Introduction to Quantum Field Theory
Quantum Electrodynamics

Rafael Juan Alvarez Reyes

Supervisor: Vicente Delgado

Universidad de La Laguna

September, 2017

Contents

1 Summary

In this work, we present an introductory study of Quantum Field Theory. Starting from the Poincaré-covariant free classical fields, the canonical quantization method is applied to develop the theory of the corresponding quantum fields. The study of interacting fields using the Interaction picture formalism is considered afterwards. Finally, particularizing to the case of spinor Quantum Electrodynamics (QED) the transition amplitudes and the corresponding Feynman diagrams of the different processes are obtained and the Feynman rules for QED are derived. As we will see, the transition amplitudes of all physical processes of QED can be obtained using these Feynman rules.

Resumen

En este trabajo presentamos un estudio introductorio de la Teoría Cuántica de Campos. Partiendo de los campos clásicos libres covariantes Poincaré, se aplica el método de cuantización canónica para desarrollar la teoría de los correspondientes campos cuánticos. Se aborda a continuación el estudio de los campos en interacción usando la imagen de Interacción. Finalmente, particularizando al caso de la Electrodinámica Cuántica espinorial (QED) se obtienen las amplitudes de transición y los correspondientes diagramas de Feynman de los distintos procesos y se derivan las reglas de Feynman de la QED. Como veremos, las amplitudes de transición de todos los procesos físicos de la QED pueden obtenerse usando estas reglas de Feynman.

2 Introduction, Objectives and Methodology

Quantum Field Theory (QFT) is an attempt of theoretical physicists to unify Quantum Mechanics and Special Relativity that dates back to the beginning of the twentieth century. The inception of QFT is usually considered to be Dirac's famous 1927 paper on "The quantum theory of the emission and absorption of radiation", where he gave a theoretical description of how photons appear in the quantization of the electromagnetic radiation field [1, 2]. Besides, this physicist derived in 1928 the equation named after him which describes the behavior of fermions and with which he predicted the existence of antimatter, obtained in a natural way the spin property of particles, and accomplished the unification of Quantum Mechanics and Special Relativity for the first time, which is the essence of QFT [3]. Nevertheless, it has long seemed to Steven Weinberg that a better starting point for QFT is Wigner's definition of particles as representations of the inhomogeneous Lorentz group, even though this work was not published until 1939 [4].

Ordinary Quantum Mechanics is not a Lorentz covariant theory, so it does not allow the study of systems in which relativistic effects can not be neglected. When the masses of the particles involved in a physical process are negligible in relation to the typical energies (as always happens in those processes due to the exchange of individual photons), it is necessary to consider the study in terms of a QFT [4, 5]. The combination of Quantum Mechanics and Special Relativity implies the existence of virtual pairs and particle fluctuations and thus the nonconservation of particle number.

In its more modern manifestly covariant formulation, Quantum Field Theory was developed in the mid-twentieth century. It constitutes the theoretical basis of the Standard Model of fundamental particles and interactions and, in spite of their short existence, it has been shown to be a useful tool not only in the field of High Energy Physics but also in the fields of Condensed Matter, Statistical Physics and Bose-Einstein Condensation. Moreover, since quantum fields are objects defined in the whole space at any instant of time, it is not surprising that they have also proved to be very useful in the development of Cosmology [6, 7].

In this work, we give a self-contained introduction to the topic of Quantum Field Theory based mainly on Refs. [8, 9, 10, 11]. Our final goal is to derive the Feynman rules for spinor Quantum Electrodynamics (QED) and use them to obtain the transition amplitudes for the different relevant physical processes. To this end, we start from the Poincaré-covariant free classical fields and formulate the theory of the corresponding quantum fields by applying the Canonical Quantization method. As we will see, according to this theory elementary particles arise from the quantization of the fields. In particular, electrons and positrons are quanta of the Dirac field just as photons are quanta of the electromagnetic field. Then, by making use of the Interaction Picture we develop the corresponding quantum theory for the interacting fields. Particularizing to spinor QED and after applying Wick's theorem we obtain the Feynman diagrams in configuration space. We also illustrate how to calculate analytically the transition amplitudes of relevant physical processes such as the Compton Scattering and derive from the obtained analytic expressions the corresponding Feynman rules and diagrams in momentum space. Finally we use these rules, which associate a specific graphical representation with every analytic contribution and vice versa, to obtain the transition amplitudes of the other physical processes relevant to QED.

3 Lagrangian Formalism

The Lagrangian formalism of Quantum Field Theory allows us to accommodate the following basic features [5]:

- Space-time symmetry in terms of Lorentz invariance (more precisely, invariance under transformations of the Poincaré symmetry group), as well as internal symmetries like gauge symmetries
- Causality
- Local interactions

Particles are described by fields that are operators on the quantum mechanical Hilbert space of the particle states, acting as creation and annihilation operators for particles and antiparticles. In the Standard Model, the following classes of particles appear, each of them described by a specific type of fields (which is a consequence of group theory):

- Spin-0 bosons, described by scalar fields $\phi(x)$
- Spin-1 bosons, described by vector fields $A_\mu(x)$
- Spin-1/2 fermions, described by spinor fields $\Psi(x)$.

In classical mechanics, for a system of n generalized coordinates q_i and velocities $\dot{q}_i$ governed by the Lagrangian $L(q_1, ..., q_n, \dot{q}_1, ..., \dot{q}_n)$, the Hamiltonian's Principle yields the Euler–Lagrange equations of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, \quad i = 1, ..., n. \tag{1}$$

Proceeding to Field Theory, one has to perform the replacement

$$q_i \to \phi(x), \quad \dot{q}_i \to \partial_\mu \phi(x), \quad L(q_1, ..., q_n, \dot{q}_1, ..., \dot{q}_n) \to \mathscr{L}(\phi(x), \partial_\mu \phi(x)), \tag{2}$$

where the dynamics of the physical system involving a set of fields, denoted here by a generic field variable $\phi(x)$, is determined by the Lorentz-invariant Lagrangian density $\mathscr{L}$. By defining the action

$$S[\phi(x)] = \int dt L(t) = \int d^4x \mathscr{L}(\phi(x), \partial_\mu \phi(x)), \tag{3}$$

and using Hamilton's principle,

$$\delta S[\phi(x)] = \int d^4x \left\{\mathscr{L}[\phi + \delta\phi, \partial_\mu \phi + \delta(\partial_\mu \phi)] - \mathscr{L}[\phi, \partial_\mu \phi]\right\} = 0, \tag{4}$$

one can obtain the equations of motion for each field (or field components),

$$\partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi(x))} - \frac{\partial \mathscr{L}}{\partial \phi(x)} = 0. \tag{5}$$

4 Noether's Theorem

Noether's theorem states that there exists a conservation law associated with every differentiable symmetry of the action of a physical system. The theorem was proven by mathematician Emmy Noether in 1915 and published in 1918 [12].

Here we are interested in the field theory version of Noether's theorem. Let $\{\phi^a(x), \quad a = 1, 2, ...\}$ be a set of differentiable fields defined over all space and time ($x = t, \mathbf{x}$) and governed by the Lagrangian density $\mathscr{L}(x)$. Consider the following infinitesimal transformation:

$$x^\mu \to x^{\mu\prime} = x^\mu + f^\mu{}_k(x)\epsilon^k \tag{6}$$

$$\phi^a(x) \to \phi^{a\prime}(x') = \phi^a(x) + F^a{}_k(x)\epsilon^k, \tag{7}$$

where $\left\{\epsilon^k\right\}$ are the set of parameters to be modified infinitesimally, $\{f^\mu{}_k(x)\}$ the set of infinitesimal generators of the spatial-time coordinates, and $\{F^a{}_k(x)\}$ the set of infinitesimal generators of the respective fields.

The condition for these infinitesimal transformations to generate a physical symmetry is that the action Eq. (3) is left invariant. This will certainly be true if the Lagrangian density of the system $\mathscr{L}$ is left invariant under these transformations ($\mathscr{L}(x) \to \mathscr{L}'(x') = \mathscr{L}(x)$). Since the equations of motions are extracted from the Lagrangian density using the Hamilton's principle, if $\mathscr{L}$ is invariant, the motion equations are invariant too.

If there exist this $\mathscr{L}$ invariance, it can be proven through Noether's theorem that there are n conserved current densities which comply with

$$\partial_\mu J^\mu{}_k = 0, \qquad k = 1, 2, ..., n, \tag{8}$$

where

$$J^\mu{}_k = \mathscr{L} f^\mu{}_k + \frac{\partial \mathscr{L}}{\partial\left(\partial_\mu \phi^a\right)}\left[F^a{}_k - \left(\partial_\nu \phi^a\right) f^\nu{}_k\right], \qquad k = 1, 2, ..., n, \tag{9}$$

and respective n conserved charges

$$Q_k = \int J^0{}_k d^3x, \qquad k = 1, 2, ..., n, \tag{10}$$

such that

$$\frac{dQ_k}{dt} = 0, \qquad k = 1, 2, ..., n. \tag{11}$$

5 Free Quantum Fields and Canonical Quantization

To construct a Quantum Field Theory we start from manifestly Poincaré-covariant classical field equations of motion. Such equations could be inferred from the mathematical properties of continuous Lie groups. However, they are already known from other disciplines of Physics. For instance, the classical equations of Electromagnetism for the four-potential is nothing but a Poincaré covariant equation governing the dynamics of a vector field $A_\mu(x)$. On the other hand, the Klein–Gordon and the Dirac equations from Relativistic Quantum Mechanics can be interpreted as Poincaré covariant equations describing classical scalar and spinor fields, respectively. Then, the need to quantify these fields arises and we will use the Canonical Quantization method for this purpose (Second Quantization).

The Canonical Quantization method is a quantization rule that essentially consists in assuming that the classical equations of motion for the canonical coordinates $\left(q^i, p^i\right)$ hold true in Quantum Mechanics provided that $\left(q^i, p^i\right)$ are reinterpreted as quantum mechanical operators in the Heisenberg picture (acting on the Hilbert space) and the classical Poisson brackets of the canonical coordinates are replaced by commutators by means of the following relation: $\{\ ,\ \} \to \frac{-i}{\hbar}[\ ,\ ]$. This way, one arrives at the quantum commutation relations

$$[q^i(t), p^j(t)] = i\hbar\delta^{ij}, \qquad i,j = 1,...,n \tag{12}$$

For a classical field theory, we must interpret the continuous variable $\mathbf{x}$ as a label that distinguishes the coordinates $\phi(t,\mathbf{x})$ (substituting the i label of the discrete case) and promote the canonical coordinates $\left(q^i, p^i\right)$ to field operators $\phi(t,\mathbf{x})$ in the Heisenberg picture and the corresponding conjugate momenta $\Pi_\phi(t,\mathbf{x})$, defined by

$$\Pi_\phi(t,\mathbf{x}) = \frac{\partial \mathscr{L}}{\partial(\partial_0\phi(t,\mathbf{x}))}. \tag{13}$$

Moreover, as we are working with a continuous variable $\mathbf{x}$, we must also extend the commutation relations by spreading the Kronecker delta δ^{ij} over all space through the Dirac delta. Thus, the equal-time commutation relations defining the quantum fields take the form

$$[\phi(t,\mathbf{x}), \Pi(t,\mathbf{y})] = i\delta^{(3)}(\mathbf{x}-\mathbf{y}), \tag{14}$$

where as usual, we consider natural units ($\hbar \equiv c \equiv 1$). Here, we have quantized fields (which are solutions of the classical equations of motion) as operators in the Heisenberg picture, substituting the respective wave functions of the first quantization [10].

The key ideas of this quantization method were introduced in 1927 by Dirac [2], and were developed, most notably, by Fock and Jordan later [13, 14].

5.1 Scalar Fields

A scalar field, by definition is a real or complex field which is invariant under Lorentz transformations

$$\phi'(x') = \phi(x), \tag{15}$$

where the space-time coordinates established by the four-position tensor $x^\mu \to \left\{x^0, x^1, x^2, x^3\right\} \equiv \{t, x, y, z\}$ allows us to redefine $(x) \equiv (t,\mathbf{x})$. Furthermore every coordinate $\phi_i(x)$ is a solution of the Klein–Gordon (KG) equation of motion

$$(\Box + m^2)\phi_i(x) = 0, \tag{16}$$

where m is the mass of the particle and $\Box$ is the D'Alembert operator which is defined as

$$\Box = \partial_\mu\partial^\mu = \partial_\mu\eta^{\mu\nu}\partial_\nu = \frac{\partial^2}{\partial t^2} - \nabla^2, \tag{17}$$

being $\eta^{\mu\nu}$ the Minkowski metric tensor for space-time coordinates

$$\eta^{\mu\nu} = \eta_{\mu\nu} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{18}$$

and $\partial_\mu = \partial/\partial x^\mu$.

The Klein–Gordon equation can be readily derived starting from the relativistic energy for free particles of mass m and momentum $\mathbf{p}$

$$E^2 = \mathbf{p}^2 + m^2, \tag{19}$$

by simply making the replacement

$$E \to i\frac{\partial}{\partial t} \qquad \text{and} \qquad \mathbf{p} \to -i\nabla. \tag{20}$$

On the other hand, from the four-momentum tensor $p^\mu \equiv (E,\mathbf{p})$, where $P_\mu\,|p\rangle = p_\mu\,|p\rangle$, being P_μ the four-momentum operator, and the particle relativistic energy it can be easily seen that $p^2 = m^2$

$$p^2 \equiv p^\mu p_\mu = p^\mu\eta_{\mu\nu}p^\nu = E^2 - \mathbf{p}^2 = m^2. \tag{21}$$

5.2 Real (or Neutral) Scalar Field

The free real scalar field is defined as a scalar field $\phi(x)$ which is an Hermitian operator

$$\phi^\dagger(x) = \phi(x) \tag{22}$$

and is solution of the KG equation

$$(\Box + m^2)\phi(x) = 0. \tag{23}$$

5.2.1 Lagrangian Density of the Real Scalar Field

It can be easily seen that the Lagrangian density for a free real scalar field is

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2, \tag{24}$$

which is a scalar (Poincaré-invariant) magnitude, as expected. Indeed, introducing this Lagrangian in the Euler–Lagrange equations, Eq. (5), and taking into account that

$$\frac{\partial \mathscr{L}}{\partial \phi} = -m^2\phi \tag{25}$$

and

$$\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} = \frac{1}{2}\left[\frac{\partial \eta^{\beta\alpha}(\partial_\alpha \phi)(\partial_\beta \phi)}{\partial(\partial_\mu \phi)}\right] = \frac{1}{2}\eta^{\beta\alpha}\left[\delta^\mu_\alpha(\partial_\beta \phi) + (\partial_\alpha \phi)\delta^\mu_\beta\right] = \frac{1}{2}\left(\partial^\mu \phi + \partial^\mu \phi\right) = \partial^\mu \phi, \tag{26}$$

we can get the KG equation

$$\partial_\mu \partial^\mu \phi + m^2\phi = (\Box + m^2)\phi = 0. \tag{27}$$

On the other hand, the hamiltonian density can be computed as

$$\mathscr{H} = \Pi\dot{\phi} - \mathscr{L} = \frac{1}{2}\left[\Pi^2 + (\nabla\phi)^2 + m^2\phi^2\right]. \tag{28}$$

5.2.2 Conjugate Momentum of the Real Scalar Field

From the definition of Eq. (13), and using Eq. (26) with the substitution $\mu \to 0$, we can evaluate the conjugate momentum

$$\Pi(x) \equiv \frac{\partial L}{\partial \dot{\phi}} = \frac{\partial L}{\partial(\partial_0 \phi)} = \partial^0 \phi = \frac{\partial \phi}{\partial t} = \dot{\phi}(t). \tag{29}$$

5.2.3 Quantization of the Real Scalar Field

To quantize the real scalar field, we impose the following commutation relations at equal time t between the fields and the conjugate momenta:

$$[\phi(\mathbf{x}, t), \phi(\mathbf{x}', t)] = [\Pi(\mathbf{x}, t), \Pi(\mathbf{x}', t)] = 0 \tag{30}$$

$$[\phi(\mathbf{x}, t), \Pi(\mathbf{x}', t)] = i\delta^{(3)}(\mathbf{x} - \mathbf{x}'), \tag{31}$$

were we have used natural units ($c = \hbar = 1$).

5.2.4 Solution of the Klein-Gordon Equation

We are now interested in expressing the quantized field operator as a function of creation and annihilation operators acting on the Hilbert space, which will allow us to define the states on which the field act. To this end we write the solution of Eq. (23) as the Fourier transformation

$$\phi(x) = \frac{1}{(2\pi)^3} \int d^4k e^{-ikx} \widetilde{\phi}(k), \tag{32}$$

where k is the four-momentum (in natural units the wavenumber and the momentum coincide $p^\mu = k^\mu$) and $\widetilde{\phi}(k)$ is the Fourier transform of $\phi(x)$.

Then, substituting Eq. (32) in Eq. (23) one obtains

$$\int \frac{d^4k}{(2\pi)^3} \left(-k^2 + m^2\right) \widetilde{\phi}(k) e^{-ikx} = 0, \tag{33}$$

and from the linear independence of the plane waves, all the coefficients must vanish

$$(-k^2 + m^2)\widetilde{\phi}(k) = 0. \tag{34}$$

Thus, the solution must satisfy the dispersion relation $k^2 = m^2$ and hence, $\widetilde{\phi}(k)$ should be of the form

$$\widetilde{\phi}(k) = \varphi(k)\delta(k^2 - m^2), \tag{35}$$

where $\varphi(k)$ is an arbitrary function of k.

