\alpha - \beta p^{\gamma} + \lambda)\,(A+1)^{\eta}\,T + hf(p)\,(A+1)^{\eta}\,T^{\theta+1}$, where

$$f(p) = \frac{(\alpha - \beta p^{\gamma})\,(\theta+\delta) + \lambda\delta(\theta+1)}{(\theta+1)(\theta+\delta)} > 0. \tag{3}$$

Therefore, the total profit during the inventory cycle is

$$(p - c - h_0)(\alpha - \beta p^{\gamma} + \lambda)\,(A+1)^{\eta}\,T - \left(K + vA + hf(p)\,(A+1)^{\eta}\,T^{1+\theta}\right). \tag{4}$$

The objective is to maximize the total profit per unit of time, that is,

$$B(A,p,T) = (p - c - h_0)(\alpha - \beta p^{\gamma} + \lambda)\,(A+1)^{\eta} - \left(\frac{K + vA}{T} + hf(p)\,(A+1)^{\eta}\,T^{\theta}\right). \tag{5}$$

So, the inventory problem consists of solving the optimization problem

$$\max_{(A,p,T)\in\Pi} B(A,p,T), \tag{6}$$

where $\Pi = \{(A,p,T) : T > 0,\ c \le p \le p_m \text{ and } A \in \mathbb{Z}, \text{ with } A \ge 0\}$.

4 Problem solution

In this section, we provide a procedure to obtain an optimal inventory policy that solves the problem (6). We start by giving some useful theoretical results in order to find the optimal inventory cycle T^* and the optimal selling price p^* when A is fixed. In this situation, we first consider the unit selling price of the item as fixed and then, the best inventory cycle for that price is determined. Thus, assuming that the advertising frequency A and the unit selling price p are fixed, with $A > 0$ and $p \in [c, p_m]$, we can consider the function $B_{A,p}(T) = B(A,p,T)$, with $T > 0$ variable. The following result gives the optimal length of the inventory cycle.

Lemma 1 *For any fixed values of A and p, the function $B_{A,p}(T)$ is strictly concave and attains its maximum at the point*

$$T^*(A,p) = \left(\frac{Av + K}{\theta hf(p)\,(A+1)^{\eta}}\right)^{1/(\theta+1)}. \tag{7}$$

Proof. See Appendix. ■

Now, evaluating the function $B_{A,p}(T)$ at the point $T^*(A,p)$ and considering p variable, we obtain the univariate function

$$G_A(p) = B(A,p,T^*(A,p)) = (p - c - h_0)(\alpha - \beta p^{\gamma} + \lambda)\,(A+1)^{\eta} - (\theta+1)hf(p)\,(A+1)^{\eta}\,(T^*(A,p))^{\theta} \tag{8}$$

Also, this function $G_A(p)$ can be expressed as

$$G_A(p) = (p - c - h_0)(\alpha - \beta p^\gamma + \lambda)(A+1)^\eta - \frac{(\theta+1)(Av+K)}{\theta T^*(A,p)} \tag{9}$$

It is evident that $G_A(p)$ is a continuous and twice-differentiable function on the interval (c, p_m). Thus, the first derivative of $G_A(p)$ is given by

$$G'_A(p) = \left[g_1(p) + \frac{\beta\gamma h}{\theta+1} p^{\gamma-1} \left(T^*(A,p)\right)^\theta \right] (A+1)^\eta, \tag{10}$$

where

$$g_1(p) = \alpha + \lambda + \beta \left[\gamma(c+h_0) - (\gamma+1)p\right] p^{\gamma-1}, \tag{11}$$

and the second derivative of the function $G_A(p)$ is:

$$G''_A(p) = \beta\gamma p^{\gamma-2}(A+1)^\eta \left[(c+h_0-p)\gamma - (c+h_0+p) + \frac{h}{\theta+1}\left(T^*(A,p)\right)^\theta \left(\gamma - 1 + \frac{\beta\gamma\theta p^\gamma}{(\theta+1)^2 f(p)} \right) \right]. \tag{12}$$

By studying the function $G_A(p)$ and its first two derivatives, we can establish a criterion to determine the selling price $p^*(A)$ that maximizes the function $G_A(p)$. To do this, below we provide an interesting property of the function $g_1(p)$.

Lemma 2 *Let $g_1(p)$ be the function defined by (11).*

1. *If $\lambda \geq \alpha\gamma(1 - (c+h_0)/p_m)$, then $g_1(p) > 0$ for all $p \in [c, p_m)$.*
2. *Otherwise, $g_1(p) > 0$ for all $p \in [c, p_1)$ and $g_1(p) < 0$ for all $p \in (p_1, p_m]$, where*

$$p_1 = \arg_{p\in(c,p_m)}\{g_1(p) = 0\}. \tag{13}$$

Proof. See Appendix. ■

Taking into account the result of the above lemma, if $\lambda \geq \alpha\gamma(1-(c+h_0)/p_m)$, then $g_1(p) > 0$ and $G'_A(p) > 0$ for all $p \in [c, p_m)$. Therefore, the maximum profit is attained at the price $p^*(A) = p_m$. However, if $\lambda < \alpha\gamma(1-(c+h_0)/p_m)$, then $G'_A(p) > 0$ and $G_A(p)$ increases for all $p \in [c, p_1)$. Thus, the maximum profit must be in the interval $(p_1, p_m]$. Hence, let $\widetilde{p}(A)$ be a solution of the equation $G'_A(p) = 0$ in the interval (p_1, p_m). From (12), that solution has to satisfy $G''(\widetilde{p}(A)) = \frac{(A+1)^\eta(\widetilde{p}(A))^{\gamma-1}}{(\theta+1)^2 f(\widetilde{p}(A))} g_2(\widetilde{p}(A))$, where

$$g_2(p) = \beta^2\gamma\theta p^{\gamma-1}\left((\gamma+1)p - \gamma(c+h_0)\right) + p^{-\gamma}(1-\gamma)(\alpha+\lambda)(\theta+1)^2 f(p) - \beta\left((\theta+1)^2(\gamma+1)f(p) + \gamma\theta(\alpha+\lambda)\right). \tag{14}$$

Next, in order to know the number of local extrema of the function $G_A(p)$ in the interval (p_1, p_m), we study the function $g_2(p)$ on such an interval.

The first derivative of the function $g_2(p)$ is given by

$$\begin{aligned} g'_2(p) \quad = \quad & \beta^2\gamma^2\theta p^{\gamma-2}\left((\gamma+1)p - (\gamma-1)(c+h_0)\right) + \frac{\gamma}{\theta+\delta}(\gamma-1)(\theta+1)(\alpha+\lambda)\left(\theta(\alpha+\delta\lambda) + (\alpha+\lambda)\delta\right)p^{-(\gamma+1)} \\ & + \beta^2\gamma(\gamma+1)(\theta+1)p^{\gamma-1} \end{aligned}$$

Since $p \in (p_1, p_m)$, we have $g_1(p) < 0$ and, from (11), it follows that $\gamma (c + h_0) < (\gamma + 1)p$ and, therefore, the derivative of $g_2(p)$ is always positive in such an interval. Consequently, the function $g_2(p)$ is strictly increasing. Thus, we can conclude that the function $G_A(p)$ has at most two local extremes in the interval (p_1, p_m).

