Decoherence induced by gravity

An educational review of "A classical channel model for gravitational decoherence [1]"

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Abstract

This is an review, with educational purpose, of the article "A classical channel model for gravitational decoherence [1]", where in order to understand the original article, it is necessary to introduce some advanced level material. Based in an equivalent model first proposed by Diosi, the original article proposes a gravitational decoherence model of two mechanical resonators linked under gravitational interaction, which is treated as a classical measurement channel. Using a previous result obtained by Kafri and Taylor [2], it implies that gravitational interactions between two resonators cannot create entanglement in a classical measurement channel. Following experimental tests implies the the gravitational rate is of the order of the normal mode splitting of the resonators induced by gravity. Finally, the original article sets a new research route, reconsidering the measurement channel as purely classical.

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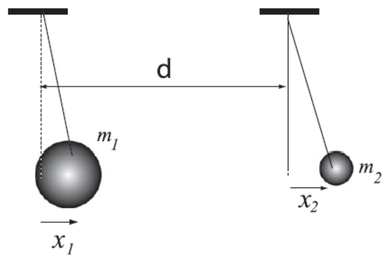

Figure 1: A gravitationally coupled system of two harmonic oscillators comprising two suspended masses m_1, m_2 [1].

1 Introduction

No one can shield against gravity. In the current state of quantum physics, we have a relative deep understanding of the others three fundamental interactions in nature. Nowadays with that knowledge, for example, it is possible to engineer and compute quantum ion traps which, with current technology limitations, almost prevents unwanted Coulomb interactions. In others words, we can engineering quantum states of opto-mechanical systems close to ground state for electromagnetic interactions.

However this is not the case for gravitational interactions. Measurement methods of a gravitational field of a large mass are well known. Even for weakest interactions, we can assume there is always an open measurement channel, affecting from gravitational sources on space-time geometry. While a measurement channel exits, necessarily remains a source of noise and decoherence, knowing or not the measurement result.

In the assumption we had a quantum theory of gravity, the additional source of noise would be bound to quantum fluctuations in the gravitational field, where the quantum mechanical degrees of freedom would interact gravitationally [3]. Such effects are mostly important at the Planck scale, and more unlikely to notice in opto-mechanical experiments at the range of Newtonian description of gravity.

In this concern, Penrose [4] and Diósi [5] propose this is not the case with opto-mechanical experiments, and with sufficient quantum control over macroscopic mechanical degrees of freedom, these systems could show gravitational decoherence, which would provide a path to study the meeting between gravitational and quantum physics.

The original article works under the proposal [2] for a two-way communication channel picture, where long range gravitational interactions are treated as a weak continuous measurement of the position of each mass. Then they compare a conventional unitary treatment of the mutual gravitational interaction of two masses with their classical measurement channel picture.

Classical measurement channel is an example of local operation with classical communication. In essence, it carries the information to each mass of the other's presence, position and the corresponding gravitational force, and cannot entangle the two masses. The continuous aspect of the record allows to control the reciprocal classical force on each mass using a feedback control, and with its reciprocity they achieve an equivalent result that Penrose and Diósi's models: the gravitational decoherence rate is completely determined by the gradient of the gravitational force between the two masses.In the original article, they estimate the magnitude of these effects for an two oscillators experiment, coupled by gravitational interactions (Figure 1).

2 Theoretical Model and Formalisms

It is mandatory to introduce the theoretical model and formalisms used in the original article. In the following subsections, we start setting the oscillators model presented in Figure 1, then a brief introduction to Quantum Measurement Theory, and finally we use the introduced concepts to consider gravity as a classical measurement channel in the two harmonic oscillators model, obtaining the dynamic equation, averaged over all measurement records, as a goal. In the following section, we discuss the results with a experimental test.

2.1 Two Harmonic Oscillators Model in Gravitational Coupled System

The main idea is setting a two harmonic oscillators model, as Figure 1 shows, drawing on QED formalisms, with a gravitational potential instead of EM potentials. Consider two masses, m_1, m_2 freely suspended so as to move almost harmonically along the x-axis, where the two masses are coupled by gravitational interaction. The displacement of mass m_k from equilibrium is denoted x_k (k=1,2). The interaction potential energy between the masses, expanded to second order in the relative displacement, may be written

$$V(x_1, x_2) = \sum_{k=1}^{2} \frac{1}{2} m_k w_k^2 x_k^2 - \frac{G m_1 m_2}{d^2}(x_1 - x_2) - \frac{G m_1 m_2}{d^3}(x_1 - x_2)^2$$

Analyzing the terms, the linear one represents a constant force between the masses that modifies the equilibrium position of both masses to $\bar{x}_1 = \frac{G m_2}{d^2 w_1^2}$ and $\bar{x}_2 = \frac{-G m_2}{d^2 w_2^2}$. Including this into the displacement coordinate definition, the linear term is neglected. The quadratic term can be incorporated into the definition of the harmonic frequency of each mass. Consider the total mechanic Hamiltonian

$$H = T + \bar{V} \quad , \quad \text{where} \quad T = \sum_{k=1}^{2} \frac{p_k^2}{2m_k} \quad , \quad \bar{V} = \sum_{k=1}^{2} \frac{1}{2} m_k w_k^2 x_k^2 - \frac{G m_1 m_2}{d^3}(x_1 - x_2)^2$$