Taking into account the property of the delta function

$$\delta\left[g(k^0)\right] = \sum_j \frac{1}{\left|g'(k_j^0)\right|} \delta(k^0 - k_j^0), \qquad g(k_j^0) = 0, \qquad g'(k_j^0) \neq 0, \tag{36}$$

$[g'(k^0)$ is the derivative of $g(k^0)$ and the k_j^0 are the simple zeros of the function $g(k^0)]$, and substituting Eq. (36) in Eq. (35), and the latter in Eq. (32), $\phi(x)$ can be expanded in terms of a complete set (basis) of plane waves $e^{\pm ikx}$ as

$$\begin{aligned} \phi(x) &= \int \frac{d^4k}{(2\pi)^3 2\omega_{\mathbf{k}}} \left[\varphi(k^0, \mathbf{k}) e^{-ikx} \delta(k^0 - \omega_{\mathbf{k}}) + \varphi(k^0, \mathbf{k}) e^{-ikx} \delta(k^0 + \omega_{\mathbf{k}})\right] \\ &= \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}} \left[\varphi(k^0 = +\omega_{\mathbf{k}}, \mathbf{k}) e^{-i(\omega_{\mathbf{k}} t - \mathbf{kx})} + \varphi(k^0 = -\omega_{\mathbf{k}}, \mathbf{k}) e^{i(\omega_{\mathbf{k}} t + \mathbf{kx})}\right], \end{aligned} \tag{37}$$

were $\omega_{\mathbf{k}} = +\sqrt{\mathbf{k}^2 + m^2}$. As can be seen from Eq. (37), the first term of the sum refers to waves of positive energy ($E \equiv k^0 = +\omega_{\mathbf{k}}$) while the second term refers to negative values ($E \equiv k^0 = -\omega_{\mathbf{k}}$).

Making the change ($\mathbf{k} \to -\mathbf{k}$) in the second term of Eq. (37), denoting $k \equiv (\omega_{\mathbf{k}}, \mathbf{k})$, and taking into account the scalar field condition $\phi^\dagger(x) = \phi(x)$ and the linear independence of the plane waves, we have $\varphi^\dagger(k) = \varphi(-k)$ and $\varphi^\dagger(-k) = \varphi(k)$. Thus, redefining $\varphi(k) \equiv a(k)$ it follows that $\varphi(-k) \equiv a^\dagger(k)$, and substituting all these changes in Eq. (37), it results

$$\phi(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}} \left[a(k) e^{-ikx} + a^\dagger(k) e^{ikx}\right]. \tag{38}$$

5.2.5 Commutation Relations of $a(k)$ and $a^\dagger(k')$:

Here we want to obtain the commutation relations of $a(k)$ and $a^\dagger(k')$ starting from the commutation relations for the field operators and their conjugate momenta, Eqs. (30) and (31). To do this, we have to write $a(k)$ and $a^\dagger(k)$ as a function of $\phi(x)$ and $\Pi(x)$.

It can be shown that

$$a(k) = i \int d^3\mathbf{x} e^{ikx} \overleftrightarrow{\partial_0} \phi(x), \tag{39}$$

where $a \overleftrightarrow{\partial_0} b \equiv a \left(\overrightarrow{\partial_0} b\right) - \left(a \overleftarrow{\partial_0}\right) b \equiv a\left(\partial_0 b\right) - \left(\partial_0 a\right) b$. Equation (39) can be readily checked by substituting $\phi(x)$ as given by Eq. (38).

Analogously, taking the Hermitian conjugate of Eq. (39)

$$a^\dagger(k) = -i \int d^3\mathbf{x} e^{-ikx} \overleftrightarrow{\partial_0} \phi(x). \tag{40}$$

Finally, we can obtain the commutation relations for $a(k)$ and $a^\dagger(k')$ using Eqs. (39)–(40) and the commutation relations of Eqs. (30)–(31)

$$[a(k), a(k')] = \left[a^\dagger(k), a^\dagger(k')\right] = 0 \tag{41}$$

$$\left[a(k), a^\dagger(k')\right] = (2\pi)^3 2\omega_{\mathbf{k}}\, \delta^{(3)}(\mathbf{k} - \mathbf{k}'). \tag{42}$$

5.2.6 Constructing the Hilbert Space ($a^\dagger(k)$ and $a(k)$ as Creation and Annihilation Operators)

Now, we are going to show that $a^\dagger(k)$ and $a(k)$ can be interpreted as operators of creation and annihilation of particles, respectively. To see this, we start from the following formula, which is a direct consequence of the invariance of the real scalar field under space-time translations (Noether's theorem):

$$i\partial_\mu \phi(x) = [\phi(x), P_\mu], \tag{43}$$

where P_μ is the four-momentum operator of the scalar field (a representation of the infinitesimal generator of the space-time translations).

Substituting Eq. (38) in Eq. (43) we find

$$k_\mu a(k) = [a(k), P_\mu], \tag{44}$$

$$-k_\mu a^\dagger(k) = \left[a^\dagger(k), P_\mu\right]. \tag{45}$$

Using Eq. (45) we can write

$$P_\mu \left(a^\dagger(k) |p\rangle\right) = \left(a^\dagger(k) P_\mu + k_\mu a^\dagger(k)\right) |p\rangle = (p_\mu + k_\mu) \left(a^\dagger(k) |p\rangle\right), \tag{46}$$

and taking into account that $k^2 = m^2$, we can interpret $a^\dagger(k)$ as an operator which creates one particle of mass m and four-momentum k_μ.

Similarly, using Eq. (44),

$$P_\mu \left(a(k) |p\rangle\right) = \left(a(k) P_\mu - k_\mu a(k)\right) |p\rangle = (p_\mu - k_\mu) \left(a(k) |p\rangle\right), \tag{47}$$

and $a(k)$ can be interpreted as an operator which annihilates one particle of mass m and four-momntum k_μ.

*Hilbert Space (**Fock Space**):*

We define the vacuum $|0\rangle$ as $P_\mu |0\rangle = 0$ and $a(k) |0\rangle = 0$ which means that this state has zero energy-momentum and no particles, respectively. Hence, we can construct a basis of the Hilbert space by applying the operators $a^\dagger(k)$ with different k onto the vacuum: $a^\dagger(k_1)...a^\dagger(k_s) |0\rangle$. The Hilbert space so constructed is called the Fock space.

Furthermore, using the commutation relations for $a^\dagger(k)$, Eq. (41), we can see that the states in this space are symmetrical under the exchange of particles. It is also possible to have particles with the same four-momentum, so we can ensure that the particles created by the scalar field obey the Bose-Einstein statistics.

On the other hand, as a consequence of the Lagrangian invariance under rotation transformations the total angular momentum is conserved. Indeed, making use of Noether's theorem it can be seen that there are six conserved currents which, using Eqs. (8)–(9), satisfy

$$\partial_\mu (\mathscr{J}^\mu)^{\rho\sigma} = 0, \qquad \rho, \sigma = 1, 2, 3, \tag{48}$$

where $(\mathscr{J}^\mu)^{\rho\sigma} \equiv X^\rho T^{\mu\sigma} - X^\sigma T^{\mu\rho}$, and X^ρ and $T^{\mu\sigma}$ are the four-position and the energy-momentum tensor, respectively. Associated with these conserved currents, there are six conserved charges, Eq. (10), three of which are the spatial components of the total angular momentum

$$J^i = \epsilon^i_{jk} \int d^3\mathbf{x}\, (\mathscr{J}^o)^{jk}\,, \tag{49}$$

where ϵ_{ijk} is the totally antisymmetric tensor. It can be shown that

$$J^i \left|p = 0\right\rangle = 0. \tag{50}$$

Thus, the quantization of the scalar field gives rise to spin 0 particles (these particles have no internal angular momentum).

5.2.7 P^μ as a Function of $a^\dagger(k)$ and $a(k)$

Under space-time translations, four-position vectors and fields transform as

$$x^\mu \to x^{\mu\prime} = x^\mu + \epsilon^\mu = x^\mu + \delta^\mu_\nu \epsilon^\nu, \tag{51}$$

$$\phi(x) \to \phi'(x') = \phi(x), \tag{52}$$

and comparing with Eqs. (6) and (7) we see that the infinitesimal generators for these transformations are $\{f^\mu{}_k(x)\} = \{\delta^\mu_\nu\}$ and $\{F^a{}_k(x)\} = 0$. As the Lagrangian (24) only depends on the fields, it remains invariant under the transformations. Hence, using Noether's theorem, Eqs. (8) and (9) with $J^\mu{}_k \equiv -T^\mu_\nu$, we can obtain four conserved currents ($n = 4$), one for each translation,

$$\partial_\mu T^\mu{}_\nu = 0, \qquad \nu = 0, 1, 2, 3, \tag{53}$$

where

$$T^\mu{}_\nu = \frac{\partial \mathscr{L}}{\partial\left(\partial_\mu \phi\right)} \partial_\nu \phi - \delta^\mu_\nu \mathscr{L} \tag{54}$$

corresponds to the energy-momentum tensor of the scalar field. Associated with the conserved currents, using Eq. (10) with $Q_k = P^\mu$, $\;k = \mu = 0, 1, 2, 3$, there also exist four conserved charges P^μ, which define the four-momentum of the field

$$P^\mu = \int T^{\mu 0} d^3\mathbf{x} \qquad / \qquad \frac{d}{dt} P^\mu = 0. \tag{55}$$

Here, using Eqs. (24) and (26), we have

$$T^{\mu 0} = \eta^{\nu 0} T^\mu{}_\nu = \frac{\partial \mathscr{L}}{\partial\left(\partial_\mu \phi\right)} \partial^0 \phi - \eta^{\mu 0} \mathscr{L} = \left(\partial^\mu \phi\right)\left(\partial^0 \phi\right) - \frac{1}{2} \eta^{\mu 0} \left[\left(\partial_\xi \phi\right)\left(\partial^\xi \phi\right) - m^2 \phi^2\right]. \tag{56}$$

From Eq. (38) we find

$$\partial^\mu \phi(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_\mathbf{k}} \left[\left(-ik^\mu\right) a(k) e^{-ikx} + \left(ik^\mu\right) a^\dagger(k) e^{ikx}\right]. \tag{57}$$

Substituting Eqs. (38) and (57) in Eq. (56), and the latter in Eq. (55) and integrating over $\int d^3\mathbf{x}$ to obtain $\delta^{(3)}(\mathbf{k} \pm \mathbf{k}')$ and then integrating over $\int d^3\mathbf{k}'$, we arrive at

$$P^\mu = \frac{1}{2} \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_\mathbf{k}}\, k^\mu \, \left[a^\dagger(k)a(k) + a(k)a^\dagger(k)\right]. \tag{58}$$

Thus, we have obtained the Hamiltonian (temporal component) and the Lineal Momentum (spatial components) of the scalar field, respectively, as functions of $a^\dagger$ and a

$$H = P^0 = \frac{1}{2} \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_\mathbf{k}}\, k^0 \, \left[a^\dagger(k)a(k) + a(k)a^\dagger(k)\right] \tag{59}$$

$$P^i = \frac{1}{2} \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_\mathbf{k}}\, k^i \, \left[a^\dagger(k)a(k) + a(k)a^\dagger(k)\right]. \tag{60}$$

- *Vacuum Momentum (Spatial Components):* $\langle 0| P_j |0\rangle$

$$\langle 0| P_j |0\rangle = \langle 0| \frac{1}{2} \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_\mathbf{k}}\, k_j \left[a(k), a^\dagger(k)\right] |0\rangle = \frac{1}{2} \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_\mathbf{k}}\, k_j \, (2\pi)^3 2\omega_\mathbf{k} \delta^{(3)}(\mathbf{k} - \mathbf{k}) = 0 \tag{61}$$

where we have used Eq. (42) to rewrite the second term of (60) in reverse order and have taken into account that $a(k)|0\rangle = 0$. Since k_j is an odd function while $\delta^{(3)}(\mathbf{k} - \mathbf{k})$ is even (although infinite), the integrand is odd and, consequently, the integral over all phase space vanishes.

- *Vacuum Energy (Temporal Component):* $\langle 0| P_0 |0\rangle$

$$\langle 0| P_0 |0\rangle = \langle 0| \frac{1}{2} \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_\mathbf{k}}\, k_0 \left[a(k), a^\dagger(k)\right] |0\rangle = \frac{1}{2} \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_\mathbf{k}}\, k_0 \, (2\pi)^3 2\omega_\mathbf{k} \delta^{(3)}(\mathbf{0}) = \infty \tag{62}$$

where we have used that $\omega_\mathbf{k} = \sqrt{\mathbf{k}^2 + m^2}$ is an even function of $\mathbf{k}$ and the fact that $\delta^{(3)}(\mathbf{k})$ is an even function whose value is infinite at $\mathbf{k}$=$\mathbf{0}$.

We define the number operator $N(k)$ as

$$N(k) = a^\dagger(k)a(k). \tag{63}$$

Then, using Eq. (42), P_μ can be rewritten as

$$P_\mu = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_\mathbf{k}}\, k_\mu \left(N(k) + \frac{1}{2}\left[a(k), a^\dagger(k)\right] \right), \tag{64}$$

where the term $\frac{1}{2}\left[a(k), a^\dagger(k)\right]$ is the only one that contributes to the divergent zero point energy.

How can we treat this infinite energy?

To eliminate the infinities we take the origin of energies in the vacuum (scale change) and focus on the energy difference with respect to this state: $P_\mu \equiv P_\mu - \langle 0| P_\mu |0\rangle$. With this redefinition, P_μ becomes

$$P_\mu = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_\mathbf{k}}\, k_\mu \, a^\dagger(k)a(k) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_\mathbf{k}}\, k_\mu \, N(k). \tag{65}$$

Since $a(k)$ and $a^\dagger(k)$ satisfy the commutation relations (42), it is clear that subtracting the mean value of P_μ in the vacuum is equivalent to rewrite the operator by replacing $a^\dagger(k)$ to the left of $a(k)$. This operation is named normal ordering (also called **Wick ordering** or normal product) and is denoted by : :

For instance

$$: aa^\dagger + a^\dagger a : \; = \; 2a^\dagger a. \tag{66}$$

We can now rewrite

$$P_\mu = \, : \frac{1}{2} \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_\mathbf{k}} \, k_\mu \left[a(k) a^\dagger(k) + a^\dagger(k) a(k) \right] : \, = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_\mathbf{k}} \, k_\mu \, a^\dagger(k) a(k) \tag{67}$$

and, in general, using the previous convention, we redefine any physical quantity A as a difference with its vacuum expectation value:

$$A \; \rightarrow \; : A : \; \equiv A - \langle 0 | A | 0 \rangle \, . \tag{68}$$

5.3 Complex (or Charged) Scalar Field

We define the complex scalar field as

$$\phi(x) = \frac{1}{\sqrt{2}} \left(\phi_1(x) + i \phi_2(x) \right), \tag{69}$$

where $\phi_1(x)$ and $\phi_2(x)$ are independent fields with real values, so

$$\phi_1^\dagger(x) = \phi_1(x), \qquad \phi_2^\dagger(x) = \phi_2(x) \tag{70}$$

and are solutions independently of the Klein-Gordon equation with the same mass m of the particle

$$\left(\Box + m^2 \right) \phi_1(x) = 0, \qquad \left(\Box + m^2 \right) \phi_2(x) = 0. \tag{71}$$

5.3.1 Lagrangian Density

Equations (71) follow from the Lagrangian density

$$\mathscr{L} = \frac{1}{2} (\partial_\mu \phi_1)(\partial^\mu \phi_1) - \frac{1}{2} m^2 \phi_1^2 + \frac{1}{2} (\partial_\mu \phi_2)(\partial^\mu \phi_2) - \frac{1}{2} m^2 \phi_2^2, \tag{72}$$

which, in terms of $\phi(x)$ and $\phi^\dagger(x)$, using Eq. (69), can be rewritten as

$$\mathscr{L} = (\partial_\mu \phi)(\partial^\mu \phi^\dagger) - m^2 \phi \phi^\dagger. \tag{73}$$

Substitution of Eq. (73) in the Euler–Lagrange equations (5), leads to the equations of motion of the complex scalar field

$$\left(\Box + m^2 \right) \phi(x) = 0, \qquad \left(\Box + m^2 \right) \phi^\dagger(x) = 0. \tag{74}$$

5.3.2 Conjugate Momenta

Using Eq. (13) and the Lagrangian of Eq. (73), the respective conjugate momenta of $\phi(x)$ and $\phi^\dagger(x)$ are given by

$$\Pi = \frac{\partial \mathscr{L}}{\partial \dot{\phi}} = \dot{\phi}^\dagger = \frac{1}{\sqrt{2}} \left(\dot{\phi}_1 - i \dot{\phi}_2 \right) \tag{75}$$

$$\Pi^\dagger = \frac{\partial \mathscr{L}}{\partial \dot{\phi}^\dagger} = \dot{\phi} = \frac{1}{\sqrt{2}} \left(\dot{\phi}_1 + i \dot{\phi}_2 \right). \tag{76}$$

5.3.3 Quantization of the Field

Following the established Canonical quantization method (see Eq. (14)), the imposed commutation relations at equal times which define the quantum fields are

$$[\phi(\mathbf{x}, t), \Pi(\mathbf{x}', t)] = i \delta^{(3)}(\mathbf{x} - \mathbf{x}'), \tag{77}$$

$$\left[\phi^\dagger(\mathbf{x}, t), \Pi^\dagger(\mathbf{x}', t) \right] = i \delta^{(3)}(\mathbf{x} - \mathbf{x}'), \tag{78}$$

the others being zero.

5.3.4 Solution of the Klein-Gordon Equation of the Complex Scalar Field

The complex scalar fields $\phi(x)$ and $\phi^\dagger(x)$ are different and independent solutions of the Klein-Gordon equations (74), so they can be expressed as a Fourier expansion in a similar way as already seen for the real scalar field, Eq.(37),

$$\phi(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_\mathbf{k}} \left[\zeta(k^0 = +\omega_\mathbf{k}, \mathbf{k})e^{-ikx} + \zeta(k^0 = -\omega_\mathbf{k}, -\mathbf{k})e^{ikx}\right], \tag{79}$$

$$\phi^\dagger(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_\mathbf{k}} \left[\xi(k^0 = +\omega_\mathbf{k}, \mathbf{k})e^{-ikx} + \xi(k^0 = -\omega_\mathbf{k}, -\mathbf{k})e^{ikx}\right], \tag{80}$$

where $\zeta(k)$ and $\xi(k)$ are functions of k, and we have changed $\mathbf{k} \to -\mathbf{k}$ in the second term of the sum of Eqs. (79)–(80), and have denoted $k \equiv (\omega_\mathbf{k}, \mathbf{k})$.