Taking into account that

$$g_2(p_1) = \frac{\beta\gamma(\theta+1)^2 \left(\gamma\left(c+h_0-p_1\right)-\left(c+h_0+p_1\right)\right) f(p_1)}{p_1} = -\frac{\beta\gamma(\theta+1)^2 f(p_1)}{p_1}\left((\gamma+1)p_1-(\gamma-1)(c+h_0)\right)$$

and $g_1(p_1) = 0$, it leads to $g_2(p_1) < 0$. Since $\lim_{p\to\infty} g_2(p) = \infty$, it follows that $g_2(p)$ has a root p_2 in the interval (p_1, ∞). In addition, if $\widetilde{p}(A) < p_2$ then $G''(\widetilde{p}(A)) < 0$ and the function $G_A(p)$ has a local maximum at the point $\widetilde{p}(A) \in (p_1, p_m)$.

Now, we can already state one of our main results in the following theorem, which gives us a criterion to determine the selling price $p^*(A)$ that maximizes the function $G_A(p)$.

Theorem 1 *Let $G_A(p)$, $G'_A(p)$, $g_1(p)$, $G''_A(p)$ and $g_2(p)$ be the functions defined by (8), (10), (11), (12) and (14), respectively. Consider $p_m = (\alpha/\beta)^{1/\gamma}$, $p_1 = \arg_{p\in(c,p_m)}\{g_1(p) = 0\}$ and $p_2 = \arg_{p\in(p_1,\infty)}\{g_2(p) = 0\}$. We have:*

1. *If $\lambda \geq \alpha\gamma\left(1-(c+h_0)/p_m\right)$, then $p^*(A) = p_m$.*
2. *If $\lambda < \alpha\gamma\left(1-(c+h_0)/p_m\right)$ and $G'_A(p_m) < 0$, then $p^*(A)$ is the only root of the equation $G'_A(p) = 0$ in the interval (p_1, p_m).*
3. *If $\lambda < \alpha\gamma\left(1-(c+h_0)/p_m\right)$ and $G'_A(p_m) \geq 0$, then three cases can occur:*
   (a) *If $p_2 \geq p_m$, then $p^*(A) = p_m$.*
   (b) *If $p_2 < p_m$ and $G'_A(p_2) \geq 0$, then $p^*(A) = p_m$.*
   (c) *Otherwise (that is, $p_2 < p_m$ and $G'_A(p_2) < 0$), let $\widetilde{p}(A) = \arg_{p\in(p_1,p_2)}\{G'_A(p) = 0\}$.*
      i. *If $G_A(\widetilde{p}(A)) \leq G_A(p_m)$, then $p^*(A) = p_m$.*
      ii. *If $G_A(\widetilde{p}(A)) > G_A(p_m)$, then $p^*(A) = \widetilde{p}(A)$.*

Proof. See Appendix. ■

To obtain an optimal inventory policy that solves the inventory problem (6), we next analyze the behavior of the function $B(A, p, T)$ with respect to the integer decision variable A. Thus, assuming that the unit selling price p and the length of the inventory cycle T are fixed, with $p \in [c, p_m]$ and $T > 0$, we can consider the univariate function $B_{p,T}(A) = B(A, p, T)$. Relaxing the integer condition and assuming A is a continuous variable, the first derivarive of $B_{p,T}(A)$ is given by

$$B'_{p,T}(A) = \left[(p-c-h_0)(\alpha-\beta p^\gamma+\lambda) - hf(p)T^\theta\right]\eta\,(A+1)^{\eta-1} - \frac{v}{T} \tag{15}$$

By analyzing the function $B'_{p,T}(A)$, we can establish the following result:

Theorem 2 *For any fixed values of p and T, we have:*

1. *If $(p-c-h_0)(\alpha-\beta p^\gamma+\lambda) - hf(p)T^\theta \leq 0$, then the function $B_{p,T}(A)$ is strictly decreasing.*

2. *If* $(p-c-h_0)(\alpha-\beta p^{\gamma}+\lambda)-hf(p)T^{\theta}>0$*, then* $B_{p,T}(A)$ *is a strictly concave function.*

Proof. See Appendix. ■

Based on Eq. (15) and Theorems 1 and 2, we can develop an algorithm to solve the model studied in this paper.

Algorithm 1

Step 1 *Set* $i=1$, $A^{(i)}=0$.

Step 2 *Set* $A=A^{(i)}$.

Using Theorem 1, calculate $p^{(i)}=p^*_A$.

From (7), obtain $T^{(i)}=T^*(A^{(i)},p^{(i)})$*, and, from (8), determine* $B^{(i)}=G_A(p^{(i)})$.

Step 3 *Set* $p=p^{(i)}$ *and* $T=T^{(i)}$.

If $(p-c-h_0)(\alpha-\beta p^{\gamma}+\lambda)-hf(p)T^{\theta}\le 0$*, then* $A^{(i+1)}=0$.

Otherwise, calculate

$$A^{(i+1)}=\left\lceil\left(\frac{v}{\eta T\left((p-c-h_0)(\alpha-\beta p^{\gamma}+\lambda)-hf(p)T^{\theta}\right)}\right)^{1/(\eta-1)}-1\right\rceil.$$

Step 4 *If* $A^{(i+1)}\neq A^{(i)}$*, set* $i=i+1$ *and go to step 2.*

Otherwise, go to step 5.

Step 5 *If* $i=1$*, then set* $(A^*,p^*,T^*,B^*)=(A^{(i)},p^{(i)},T^{(i)},B^{(i)})$.

Otherwise:

If $B^{(i)}<B^{(i-1)}$*, then set* $(A^*,p^*,T^*,B^*)=(A^{(i-1)},p^{(i-1)},T^{(i-1)},B^{(i-1)})$.

Otherwise:

Set $(A^*,p^*,T^*,B^*)=(A^{(i)},p^{(i)},T^{(i)},B^{(i)})$.

Step 6 *The optimal solution is* (A^*,p^*,T^*) *and the optimal profit per unit time is* B^*.

Step 7 *From (2), calculate the optimal lot size* $Q^*=\left[\alpha-\beta\left(p^*\right)^{\gamma}+\lambda\right]\left(A^*+1\right)^{\eta}T^*$*. Stop.*

Note that Theorem 2 ensures that Algorithm 1 converges and gives an optimal inventory policy.

Let us present some interesting consequences of the previous results.