Taking into account that $\phi^\dagger(x)$ is the complex conjugate of $\phi(x)$, we can see that

$$\xi(k^0 = +\omega_\mathbf{k}, \mathbf{k}) = \left[\zeta(k^0 = -\omega_\mathbf{k}, -\mathbf{k})\right]^\dagger, \tag{81}$$

$$\xi(k^0 = -\omega_\mathbf{k}, -\mathbf{k}) = \left[\zeta(k^0 = +\omega_\mathbf{k}, \mathbf{k})\right]^\dagger. \tag{82}$$

Then, redefining $\zeta(k^0 = +\omega_\mathbf{k}, \mathbf{k}) \equiv a(k)$ and $\xi(k^0 = +\omega_\mathbf{k}, \mathbf{k}) \equiv b(k)$, we can rewrite $\phi(x)$ and $\phi^\dagger(x)$ in the form

$$\phi(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_\mathbf{k}} \left[a(k)e^{-ikx} + b^\dagger(k)e^{ikx}\right], \tag{83}$$

$$\phi^\dagger(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_\mathbf{k}} \left[b(k)e^{-ikx} + a^\dagger(k)e^{ikx}\right]. \tag{84}$$

5.3.5 Commutation Relations among $a(k)$, $a^\dagger(k)$, $b(k)$ and $b^\dagger(k)$

Taking into account Eq. (38), the Fourier expansions for the real components of the complex scalar field are

$$\phi_i(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_\mathbf{k}} \left[a_i(k)e^{-ikx} + {a_i}^\dagger(k)e^{ikx}\right], \qquad i = 1, 2 \tag{85}$$

and using Eq. (69), we readily arrive at

$$a(k) = \frac{1}{\sqrt{2}}\left[a_1(k) + ia_2(k)\right] \quad \longrightarrow \quad a^\dagger = \frac{1}{\sqrt{2}}\left[a_1^\dagger(k) - ia_2^\dagger(k)\right] \tag{86}$$

$$b(k) = \frac{1}{\sqrt{2}}\left[a_1(k) - ia_2(k)\right] \quad \longrightarrow \quad b^\dagger = \frac{1}{\sqrt{2}}\left[a_1^\dagger(k) + ia_2^\dagger(k)\right]. \tag{87}$$

Making use of Eqs. (41)–(42), the commutation relations for the creation and the annihilation operators of the real components of the complex scalar field are

$$\left[a_i(k), {a_j}^\dagger(k')\right] = (2\pi)^3 2\omega_\mathbf{k}\delta_{ij}\delta^{(3)}(\mathbf{k} - \mathbf{k}'), \qquad i, j = 1, 2 \tag{88}$$

with the others being zero. Thus, taking into account Eqs. (86)–(87) we can finally obtain

$$\left[a(k), a^\dagger(k')\right] = (2\pi)^3 2\omega_\mathbf{k}\delta^{(3)}(\mathbf{k} - \mathbf{k}'), \tag{89}$$

$$\left[b(k), b^\dagger(k')\right] = (2\pi)^3 2\omega_\mathbf{k}\delta^{(3)}(\mathbf{k} - \mathbf{k}'). \tag{90}$$

and all the others are zero.

5.3.6 Determination of P^μ as a Function of $a^\dagger(k)$ and $a(k)$

P_μ can be obtained as a function of $a(k)$ and $a^\dagger(k)$ in a similar way to that seen previously for the real scalar field (see Section 5.2.7). Using Noether's theorem and the Lagrangian invariance under space-time translations, we get the four conserved currents $T^\mu{}_\nu$, one for each transformation, which satisfy the condition (53). This is the energy-momentum tensor of the complex scalar field whose value is obtained from Eq. (9) with $\{\phi^a(x)\} = \{\phi(x), \phi^\dagger(x)\}$ and $J^\mu{}_k = T^\mu{}_\nu$, and using the same infinitesimal generators as for the real scalar field

$$T^\mu{}_\nu = \frac{\partial \mathscr{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi + \frac{\partial \mathscr{L}}{\partial(\partial_\mu\phi^\dagger)}\partial_\nu\phi^\dagger - \delta^\mu_\nu\mathscr{L}. \tag{91}$$

Associated to these four conserved currents, there are the respective four conserved charges (which define the four-momentum P^μ), Eq. (55), where $T^{\mu 0}$ is extracted from Eq. (91) using the Lagrangian density of Eq. (73)

$$T^{\mu 0} = (\partial^\mu\phi^\dagger)(\partial^0\phi) + (\partial^\mu\phi)(\partial^0\phi^\dagger) - \eta^{\mu 0}\left[(\partial_\mu\phi)(\partial^\mu\phi^\dagger) - m^2\phi\phi^\dagger\right]. \tag{92}$$

Then, substituting Eqs. (83) and (84) in Eq. (92), and the latter in Eq. (55) and integrating over $\int d^3\mathbf{x}$ to obtain $\delta^{(3)}(\mathbf{k}\pm\mathbf{k}')$ and then integrating over $\int d^3\mathbf{k}'$, we arrive at

$$P_\mu = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_\mathbf{k}}\frac{1}{2}k_\mu\left[a^\dagger(k)a(k) + b(k)b^\dagger(k)\right], \tag{93}$$

and considering the Wick ordering to take out the vacuum infinite energy from P_μ we finally get

$$P_\mu = \; :\int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_\mathbf{k}}\frac{1}{2}k_\mu\left[a^\dagger(k)a(k) + b(k)b^\dagger(k)\right]: \; = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_\mathbf{k}}k_\mu\left[a^\dagger(k)a(k) + b^\dagger(k)b(k)\right]. \tag{94}$$

On the other hand, using the Wick ordering to take out the vacuum infinite energy too from $\mathscr{L}$, Eq. (73), it results

$$\mathscr{L} = \; :(\partial_\mu\phi)(\partial^\mu\phi^\dagger) - m^2\phi\phi^\dagger: . \tag{95}$$

5.3.7 Constructing the Hilbert Space ($a^\dagger(k), b^\dagger(k)$ and $a(k), b(k)$ as Creation and Annihilation operators)

From Eq. (94) and the commutation relations between the annihilation and creation operators of Eqs. (89) and (90), we can easily obtain the commutation relations between P^μ and $a(q)$, $a^\dagger(q)$, $b(q)$ and $b^\dagger(q)$:

$$[a(q), P^\mu] = a(q)q^\mu \tag{96}$$

$$[b(q), P^\mu] = b(q)q^\mu \tag{97}$$

$$\left[a^\dagger(q), P^\mu\right] = -a^\dagger(q)q^\mu \tag{98}$$

$$\left[b^\dagger(q), P^\mu\right] = -b^\dagger(q)q^\mu. \tag{99}$$

In a similar manner as already done for the real scalar field, the above commutation relations enable us to give an interpretation for the $a(q)$, $b(q)$, $a^\dagger(q)$, and $b^\dagger(q)$ operators as annihilation and creation operators. Indeed, let $|p\rangle$ be an arbitrary eigenvector of P^μ: $\quad P^\mu|p\rangle = p^\mu|p\rangle$. Then, using Eq. (96) we can see that

$$P^\mu a(q)|p\rangle = (a(q)P^\mu + [P^\mu, a(q)])|p\rangle = (p^\mu - q^\mu)\,a(q)|p\rangle. \tag{100}$$

Similar expressions can be obtained for the other operators, by making use of Eqs. (97)–(99). Thus, one can interpret that $a(q)$ and $a^\dagger(q)$ annihilates and creates, respectively, one particle of type a with four-momentum q^μ and mass m, where $m^2 = q^2$. Also, $b(q)$ and $b^\dagger(q)$ annihilates and creates, respectively, one particle of the type b, four-momentum q^μ, and mass m.

5.3.8 Invariance of $\mathscr{L}$ under $U(1)$ Global

In what follows we will apply Noether's theorem to calculate the conserved charges associated to the invariance of the Lagrangian (73) under global $U(1)$ transformations.

Consider the following global $U(1)$ (phase) transformation of the fields:

$$\phi(x) \to \phi'(x') = e^{-i\theta}\phi(x) \simeq (1 - i\epsilon)\,\phi(x) \quad \longrightarrow \quad F(x) = -i\phi(x) \tag{101}$$

$$\phi^\dagger(x) \to \phi^{\dagger\prime}(x') = e^{i\theta}\phi^\dagger(x) \simeq (1 + i\epsilon)\,\phi^\dagger(x) \quad \longrightarrow \quad F^\dagger(x) = +i\phi^\dagger(x), \tag{102}$$

where we have set $\theta = \epsilon \simeq 0$ to write the corresponding infinitesimal transformations. Using Eq. (7), we can identify $\{F^a{}_k(x)\} = \left\{F(x), F^\dagger(x)\right\}$ for the respective fields $\{\phi^a(x)\} = \left\{\phi(x), \phi^\dagger(x)\right\}$.

On the other hand, since this is an internal symmetry transformation (spatial-time coordinates remain unchanged), we see from Eq. (6) that the infinitesimal generators associated to the coordinates are zero, $f^\mu{}_k(x) = 0$.

The Lagrangian $\mathscr{L}$ of the complex scalar field, Eq. (73), is invariant under $U(1)$

$$\mathscr{L} \to \mathscr{L}' = (\partial_\mu\phi e^{-i\theta})(\partial^\mu\phi^\dagger e^{i\theta}) - m^2\phi e^{-i\theta}\phi^\dagger e^{i\theta} = e^{-i\theta}e^{i\theta}\left[(\partial_\mu\phi)(\partial^\mu\phi^\dagger) - m^2\phi\phi^\dagger\right] = \mathscr{L}, \tag{103}$$

were we have used that θ does not depend on space-time coordinates ($U(1)$ is a global gauge transformation).

Using Noether's theorem, Eqs. (8)–(9) with $k = 1$, one obtains the following conserved current density:

$$J^\mu =: \frac{\partial\mathscr{L}}{\partial\left(\partial_\mu\phi\right)}F + F^\dagger\frac{\partial\mathscr{L}}{\partial\left(\partial_\mu\phi^\dagger\right)} : \quad = \quad : -i\phi\left(\partial^\mu\phi^\dagger\right) + i\phi^\dagger\left(\partial^\mu\phi\right) : \quad = \quad : i\phi^\dagger\overleftrightarrow{\partial^\mu}\phi : \tag{104}$$

which satisfies $\partial_\mu J^\mu = 0$. Note that we have extracted the vacuum value from J^μ by using the Wick ordering.

From Eq. (10) and taking into account Eqs. (83)–(84), the corresponding conserved charge is

$$\begin{aligned} Q = \int d^3\mathbf{x} J^0 = \quad &: \int d^3\mathbf{x} \quad i\phi^\dagger\overleftrightarrow{\partial^0}\phi : \quad = \quad \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_\mathbf{k}} : \left[a^\dagger(k)a(k) - b(k)b^\dagger(k)\right] : \quad = \\ = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_\mathbf{k}} \left[a^\dagger(k)a(k) - b^\dagger(k)b(k)\right] &= \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_\mathbf{k}} N_a(k) - \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_\mathbf{k}} N_b(k) = N_a - N_b, \end{aligned} \tag{105}$$

where $N_a(k)$ and $N_b(k)$ are the density number operators for particles of type a and b, respectively (number of type a or b particles per unit four-momentum). On the other hand, N_a and N_b are the total number of particles of type a and b, respectively. In particular, the total charge Q can be written as

$$Q = (+1)N_a + (-1)N_b. \tag{106}$$

Using Eq. (105) and the commutation relations (89)–(90), one can obtain the following commutation relations involving the charge operator:

$$[a(q), Q] = a(q) \tag{107}$$

$$\left[a^\dagger(q), Q\right] = -a^\dagger(q) \tag{108}$$

$$[b(q), Q] = -b(q) \tag{109}$$

$$\left[b^\dagger(q), Q\right] = b^\dagger(q) \tag{110}$$

Let $|e\rangle$ be an eigenstate of the charge operator satisfying the eigenvalue equation $Q\,|e\rangle = e\,|e\rangle$. Then, making use of Eqs. (107)–(110) one finds

$$Q\left(a(q)\,|e\rangle\right) = \left(a(q)Q - [a(q), Q]\right)|e\rangle = (e-1)\left(a(q)\,|e\rangle\right) \tag{111}$$

$$Q\left(a^\dagger(q)\left|e\right\rangle\right) = \left(a^\dagger(q)Q - \left[a^\dagger(q), Q\right]\right)\left|e\right\rangle = (e+1)\left(a^\dagger(q)\left|e\right\rangle\right) \tag{112}$$

$$Q\left(b(q)\left|e\right\rangle\right) = \left(b(q)Q - \left[b(q), Q\right]\right)\left|e\right\rangle = (e+1)\left(b(q)\left|e\right\rangle\right) \tag{113}$$

$$Q\left(b^\dagger(q)\left|e\right\rangle\right) = \left(b^\dagger(q)Q - \left[b^\dagger(q), Q\right]\right)\left|e\right\rangle = (e-1)\left(b^\dagger(q)\left|e\right\rangle\right) \tag{114}$$

Thus, $a(q)$ and $a^\dagger(q)$ annihilates and creates, respectively, a positive quantum of charge, while $b(q)$ and $b^\dagger(q)$ annihilates and creates, respectively, a negative quantum of charge.

5.3.9 Example: Application to Pion Particles $\left\{\pi^+, \pi^-, \pi^0\right\}$

Consider three real scalar fields $\{\phi_j(x),\ \ j = 1, 2, 3\}$ governed by the Lagrangian

$$\mathscr{L} = \sum_{j=1}^{3} \frac{1}{2}\left[(\partial_\mu\phi_j)(\partial^\mu\phi_j) - m^2{\phi_j}^2\right]. \tag{115}$$

The pions can be described by the scalar fields

$$\pi^+(x) \equiv \frac{1}{\sqrt{2}}\left[\phi_1(x) + i\phi_2(x)\right], \qquad \pi^-(x) \equiv \frac{1}{\sqrt{2}}\left[\phi_1(x) - i\phi_2(x)\right], \qquad \pi^0(x) \equiv \phi_3(x). \tag{116}$$

In terms of the above fields, the Lagrangian of Eq. (115) becomes

$$\mathscr{L} = (\partial_\mu\pi^-)(\partial^\mu\pi^+) - m^2\pi^-\pi^+ + \frac{1}{2}(\partial_\mu\pi^0)(\partial^\mu\pi^0) - \frac{1}{2}m^2\left(\pi^0\right)^2, \tag{117}$$

which is invariant under transformations of the $U(1)$ symmetry group in the space $\left\{\pi^+, \pi^-, \pi^0\right\}$

$$\pi^+ \to e^{-i\theta}\pi^+, \qquad \pi^- \to e^{+i\theta}\pi^-, \qquad \pi^0 \to \pi^0. \tag{118}$$

5.3.10 Chronologically Ordered Product of Fields

We define the chronologically ordered product of fields by

$$T(\phi(x), \phi^\dagger(y)) = \begin{cases} \phi(x)\phi^\dagger(y), & x^0 > y^0 \\ \phi^\dagger(y)\phi(x), & y^0 > x^0 \end{cases} \tag{119}$$

or equivalently,

$$T\{\phi(x), \phi^\dagger(y)\} = \theta(x^0 - y^0)\phi(x)\phi^\dagger(y) + \theta(y^0 - x^0)\phi^\dagger(y)\phi(x), \tag{120}$$

where $\theta(x)$ is the Heaviside function.

5.3.11 Feynman Propagator of the Complex Scalar Field

The Feynman Propagator of the complex scalar field is defined as

$$G_F(x-y) = i\left\langle 0\right| T(\phi(x), \phi^\dagger(y))\left|0\right\rangle, \tag{121}$$

where for free fields, it can be shown that this is a Green function which satisfies

$$(\Box_x + m^2)\; i\left\langle 0\right| T(\phi(x), \phi^\dagger(y))\left|0\right\rangle = \delta^{(4)}(x-y). \tag{122}$$

By substituting in Eq. (122) the Fourier transform

$$G_F(x-y) = \frac{1}{(2\pi)^4}\int d^4k\; e^{-ik(x-y)}\widetilde{G}_F(k), \tag{123}$$

we can compute the solution in the Fourier space

$$(\Box_x + m^2)G_F(x-y) = \delta^{(4)}(x-y) \quad \longrightarrow \quad -\left(k^2 - m^2\right)\widetilde{G}_F(k) = 1 \quad \longrightarrow \quad \widetilde{G}_F(k) = \frac{-1}{k^2 - m^2}. \tag{124}$$

Thus, substituting in Eq. (123) we find

$$G_F(x-y) = \frac{-1}{(2\pi)^4}\int d^4k\, e^{-ik(x-y)}\frac{1}{k^2 - m^2 + i\epsilon}, \tag{125}$$

Since $k^2 - m^2 = \left(k^0\right)^2 - \mathbf{k}^2 - m^2 = \left(k^0\right)^2 - \omega_{\mathbf{k}}^2$, we can see that for $\epsilon = 0$ the integrand of Eq. (125) have poles at $k^0 = \pm\omega_{\mathbf{k}}$, so that, for carrying out the k^0 integral in the complex plane it is necessary to indicate how to avoid the above singularities. The prescription $+i\epsilon$ that we have introduced in the integrand of Eq. (125) does this work by moving the poles off the real k^0 axis to new positions located at $k^0 = \pm(\omega_{\mathbf{k}} - i\epsilon)$.

Using the Residue theorem to perform the integral over k^0, we finally arrive at the Feynman propagator

$$G_F(x-y) = i\int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}}\left[\theta(x^0 - y^0)e^{-ik(x-y)} + \theta(y^0 - x^0)e^{ik(x-y)}\right], \tag{126}$$

where $k(x-y) = \omega_{\mathbf{k}}(x^0 - y^0) - \mathbf{k}(\mathbf{x} - \mathbf{y})$ and we have made the change $\mathbf{k} \to -\mathbf{k}$ in the second term of the sum. Here, we can see that for propagation toward the future (i.e., $x^0 - y^0 > 0$) only positive energies ($e^{-ik(x-y)}$) contribute, while for propagation toward the past (i.e., $y^0 - x^0 > 0$) only negative energies ($e^{ik(x-y)}$) contribute.

5.4 Free Dirac (Spinor) Field

The free Dirac field can be described by a 4-component spinor $\psi(x)$, that in natural units satisfies

$$i\frac{\partial\psi}{\partial t} = H\psi, \qquad H = \beta m - i\alpha\nabla, \qquad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}, \tag{127}$$

where I and σ^i are the identity and the Pauli matrices, respectively.