Corollary 1 *Let* (A^*,p^*,T^*) *be the optimal inventory policy given by Algorithm 1. If* $p^*=p_m$*, then the optimal inventory cycle is*

$$T_0^*=T^*(A^*,p_m)=\left(\frac{(A^*v+K)(\theta+\delta)}{\delta\theta\lambda h\left(A^*+1\right)^{\eta}}\right)^{1/(1+\theta)},\tag{16}$$

the economic ordering quantity is $Q_0^*=\lambda\left(A^*+1\right)^{\eta}T_0^*$ *and the maximum profit per unit of time is*

$$B_0^*=G_{A^*}(p_m)=(p_m-c-h_0)\lambda\left(A^*+1\right)^{\eta}-(\theta+1)(A^*v+K)/(\theta T_0^*).\tag{17}$$

Proof. It follows directly from (2), (3), (7) and (8). ■

Corollary 2 *Let (A^*, p^*, T^*) be the optimal inventory policy given by Algorithm 1. If $p^* < p_m$, then the optimal inventory cycle is*

$$T_1^* = \left(\frac{-(1+\theta)g_1\left(p^*\right)}{\beta\gamma h\left(p^*\right)^{\gamma-1}}\right)^{1/\theta} = \left(\frac{(1+\theta)\left\{\beta\left(p^*\right)^{\gamma-1}\left[(1+\gamma)p^* - \gamma(c+h_0)\right] - \alpha - \lambda\right\}}{\beta\gamma h\left(p^*\right)^{\gamma-1}}\right)^{1/\theta}, \tag{18}$$

the economic ordering quantity is $Q_1^* = (A^*+1)^{\eta}\left(\alpha+\lambda-\beta\left(p^*\right)^{\gamma}\right)T_1^*$ and the maximum profit per unit of time is

$$B_1^* = (A^*+1)^{\eta}\left\{\left(p^*-c-h_0\right)\left(\alpha+\lambda-\beta\left(p^*\right)^{\gamma}\right) + \frac{(\theta+1)^2 f(p^*)\left[(\alpha+\lambda)\,p^* + \beta\left(p^*\right)^{\gamma}\left(\gamma\left(c+h_0\right)-(\gamma+1)\,p^*\right)\right]}{\beta\gamma\left(p^*\right)^{\gamma}}\right\} \tag{19}$$

Proof. See Appendix. ■

4.1 Special models

In this subsection, we show how some inventory models developed by other authors can be considered as particular cases of the inventory model studied here.

(a) If we assume that $\delta = 1$, $h_0 = 0$ and $\alpha, \beta, \eta \to 0$, we have the inventory model developed by Weiss (1982) and Ferguson et al. (2007).

(b) If we suppose $\theta = 1$, $h_0 = 0$ and $\alpha, \beta, \eta \to 0$, we obtain the inventory system studied by Sicilia et al (2012) for an item with power demand pattern where shortages are not allowed.

(c) If one considers $\delta = 1$, $\theta = 1$, $\gamma = 1$, $h_0 = 0$ and $\eta \to 0$, then we derive in the model proposed by Kunreuther and Richard (1971) and Smith et al. (2007) when a linear demand curve is considered in their models.

(d) If $\delta = 1$, $\theta = 1$, $\gamma = 1$, $h_0 = 0$ and $\eta \to 0$ are assumed, then we obtain the model developed by Kabirian (2012) when it is supposed that the production cost is constant, the demand rate is linear and the production rate tends to infinity.

(e) If $\delta = 1$, $\gamma = 1$, $h_0 = 0$ and $\eta \to 0$, we have the inventory model proposed by Alfares and Ghaithan (2016), assuming in their model that the time-varying holding cost coefficient is zero and quantity discounts are not considered in the unit purchasing cost.

5 Numerical examples

In this section several numerical examples are presented to illustrate how the algorithm works.

Example 1 Consider the following values for the parameters of the model: $\alpha = 243$, $\beta = 1$, $\gamma = 1.25$, $\lambda = 10$, $\delta = 2$, $K = 200$, $c = 20$, $h_0 = 1$, $h = 0.6$, $\theta = 1.5$, $\eta = 0.04$ and $v = 120$. Applying Algorithm 1, the result of the solution procedure is shown in Table 3. Thus, the optimal frequency of advertisement per cycle is $A^* = A^{(2)} = 2$, the optimal selling price is $p^* = p^{(2)} = 53.7419$, the optimal inventory cycle is $T^* = T^{(2)} = 2.55792$ and the maximum profit is $B^* = B^{(2)} = 3390.86$. From (2), the economic lot size is $Q^* = 287.304$.

Table 3. Computational results of Example 1

Iteration	$A^{(i)}$	$p^{(i)}$	$T^{(i)}$	$B^{(i)}$	$A^{(i+1)}$
$i = 1$	0	53.5764	1.89532	3344.02	$\lceil 1.251 \rceil = 2$
$i = 2$	**2**	**53.7419**	**2.55792**	**3390.86**	$\lceil 2.039 \rceil = 3$
$i = 3$	3	53.8101	2.80634	3387.28	$\lceil 2.331 \rceil = 3$

Example 2 Assume the same parameters as in Example 1, but now change the values of β and λ to $\beta = 3.5$ and $\lambda = 100$, respectively. Now, applying Algorithm 1, the result of the solution procedure is shown in Table 4. Thus, the optimal frequency of advertisement per cycle is $A^* = A^{(1)} = 0$, the optimal selling price is $p^* = p^{(1)} = 29.7324$, the optimal inventory cycle is $T^* = T^{(1)} = 1.72159$ and the maximum profit is $B^* = B^{(1)} = 679.625$. From (2), the economic lot size is $Q^* = 172.159$.

Table 4. Computational results of Example 2

Iteration	$A^{(i)}$	$p^{(i)}$	$T^{(i)}$	$B^{(i)}$	$A^{(i+1)}$
$i = 1$	**0**	**29.7324**	**1.72159**	**679.625**	$\lceil -0.5580 \rceil = 0$

Example 3 Now, let us assume the following parameters of the model: $\alpha = 243$, $\beta = 1$, $\gamma = 1.25$, $\lambda = 80$, $\delta = 0.01$, $K = 1000$, $c = 15$, $h_0 = 0$, $h = 1$, $\theta = 2$, $\eta = 0.04$ and $v = 600$. Executing Algorithm 1, the result of the solution procedure is shown in Table 5. Thus, the optimal frequency of advertisement per cycle is $A^* = A^{(1)} = 0$, the optimal selling price is $p^* = p^{(1)} = 61.0694$, the optimal inventory cycle is $T^* = T^{(1)} = 2.73306$ and the maximum profit is $B^* = B^{(1)} = 6466.70$. From (2), the economic lot size is $Q^* = 416.195$.