Defining the Dirac representation,

$$\{\gamma^\mu, \quad \mu = 0,1,2,3\}, \qquad \gamma^0 \equiv \beta, \qquad \gamma^i \equiv \beta\alpha^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{128}$$

and taking into account that $\left(\gamma^0\right)^2 = I$, it can be shown that the equation of motion (127) can be written in the manifestly covariant form

$$(i\partial\!\!\!/ - m)\psi(x) = 0, \tag{129}$$

where $\partial\!\!\!/ \equiv \gamma^\mu\partial_\mu$. On the other hand, taking the Hermitian conjugate of Eq. (129), using that $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$, and that $\left(\gamma^0\right)^2 = I$, we obtain

$$\overline{\psi}(x)(i\overleftarrow{\partial\!\!\!/} + m) = 0, \tag{130}$$

where $\overline{\psi}(x)\overleftarrow{\partial\!\!\!/} \equiv \partial_\mu\overline{\psi}(x)\gamma^\mu$, and

$$\overline{\psi}(x) = \psi^\dagger(x)\gamma^0, \tag{131}$$

with $\psi^\dagger(x)$ being the adjoint of $\psi(x)$. From group theory, it can be shown that the Dirac fields $\psi(x)$ and $\overline{\psi}(x)$ describe, respectively, particles and antiparticles of spin 1/2 and mass m.

5.4.1 Lagrangian Density

The Lagrangian of the Dirac field is given by

$$\mathscr{L} = \overline{\psi}(x)(\frac{i}{2}\overleftrightarrow{\not\partial} - m)\psi(x), \tag{132}$$

where $a\overleftrightarrow{\not\partial}b \equiv a(\overrightarrow{\not\partial} - \overleftarrow{\not\partial})b \equiv a(\not\partial b) - (\not\partial a)b$. It can be shown that $\mathscr{L}$ can also be expressed as

$$\mathscr{L} = \overline{\psi}(x)(i\not\partial - m)\psi(x) - \frac{i}{2}\partial_\mu\left(\overline{\psi}(x)\gamma^\mu\psi(x)\right), \tag{133}$$

where $\partial_\mu\left(\overline{\psi}(x)\gamma^\mu\psi(x)\right) \equiv \partial_\mu J^\mu$ is a four-divergence of a four-current that does not affect the equations of motion, because when substituted in Eq. (3), by virtue of the four-divergence theorem the integral over the infinite volume of space-time is equal to the outward flux of J^μ through the closed surface at infinite (which must be zero). So, we can rewrite Eq. (133) as

$$\mathscr{L} = \overline{\psi}(x)(i\not\partial - m)\psi(x). \tag{134}$$

Substituting Eq. (134) in Eq. (5) it can be easily seen that we can get Eq. (129).

5.4.2 Canonical Momentum of the Field

The conjugate momentum of $\psi(x)$ follows from the definition (13) and the Lagrangian (134)

$$\Pi(x) = \frac{\partial\mathscr{L}}{\partial(\partial_0\psi)} = i\overline{\psi}(x)\gamma^0 = i\psi^\dagger(x). \tag{135}$$

5.4.3 Canonical Quantization of Fermionic Fields

In order to be consistent with the Fermi-Dirac statistics (antisymmetric states under the exchange of particles), the commutation relations of the Canonical Quantization Method must be replaced by corresponding anticommutation relations. Thus, denoting the anticommutators by { , } and using Eq. (135), we have

$$\{\psi_a(\mathbf{x},t), \Pi_b(\mathbf{x}',t)\} = i\left\{\psi_a(\mathbf{x},t), \psi_b^\dagger(\mathbf{x}',t)\right\} = i\delta_{ab}\delta^{(3)}(\mathbf{x}-\mathbf{x}'), \tag{136}$$

where a and b are spinor indices and all the other anticommutators are zero.

5.4.4 Solutions of the Free Dirac Equation

To express the Dirac fields as a Fourier expansion, we focus first on the independent solutions of the Dirac equation, which constitute a complete basis set.

- *Positive energy solutions ($E > 0$):*

The Dirac equation (129) admits plane wave solutions with $E > 0$, of the form

$$\psi^{(+)}(x) = e^{-ikx}u(k), \tag{137}$$

where $u(k)$ is a four-spinor satisfying

$$(\not k - m)u(k) = 0, \qquad \not k = k_\mu\gamma^\mu \tag{138}$$

As is well known, the helicity is the projection of the spin onto the direction of momentum. Dirac particles (particles with spin 1/2) can have helicities +1/2 and −1/2. It can be shown that there are two independent solutions of Eq. (138), corresponding to helicity +1/2 ($\alpha = 1$) and −1/2 ($\alpha = 2$)

$$u^{(\alpha)}(k) = \begin{pmatrix} \sqrt{m+k^0}\, I\varphi_0^{(\alpha)} \\ \\ \frac{k_i\sigma^i}{\sqrt{m+k^0}}\,\varphi_0^{(\alpha)} \end{pmatrix}, \quad \alpha = 1,2, \qquad \varphi_0^{(1)} \equiv \begin{pmatrix}1\\0\end{pmatrix}, \quad \varphi_0^{(2)} \equiv \begin{pmatrix}0\\1\end{pmatrix}, \tag{139}$$

where $\left\{\sigma^i,\ \ i=1,2,3\right\}$ are the Pauli matrices.

Introducing the notation

$$\left\{\varphi_0^{(1)} \equiv \begin{pmatrix}1\\0\end{pmatrix}, \varphi_0^{(2)} \equiv \begin{pmatrix}0\\1\end{pmatrix}\right\} \equiv \left\{+\frac{1}{2}, -\frac{1}{2}\right\} \equiv \{\uparrow, \downarrow\}, \tag{140}$$

the positive energy solutions of the free Dirac equation can be written as

$$\{E>0\uparrow, E>0\downarrow\} \equiv \left\{u^{(1)}(k)e^{-ikx}, u^{(2)}(k)e^{-ikx}\right\}. \tag{141}$$

- <u>*Negative energy solutions ($E<0$):*</u>

The Dirac equation (129) also admits negative energy plane-wave solutions of the form

$$\psi^{(-)}(x) = e^{ikx}v(k), \tag{142}$$

where the 4-spinor $v(k)$ is a solution of the equation

$$(\not{k}+m)v(k) = 0. \tag{143}$$

As before, there are two independent solutions, $\alpha = 1,2$, corresponding to helicities $+1/2$ and $-1/2$, respectively

$$v^{(\alpha)}(k) = \begin{pmatrix} \frac{k_i\sigma^i}{\sqrt{m+k^0}}\,\varphi_0^{(\alpha)} \\ \\ \sqrt{m+k^0}\,I\varphi_0^{(\alpha)} \end{pmatrix}, \quad \alpha = 1,2, \tag{144}$$

In summary, the Dirac equation (129) admits the following set of independent solutions

$$\{E>0\uparrow, E>0\downarrow, E<0\uparrow, E<0\downarrow\} \equiv \left\{u^{(1)}(k)e^{-ikx}, u^{(2)}(k)e^{-ikx}, v^{(1)}(k)e^{+ikx}, v^{(2)}(k)e^{+ikx}\right\}. \tag{145}$$

Fourier Expansion of the Field

We can now expand the Dirac fields $\psi(x)$ and $\overline{\psi}(x)$ in the basis set (145) as

$$\psi(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}} \sum_{\lambda=1}^{2} \left[b^{(\lambda)}(k)u^{(\lambda)}(k)e^{-ikx} + d^{\dagger(\lambda)}(k)v^{(\lambda)}(k)e^{ikx}\right] \tag{146}$$

$$\overline{\psi}(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}} \sum_{\lambda=1}^{2} \left[d^{(\lambda)}(k)\overline{v}^{(\lambda)}(k)e^{-ikx} + b^{\dagger(\lambda)}(k)\overline{u}^{(\lambda)}(k)e^{ikx}\right], \tag{147}$$

where $\lambda = 1,2$ are the different helicities, and $\overline{u}(k) \equiv u^\dagger(k)\gamma^0$ and $\overline{v}(k) \equiv v^\dagger(k)\gamma^0$.

5.4.5 Commutation Relations among $b(k)$, $b^\dagger(k)$, $d(k)$ and $d^\dagger(k)$

From the above equations and Eq. (136), it can be shown that

$$\left\{b^{(\lambda)}(q), b^{\dagger(\lambda')}(k)\right\} = (2\pi)^3 2\omega_{\mathbf{k}}\delta^{\lambda\lambda'}\delta^{(3)}(\mathbf{k}-\mathbf{q}), \tag{148}$$

$$\left\{d^{(\lambda)}(q), d^{\dagger(\lambda')}(k)\right\} = (2\pi)^3 2\omega_{\mathbf{k}}\delta^{\lambda\lambda'}\delta^{(3)}(\mathbf{k}-\mathbf{q}), \tag{149}$$

with all the others being zero.

5.4.6 Four-Momentum of the Field

Following a similar procedure to that already seen in the cases of the scalar fields, from the Lagrangian invariance under space-time translations and Noether's theorem one can obtain the following expression for the 4-momentum of the Dirac field

$$P^\mu = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_\mathbf{k}} k^\mu \sum_{\lambda=1}^{2} \Big[b^{\dagger(\lambda)}(k) b^{(\lambda)}(k) - d^{(\lambda)}(k) d^{\dagger(\lambda)}(k) \Big]. \tag{150}$$

Using Eq. (149), the expectation value of P^μ in the vacuum takes the form

$$\langle 0| P^\mu |0\rangle = \langle 0| \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_\mathbf{k}} k^\mu \sum_{\lambda=1}^{2} \Big\{ d^{(\lambda)}(k), d^{\dagger(\lambda)}(k) \Big\} |0\rangle , \tag{151}$$

and, as occurs for the scalar fields, $\langle 0| P^j |0\rangle = 0$ because of the odd integrand, while $\langle 0| P^0 |0\rangle = \infty$ as a consequence of the even and divergent integrand.

Taking into account that $d^{(\lambda)}(k)d^{\dagger(\lambda)}(k) = \big\{d^{(\lambda)}(k), d^{\dagger(\lambda)}(k)\big\} - d^{\dagger(\lambda)}(k)d^{(\lambda)}(k)$, we can extract the infinite energy value of the vacuum from the P^μ redefining the latter as

$$P^\mu \equiv P^\mu - \langle 0| P^\mu |0\rangle = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_\mathbf{k}} k^\mu \sum_{\lambda=1}^{2} \Big[b^{\dagger(\lambda)}(k) b^{(\lambda)}(k) + d^{\dagger(\lambda)}(k) d^{(\lambda)}(k) \Big]. \tag{152}$$

Defining the Wick ordering for fermionic operators

$$: dd^\dagger : \equiv -d^\dagger d, \qquad : bb^\dagger : \equiv -b^\dagger b, \tag{153}$$

Eq. (152) can be rewritten in the form

$$P^\mu \equiv \; : \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_\mathbf{k}} k^\mu \sum_{\lambda=1}^{2} \Big[b^{\dagger(\lambda)}(k) b^{(\lambda)}(k) - d^{(\lambda)}(k) d^{\dagger(\lambda)}(k) \Big] : \tag{154}$$

where we have used that $: b^\dagger b - dd^\dagger : = b^\dagger b + d^\dagger d$.

5.4.7 Invariance of $\mathscr{L}$ under U(1) Global

It can be easily seen that the Lagrangian (132) is invariant under global $U(1)$ gauge transformations

$$\psi(x) \to e^{-i\theta}\psi(x) \simeq (1 - i\epsilon)\psi(x) \quad \longrightarrow \quad F = -i\psi(x) \tag{155}$$

$$\overline{\psi}(x) \to e^{i\theta}\overline{\psi}(x) \simeq \overline{\psi}(x)(1 + i\epsilon) \quad \longrightarrow \quad \overline{F} = i\overline{\psi}(x). \tag{156}$$

Conserved Quantities

In a similar manner to that already discussed for the complex scalar field, using Noether's theorem one finds the following conserved current associated to this internal symmetry:

$$J^\mu = \; : \frac{\partial \mathscr{L}}{\partial(\partial_\mu \psi)} F + \overline{F} \frac{\partial \mathscr{L}}{\partial(\partial_\mu \overline{\psi})} : \; = \; : \frac{\partial \mathscr{L}}{\partial(\partial_\mu \psi)} (-i\psi) + (i\overline{\psi}) \frac{\partial \mathscr{L}}{\partial(\partial_\mu \overline{\psi})} : \; = \; : \overline{\psi}\gamma^\mu\psi : \qquad / \; \partial_\mu J^\mu = 0 \tag{157}$$

where we have used Eq. (134) and have extracted the vacuum value with the Wick ordering. Moreover, from Eq. (10), the corresponding conserved charge is

$$Q = \; : \int d^3x J^0 : \; = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_\mathbf{k}} \sum_{\lambda=1}^{2} \Big[b^{\dagger(\lambda)}(k) b^{(\lambda)}(k) - d^{\dagger(\lambda)}(k) d^{(\lambda)}(k) \Big] = (+1)N_b + (-1)N_d, \tag{158}$$

where the positive charge (+1) is given in units of the electron charge.

Using Eq. (158) and the anticommutation relations (148)–(149), it can be shown that

$$\left[Q, b^{\dagger(\lambda)}(k)\right] = b^{\dagger(\lambda)}(k) \tag{159}$$

$$\left[Q, d^{\dagger(\lambda)}(k)\right] = -d^{\dagger(\lambda)}(k). \tag{160}$$

Consider an eigenstate $|e\rangle$ of the charge operator Q satisfying the eigenvalue equation $Q\,|e\rangle = e\,|e\rangle$. Then, making use of Eqs. (159)–(160) we have

$$Q\; b^{\dagger(\lambda)}(k)\,|0\rangle = (e+1)\; b^{\dagger(\lambda)}(k)\,|0\rangle\,, \qquad \lambda = 1,2. \tag{161}$$

$$Q\; d^{\dagger(\lambda)}(k)\,|0\rangle = (e-1)\; d^{\dagger(\lambda)}(k)\,|0\rangle\,, \qquad \lambda = 1,2. \tag{162}$$

Thus, $b^{\dagger(\lambda)}(k)$ can be interpreted as an operator that creates a Dirac particle ($s = \frac{1}{2}$) with helicity λ, four-momentum k^μ, mass m ($k^2 = m^2$), and charge (+1) (in units of the electron charge). Similarly, the operator $d^{\dagger(\lambda)}(k)$ creates the corresponding antiparticle (with charge (−1)).

5.4.8 Chronologically Ordered Product of Dirac Fields

In a similar manner as has been done for the complex scalar field, we define the chronologically ordered product of Dirac fields by

$$T\{\psi_a(x), \overline{\psi}_b(y)\} = \theta(x^0 - y^0)\psi_a(x)\overline{\psi}_b(y) - \theta(y^0 - x^0)\overline{\psi}_b(y)\psi_a(x) \tag{163}$$

where $\theta(x)$ is the Heaviside function.

5.4.9 Feynman Propagator for the Dirac Field

Following a similar procedure to that used in the case of the complex scalar field (Section 5.3.11) it can be shown that the Feynman propagator of the Dirac field is given by

$$S_F(x-y)_{ab} = -i\,\langle 0|\,T\left\{\psi_a(x), \overline{\psi}_b(y)\right\}|0\rangle = \frac{1}{(2\pi)^4}\int d^4k e^{-ik(x-y)}\left(\frac{1}{\not{k} - m + i\epsilon}\right)_{ab}. \tag{164}$$

5.5 Free Electromagnetic Field

The electromagnetic field can be quantized by two methods. One starts from the equations of Classical Electrodynamics and it is not a manifestly covariant method. The other one is the Gupta-Bleuler indefinite metric quantization method, which is manifestly covariant.

The Classical Electrodynamics Method

The sources of the electromagnetic field can be combined in a four-current $j^\mu = (\rho, \mathbf{j})$, where ρ and $\mathbf{j}$ are the charge and the current densities, respectively. Similarly, the scalar and vector potentials can be considered as the components of the electromagnetic four-potential $A^\mu = (\phi, \mathbf{A})$.

The electromagnetic field strength tensor $F^{\mu\nu}$ is defined by

$$F^{\mu\nu} = -F^{\nu\mu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix}, \tag{165}$$

where $(E^1, E^2, E^3) = (E_x, E_y, E_z)$ are the electric field components and $(B^1, B^2, B^3) = (B_x, B_y, B_z)$ are the magnetic field components.

On the other hand, the dual electromagnetic field strength tensor $\widetilde{F}^{\mu\nu}$ is given by

$$\widetilde{F}^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\varsigma} F_{\rho\varsigma} = -\widetilde{F}^{\mu\nu} = \begin{pmatrix} 0 & -B^1 & -B^2 & -B^3 \\ B^1 & 0 & E^3 & -E^2 \\ B^2 & -E^3 & 0 & E^1 \\ B^3 & E^2 & -E^1 & 0 \end{pmatrix}, \tag{166}$$

where $\epsilon^{\mu\nu\rho\varsigma}$ is the antisymmetric tensor. Note that the dual transformation involved in Eq. (166) consists in making the changes: $\mathbf{E} \to \mathbf{B}$ and $\mathbf{B} \to -\mathbf{E}$.

The electric and magnetic fields can be obtained from the scalar and vector potentials

$$\left.\begin{array}{l} \mathbf{E} \ = -\nabla\phi - \frac{\partial}{\partial t}\mathbf{A} \\ \\ \mathbf{B} = \nabla \times \mathbf{A} \end{array}\right\} \iff F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{167}$$

and using Eq. (167), the Maxwell equations can be rewritten in a manifestly covariant form

$$\left.\begin{array}{l} \nabla \times \mathbf{E} + \frac{\partial}{\partial t}\mathbf{B} = \mathbf{0} \\ \\ \nabla \cdot \mathbf{B} = 0 \end{array}\right\} \iff \partial_\mu \widetilde{F}^{\mu\nu} = 0 \tag{168}$$

$$\left.\begin{array}{l} \nabla \cdot \mathbf{E} = \rho \\ \\ \nabla \times \mathbf{B} - \frac{\partial}{\partial t}\mathbf{E} = \mathbf{j} \end{array}\right\} \iff \partial_\mu F^{\mu\nu} = j^\nu. \tag{169}$$

Noting that the multiplication of a symmetric tensor $(\partial_\nu\partial_\mu)$ and an antisymmetric one $(F^{\mu\nu})$ gives zero, from Eq. (169) it immediately follows the continuity equation

$$\partial_\mu F^{\mu\nu} = j^\nu \quad \longrightarrow \quad \partial_\nu j^\nu = \partial_\nu \partial_\mu F^{\mu\nu} = 0. \tag{170}$$

While the equations of motion of the electromagnetism (168)–(169) are first order differential equations, when are expressed in terms of A^ν they become second order differential equations

$$j^\nu = \partial_\mu F^{\mu\nu} = \partial_\mu \left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) = \Box A^\nu - \partial^\nu \left(\partial_\mu A^\mu\right). \tag{171}$$

We will consider $A^\mu(x)$ as the quantum field associated to the photon.