Table 5. Computational results of Example 3

Iteration	$A^{(i)}$	$p^{(i)}$	$T^{(i)}$	$B^{(i)}$	$A^{(i+1)}$
$i = 1$	**0**	**61.0694**	**2.73306**	**6466.70**	$\lceil 0.2563 \rceil = 1$
$i = 2$	1	61.4821	3.18821	6455.00	$\lceil 0.4603 \rceil = 1$

Example 4 Consider the same parameters as in Example 3, but modifying the values of β and λ to $\beta = 2$ and $\lambda = 120$, respectively. Applying Algorithm 1, the result of the solution procedure is shown in Table 6. Thus, the optimal frequency of advertisement per cycle is $A^* = A^{(2)} = 2$, the optimal selling price is $p^* = p^{(2)} = 46.5223$, the optimal inventory cycle is $T^* = T^{(2)} = 12.0811$ and the maximum profit is $B^* = B^{(2)} = 3679.45$. From (2), the economic lot size is $Q^* = 1514.86$.

Table 6. Computational results of Example 4

Iteration	$A^{(i)}$	$p^{(i)}$	$T^{(i)}$	$B^{(i)}$	$A^{(i+1)}$
$i = 1$	0	46.5223	9.42602	3623.54	$\lceil 1.428 \rceil = 2$
$i = 2$	**2**	**46.5223**	**12.0811**	**3679.45**	$\lceil 2.115 \rceil = 3$
$i = 3$	3	46.5223	13.0423	3676.33	$\lceil 2.360 \rceil = 3$

Example 5 Assume the same parameters as in Example 4, but modifying the values of δ and λ to $\delta = 0.1$ and $\lambda = 160$, respectively. Executing Algorithm 1, the result of the solution procedure is shown in Table 7. Thus, the optimal frequency of advertisement per cycle is $A^* = A^{(1)} = 0$, the optimal selling price is $p^* = p^{(1)} = 45.3263$, the optimal inventory cycle is $T^* = T^{(1)} = 3.65817$ and the maximum profit is $B^* = B^{(1)} = 4678.21$. From (2), the economic lot size is $Q^* = 613.781$.

Table 7. Computational results of Example 5

Iteration	$A^{(i)}$	$p^{(i)}$	$T^{(i)}$	$B^{(i)}$	$A^{(i+1)}$
$i = 1$	**0**	**45.3263**	**3.65817**	**4678.21**	$\lceil 0.2171 \rceil = 1$
$i = 2$	1	46.5223	4.67429	4671.91	$\lceil 0.5465 \rceil = 1$

5.1 Numerical sensitivity analysis

In this subsection, we analyze the sensitivity of the inventory model proposed in this paper. For that, let us consider an inventory system that satisfies the assumptions described in Section 2 and has the following input data: $\alpha = 2000$, $\beta = 1$, $\gamma = 2$, $\lambda = 1000$, $\delta = 0.8$, $K = 200$, $c = 10$, $h_0 = 1$, $h = 0.5$, $\theta = 2$, $\eta = 0.04$ and $v = 1000$. From Algorithm 1, it follows that the optimal frequency of advertisement per cycle is $A^* = 2$, the optimal selling price is $p^* = 35.6573$, the optimal inventory cycle is $T^* = 1.58518$, the economic lot size is $Q^* = 2863.18$ and the optimal profit is $B^* = 42454.51$.

In order to know the effect of the parameters of the demand rate function on the optimal inventory policy, we summarize in Table 8 the solutions of the inventory problem for different values of the parameters α, β, γ, λ, δ and η.

From the computational results, we can establish the following managerial insights:

1. The optimal frequency of advertisement per cycle A^*, the best selling price p^*, the optimal lot size Q^* and the maximum profit B^* do not decrease as the parameter α or the parameter λ increase. However, the optimal inventory cycle T^* does not follow a pattern of increasing or decreasing.
2. The optimal frequency of advertisement per cycle A^*, the optimal selling price p^*, the economic order quantity Q^* and the optimal profit B^* do not increase as the parameter β or the parameter γ increase. However, the optimal inventory cycle T^* does not follow a pattern of increasing or decreasing.
3. The optimal inventory policy is not very sensitive to changes in the parameter δ. Even so, if the value of δ increases then the optimal advertising frequency per cycle A^*, the best selling price p^*, the economic order quantity Q^*, the maximum profit B^* and the optimal inventory cycle T^* do not increase.

Table 8. Effects of α, β, γ, λ, δ and η on optimal policy

α		1000	1500	1800	1900	2100	2200	2500	3000
	A^*	0	0	1	1	2	2	3	4
	p^*	29.7917	32.8039	34.5478	35.0815	36.1723	36.6794	38.1943	40.5522
	T^*	0.852157	0.776596	1.33746	1.31885	1.56473	1.54527	1.68502	1.74783
	Q^*	947.988	1105.80	2208.96	2263.43	2929.24	2994.64	3635.56	4390.81
	B^*	20552.87	30660.36	37546.02	39964.29	45014.80	47629.57	55825.26	70635.06
β		0.5	0.75	0.9	0.95	1.05	1.1	1.25	1.5
	A^*	5	3	2	2	1	1	1	0
	p^*	48.7931	40.5581	37.3552	36.4726	34.8500	34.1462	32.2956	29.7796
	T^*	2.05767	1.77522	1.58003	1.58262	1.30325	1.30526	1.31117	0.732453
	Q^*	4000.25	3314.32	2879.57	2871.30	2310.98	2304.74	2286.60	1223.02
	B^*	69681.86	52480.80	45943.36	44128.84	40910.63	39490.78	35765.24	30947.90
γ		1	1.5	1.8	1.9	2.1	2.2	2.5	3
	A^*	1651	29	6	3	0	0	0	0
	p^*	1521.89	118.274	52.5931	42.7682	30.2555	26.2243	18.6060	12.5992
	T^*	14.0242	3.65112	2.20504	1.78357	0.725717	0.731101	0.760544	0.887904
	Q^*	27880.9	7168.90	4165.94	3287.14	1242.92	1226.97	1145.94	887.904
	B^*	2827115.22	198633.77	74363.52	55857.90	32564.98	25139.87	11065.88	1261.34
λ		500	750	900	950	1050	1100	1250	1500
	A^*	0	1	1	1	2	2	2	3
	p^*	32.8025	34.2756	35.0808	35.3447	35.9163	36.17321838	36.9322	38.1989
	T^*	0.762660	1.33611	1.31475	1.30791	1.57716	1.569293561	1.54657	1.70681
	Q^*	1086.02	2163.78	2256.46	2286.98	2900.54	2937.686426	3047.90	3681.94
	B^*	30653.31	36346.88	39960.04	41192.10	43730.88	45020.91	48971.89	55861.45
δ		0.4	0.6	0.72	0.76	0.84	0.88	1	1.2
	A^*	2	2	2	2	2	2	1	1
	p^*	35.6865	35.6692	35.6616	35.6594	35.6553	35.6535	35.6005	35.5959
	T^*	1.72690	1.64458	1.60687	1.59572	1.57521	1.56574	1.26431	1.23522
	Q^*	3115.39	2969.00	2901.83	2881.96	2845.39	2828.51	2252.14	2200.73
	B^*	42624.36	42529.32	42482.47	42468.19	42441.39	42428.79	42397.53	42364.08
η		0.02	0.03	0.036	0.038	0.042	0.044	0.05	0.06
	A^*	0	0	1	1	2	2	3	5
	p^*	35.5336	35.5336	35.6065	35.6064	35.6571	35.6568	35.6985	35.7678
	T^*	0.721952	0.721952	1.30242	1.30182	1.58402	1.58285	1.78204	2.07075
	Q^*	1254.29	1254.29	2313.02	2315.17	2867.39	2871.61	3295.83	3967.45
	B^*	42208.24	42208.24	42317.74	42377.72	42550.95	42647.61	42985.61	43687.23