As is well known, because of the gauge invariance of the electromagnetism, there is an arbitrariness in the definition of $A^\mu(x)$. Indeed, consider the following gauge transformation

$$\left.\begin{array}{l} \phi'(x) = \phi(x) + \frac{\partial}{\partial t}\Lambda(x) \\ \\ \mathbf{A}'(x) = \mathbf{A}(x) - \nabla\Lambda(x) \end{array}\right\} \iff A^{\mu\prime}(x) = A^\mu(x) + \partial^\mu \Lambda(x), \tag{172}$$

where $\Lambda(x)$ is an arbitrary function of $(\mathbf{x}, t)$. As can be easily shown $F^{\mu\nu\prime}(x) = F^{\mu\nu}(x)$, hence we can find different four-potentials which lead to the same physical fields $\mathbf{E}$ and $\mathbf{B}$. Despite this question, it has been experimentally shown that changes in $A^\mu(x)$ have physical (measurable) effects on the system (Aharonov–Bohm effect) [15]. This enables us to consider $A^\mu(x)$ as the quantum field of the photon.

5.5.1 Lagrangian Density

The Lagrangian density of the electromagnetic field is known from Classical Electrodynamics:

$$\mathscr{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - j_\nu A^\nu, \tag{173}$$

where the first term of the sum is the free Lagrangian density and the second one accounts for the interaction of the field with the four-current.

Substitution of Eq. (173) in the Euler–Lagrange equations (5) leads to the equations of motion (171). And, for the free electromagnetic field ($j^\nu(x) = 0$), one obtains

$$\Box A^\nu - \partial^\nu(\partial_\mu A^\mu) = 0. \tag{174}$$

In Classical Electrodynamics one usually works in the Lorentz gauge ($\partial_\nu A^\nu = 0$), in which case the above equations of motion reduce to

$$\Box A^\nu(x) = 0. \tag{175}$$

However, as we will see, in Quantum Field Theory it is convenient to work in the Radiation gauge which, unlike the Lorentz gauge, it is not manifestly covariant.

5.5.2 Canonical Momentum

Substituting the Lagrangian (173) in the definition of the canonical momentum, Eq. (13), we obtain

$$A_\mu(x) \longrightarrow \Pi^\mu(x) = \frac{\partial \mathscr{L}}{\partial(\partial_0 A_\mu)} = F^{\mu 0}. \tag{176}$$

Then,

$$\begin{cases} \Pi^i(x) = \partial^i A^0 - \partial^0 A^i \longrightarrow \mathbf{\Pi}(x) = -\nabla A^0 - \dot{\mathbf{A}} = -\nabla\phi - \frac{\partial}{\partial t}\mathbf{A} = \mathbf{E} \\ \Pi^0(x) = F^{00} = 0 \end{cases} \tag{177}$$

5.5.3 Quantization of the Field

Now, we want to quantize the free electromagnetic field using the Canonical quantization method. To this end, we impose the following equal-time commutation relations (see Eq. (14)):

$$\left[A_j(\mathbf{x},t), \Pi^k(\mathbf{x}',t)\right] = i\delta_j^k \delta^{(3)}(\mathbf{x}-\mathbf{x}') \iff \left[A^j(\mathbf{x},t), \Pi^k(\mathbf{x}',t)\right] = i\eta^{jk}\delta^{(3)}(\mathbf{x}-\mathbf{x}') \tag{178}$$

$$\left[A_0(\mathbf{x},t), \Pi^0(\mathbf{x}',t)\right] = 0 \tag{179}$$

and all the other commutation relations vanish.

There are some problems that can be noticed from the above commutation relations:

1. $A_0(\mathbf{x},t)$ is a c-number and thus becomes singularized with respect to spatial components (non manifestly covariant quantization)
2. The commutation relation (178) is inconsistent with the Maxwell equation $\nabla\cdot\mathbf{E} = 0$. Indeed,

$$\nabla\cdot\mathbf{E} = \partial_k E^k = 0 \implies \frac{\partial}{\partial x'^k}\left[A^j(\mathbf{x},t), E^k(\mathbf{x}',t)\right] = 0. \tag{180}$$

   However, using Eq. (178) and substituting the expression for $\mathbf{\Pi}(x)$ which we computed in Eq. (177), we arrive at

$$\frac{\partial}{\partial x'^k}\left[A^j(\mathbf{x},t), E^k(\mathbf{x}',t)\right] = \frac{\partial}{\partial x^{k\prime}}\left[i\eta^{jk}\delta^{(3)}(\mathbf{x}-\mathbf{x}')\right] = -\int \frac{d^3\mathbf{k}}{(2\pi)^3}\, k^j e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')} \neq 0, \tag{181}$$

   which is clearly inconsistent with Eq. (180).

We can address these questions using the following procedure:

1. We eliminate the c-number field $A^0(x)$ by working in the Radiation gauge

$$A^0(x) = \phi(x) = 0, \qquad \nabla \cdot \mathbf{A} = 0 \tag{182}$$

Since the temporal and spatial components of $A^\mu(x)$ does not enter on equal footing, this procedure leads to a non-manifestly covariant formulation. An interesting consequence of working in the Radiation gauge is that from the four degrees of freedom of $A^\mu(x)$, only two them remain after imposing the conditions (182). Such degrees of freedom correspond to the two transverse photons.

2. We modify the commutation relation (178) by replacing the delta function on the right-hand side by a transverse delta function $D^{jk}_{trans}(\mathbf{x}-\mathbf{x}')$ with zero four-divergence

$$\eta^{jk}\delta^{(3)}(\mathbf{x}-\mathbf{x}') \longrightarrow \int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')} \left(\eta^{jk} + \frac{k^j k^k}{\mathbf{k}^2}\right) \equiv D^{jk}_{trans}(\mathbf{x}-\mathbf{x}'). \tag{183}$$

With this substitution, Eq. (178) becomes

$$\left[A^j(\mathbf{x},t), \Pi^k(\mathbf{x}',t)\right] \equiv iD^{jk}_{trans}(\mathbf{x}-\mathbf{x}'). \tag{184}$$

5.5.4 Equations of Motion in the Radiation Gauge

In the Radiation gauge (Eqs. (182)), the equations of motion (174) of the electromagnetic field reduce to

$$\Box \mathbf{A}(x) = 0, \qquad A^0(x) = \mathbf{0}. \tag{185}$$

As is apparent, the first of the above equations is similar to the equation of motion (23) of the real scalar field, but with the differences that now we have a vector field $\mathbf{A}(x)$ instead of the scalar field $\phi(x)$, and also $k^2 = m^2 = 0$. Thus, taking into account the expansion (38), we can write the electromagnetic field in the Radiation gauge as

$$\mathbf{A}(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}} \left[\mathbf{a}(k)e^{-ikx} + \mathbf{a}^\dagger(k)e^{ikx}\right]. \tag{186}$$

For every normal mode $\mathbf{k}$, we define an orthonormal basis set $\left\{\boldsymbol{\varepsilon}^{(\lambda)}(k), \quad \lambda = 1,2,3\right\}$,

$$\boldsymbol{\varepsilon}^{(\lambda)}(k) \cdot \boldsymbol{\varepsilon}^{(\lambda')}(k) = \delta^{\lambda\lambda'}, \qquad \lambda,\lambda' = 1,2,3 \tag{187}$$

satisfying the condition

$$\boldsymbol{\varepsilon}^{(\lambda)}(k) \cdot \mathbf{k} = 0, \quad \lambda = 1,2. \tag{188}$$

Thus, $\boldsymbol{\varepsilon}^{(1)}(k)$ and $\boldsymbol{\varepsilon}^{(2)}(k)$ are transverse polarization vectors while the longitudinal vector $\boldsymbol{\varepsilon}^{(3)}(k)$ is directed along the field propagation direction $\mathbf{k}$.

In terms of the above basis, the annihilation and creation vector operators can be expressed as

$$\mathbf{a}(k) = \sum_{\lambda=1}^{3} a^{(\lambda)}(k)\boldsymbol{\varepsilon}^{(\lambda)}(k), \qquad \mathbf{a}^\dagger(k) = \sum_{\lambda=1}^{3} a^{\dagger(\lambda)}(k)\boldsymbol{\varepsilon}^{(\lambda)}(k), \tag{189}$$

and imposing the Radiation gauge, Eqs. (182), we get

$$0 = \nabla \cdot \mathbf{A} = i\int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}} \left[\mathbf{k}\cdot\mathbf{a}(k)e^{-ikx} - \mathbf{k}\cdot\mathbf{a}^\dagger(k)e^{ikx}\right], \tag{190}$$

from which, using the independence of the plane waves and the fact that $\mathbf{k} = k\boldsymbol{\varepsilon}^{(3)}(k)$, we find

$$\mathbf{k}\cdot\mathbf{a}(k) = 0 \longrightarrow a^{(3)}(k) = 0, \qquad \mathbf{k}\cdot\mathbf{a}^\dagger(k) = 0 \longrightarrow a^{\dagger(3)}(k) = 0. \tag{191}$$

This result indicates that there are no longitudinal photons. Thus, the electromagnetic field can be finally expressed as

$$\mathbf{A}(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}} \sum_{\lambda=1}^{2} \left[a^{(\lambda)}(k)\boldsymbol{\varepsilon}^{\lambda}(k)e^{-ikx} + a^{\dagger(\lambda)}(k)\boldsymbol{\varepsilon}^{\lambda}(k)e^{ikx}\right] \tag{192}$$

where the sum runs only over transverse (physical) photons ($\lambda = 1,2$).

The Gupta-Bleuler Quantization Method (Manifestly Covariant Method)

We have just seen the quantization of the electromagnetic field postulating the canonical commutation relations only for the dynamical degrees of freedom (spatial components). The price has been to lose the desired manifest covariance in the formalism. For this reason a manifestly covariant quantization procedure has been devised which will be seen in this Section. The idea is to treat the four components A^μ on equal footing as dynamical variables and, only after the electromagnetic field has been quantized, it imposed the constraint that characterizes physical photon states.

To achieve this goal, we start from the following Lagrangian density (which is different from that of the electromagnetism, Eq. (173)):

$$\mathscr{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{\lambda}{2}(\partial_\rho A^\rho)^2, \qquad (\lambda \neq 0, \;\; \partial_\rho A^\rho \neq 0). \tag{193}$$

Then, we will recover the electromagnetism by imposing conditions onto the physical states.

Note that since $\partial_\rho A^\rho \neq 0$, λ is not a Lagrangian multiplier and, unlike what happened before (see Eq. (177)), now the time component of the canonical momentum is not zero

$$\Pi^0(x) = -\lambda(\partial_\rho A^\rho) \neq 0. \tag{194}$$

The Euler–Lagrange equations corresponding to the Lagrangian density (193) take the form

$$\Box A^\mu - (1-\lambda)\partial^\mu(\partial_\rho A^\rho) = 0. \tag{195}$$

It can be proven that the physical results are independent of λ (provided that $\lambda \neq 0$). Therefore, for simplicity we shall henceforth take $\lambda = 1$, which (by abuse of language) is called the Feynman gauge. In this gauge the equations of motion (195) coincide with the Maxwell equations in the Lorentz gauge

$$\Box A^\mu(x) = 0. \tag{196}$$

Now, the Canonical Quantization Method leads to the following commutation relations:

$$[A_\mu(\mathbf{x},t), \Pi^\nu(\mathbf{x}',t)] = i\delta^\nu_\mu \delta^{(3)}(\mathbf{x}-\mathbf{x}') \iff [A^\mu(\mathbf{x},t), \Pi^\nu(\mathbf{x}',t)] = i\eta^{\mu\nu}\delta^{(3)}(\mathbf{x}-\mathbf{x}'), \tag{197}$$

and the other ones are zero.

Notice that the above is a manifestly covariant formulation, where the time and spatial components of A^μ enter on equal footing.

Fourier Descomposition in the Feynman Gauge

By noting that Eq. (196) is nothing but a generalization of Eq. (185), taking into account Eq. (192) it is not hard to see that $A_\mu(x)$ can be expanded as

$$A_\mu(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}} \sum_{\lambda=0}^{3} \Big[a^{(\lambda)}(k)\varepsilon^{(\lambda)}_\mu(k)e^{-ikx} + a^{\dagger(\lambda)}(k)\varepsilon^{(\lambda)}_\mu(k)e^{ikx}\Big], \tag{198}$$

where we have introduced the basis of polarization vectors $\big\{\varepsilon^{(\lambda)}(k), \quad \lambda = 0,1,2,3\big\}$ in the Minkowski space, with $\lambda = 1,2$ corresponding to transverse, $\lambda = 3$ to longitudinal, and $\lambda = 0$ to temporal polarizations, respectively. It is thus clear that in this formulation there appear spurious degrees of freedom (temporal and longitudinal photons). It can be shown that such unphysical states can be eliminated from the Hilbert space by defining the physical states as those satisfying the Lorentz gauge condition in mean value

$$\langle\psi_{\rm phys}|\,\partial_\mu A^\mu(x)\,|\psi_{\rm phys}\rangle = 0. \tag{199}$$

Indeed, for such states the contribution from temporal photons cancels with that from longitudinal ones, so that, only the contribution from transverse photons survives and, in practice, the sum over polarizations in Eq. (198)) can be restricted to $\lambda = 1,2$.

Why Indefinite Metric?

It can be shown that

$$\left[a^{(\lambda)}(k), a^{\dagger(\lambda')}(k')\right] = -\eta^{\lambda\lambda'} 2\omega_{\mathbf{k}}(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}') \tag{200}$$

and the other commutation relation are zero.

The fact that $\eta^{00} = 1$ while $\eta^{ii} = -1$ implies that the squared norm of temporal photons has opposite sign to that of spatial photons. Hence, the Fock space has an indefinite metric.

5.5.5 Chronologically Ordered Product of Fields

As in previous cases, we define the chronologically ordered product of $A_\mu(x)$ fields by

$$T\{A_\mu(x), A_\nu(y)\} = \theta(x^0 - y^0)A_\mu(x)A_\nu(y) + \theta(y^0 - x^0)A_\nu(y)A_\mu(x), \tag{201}$$

where $\theta(x)$ is the Heaviside function.

5.5.6 Feynman Propagator of the Electromagnetic Field in the Feynman Gauge

We define the Feynman propagator of the electromagnetic field by

$$G_{\mu\nu}(x-y) = \langle 0| T\left(A_\mu(x), A_\nu(y)\right)|0\rangle\,. \tag{202}$$

Substituting Eq. (198) in Eq. (202), it can be shown that

$$G_{\mu\nu}(x-y) = i\eta_{\mu\nu}G_F(x-y)|_{m^2=0}, \tag{203}$$

where $G_F(x-y)$ is the Feynman propagator of the complex scalar field given in Eq. (125). Thus, substituting the latter equation one finally obtains

$$G_{\mu\nu}(x-y) = \frac{-i}{(2\pi)^4}\int d^4k e^{-ik(x-y)}\left(\frac{\eta_{\mu\nu}}{k^2+i\epsilon}\right). \tag{204}$$

6 Perturbation Theory and Feynman Diagrams

Now we are going to study the behavior of the fields in interaction.

The commutation relations at equal time of the Canonical Quantization method remain valid for interacting fields, but in general $a^\dagger$ and a do not admit an interpretation in terms of creation and annihilation operators, and in general their time evolution are not solvable in the Heisenberg picture. We will solve this problem using perturbation theory in terms of free fields making use of the Interaction picture.

6.1 Interaction Picture

Consider the Schrödinger equation for a state vector $|\phi_S(t)\rangle$ governed by a Hamiltonian H_S that can be separated into free (H_S^0) and interaction (H_S^{int}) contributions,

$$i\frac{d}{dt}|\phi_S(t)\rangle = H_S|\phi_S(t)\rangle\,, \qquad H_S = H_S^0 + H_S^{int}. \tag{205}$$

State vectors $|\psi_I(t)\rangle$ and operators $A_I(t)$ are defined in the Interaction picture by means of the following unitary transformations

$$|\psi_I(t)\rangle \equiv e^{iH_S^0 t}|\phi_S(t)\rangle \tag{206}$$

$$A_I(t) \equiv e^{iH_S^0 t} A_S\, e^{-iH_S^0 t}. \tag{207}$$

In particular, since H_S^0 commutes with any function of itself, Eq. (207) implies that $H_I^0 = H_S^0 \equiv H^0$.

By differentiating Eq. (206) with respect to t, it is easy to see that the equation of motion for the states in the Interaction picture is given by

$$i\frac{d}{dt}\left|\psi_I(t)\right\rangle = H_I(t)\left|\psi_I(t)\right\rangle, \qquad H_I(t) \equiv H_I^{int}(t) = e^{iH^0 t} H_S^{int}\, e^{-iH^0 t} \tag{208}$$

where $H_I(t)$ is the interaction term of the Schrödinger Hamiltonian in the Interaction picture ($H_I \equiv H_I^{int}$). Thus interacting states evolve according to the interaction term of the Hamiltonian, H_I^{int}(t).

Similarly, by differentiating Eq. (207) with respect to t, one obtains the equation of motion for the operators in the Interaction picture

$$i\frac{d}{dt} A_I(t) = \left[A_I(t), H^0\right], \tag{209}$$

where we have supposed $A_S \neq A_S(t)$. Therefore, in the Interaction picture the operators behave as free fields, evolving according to the free Hamiltonian H^0.

Since the different pictures are connected by unitary transformations, the physical properties, i.e. mean values and transition amplitudes, are the same in all pictures (representations)

$$\langle \phi_S(t)|A_S|\phi_S(t)\rangle = \langle \psi_H|A_H(t)|\psi_H\rangle = \langle \psi_I(t)|A_I(t)|\psi_I(t)\rangle \equiv \langle A\rangle(t), \tag{210}$$

where ψ_H and $A_H(t)$ are, respectively, the states and the operators in the Heisenberg picture,

$$\left|\psi_H\right\rangle = e^{iHt}\left|\phi_S(t)\right\rangle = \left|\phi_S(0)\right\rangle, \qquad A_H(t) = e^{iH_S t} A_S\, e^{-iH_S t}, \tag{211}$$

and we have used that the states in the Schrödinger picture evolve as $\left|\phi_S(t)\right\rangle = e^{-iHt}\left|\phi_S(0)\right\rangle$.

From Eqs. (206), (207), and (211), we can see that at $t = 0$ the three representations coincide. Moreover, if $H_S^{int} = 0$ the Interaction and the Heisenberg pictures become indistinguishable.

The interaction term of the Hamiltonian is given by

$$H_I^{int}(x) = e^{iH^0 t} H_S^{int}(\mathbf{x})\, e^{-iH^0 t}, \tag{212}$$

where $H_S^{int}(\mathbf{x})$, in general, takes the form

$$H_S^{int}(\mathbf{x}) = \prod_\beta \phi_{\beta\, S}(\mathbf{x}), \tag{213}$$

with $\{\phi_{\beta\, S}(\mathbf{x})\}$ being (interacting) quantum fields (operators) in the Schrödinger picture.

Then, substituting Eq. (213) in Eq. (212) and using Eq. (211) we get

$$H_I^{int}(x) = e^{iH^0 t}\prod_\beta \left[\phi_{\beta\, H}(\mathbf{x}, t=0)\right] e^{-iH^0 t} = \prod_\beta e^{iH^0 t}\phi_{\beta\, H}(\mathbf{x}, t=0) e^{-iH^0 t} = \prod_\beta \phi_{\beta\, H}(x), \tag{214}$$

Therefore, the interaction Hamiltonian $H^{int}(x)$, which in general depends on interacting fields, in the Interaction picture is a function of free fields in the Heisenberg picture (those studied in the previous Sections).

6.1.1 Evolution Operator in the Interaction Picture ($U_I(t, t_0) \equiv U(t, t_0)$)

From now on, we will always work in the Interaction picture. In this picture, the evolution operator $U(t, t_0)$ is defined as

$$\left|\psi_I(t)\right\rangle \equiv U(t, t_0)\left|\psi_I(t_0)\right\rangle, \tag{215}$$

where

$$U(t,t) = I, \qquad U(t_1,t_2)U(t_2,t_3) = U(t_1,t_3). \tag{216}$$

Differentiating Eq. (215) with respect to t and using Eq. (208), we find the equation of motion

$$i\frac{\partial U(t,t_0)}{\partial t} = H_I(t)U(t,t_0), \tag{217}$$

which can be expressed in an integral form

$$U(t,t_0) = I - i\int_{t_0}^{t} dt' H_I(t')U(t',t_0). \tag{218}$$

The above equation can be solved iteratively

$$U(t,t_0) = I - i\int_{t_0}^{t} dt' H_I(t') + (-i)^2\int_{t_0}^{t} dt' H_I(t')\int_{t_0}^{t'} dt'' H_I(t'') + \cdots, \tag{219}$$

where the integrands are chronologically ordered: $\quad \cdots < t'' < t' < t.$

It can be shown that Eq. (219) can be rewritten as

$$U(t,t_0) = \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_{t_0}^{t} dt_1\cdots\int_{t_0}^{t} dt_n\, T(H_I(t_1),...,H_I(t_n)) \equiv T\mathrm{exp}\left\{-i\int_{t_0}^{t} dt' H_I(t')\right\}. \tag{220}$$

Let $|\psi_I(-\infty)\rangle$ and $|\psi_I(+\infty)\rangle$ be asymptotic free states defined by

$$|\psi_I(-\infty)\rangle = \lim_{t\to-\infty}|\psi_I(t)\rangle, \qquad |\psi_I(+\infty)\rangle = \lim_{t\to+\infty}|\psi_I(t)\rangle \tag{221}$$

The transition amplitude from the initial state $|\alpha\rangle \equiv |\psi_I(-\infty)\rangle$ to the final state $|\beta\rangle$ is given by

$$\lim_{t_2\to+\infty}\langle\beta|\psi_I(t_2)\rangle = \lim_{\substack{t_1\to-\infty\\ t_2\to+\infty}}\langle\beta|U(t_2,t_1)|\alpha\rangle \equiv \langle\beta|S|\alpha\rangle \equiv S_{\alpha\beta} \tag{222}$$

where we have made use of Eqs. (215) and (220) and have defined the S operator containing all the information about the physical process by

$$S \equiv \lim_{\substack{t_1\to-\infty\\ t_2\to+\infty}} U(t_2,t_1) = T\mathrm{exp}\left\{-i\int d^4x\mathscr{H}_I(x)\right\} \tag{223}$$

with $\mathscr{H}_I(x)$ being the Hamiltonian density.

As already seen for the interaction Hamiltonian, when the S operator is expressed in the Interaction picture it is a function of free fields in the Heisenberg picture.

6.2 Transition Amplitude of the Process $|\alpha\rangle \to |\beta\rangle$

It can be shown that $\mathscr{L}_I = -\mathscr{H}_I$. Using this result, we can rewrite the S operator of Eq. (223) as

$$S = T\mathrm{exp}\left\{i\int d^4x\mathscr{L}_I(x)\right\} = \sum_{n=0}^{\infty}\frac{(+i)^n}{n!}\int d^4x_1\cdots\int d^4x_n\, T(\mathscr{L}_I(x_1),...,\mathscr{L}_I(x_n)). \tag{224}$$

The chronologically ordered product of fields appearing in the above equation can be more easily calculated by using Wick's theorem, which we consider in the next Section.

6.2.1 Wick's Theorem for Bosonic Fields

Wick's theorem, that will be given without proof, establishes of a relationship between the chronologically ordered product of n fields $(A_1(x_1), ..., A_n(x_n))$ and a sum of normal ordered products for these fields.

It states that

$$\begin{aligned} T(A_1(x_1), ..., A_n(x_n)) = \quad & : A_1(x_1) \cdots A_n(x_n) : + \sum_{k<l} : A_1(x_1) \cdots \hat{A}_k(x_k) \cdots \hat{A}_l(x_l) \cdots A_n(x_n) : \times \\ \overbrace{A_k(x_k) A_l(x_l)} + & \sum_{k_1<k_2<k_3<k_4} : A_1(x_1) \cdots \hat{A}_{k_1}(x_{k_1}) \cdots \hat{A}_{k_4}(x_{k_4}) \cdots A_n(x_n) : \sum_P \overbrace{A_{k_1}(x_{k_1}) A_{k_2}(x_{k_2})} \times \\ & \overbrace{A_{k_3}(x_{k_3}) A_{k_4}(x_{k_4})} + \cdots + \sum_{k_1<k_2<\cdots<k_{2p}} : A_1(x_1) \cdots \hat{A}_{k_1}(x_{k_1}) \cdots \hat{A}_{k_{2p}}(x_{k_{2p}}) \cdots A_n(x_n) : \times \\ & \sum_P \overbrace{A_{k_1}(x_{k_1}) A_{k_2}(x_{k_2})} \cdots \overbrace{A_{k_{2p-1}}(x_{k_{2p-1}}) A_{k_{2p}}(x_{k_{2p}})} + \ldots, \end{aligned} \tag{225}$$

where $\hat{A}$ indicates that A has been eliminated, and $\sum_P$ is the sum over all the possible permutations that give rise to different contractions $(\overbrace{A_k(x_k) A_l(x_l)})$. These contractions are c-number propagators defined as

$$\overbrace{A_k(x_k) A_l(x_l)} = \langle 0 | T(A_k(x_k), A_l(x_l)) | 0 \rangle \tag{226}$$

Note that for an odd number of fields, one has

$$\langle 0 | T(A_1(x_1), ..., A_{2n-1}(x_{2n-1})) | 0 \rangle = 0, \tag{227}$$

while for an even number of fields

$$\langle 0 | T(A_1(x_1), ..., A_{2n}(x_{2n})) | 0 \rangle = \sum_P \overbrace{A_1(x_1) A_2(x_2)} \cdots \overbrace{A_{2n-1}(x_{2n-1}) A_{2n}(x_{2n})}. \tag{228}$$

6.2.2 Wick's Theorem for Fermionic Fields

The above equations remain valid for fermionic fields with the following differences:

- A minus sign appears for every jump of fermionic fields that has to be done for performing a contraction, e.g., $T(\psi_1, \psi_2, \psi_3, \overline{\psi}_4) \sim : \psi_2 \psi_3 : (-1)^S \overbrace{\psi_1 \overline{\psi}_4} + \cdots$, where S is the number of jumps necessary to put ψ_1 next to $\overline{\psi}_4$ $(S = 2)$.

- The only contractions that are different from zero are those of the form $\overbrace{\psi_\alpha \overline{\psi}_\beta} = \langle 0 | T(\psi_\alpha, \overline{\psi}_\beta) | 0 \rangle \neq 0$ (the others vanish: $\overbrace{\psi_\alpha \psi_\beta} = \overbrace{\overline{\psi}_\alpha \overline{\psi}_\beta} = 0$).

In summary, Wick's theorem is applicable to both bosonic and fermionic fields, taking into account that different fermionic fields anticommute, different bosonic fields commute, and fermionic and bosonic fields commute with each other.

6.3 Scattering Cross Section

Suppose a particle ("1") that collides with another one (target at rest, "2"), giving rise to a final state in which we have n particles ("f_1, ..., f_n").

The incident flux $d\phi_i$, which is the number of particles "1" that perpendicularly cross the unit area in a given direction per unit time, is

$$d\phi_i = 2\,|\mathbf{p}_1|\,, \tag{229}$$

and the number of target particles per unit of volume is $(dn_2/dV) = 2m_2$.

The differential scattering cross section is defined as the transition probability from $|i_1, i_2\rangle$ to $|f_1, ..., f_n\rangle$ per unit time, per unit of target particles and per unit incident flux,

$$d\sigma(i \to f) = \frac{dW_{i\to f}}{dt\, dn_2\, d\phi_i} = \frac{1}{4m_2\,|\mathbf{p}_1|}(2\pi)^4\delta^{(4)}(P_f - P_i)|\langle f_1, ..., f_2|\mathscr{T}|i_1, i_2\rangle|^2 \prod_{k=1}^{n} \frac{d^3\mathbf{p}_{f_k}}{(2\pi)^3 2p^0_{f_k}}, \tag{230}$$

where $P_i = p_1 + p_2$, $P_f = \sum_k p_{f_k}$, and the factors $\frac{d^3\mathbf{p}_{f_k}}{(2\pi)^3 2p^0_{f_k}}$ (which appear due to the continuum distribution of linear momenta) correspond to the number of states in the volume element $d^3\mathbf{p}_{f_k}$ of momentum space. On the other hand, $\mathscr{T}$ is the reduced transition operator defined by

$$\langle f|T|i\rangle = (2\pi)^4\delta^{(4)}(P_f - P_i)\,\langle f|\mathscr{T}|i\rangle\,, \tag{231}$$

where T is the transition operator, which is related to the S operator of Eq. (224) by

$$S = I + iT, \tag{232}$$

with I being the identity operator.

6.4 Quantum Electrodynamics (QED)

Quantum electrodynamics (QED) is the relativistic quantum field theory of electrodynamics. In essence, it describes how light and matter interact and is the first theory where full agreement between quantum mechanics and special relativity was achieved.

The QED Lagrangian describing the interaction between the electromagnetic field and Dirac particles is

$$\mathscr{L}_{QED} = \mathscr{L}_{em} + \mathscr{L}_{Dirac} + \mathscr{L}_{int} = \; : -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(x)(i\partial\!\!\!/ - m)\psi(x) - j^\mu(x)A_\mu(x) : . \tag{233}$$

As already seen, in the Interaction picture $\mathscr{L}_{QED}$ is a function of free fields in the Heisenberg picture. In particular, the above Lagrangian is a function of the free Dirac field $\psi(x)$ and the electromagnetic field $A_\mu(x)$ studied in previous Sections, and $j^\mu(x)$ is the conserved current of the Dirac field, Eq. (157). Thus, the interaction term ($\mathscr{L}_{int} = \mathscr{L}_I^{int} \equiv \mathscr{L}_I$) can be written as

$$\mathscr{L}_I = \; : -j^\mu(x)A_\mu(x) : \; = \; : -e\overline{\psi}(x)\gamma^\mu\psi(x)A_\mu(x) : \; = \; : -e\overline{\psi}(x)A\!\!\!/_\mu(x)\psi(x) : . \tag{234}$$

Making a perturbative expansion of the S operator in terms of Eq. (224) up to second order, we have

$$S = I + i\int d^4x\,\mathscr{L}_I(x) + \frac{i^2}{2!}\int d^4x_1 \int d^4x_2\, T(\mathscr{L}_I(x_1), \mathscr{L}_I(x_2)) + ... \equiv I + S^{(1)} + S^{(2)} + ..., \tag{235}$$

and taking into account Eq. (232), we can identify

$$i\,\langle f|T|i\rangle = \langle f|S^{(1)}|i\rangle + \langle f|S^{(2)}|i\rangle + ... \tag{236}$$

It can be shown that $\langle f|S^{(1)}|i\rangle$ does not conserve the four-momentum and thus vanishes. On the other hand, $S^{(2)}$ has the expression

$$S^{(2)} = \frac{(-ie)^2}{2!}\int d^4x_1 \int d^4x_2 \;\; T\big(: \overline{\psi}(x_1)\gamma_\mu\psi(x_1)A^\mu(x_1) : , : \overline{\psi}(x_2)\gamma_\nu\psi(x_2)A^\nu(x_2) :\big). \tag{237}$$

Using Wick's theorem to calculate the chronologically ordered product appearing in $S^{(2)}$, we get

$$
\begin{aligned}
&T(:\overline{\psi}_a(x_1)\gamma_\mu^{ab}\psi_b(x_1)A^\mu(x_1):,:\overline{\psi}_c(x_2)\gamma_\nu^{cd}\psi_d(x_2)A^\nu(x_2):) = \\
&= \boxed{1}\ :\overline{\psi}_a(x_1)\gamma_\mu^{ab}\psi_b(x_1)A^\mu(x_1)\,\overline{\psi}_c(x_2)\gamma_\nu^{cd}\psi_d(x_2)A^\nu(x_2): + \\
&+\boxed{2}\ :\gamma_\mu^{ab}\psi_b(x_1)A^\mu(x_1)\,\overline{\psi}_c(x_2)\gamma_\nu^{cd}A^\nu(x_2):\ \overbrace{\overline{\psi}_a(x_1)\psi_d(x_2)} + \\
&+\boxed{3}\ :\overline{\psi}_a(x_1)\gamma_\mu^{ab}A^\mu(x_1)\,\gamma_\nu^{cd}\psi_d(x_2)A^\nu(x_2):\ \overbrace{\psi_b(x_1)\overline{\psi}_c(x_2)} + \\
&+\boxed{4}\ :\overline{\psi}_a(x_1)\gamma_\mu^{ab}\psi_b(x_1)\,\overline{\psi}_c(x_2)\gamma_\nu^{cd}\psi_d(x_2):\ \overbrace{A^\mu(x_1)A^\nu(x_2)} + \\
&+\boxed{5}\,\gamma_\mu^{ab}\gamma_\nu^{cd}\ :A^\mu(x_1)A^\nu(x_2):\ \overbrace{\overline{\psi}_a(x_1)\psi_d(x_2)}\ \overbrace{\psi_b(x_1)\overline{\psi}_c(x_2)} + \\
&+\boxed{6}\,\gamma_\mu^{ab}\gamma_\nu^{cd}\ :\psi_b(x_1)\overline{\psi}_c(x_2):\ \overbrace{\overline{\psi}_a(x_1)\psi_d(x_2)}\ \overbrace{A^\mu(x_1)A^\nu(x_2)} + \\
&+\boxed{7}\,\gamma_\mu^{ab}\gamma_\nu^{cd}\ :\overline{\psi}_a(x_1)\psi_d(x_2):\ \overbrace{\psi_b(x_1)\overline{\psi}_c(x_2)}\ \overbrace{A^\mu(x_1)A^\nu(x_2)} + \\
&\qquad+\boxed{8}\,\gamma_\mu^{ab}\gamma_\nu^{cd}\ \overbrace{\overline{\psi}_a(x_1)\psi_d(x_2)}\ \overbrace{\psi_b(x_1)\overline{\psi}_c(x_2)}\ \overbrace{A^\mu(x_1)A^\nu(x_2)},
\end{aligned}
\tag{238}
$$

where summation over repeated indexes a, b, c, d, μ, ν is assumed and we have used the label $\boxed{\textit{number}}$ to mark every term in the sum.

6.4.1 Feynmam Diagrams in Configuration Space

Now we are going to identify each term in the sum (238) with a Feynmam diagram in configuration space. To this end we will apply the following graphical rules:

- A photon line $A^\mu(x)$ is represented by $\sim\!\sim\!\sim\!\sim\bullet\ x$
- A fermion line $\psi(x)$ is represented by $\longrightarrow\!\bullet\ x$
- An antifermion line $\overline{\psi}(x)$ is represented by $x\ \bullet\!\longrightarrow$
- A photon propagator $\overbrace{A^\mu(x_1)A^\nu(x_2)}$ is represented by $x_1\ \bullet\!\sim\!\sim\!\sim\!\sim\!\bullet\ x_2$
- A fermion propagator $\overbrace{\overline{\psi}_c(x_1)\psi_b(x_2)}$ is represented by $x_1\ \bullet\!\longrightarrow\!\bullet\ x_2$

The arrows are assumed to represent the direction of the (electron) charge propagation. With the above rules one can associate the different terms in Eq. (238) with the Feynman diagrams shown in Fig. 1. In fact, it can be easily verified that following the fermion lines in a given diagram in opposite direction to that of charge propagation one recovers the corresponding term in Eq. (238). On the other hand, it can be shown that a given term in Eq. (238) can only connect states $|i\rangle$ and $|f\rangle$ containing as many fermions/photons as external fermion/photon lines appear in its corresponding Feynman diagram.