4. The optimal frequency of advertisement per cycle A^*, the economic order quantity Q^* and the optimal profit B^* do not decrease as the parameter η increases. However, the optimal inventory cycle T^* and the optimal selling price p^* do not follow a pattern of increasing or decreasing.

Table 9 shows the results of the sensitivity analysis, in which we explore the impact on the optimal policy and the maximum profit of changing the parameters of the inventory system h_0, h, θ, v, K and c. According to the obtained results, we can establish the following issues:

1. The optimal frequency of advertisement per cycle A^* is not sensitive to changes in the parameters h_0 or K.
2. The optimal selling price p^* and the optimal inventory cycle T^* increase, while the optimal lot size Q^* and the maximum profit B^* decrease as the parameter h_0 increases.
3. The optimal frequency of advertisement per cycle A^*, the economic order quantity Q^*, the optimal inventory cycle T^* and the optimal profit B^* do not increase as the parameter h increases. However, the optimal selling price p^* does not follow a pattern of increasing or decreasing.
4. The optimal frequency of advertisement per cycle, the best selling price, the economic order quantity and the inventory cycle do not increase as the parameter θ increases. However, the maximum profit is first decreasing and, later, increasing.
5. The optimal frequency of advertisement per cycle A^* and the maximum profit B^* do not increase as the parameter v increases. However, the optimal inventory cycle, the best selling price and the lot size do not follow a pattern of increasing or decreasing.
6. The optimal selling price p^*, the economic order quantity Q^* and the optimal inventory cycle T^* increase, while the optimal profit B^* decreases as the parameter K increases.
7. The optimal frequency of advertisement per cycle A^*, the optimal lot size Q^* and the maximum profit B^* do not increase, while the best selling price p^* increases as the parameter c increases. However, the optimal inventory cycle T^* does not follow a pattern of increasing or decreasing.

The impact of deviations in the estimation of the parameters of the demand rate funcion on the optimal solution and the maximum profit is shown in Table 10. In this table, A', p', T', Q' and B' are the optimal values calculated under the wrong parameter values, and A^*, p^*, T^*, Q^* and B^* are the optimal values computed under the correct parameter values. Thus, we have considered the effect of 50% over and under-estimation of the parameters α, β, γ, λ, δ and η.

From the results shown in Table 10, we can make the following inferences:

1. The optimal frequency of advertisement, the best selling price p^*, the economic lot size Q^* and the maximum profit B^* are much more sensitive to the parameter γ as compared to other parameters considered. Thus, for deviations in estimating the parameter γ, the ratio $(A'+1)/(A^*+1)$ varies from 0.333333 to 550.667, the ratio p'/p^* varies from 0.353342 to 42.6810 and the ratio B'/B^* varies from 0.029710 to 66.5916.

Table 9. Effects of h_0, h, θ, v, K and c on optimal policy

h_0		0.5	0.75	0.9	0.95	1.05	1.1	1.25	1.5
	A^*	2	2	2	2	2	2	2	2
	p^*	35.4703	35.5637	35.6198	35.6385	35.6760	35.6948	35.7511	35.8452
	T^*	1.58077	1.58297	1.58429	1.58474	1.58563	1.58608	1.58742	1.58969
	Q^*	2877.19	2870.20	2865.99	2864.58	2861.77	2860.35	2856.11	2848.99
	B^*	43361.10	42906.94	42635.27	42544.86	42364.24	42274.03	42003.84	41554.91
h		0.25	0.375	0.45	0.475	0.525	0.55	0.625	0.75
	A^*	3	2	2	2	2	1	1	1
	p^*	35.6590	35.6429	35.6519	35.6546	35.6599	35.6098	35.6145	35.6216
	T^*	2.25429	1.74434	1.64171	1.61245	1.55967	1.26059	1.20809	1.13698
	Q^*	4118.56	3152.52	2965.94	2912.75	2816.80	2244.65	2150.75	2023.55
	B^*	42922.42	42644.89	42526.36	42489.81	42420.38	42393.14	42330.98	42237.62
θ		1	1.5	1.8	1.9	2.1	2.2	2.5	3
	A^*	5	2	2	2	2	1	1	1
	p^*	35.8253	35.6825	35.6666	35.6618	35.6530	35.5995	35.5906	35.5788
	T^*	3.47274	1.80816	1.65824	1.61962	1.55428	1.26856	1.23039	1.18542
	Q^*	6404.04	3262.53	2993.98	2924.83	2807.87	2259.79	2192.60	2113.48
	B^*	42785.32	42507.60	42472.22	42462.89	42446.96	42445.31	42455.99	42471.87
v		500	750	900	950	1050	1100	1250	1500
	A^*	5	3	2	2	1	1	1	0
	p^*	35.6770	35.6677	35.6476	35.6525	35.6093	35.6121	35.6205	35.5336
	T^*	1.68206	1.63705	1.53539	1.56068	1.31910	1.33652	1.38623	0.721952
	Q^*	3121.03	2989.80	2774.35	2819.47	2348.88	2379.62	2467.29	1254.29
	B^*	43379.94	42806.52	42582.69	42518.09	42399.63	42361.97	42251.80	42208.24
K		100	150	180	190	210	220	250	300
	A^*	2	2	2	2	2	2	2	2
	p^*	35.6525	35.6549	35.6563	35.6568	35.6578	35.6582	35.6597	35.6620
	T^*	1.56068	1.57303	1.58034	1.58277	1.58759	1.58999	1.59716	1.60896
	Q^*	2819.47	2841.50	2854.54	2858.87	2867.47	2871.75	2884.52	2905.56
	B^*	42518.09	42486.18	42467.15	42460.83	42448.21	42441.92	42423.09	42391.90
c		5	7.5	9	9.5	10.5	11	12.5	15
	A^*	3	2	2	2	2	1	1	1
	p^*	33.8662	34.7311	35.2841	35.4703	35.8452	35.9820	36.5520	37.5195
	T^*	1.74466	1.56404	1.57646	1.58077	1.58969	1.30864	1.32039	1.34180
	Q^*	3417.35	2931.52	2891.02	2877.19	2848.99	2294.36	2258.83	2196.60
	B^*	51831.38	47055.91	44274.60	43361.10	41554.91	40670.68	38072.54	33887.07

Table 10. Sensitivity analysis of α, β, γ, λ, δ and η on optimal policy

Parameter		Percentage of under -or over- estimation of parameter							
		-50	-25	-10	-5	5	10	25	50
α	$\frac{A'+1}{A^*+1}$	0.333333	0.333333	0.666667	0.666667	1	1	1.33333	1.66667
	p'/p^*	0.835500	0.919976	0.968885	0.983852	1.01444	1.02867	1.07115	1.13728
	T'/T^*	0.537576	0.489909	0.843725	0.831985	0.987096	0.974823	1.06298	1.10260
	Q'/Q^*	0.331097	0.386215	0.771508	0.790532	1.02307	1.04592	1.26977	1.53354
	B'/B^*	0.484115	0.722193	0.884382	0.941344	1.06031	1.12190	1.31494	1.66378
β	$\frac{A'+1}{A^*+1}$	2	1.33333	1	1	0.666667	0.666667	0.666667	0.333333
	p'/p^*	1.36839	1.13744	1.04762	1.02286	0.977361	0.957621	0.905721	0.835162
	T'/T^*	1.29806	1.11988	0.996748	0.998385	0.822146	0.823415	0.827144	0.46206
	Q'/Q^*	1.39714	1.15757	1.00573	1.00284	0.807137	0.804959	0.798624	0.427156
	B'/B^*	1.64133	1.23617	1.08218	1.03944	0.963634	0.930190	0.842437	0.728966
γ	$\frac{A'+1}{A^*+1}$	550.667	10	2.33333	1.33333	0.333333	0.333333	0.333333	0.333333
	p'/p^*	42.6810	3.31697	1.47496	1.19942	0.84851	0.735455	0.521802	0.353342
	T'/T^*	8.84707	2.30328	1.39103	1.12515	0.457813	0.461209	0.47978	0.560127
	Q'/Q^*	9.7377	2.50383	1.45501	1.14807	0.434104	0.428535	0.400235	0.310112
	B'/B^*	66.5916	4.67874	1.75160	1.31571	0.767056	0.592160	0.260653	0.029710
λ	$\frac{A'+1}{A^*+1}$	0.333333	0.666667	0.666667	0.666667	1	1	1	1.33333
	p'/p^*	0.919939	0.96125	0.983833	0.991233	1.00726	1.01447	1.03575	1.07128
	T'/T^*	0.48112	0.842873	0.829397	0.825086	0.994939	0.989977	0.975644	1.07673
	Q'/Q^*	0.379307	0.755727	0.78810	0.798755	1.01305	1.02602	1.06452	1.28597
	B'/B^*	0.722027	0.856137	0.941244	0.970264	1.03006	1.06045	1.15351	1.31580
δ	$\frac{A'+1}{A^*+1}$	1	1	1	1	1	1	0.666667	0.666667
	p'/p^*	1.00082	1.00033	1.00012	1.00006	0.999945	0.999893	0.998406	0.998280
	T'/T^*	1.08940	1.03747	1.01368	1.00665	0.993706	0.987737	0.797580	0.779228
	Q'/Q^*	1.08809	1.03696	1.01350	1.00656	0.993787	0.987893	0.786587	0.768631
	B'/B^*	1.00400	1.00176	1.00066	1.00032	0.999691	0.999394	0.998658	0.997870
η	$\frac{A'+1}{A^*+1}$	0.333333	0.333333	0.666667	0.666667	1	1	1.33333	2
	p'/p^*	0.996532	0.996532	0.998577	0.998574	0.999994	0.999987	1.00116	1.00310
	T'/T^*	0.455438	0.455438	0.821622	0.821242	0.999264	0.998529	1.12419	1.30632
	Q'/Q^*	0.438077	0.438077	0.807853	0.808602	1.00147	1.00295	1.15111	1.38568
	B'/B^*	0.994199	0.994199	0.996778	0.998191	1.00227	1.00455	1.01251	1.02904

2. The maximum profit B^* and the optimal selling price p^* are almost insensitive to the parameters δ or η. Thus, p'/p^* varies from 0.998280 to 1.00082 for δ, and from 0.996532 to 1.00310 for η. The ratio B'/B^* varies from 0.997870 to 1.00400 for δ, and from 0.994199 to 1.02904 for η.
3. In general, the optimal inventory policy is less sensitive to the variation of the parameter λ as compared to the parameters α or β.
4. The economic order quantity Q^* and the maximum profit B^* are more sensitive to the parameter α than to the parameter β. However, the optimal frequency of advertisement A^*, the optimal selling price p^* and the optimal inventory cycle T^* are more sensitive to β than to α.
5. The effect of over or under estimating the parameter δ on the economic order quantity Q^* or on the optimal inventory cycle T^* is quite similar. The same can be said with respect to the parameter η.
6. The under-estimation of the parameters α and λ produces a lower frequency of advertisement A', a lower selling price p' and a lower profit B', while the under-estimation of the parameters β, γ and δ produces a higher frequency of advertisement A', a higher selling price p', a higher lot size Q' and a higher profit B'. However, the under-estimation of the parameter η produces a lower frequency of advertisement A', a lower lot size Q' and a lower profit B'.

Table 11 shows the impact of deviations in the estimation of the parameters h_0, h, θ, v, K and c on the optimal solution and the maximum profit. As in Table 10, A', p', T', Q' and B' are the optimal values calculated under the wrong parameter values, and A^*, p^*, T^*, Q^* and B^* are the optimal values computed under the correct parameter values. From these results, we can make the following observations:

1. In general, the optimal inventory solution is almost insensitive to the parameters h_0 or K. Thus, the optimal frequency of advertisement does not change for variarions of these parameters. However, p'/p^* varies from 0.994755 to 1.00527 for h_0, and from 0.999866 to 1.00013 for K; Q'/Q^* varies from 0.995046 to 1.00489 for h_0, and from 0.984737 to 1.01480 for K. The maximum profit is a little more sensitive, since the ratio B'/B^* varies from 0.978810 to 1.02135 for h_0 and from 0.998525 to 1.00150 for K.
2. The optimal frequency of advertisement A^* is more sensitive to the parameters θ and v than the other parameters considered in this table.
3. As expected, the optimal selling price p^* and the maximum profit B^* are more sensitive to the parameter c than the other parameters considered here.
4. The economic order quantity Q^* and the optimal inventory cycle T^* are more sensitive to the parameters θ and h than the other parameters considered in this study. Moreover, the under-estimation of θ and h is more sensitive than the over-estimation.
5. The effect of over or under estimation of the parameter h_0 on the optimal policy or on the maximum profit is quite similar. Also, the effect of over or under estimation of the parameter c on the optimal selling price is almost the same.