Therefore, the first two diagrams (corresponding to $\boxed{1}$) may only contribute to processes involving one photon and two fermions. Since such processes do not conserve the four-momentum, the corresponding transition amplitude vanishes for any initial and final states. The diagrams $\boxed{2}$ and $\boxed{3}$ describe processes like: $e^\pm+\gamma \to e^\pm+\gamma$, and $e^-+e^+ \rightleftarrows \gamma+\gamma$. The diagram $\boxed{4}$ contributes to the process $e^-e^+ \to e^-e^+$. The one-loop diagram $\boxed{5}$ corresponds to vacuum polarization, while $\boxed{6}$ and $\boxed{7}$ represent the electron and positron self-energies. The two-loop diagram $\boxed{8}$ is a vacuum diagram.

6.5 Compton Scattering

Compton scattering, discovered by Arthur Holly Compton, is the scattering of a photon by a charged particle, usually an electron. It results in a decrease in energy (increase in wavelength) of the photon

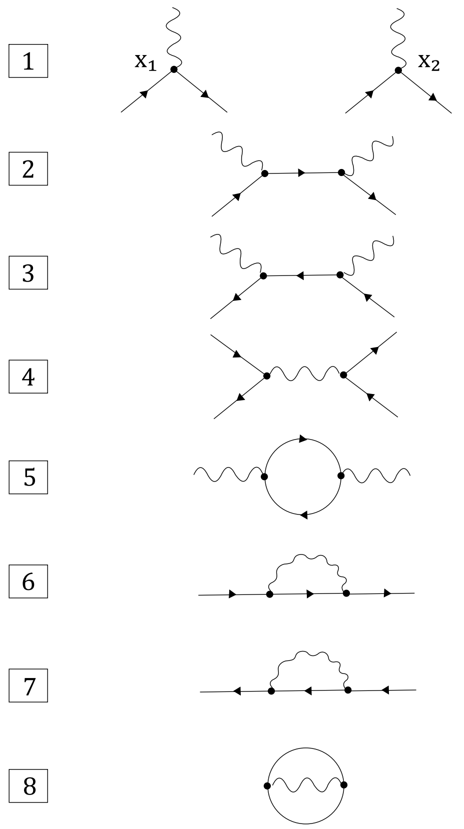

Figure 1: Feynman diagrams in configuration space corresponding to the different terms of Eq. (238)

(which may be an X-ray or gamma ray photon), called the Compton effect, where part of the energy of the photon is transferred to the recoiling electron.

Let's calculate the transition amplitude (S_{fi}) of the elastic Compton process

$$\gamma(k_1, \sigma_1) + e^-(p_1, \lambda_1) \to \gamma(k_2, \sigma_2) + e^-(p_2, \lambda_2), \tag{239}$$

where a photon with four-momentum k_1 and polarization σ_1 collides with an electron with four-momentum p_1 and helicity λ_1 giving rise to a photon with four-momentum k_2 and polarization σ_2 plus an electron with four-momentum p_2 and helicity λ_2. Thus,

$$|i\rangle = |p_1, \lambda_1, k_1, \sigma_1\rangle = b^{\dagger(\lambda_1)}(p_1) a^{\dagger(\sigma_1)}(k_1) \left|0\right\rangle, \tag{240}$$

$$|f\rangle = |p_2, \lambda_2, k_2, \sigma_2\rangle = b^{\dagger(\lambda_2)}(p_2) a^{\dagger(\sigma_2)}(k_2) \left|0\right\rangle. \tag{241}$$

Knowing that to second order we have

$$S_{fi} \cong \delta_{fi} + S^{(2)}_{fi} = \delta_{fi} + \langle f|S^{(2)}|i\rangle, \tag{242}$$

in what follows we are going to obtain $S^{(2)}_{fi}$ analytically and then, based on this result, we will derive the corresponding Feynman rules, which, as will be seen in the next Sections, will enable us to predict the transition amplitudes of other physical processes relevant to QED.

Recalling Eq. (237) and introducing the notation $a(i) \equiv a^{(\sigma_i)}(k_i)$ and $b^\dagger(j) \equiv b^{\dagger(\lambda_j)}(p_j)$, we have

$$S^{(2)}_{fi} = \frac{(-ie)^2}{2!} \int d^4x_1 \int d^4x_2 \; \langle 0|\; a(2) b(2)\; T\left\{\mathscr{L}_I(x_1), \mathscr{L}_I(x_2)\right\}\; b^\dagger(1) a^\dagger(1)\; |0\rangle, \tag{243}$$

and making use of Wick's theorem (238) to develop $T\left\{\mathscr{L}_I(x_1), \mathscr{L}_I(x_2)\right\}$ we see that the only terms having a nonzero contribution to $S^{(2)}_{fi}$ are those corresponding to the 2 and 3 diagrams of Fig. 1, which are the only ones that have two external fermionic fields and two external bosonic fields, coinciding with de sum of creation and annihilation operators in the initial and final states. Then

$$\begin{aligned} T\left\{L_I(x_1), L_I(x_2)\right\} \sim \boxed{2} \; : \gamma^{ab}_{\mu} \psi_b(x_1) A^{\mu}(x_1)\, \overline{\psi}_c(x_2) \gamma^{cd}_{\nu} A^{\nu}(x_2) : \; \overbrace{\overline{\psi}_a(x_1) \psi_d(x_2)} + \\ + \boxed{3} \; : \overline{\psi}_a(x_1) \gamma^{ab}_{\mu} A^{\mu}(x_1)\, \gamma^{cd}_{\nu} \psi_d(x_2) A^{\nu}(x_2) : \; \overbrace{\psi_b(x_1) \overline{\psi}_c(x_2)} . \end{aligned} \tag{244}$$

Let's firstly consider the contribution of the $\boxed{2}$ term to Eq. (243). As A^ν and ψ_b commute and taking into account that $|p_i, \lambda_i, k_i, \sigma_i\rangle = |p_i, \lambda_i\rangle \otimes |k_i, \sigma_i\rangle$ the problem reduces to compute

$$\langle k_2, \sigma_2| : A^{\mu}(x_1) A^{\nu}(x_2) : |k_1, \sigma_1\rangle \otimes \langle p_2, \lambda_2| : \psi_b(x_1) \overline{\psi}_c(x_2) : |p_1, \lambda_1\rangle \,. \tag{245}$$

We focus now on $\langle k_2, \sigma_2| : A^{\mu}(x_1) A^{\nu}(x_2) : |k_1, \sigma_1\rangle$. Using Eq. (198) with $\lambda = 1, 2$, we find

$$\begin{aligned} \langle k_2, \sigma_2| : A^{\mu}(x_1) A^{\nu}(x_2) : |k_1, \sigma_1\rangle = \langle 0|\, a(2) \int \frac{d^3\mathbf{k}_3}{(2\pi)^3 2\omega_{\mathbf{k}_3}} \int \frac{d^3\mathbf{k}_4}{(2\pi)^3 2\omega_{\mathbf{k}_4}} \cdot \\ \cdot \sum_{\sigma_3, \sigma_4 = 1}^{2} : \left[a(3) \varepsilon^{\mu}(3) e^{-ik_3 x_1} + a^{\dagger}(3) \varepsilon^{\mu}(3) e^{ik_3 x_1}\right] \cdot \left[a(4) \varepsilon^{\nu}(4) e^{-ik_4 x_2} + a^{\dagger}(4) \varepsilon^{\nu}(4) e^{ik_4 x_2}\right] : a^{\dagger}(1)\, |0\rangle \,. \end{aligned} \tag{246}$$

Using the commutation relations (200) and integrating we have

$$\langle k_2, \sigma_2| : A^{\mu}(x_1) A^{\nu}(x_2) : |k_1, \sigma_1\rangle = \varepsilon^{\mu}(1) \varepsilon^{\nu}(2) e^{-ik_1 x_1 + ik_2 x_2} + \varepsilon^{\mu}(2) \varepsilon^{\nu}(1) e^{ik_2 x_1 - ik_1 x_2}. \tag{247}$$

Let's now evaluate $\langle p_2, \lambda_2| : \psi_b(x_1) \overline{\psi}_c(x_2) : |p_1, \lambda_1\rangle$. From Eqs. (146)–(147) we get

$$\begin{aligned} \langle p_2, \lambda_2| : \psi_b(x_1) \overline{\psi}_c(x_2) : |p_1, \lambda_1\rangle = \langle 0|\, b(2) \int \frac{d^3\mathbf{p}_3}{(2\pi)^3 2\omega_{\mathbf{p}_3}} \int \frac{d^3\mathbf{p}_4}{(2\pi)^3 2\omega_{\mathbf{p}_4}} \cdot \\ \cdot \sum_{\lambda_3, \lambda_4 = 1}^{2} : \left[u_b(3) b(3) e^{-ip_3 x_1} + v_b(3) d^{\dagger}(3) e^{ip_3 x_1}\right] \cdot \left[\overline{v}_c(4) d(4) e^{-ip_4 x_2} + \overline{u}_c(4) b^{\dagger}(4) e^{ip_4 x_2}\right] : b^{\dagger}(1)\, |0\rangle \,, \end{aligned} \tag{248}$$

and using the anticommutation relations of Eqs. (148)–(149), and integrating we have

$$\langle p_2, \lambda_2| : \psi_b(x_1) \overline{\psi}_c(x_2) : |p_1, \lambda_1\rangle = -u_b(1) \overline{u}_c(2) e^{-ip_1 x_1 + ip_2 x_2}. \tag{249}$$

It is not difficult to see that the second term $\boxed{3}$ of Eq. (244) can be obtained from the first one $\boxed{2}$ by interchanging x_1 and x_2. Therefore, after substituting in Eq. (243), since x_1 and x_2 are dummy integration variables, it results that this second term gives the same contribution as the first one, and the net effect is that the factor $1/2!$ disappears from the equation.

Then, taking into account that the fermion propagator $\overbrace{\overline{\psi}_a(x_1) \psi_d(x_2)} = -\overbrace{\psi_d(x_2) \overline{\psi}_a(x_1)}$ of Eq. (244) is defined by Eqs. (226) and (164), and substituting Eqs. (247) and (249) in Eq. (243), we obtain

$$\begin{aligned} S^{(2)}_{fi} = (-ie)^2 \int d^4x_1 \int d^4x_2 \quad \left\{+u_b(1) \overline{u}_c(2) e^{-ip_1 x_1 + ip_2 x_2}\right\} \gamma^{ab}_{\mu} \gamma^{cd}_{\nu} \cdot \\ \cdot \left\{\varepsilon^{\mu}(1) \varepsilon^{\nu}(2) e^{-ik_1 x_1 + ik_2 x_2} + \varepsilon^{\mu}(2) \varepsilon^{\nu}(1) e^{ik_2 x_1 - ik_1 x_2}\right\} \int \frac{d^4k}{(2\pi)^4} e^{-ik(x_2 - x_1)} \left(\frac{i}{\not{k} - m}\right)_{da} , \end{aligned} \tag{250}$$

and integrating over x_1 and x_2 we have

$$\begin{aligned} S^{(2)}_{fi} = (-ie)^2 \int \frac{d^4k}{(2\pi)^4} u_b(1) \overline{u}_c(2) \left\{\varepsilon^{\mu}(1) \varepsilon^{\nu}(2) (2\pi)^8 \delta^{(4)}(p_1 + k_1 - k) \delta^{(4)}(p_2 + k_2 - k) + \right. \\ \left. \varepsilon^{\mu}(2) \varepsilon^{\nu}(1) (2\pi)^8 \delta^{(4)}(p_1 - k_2 - k) \delta^{(4)}(p_2 - k_1 - k)\right\} \gamma^{ab}_{\mu} \gamma^{cd}_{\nu} \left(\frac{i}{\not{k} - m}\right)_{da} . \end{aligned} \tag{251}$$

Moreover, taking into account that

$$\overline{u}_c(2) \gamma^{cd}_{\nu} \left(\frac{i}{\not{k} - m}\right)_{da} \gamma^{ab}_{\mu} u_b(1) = \overline{u}(2) \gamma_{\nu} \left(\frac{i}{\not{k} - m}\right) \gamma_{\mu} u(1), \tag{252}$$

and using

$$\int \frac{d^4k}{(2\pi)^4}\left(\frac{i}{\not{k}-m}\right)\delta^{(4)}(p_1+k_1-k)\delta^{(4)}(p_2+k_2-k) = \frac{1}{(2\pi)^4}\left(\frac{i}{\not{p}_1+\not{k}_1-m}\right)\delta^{(4)}(p_1+k_1-p_2-k_2), \tag{253}$$

$$\int \frac{d^4k}{(2\pi)^4}\left(\frac{i}{\not{k}-m}\right)\delta^{(4)}(p_1-k_2-k)\delta^{(4)}(p_2-k_1-k) = \frac{1}{(2\pi)^4}\left(\frac{i}{\not{p}_1-\not{k}_2-m}\right)\delta^{(4)}(p_1+k_1-p_2-k_2), \tag{254}$$

Eq. (251) finally reads

$$S_{fi}^{(2)} = (-ie)^2(2\pi)^4\delta^{(4)}(p_1+k_1-p_2-k_2)\overline{u}(2)\gamma_\nu\left\{\frac{i}{\not{p}_1+\not{k}_1-m}\varepsilon^\mu(1)\varepsilon^\nu(2)+\frac{i}{\not{p}_1-\not{k}_2-m}\varepsilon^\mu(2)\varepsilon^\nu(1)\right\}\gamma_\mu u(1). \tag{255}$$

Therefore, taking into account Eqs. (231) and (236), the transition amplitude is given by

$$i\mathscr{T}_{fi}^{(2)} = (-ie)^2\overline{u}(2)\gamma_\nu\left\{\frac{i}{\not{p}_1+\not{k}_1-m}\varepsilon^\mu(1)\varepsilon^\nu(2)+\frac{i}{\not{p}_1-\not{k}_2-m}\varepsilon^\mu(2)\varepsilon^\nu(1)\right\}\gamma_\mu u(1). \tag{256}$$

The two terms in the above sum can be associated to respective Feynman diagrams in momentum space (the only ones that contribute to the Compton Scattering to second order). Now we are going to consider these diagrams.

A- **First term of Eq. (256)**:

We can interpret this term as representing an incident electron with p_1 and λ_1 that interacts (at a coupling vertex $-ie\gamma_\mu$) with an incoming photon with k_1 and σ_1 and space-time index μ. Then, there is a propagation of a virtual fermion with momentum $p_1 + k_1$ towards the coupling vertex $-ie\gamma_\nu$, from where an outgoing electron with p_2 and λ_2 and an outgoing photon with k_2 and σ_2 and space-time index ν emerge (see Fig. 2).

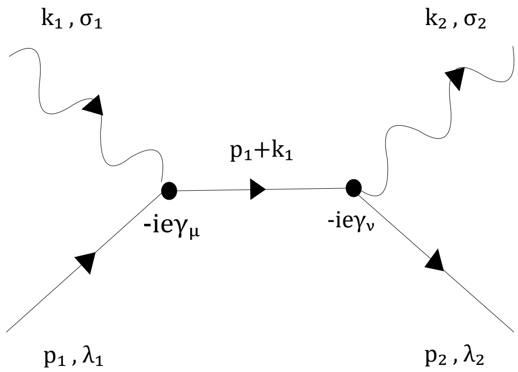

Figure 2: Feynman diagram corresponding to the first term of Eq. (256).

B- **Second term of Eq. (256)**:

We can interpret this term as representing an incident electron with p_1 and λ_1 that interacts (at the coupling vertex $-ie\gamma_\mu$) with an outgoing photon with k_2 and σ_2 and space-time index μ. Then, there is a propagation of a virtual fermion with momentum $p_1 - k_2$ towards the coupling vertex $-ie\gamma_\nu$, where it interacts with an incoming photon with k_1 and σ_1 and space-time index ν

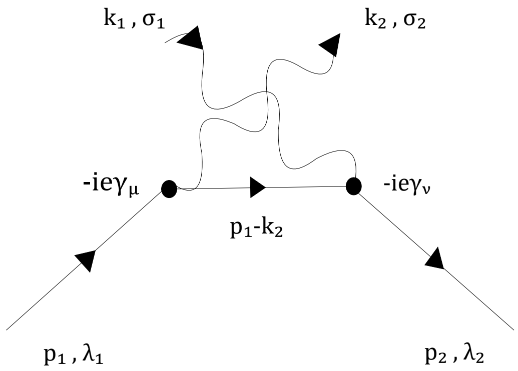

Figure 3: Feynman diagram corresponding to the second term of Eq. (256).

giving rise to an outgoing electron with p_2 and λ_2 (see Fig. 3).

Note that the order of the expansion of $S_{fi}^{(n)}$ gives the number of vertices (n) of the corresponding Feynman diagrams and four-momentum conservation must be fulfilled at each vertex. Moreover, the number of fields that appear in the interaction Lagrangian is equal to the number of lines that converge at each vertex. Since the interaction Lagrangian is proportional to $\overline{\psi}\gamma_\mu\psi A^\mu$, the interaction vertices must involve a photon line, a fermion line and an antifermion line.

6.5.1 Feynman Rules Obtained from the Compton Scattering (Valid for Tree Diagrams)

From the Compton scattering analytic solution it can be derived practical rules that enable one to obtain the Feynmam diagrams from the knowledge of $\mathscr{T}_{fi}$ and vice versa. Such a graphical method is equivalent to the analytic one and equally generalizable. The complete set of rules will be given in the next Sections. The Feynman rules considered here are only valid for tree diagrams (diagrams with no loops):

1. Each vertex is associated with a factor $(-ie\gamma_\mu)$.
2. Every incoming photon with (k,σ) ∼∼∼• μ is associated with $\varepsilon_\mu^{(\sigma)}(k)$.
3. Every outgoing photon with (k,σ) ν •∼∼∼ is associated with $\varepsilon_\nu^{(\sigma)*}(k)$ (as we have chosen a real basis $\varepsilon_\nu^{(\sigma)*}(k) \equiv \varepsilon_\nu^{(\sigma)}(k)$).
4. Every incoming electron with (p,λ) $\longrightarrow\bullet$ is associated with $u^{(\lambda)}(p)$.
5. Every outgoing electron with (p,λ) $\bullet\!\longrightarrow$ is associated with $\overline{u}^{(\lambda)}(p)$.
6. Every fermion propagator with four-momentum p $\bullet\!\longrightarrow\!\bullet$ is associated with $\frac{i}{\not{p}-m+i\epsilon}$.