Table 11. Sensitivity analysis of h_0, h, θ, v, K and c on optimal policy

Parameter		Percentage of under -or over- estimation of parameter							
		-50	-25	-10	-5	5	10	25	50
h_0	$\frac{A'+1}{A^*+1}$	1	1	1	1	1	1	1	1
	p'/p^*	0.994755	0.997374	0.998949	0.999474	1.00053	1.00105	1.00263	1.00527
	T'/T^*	0.997219	0.998602	0.999439	0.999719	1.00028	1.00056	1.00141	1.00284
	Q'/Q^*	1.00489	1.00245	1.00098	1.00049	0.999507	0.999014	0.997530	0.995046
	B'/B^*	1.02135	1.01066	1.00426	1.00213	0.997874	0.995749	0.989384	0.978810
h	$\frac{A'+1}{A^*+1}$	1.33333	1	1	1	1	0.666667	0.666667	0.666667
	p'/p^*	1.00005	0.999598	0.999848	0.999925	1.00007	0.998667	0.998800	0.999000
	T'/T^*	1.42210	1.10041	1.03566	1.01720	0.983906	0.795233	0.762113	0.717252
	Q'/Q^*	1.43846	1.10106	1.03589	1.01732	0.983802	0.783973	0.751175	0.706751
	B'/B^*	1.01102	1.00448	1.00169	1.00083	0.999196	0.998554	0.997090	0.994891
θ	$\frac{A'+1}{A^*+1}$	2	1	1	1	1	0.666667	0.666667	0.666667
	p'/p^*	1.00471	1.00071	1.00026	1.00013	0.999879	0.998378	0.998129	0.997798
	T'/T^*	2.19075	1.14067	1.04609	1.02173	0.980508	0.800259	0.776181	0.747810
	Q'/Q^*	2.23669	1.13948	1.04568	1.02153	0.980683	0.789261	0.765794	0.738160
	B'/B^*	1.00779	1.00125	1.00042	1.00020	0.999822	0.999783	1.00003	1.00041
v	$\frac{A'+1}{A^*+1}$	2	1.33333	1	1	0.666667	0.666667	0.666667	0.333333
	p'/p^*	1.00055	1.00029	0.999729	0.999866	0.998653	0.998733	0.998969	0.996532
	T'/T^*	1.06111	1.03272	0.968589	0.984542	0.832143	0.843130	0.874495	0.455438
	Q'/Q^*	1.09006	1.04423	0.968975	0.984737	0.820377	0.831111	0.861730	0.438077
	B'/B^*	1.02180	1.00829	1.00302	1.00150	0.998707	0.997820	0.995225	0.994199
K	$\frac{A'+1}{A^*+1}$	1	1	1	1	1	1	1	1
	p'/p^*	0.999866	0.999933	0.999973	0.999987	1.00001	1.00003	1.00007	1.00013
	T'/T^*	0.984542	0.992331	0.996946	0.998475	1.00152	1.00304	1.00755	1.01500
	Q'/Q^*	0.984737	0.992428	0.996985	0.998495	1.00150	1.00300	1.00746	1.01480
	B'/B^*	1.00150	1.00075	1.00030	1.00015	0.999852	0.999703	0.999260	0.998525
c	$\frac{A'+1}{A^*+1}$	1.33333	1	1	1	1	0.666667	0.666667	0.666667
	p'/p^*	0.949769	0.974025	0.989535	0.994755	1.00527	1.00911	1.02509	1.05223
	T'/T^*	1.10061	0.986660	0.994497	0.997219	1.00284	0.825544	0.832957	0.846465
	Q'/Q^*	1.19355	1.02387	1.00973	1.00489	0.995046	0.801332	0.788925	0.767189
	B'/B^*	1.22087	1.10838	1.04287	1.02135	0.978810	0.957982	0.896784	0.798197

5.2 Management insights

In this section, some comments or suggestions to inventory systems managers that could help in improving the effectiveness of the inventory control practices are proposed. The sensitivity analysis reveals that the maximum profit is almost insensitive to variations of the index δ of the power demand pattern or changes in the power η of the advertising frequency in the demand rate. Furthermore, the maximum profit is not very sensitive to movements of the scale parameter λ of the part of the time-dependent demand or changes in the parameter β of the price-dependent demand. Thus, the inventory manager should not worry about those parameters. However, the manager should be aware of the parameters α and γ. In relation to the scale parameter α of the price-dependent demand, a 10% or 25% decrease in its value results in a profit drop of almost 12% or 28%. Therefore, the decision maker should boost the price-dependent demand by implementing policies that increase the scale parameter α of the demand (e.g., increasing the marketing policies such as sales or quantity discount).

With respect to the exponent γ of the selling price in the demand rate, this has the greatest impact on the total profit per unit time among the parameters associated with customer demand. Thus, the manager should reduce this parameter as much as possible and this would significantly increase the profit. One possibility would be to increase the maximum selling price, keeping the parameters α and β constant. With this, the parameter γ would decrease.

The variation in the parameters that appear in the holding cost do not have any great influence on the behavior of the maximum profit. The effect of h_0, h or θ on the profit is almost negligible. Thus, a 25% decrease in the value of the parameter h_0, h or θ leads to an increase in profit of less than 1.1%, 0.5% and 0.2%, respectively. Thus, to obtain a major benefit, the manager should fundamentally try to reduce the fixed unit cost of storage.

From the findings obtained with the sensitivity analysis, we can deduce that the impact of the ordering cost K on the total profit per unit time is almost negligible. A 25% increase in the order cost leads to a decrease in the profit of only 0.074%. Moreover, the cost v of an advertisement has very little impact on profit. Thus, if v decreases by 50%, that leads to the profit increasing by only 2.18%. Hence, the manager should not worry about possible changes in the K and v parameters, since the effect they would have on profit is relatively small.

Finally, an increment in the unit purchasing cost c has a negative effect on the maximum profit per unit time. Thus, a 10% or 25% increase in the value of c leads to a decrease of less than 4.21% or 10.33% of the maximum profit, respectively. For this reason, the decision maker should try to reduce the unit purchasing cost as much as possible. One way to reduce this cost would be to negotiate a reduction in the purchase price of the product with the supplier, promising, in exchange for a price reduction, an increase in the quantity requested to replenish the inventory.

6 Conclusions

We have developed an inventory model for items whose demand depends on marketing strategies, the selling price and time. More specifically, we assume that the demand rate is the product of a power function dependent on the frequency of advertisement and a function that is the sum of two power functions, one depending on the unit selling price and the other on time. In addition, we assume that the maintenance cost per unit of product has two components: a fixed cost and a variable cost that is a power function of the time spent in inventory. The aim is to maximize the average profit per unit of time. This objective function can have, depending on the parameters of the model, several local optima. To solve the problem, we present an efficient algorithm that analyzes all the possible cases that may occur in the inventory system and finds a maximum. Although, in general, optimal solutions cannot be expressed in closed form, they can easily be obtained using some numerical method, such as the bisection method, in solving the equations that help us identify the best inventory policy.