From the study of the transition amplitude of the Bhabha scattering (see next Section), one can obtain the following additional rules:

7. Every incoming positron with (p,λ) $\longleftarrow\bullet$ (the arrows indicate the direction of charge propagation) is associated with $\overline{v}^{(\lambda)}(p)$.
8. Every outgoing positron with (p,λ) $\bullet\!\longleftarrow$ is associated with $v^{(\lambda)}(p)$.
9. Every photon propagator with four-momentum q •∼∼∼• is associated (in the Feynman gauge) with $\frac{-i\eta^{\mu\nu}}{q^2}$.

As an example, in the next Section we will compute the transition amplitude for the Bhabha scattering by applying the above rules to the topologically distinct diagrams contributing to the process.

6.6 Bhabha Scatering

Bhabha scattering is the electron-positron scattering process

$$e^+ + e^- \to e^+ + e^- . \tag{257}$$

To second order of perturbation, there are two topologically distinct Feynman diagrams which contribute to this process: an annihilation and a scattering diagrams.

Now, let's calculate the transition amplitude from the knowledge of these diagrams. To do that, we follow the fermion lines in the opposite direction of the arrows.

- Transition amplitude associated with the annihilation process (see Fig. 4).

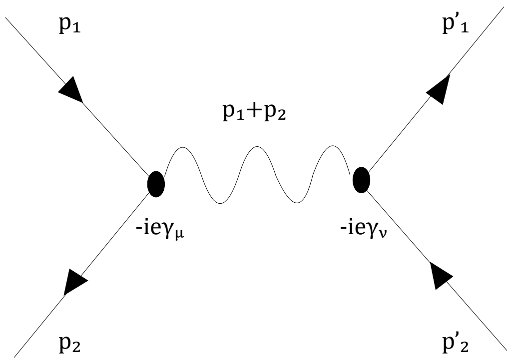

Figure 4: Annihilation Process

The transition amplitude is

$$i\mathscr{T}_a = \{\overline{u}(p'_1)(-ie\gamma_\nu)v(p'_2)\} \cdot \left\{ \frac{-i\eta^{\mu\nu}}{(p_1+p_2)^2} \right\} \cdot \{\overline{v}(p_2)(-ie\gamma_\mu)u(p_1)\} . \tag{258}$$

where the first product in curly brackets corresponds to the outgoing fermion lines. The second curly bracket corresponds to the photon propagator and the last one to the incoming fermion lines.

- Amplitude associated with the scattering process (see Fig. 5).

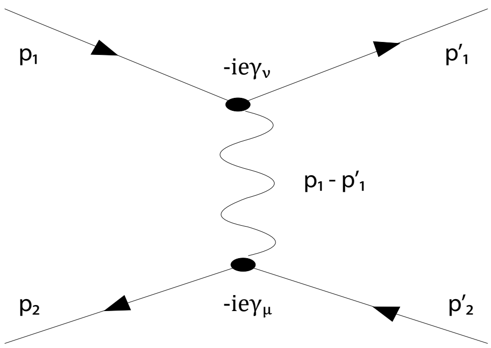

Figure 5: Scattering Process

The transition amplitude is

$$i\mathscr{T}_s = \{\overline{u}(p'_1)(-ie\gamma_\nu)u(p_1)\} \cdot \left\{ \frac{-i\eta^{\mu\nu}}{(p_1-p'_1)^2} \right\} \cdot \{\overline{v}(p_2)(-ie\gamma_\mu)v(p'_2)\} \tag{259}$$

Thus, the total transition amplitude $i\mathscr{T} = i\mathscr{T}_a + i\mathscr{T}_s$ for the Bhabha scattering is

$$i\mathscr{T} = (-ie)^2 \left\{ \overline{v}(p_2)\gamma_\mu u(p_1)\overline{u}(p'_1)\gamma_\nu v(p'_2)\frac{-ig^{\mu\nu}}{(p_1+p_2)^2} + \overline{u}(p'_1)\gamma_\nu u(p_1)\overline{v}(p_2)\gamma_\mu v(p'_2)\frac{-ig^{\mu\nu}}{(p_1-p'_1)^2} \right\}. \quad (260)$$

6.7 Elastic Scattering of Two Electrons

This process is of the form

$$e^- + e^- \rightarrow e^- + e^-. \quad (261)$$

We draw the topologically distinct diagrams contributing to the process and apply the Feynman rules to compute the corresponding probability amplitude. In this case, there are two possible Feynman diagrams.

The first one is the process which corresponds to the diagram of Fig. 6

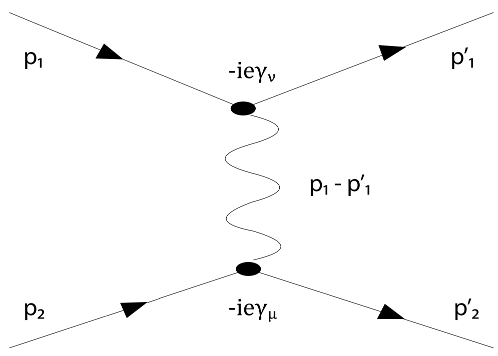

Figure 6: First diagram of the scattering of two electrons

The corresponding transition amplitude ($i\mathscr{T}_1$) in the Feynman gauge is

$$i\mathscr{T}_1 = \overline{u}(p'_2)(-ie\gamma_\mu)u(p_2)\overline{u}(p'_1)(-ie\gamma_\nu)u(p_1)\frac{-i\eta^{\mu\nu}}{(p_1-p'_1)^2}. \quad (262)$$

The other possible diagram is that shown in Fig. 7.

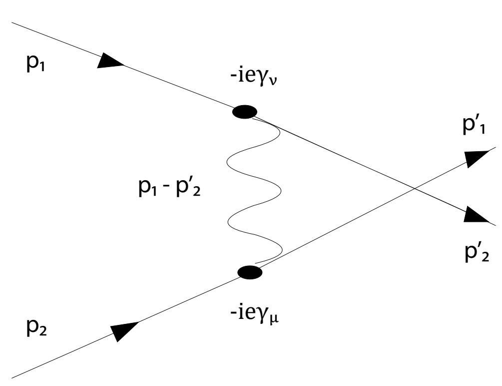

Figure 7: Second diagram of the scattering of two electrons

The corresponding transition amplitude ($i\mathscr{T}_2$) in the Feynman gauge is

$$i\mathscr{T}_2 = \overline{u}(p'_1)(-ie\gamma_\mu)u(p_2)\overline{u}(p'_2)(-ie\gamma_\nu)u(p_1)\frac{-i\eta^{\mu\nu}}{(p_1 - p'_2)^2}. \tag{263}$$

In principle, the total probability amplitude would be the sum of the amplitudes corresponding to each diagram. However, from a detailed analytic calculation we can see that there is a relative minus sign, so that $i\mathscr{T} \equiv i\mathscr{T}_1 - i\mathscr{T}_2$. This is so because the two diagrams interchange two identical fermionic particles in the final state. Thus, we must complement the Feynman rules obtained previously with the following rule:

Additional Feynman Rule: When two diagrams differ from each other by exchanging two identical fermionic particles in the final state, their relative amplitudes must have a relative minus sign.

Thus, the total transition amplitude of the process is finally given by

$$\begin{aligned} i\mathscr{T}_{fi} = {} & \overline{u}(p'_2)(-ie\gamma_\mu)u(p_2)\overline{u}(p'_1)(-ie\gamma_\nu)u(p_1)\frac{-ig^{\mu\nu}}{(p_1 - p'_1)^2} - \\ & -\overline{u}(p'_1)(-ie\gamma_\mu)u(p_2)\overline{u}(p'_2)(-ie\gamma_\nu)u(p_1)\frac{-ig^{\mu\nu}}{(p_1 - p'_2)^2}. \end{aligned} \tag{264}$$

6.8 Transition Amplitude for Diagrams with Loops

Diagrams $\boxed{5}$ to $\boxed{8}$ in Fig. 1 are Feynman diagrams (in configuration space) with loops. They contribute to processes that can be represented by Feynman diagrams in momentum space having essentially the same look. These type of diagrams have significant differences with respect to the tree diagrams previously seen, which force us to add new Feynman rules.

To see these differences, we are going to analyze the $\boxed{5}$ term of the sum of Eq. (238), which is the Vacuum Polarization (photon self-energy) where a virtual fermion-antifermion pair is created. If we calculate analytically the transition amplitude associated with this process, we obtain

$$i\mathscr{T}_{fi} = -\varepsilon_\nu^{(\sigma)}(p)\int\frac{d^4k}{(2\pi)^4}\,\mathrm{tr}\left\{(-ie\gamma^\nu)\frac{i}{\not{k} + \not{p} - m + i\varepsilon}(-ie\gamma^\mu)\frac{i}{\not{k} - m + i\varepsilon}\right\}\varepsilon_\mu^{(\sigma)}(p), \tag{265}$$

where there is a free four-momentum k conserving the four-momentum in the process. Equation (265) describes an incoming photon with momentum p that gives rise to a virtual electron-positron pair with momenta k and $k + p$ which finally annihilates into an outgoing photon of momentum p.

From Eq. (265), three differences with respect to the previous results for tree diagrams are evident: the total minus sign, the integral over k, and the trace over the matrices (Dirac indexes). Thus, for treating diagrams with loops we have to consider these differences and add the following Feynman rules:

I. For every loop appearing in the Feynman diagram we have to carry out an integration $\int \frac{d^4k}{(2\pi)^4}$ over all free internal four-momenta k of the loop (k is free when it can take any value and four-momentum conservation is still true).

II. If there are *closed* fermion loops, we have to put a minus sign for each loop in the corresponding contribution to the transition amplitude $i\mathscr{T}_{fi}$. If the loop is not closed (i.e., it is a mixture of bosons and fermions), there is no change in $i\mathscr{T}_{fi}$.

III. For every *closed* fermion loop we have to take the trace over all Dirac matrices (fermion propagators and γ^ν matrices). Because of the cyclic property of the trace, one can follow the loop starting from any vertex.

As an example, we are going to apply these rules to the process corresponding to the $\boxed{6}$ term of Eq. (238), which represents the electron self-energy and involves two contracted bosonic fields, two contracted fermionic fields and the normal product of two external fermionic fields. The corresponding transition amplitude is

$$i\mathscr{T}_{fi} = \overline{u}^{(\lambda)}(p)(-ie\gamma^{\mu}) \int \frac{d^4k}{(2\pi)^4} \frac{i}{\not{k} + \not{p} - m + i\varepsilon}(-ie\gamma^{\nu})u^{(\lambda)}(p)\frac{-i\eta_{\mu\nu}}{k^2}. \tag{266}$$

6.9 Feynman Rules for Spinor QED

Consider a physical process that starts from an initial state $|i\rangle$ and gives rise to a final state $|f\rangle$. We postulate the following Feynman rules which identify every part of the corresponding analytic transition amplitude with a certain part in the Feynman diagram associated with the process.

1. Every interaction vertex is determined by $\mathscr{L}_I(x)$, so that every vertex, which is associated with the factor $(-ie\gamma_{\mu})$, couples a fermion field, an antifermion field and a photon field [see Fig. (8)]

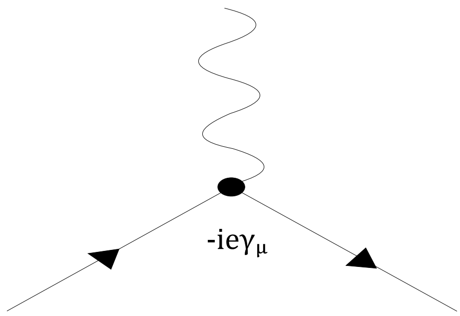

Figure 8: Vertex diagram for spinor QED.

2. Charge and four-momentum are conserved at every vertex. The four-momentum flows in the direction in which particles move and not in that of the fermion arrows (that represent charge flow). In closed fermion loops the four-momentum is assumed to flow in the direction of arrows.
3. An incoming fermion with (p, λ) is represented in the diagram by $\xrightarrow{p,\lambda}\bullet$ and it is associated with the term $u^{(\lambda)}(p)$ in the analytic transition amplitude.
4. An outgoing fermion with (p, λ) is represented by $\bullet\xrightarrow{p,\lambda}$ and it is associated with $\overline{u}^{(\lambda)}(p)$.
5. An incoming antifermion with (p, λ) is represented by $\xleftarrow{p,\lambda}\bullet$ and is associated with $\overline{v}^{(\lambda)}(p)$.
6. An outgoing antifermion with (p, λ) is represented by $\bullet\xleftarrow{p,\lambda}$ and is associated with $v^{(\lambda)}(p)$.
7. An incoming photon with (k, σ) is represented by $\overset{k,\sigma}{\sim\!\sim\!\sim\!\sim}\bullet\ \mu$ and is associated with $\varepsilon_{\mu}^{(\sigma)}(k)$.
8. An outgoing photon with (k, σ) is represented by $\nu\ \bullet\overset{k,\sigma}{\sim\!\sim\!\sim\!\sim}$ and is associated with $\varepsilon_{\nu}^{(\sigma)*}(k)$ (as we have chosen a real basis $\varepsilon_{\nu}^{(\sigma)*}(k) \equiv \varepsilon_{\nu}^{(\sigma)}(k)$).
9. A fermion propagator with four-momentum p is represented by $\bullet\xrightarrow{p}\bullet$ and is associated with $\frac{i}{\not{p}-m+i\epsilon}$, where p is assumed to evolve in the arrow direction.
10. A photon propagator with four-momentum k is represented by $\mu\ \bullet\overset{k}{\sim\!\sim\!\sim\!\sim}\bullet\ \nu$ and is associated (in the Feynman gauge) with $\frac{-i\eta^{\mu\nu}}{k^2+i\epsilon}$.

11. When two Feynman diagrams differ from each other by exchanging two identical fermionic particles in the final state, their relative amplitudes must have a relative minus sign.

12. For every loop, we have to carry out an integration $\int \frac{d^4k}{(2\pi)^4}$ over all free internal four-momenta k.

13. For every closed fermion loop, the corresponding analytic term has to be multiplied by (-1).

14. For every closed fermion loop, a trace over all Dirac matrices has to be taken.

To obtain the analytic terms from the Feynman diagrams one has to draw all the topologically distinct Feynman diagrams compatible with the initial and final states considered and having a number of vertices determined by the order of perturbation theory. Then, using the Feynman rules above, we can obtain the transition amplitude of the physical process as the sum of the amplitudes corresponding to each contributing diagram.

7 Conclusions

- We have presented a detailed introduction to Quantum Field Theory.
- By applying the Canonical Quantization method, we have developed the theory of the free Quantum Fields in the Heisenberg picture from the knowledge of the corresponding Poincaré-covariant free classical fields. In particular, we have formulated the quantum theory of scalar, electromagnetic and Dirac fields.
- We have shown that within the formalism of Quantum Field Theory particles arise from the quantization of the fields: They are quanta of the corresponding fields just as photons are quanta of the electromagnetic field.
- We have seen that when expressed in the Interaction picture, interacting Quantum Fields are formally identical to free Quantum Fields in the Heisenberg picture and have used this fact to develop a perturbative quantum theory of interacting fields.
- We have particularized the previous results to the case of spinor Quantum Electrodynamics (QED) and have applied Wick's theorem to obtain the corresponding Feynman diagrams in configuration space.
- We have calculated analytically the transition amplitude for the Compton Scattering, which is a paradigmatic example of a physical process contributing to QED, and have used this result to illustrate how to derive the Feynman diagrams and rules of QED in momentum space. These rules enable us to associate a pictorial representation with every analytic contribution and vice versa.
- Finally, we have also illustrated how to obtain the transition amplitudes of the other physical processes relevant to spinor QED starting from the topologically distinct Feynman diagrams that contribute.

A natural continuation of the present work would be the study of topics like Renormalization, Quantum Cromodynamics (QCD), Gauge Theories and The Standard Model.

References

[1] Helge Kragh (1990). "Dirac: A Scientific Biography". Cambridge University Press.

[2] P. A. M. Dirac, (1927). "The Quantum Theory of the Emission and Absorption of Radiation". Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences **114**, 243; doi:10.1098/rspa.1927.0039.

[3] P. W. Atkins (1974). "Quanta: A handbook of concepts". Oxford University Press.

[4] S. Weinberg (1995). "The Quantum Theory of Fields", Volume 1. Cambridge University Press.

[5] W. Hollik (2010). "Quantum Field Theory and The Standard Model", arXiv:1012.3883 [hep-ph].

[6] A. R. Liddle and D. H. Lyth (2000). "Cosmological Inflation and Large-Scale Structure". Cambridge University Press.

[7] V. Mukhanov and S. Winitzki (2007). "Introduction to Quantum Effects in Gravity". Cambridge University Press.

[8] V. Delgado (2010). Lecture Notes on Quantum Field Theory. Universidad de La Laguna.

[9] C. Itzykson and J.-B. Zuber (1985). "Quantum Field Theory". McGraw-Hill.

[10] M. Maggiore (2005). "Modern Introduction to Quantum Field Theory", Oxford University Press.

[11] D. Tong (2007). Lecture Notes on Quantum Field Theory. University of Cambridge Part III Mathematical Tripos.

[12] E. Noether (1918). "Invariante Variationsprobleme". Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse **1918**, 235; arXiv:physics/0503066 [physics.hist-ph] (M. A. Tavel's English translation).

[13] V. Fock (1928). "Bemerkung zur Quantelung des harmonischen Oszillators im Magnetfeld". Zeitschrift für Physik **47**, 446.

[14] Maurice A. de Gosson (2013). "Born-Jordan Quantization and the Uncertainty Principle". arXiv:1303.2590.

[15] Y. Aharonov and D. Bohm (1959). "Significance of electromagnetic potentials in quantum theory". Phys. Rev. **115**, 485.