The numerical sensitivity analysis shows that the optimal inventory solution and the maximum profit are very sensitive to changes in the exponent of the selling price in the demand rate function. Nevertheless, they are hardly sensitive to changes in the ordering cost.

Some possible extensions of the model that can be future research lines are: (i) to consider discounts in the unit purchasing cost; (ii) to analyze the case of perishable or deteriorating items; (iii) to allow shortages in the inventory system; (iv) to consider other functions for the unit holding cost; (v) to assume a finite replenishment rate and (vi) to study the same inventory system under stochastic demand.

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Appendix

Proof of Lemma 1.

The first derivative of $B_{A,p}(T)$ is

$$B'_{A,p}(T) = \frac{Av + K}{T^2} - \theta h f(p) \, (A+1)^{\eta} \, T^{\theta-1}$$

and the second derivative is given by:

$$B''_{A,p}(T) = -\left(\frac{2(Av + K)}{T^3} + \theta(\theta - 1) h f(p) \, (A+1)^{\eta} \, T^{\theta-2} \right).$$

As $\theta \geq 1$, then $B''_{A,p}(T) < 0$ for all $T > 0$ and, therefore, $B_{A,p}(T)$ is a strictly concave function. Since $\lim_{T\uparrow 0} B_{A,p}(T) = \lim_{T\to\infty} B_{A,p}(T) = -\infty$, the maximum of $B_{A,p}(T)$ is reached at the point $T^*(A,p)$ given by (7), which is the solution of the equation $B'_{A,p}(T) = 0$. ■

Proof of Lemma 2.

Let $g_1(p)$ be the function given in (11) defined for all $p > 0$. The derivative of $g_1(p)$ is

$$g_1'(p) = \beta\gamma p^{\gamma-2}\left[(\gamma-1)(c+h_0) - (\gamma+1)p\right].$$

Then:

(a) If $\gamma = 1$, then $g_1'(p) < 0$ and $g_1(p)$ is a strictly decreasing function.

(b) If $\gamma > 1$, then $g_1(p)$ has a unique local maximum at the point $p_0 = (c+h_0)(\gamma-1)/(\gamma+1)$. This is because $g_1'(p) > 0$ for all $p < p_0$ and $g_1'(p) < 0$ for all $p > p_0$. Also, the second derivative of g_1 at the point p_0 is $g_1''(p_0) = -\beta\gamma(\gamma+1)^{3-\gamma}\left[(\gamma-1)(c+h_0)\right]^{\gamma-2} < 0$.

Since $g_1(c) = \alpha - \beta c^\gamma + \lambda + \beta\gamma h_0 c^{\gamma-1} > 0$ and $g_1(p_m) = \lambda - \alpha\gamma(1-(c+h_0)/p_m)$, we can conclude that if $\lambda \geq \alpha\gamma(1-(c+h_0)/p_m)$, then $g_1(p) > 0$ for all $p \in [c, p_m)$. Note that this is true when $\gamma = 1$ or when $\gamma > 1$.

Otherwise, i.e., $\lambda < \alpha\gamma(1-(c+h_0)/p_m)$, then $g_1(c) > 0$ and $g_1(p_m) < 0$. Therefore, the function $g_1(p)$ has a single root p_1 in the interval (c, p_m). ■

Proof of Theorem 1.

1. It is immediate because $g_1(p) > 0$ for all $p \in [c, p_m)$ and, therefore, $G_A(p)$ is a strictly increasing function on such an interval.
2. In this case, $G_A(p)$ has a single local extremum $\widetilde{p}(A)$ in the interval (p_1, p_m). Thus, $G_A(p)$ is a strictly increasing function on $(c, \widetilde{p}(A))$ and strictly decreasing on $(\widetilde{p}(A), p_m)$. Therefore, $G_A(p)$ reaches its maximum value at the point $\widetilde{p}(A) = \arg_{p\in(p_0,p_1)}\{G_A'(p) = 0\}$.
3. Note that, in this case, the function $G_A(p)$ either has no local extrema or has two local extrema in the interval (p_1, p_m). We can consider the following situations:

   (a) If $p_2 \geq p_m$, then the function $G_A'(p)$ has no roots in the considered interval (p_1, p_m). Therefore, the function $G_A(p)$ is strictly increasing on that interval and the maximum benefit is obtained at the point $p^*(A) = p_m$.

   (b) Otherwise, we can divide the proof into two sections:

      i. If $p_2 < p_m$ and $G_A'(p_2) > 0$, then $G_A'(p)$ has no roots in the interval (p_1, p_m). Thus, $G_A(p)$ is strictly increasing on (p_1, p_m) and $p^*(A) = p_m$.

      ii. If $p_2 < p_m$ and $G_A'(p_2) = 0$, then $G_A(p)$ is a non-decreasing function on (p_1, p_m). Consequently, $G_A(p)$ reaches its maximum at the point $p^*(A) = p_m$.

   (c) Finally, if $p_2 < p_m$ and $G_A'(p_2) < 0$, then $G_A(p)$ has two local extremes in the interval (p_1, p_m): $\widetilde{p}(A)$ and $\widetilde{p}_1(A)$, with $\widetilde{p}(A) < p_2 < \widetilde{p}_1(A)$. Now the function $G_A(p)$ is strictly increasing on $(c, \widetilde{p}(A))$, strictly decreasing on $(\widetilde{p}(A), \widetilde{p}_1(A))$ and strictly increasing on $(\widetilde{p}_1(A), p_m)$. Therefore, $G_A(p)$ reaches its maximum at the point $p^*(A) = \widetilde{p}(A)$ or at the point $p^*(A) = p_m$, depending on the value that the function $G_A(p)$ takes at both points. ■

Proof of Theorem 2.

From (15), the second derivative is

$$B''_{p,T}(A) = \left[(p - c - h_0)(\alpha - \beta p^\gamma + \lambda) - hf(p)T^\theta\right] \eta(\eta - 1)\,(A + 1)^{\eta-2}\,.$$

Taking into account again Eq. (15), if $(p-c-h_0)(\alpha - \beta p^\gamma + \lambda) - hf(p)T^\theta \leq 0$, then $B'_{p,T}(A) < 0$ and $B_{p,T}(A)$ is a strictly decreasing function. However, if $(p - c - h_0)(\alpha - \beta p^\gamma + \lambda) - hf(p)T^\theta > 0$, then $B''_{p,T}(A) < 0$ and the function $B_{p,T}(A)$ is strictly concave. ■

Proof of Corollary 2.

In this case, we have $G'_A(p^*) = 0$. As $A \geq 0$, from (10), we obtain the optimal inventory cycle T_1^* given by (18) and, from (2), we get the optimal lot size Q_1^* . Substituting T_1^* into Eq. (8), we deduce the expression for the maximum profit B_1^* given by (19). ■

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