Considering $K = 2 G m_1 m_2 / d^3$

$$\begin{aligned} H &= \sum_{k=1}^{2} \left[\frac{p_k^2}{2m_k} + \frac{1}{2} m_k w_k^2 x_k^2 \right] - \frac{K}{2} \left(x_1^2 + x_2^2 - 2 x_1 x_2 \right) = \\ &= \sum_{k=1}^{2} \left[\frac{p_k^2}{2m_k} + \frac{1}{2} m_k x_k^2 \left(w_k^2 - \frac{K}{m_k} \right) \right] + K x_1 x_2 = \\ &= \sum_{k=1}^{2} \left[\frac{p_k^2}{2m_k} + \frac{1}{2} m_k x_k^2 \Omega_k^2 \right] + K x_1 x_2 = \\ &= H_0 + K x_1 x_2 \end{aligned}$$

We proceed to calculate the associated normal modes. Consider the Hamiltonian, the dynamic equation is given by

$$m_k \ddot{x}_k = -\frac{\partial H}{\partial x_k} \quad => \begin{cases} m_1 \ddot{x}_1 = -m_1 \Omega_1^2 x_1 - K x_2 \\ m_2 \ddot{x}_2 = -m_2 \Omega_1^2 x_2 - K x_1 \end{cases}$$

As we expect oscillatory motion of a normal mode, in the particular case of same frequency for both masses,

$$x_1 = A_1 e^{iwt} \quad , \quad x_2 = A_2 e^{iwt}$$

Replacing and operating,

$$\left\{\begin{array}{l} m_1A_1(iw)^2 = -m_1\Omega_1^2 A_1 - KA_2 \\ m_2A_2(iw)^2 = -m_2\Omega_1^2 A_2 - KA_1 \end{array}\right. \quad => \left\{\begin{array}{l} m_1\ddot{x}_1 = -m_1\Omega_1^2 x_1 - Kx_2 \\ m_2\ddot{x}_2 = -m_2\Omega_1^2 x_2 - Kx_1 \end{array}\right. \quad =>$$

$$=> \left\{\begin{array}{l} \left(w^2 - \Omega_1^2\right) A_1 - \frac{K}{m_1} A_2 = 0 \\ \left(w^2 - \Omega_2^2\right) A_2 - \frac{K}{m_2} A_1 = 0 \end{array}\right.$$

Expressing as a matrix equation,

$$\begin{bmatrix} w^2 - \Omega_1^2 & -\frac{K}{m_1} \\ -\frac{K}{m_2} & w^2 - \Omega_2^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The resulting equations system has a solution when

$$\begin{vmatrix} w^2 - \Omega_1^2 & -\frac{K}{m_1} \\ -\frac{K}{m_2} & w^2 - \Omega_2^2 \end{vmatrix} = 0 \quad => \quad \left(w^2 - \Omega_1^2\right)\left(w^2 - \Omega_2^2\right) - \frac{K^2}{m_1 m_2} =$$

$$= w^4 - \left(\Omega_1^2 + \Omega_2^2\right) w^2 - \frac{K^2}{m_1 m_2} + \Omega_1 \Omega_2 = 0$$

With variable change, the resolution is reduced to a second grade equation. Their solutions are the two frequencies of the normal modes,

$$w_\pm^2 = \frac{1}{2}\left(\Omega_1^2 + \Omega_2^2\right) \pm \frac{1}{2}\left[\left(\Omega_1^2 - \Omega_2^2\right) + \frac{4K^2}{m_1 m_2}\right]^{1/2}$$

We choose symmetric masses case: $m_1 = m_2 = m => \Omega_1 = \Omega_2 = \Omega$

$$w_+^2 = \Omega^2 + \frac{1}{2}\left[\frac{4K^2}{m^2}\right]^{1/2} = \Omega^2 + \frac{K}{m} = w^2$$

$$w_-^2 = \Omega^2 - \frac{K}{m} = w^2 - \frac{2K}{m} = w^2\left(1 - \frac{2K}{mw^2}\right)$$

In most situations of laboratory relevance, the gravitational coupling is weak and the difference in frequency between the two normal modes, the normal mode splitting, can be written

$$\Delta \equiv w_+ - w_- \approx \frac{K}{mw}$$

Therefore, the resulting classical and quantum dynamics is then described as two independent simple harmonic oscillators, the normal modes, which are linear combinations of the local co-ordinates

$$q_+ = \frac{x_1 + x_2}{\sqrt{2}} \quad , \quad q_- = \frac{x_1 - x_2}{\sqrt{2}}$$

where q_+ is the center-of-mass mode, and q_- is the breathing mode.

The ground state of two normal modes is a superposition state of the configuration space variables of two centre-of-mass degrees of freedom (the local modes) and as such the ability to prepare such a state through purely gravitational interactions would be a test of gravitational decoherence. To determinate the wave function of the normal mode ground states, $|0\rangle_+ \bigotimes |0\rangle_-$, we shall introduce first a few concepts about quantized electromagnetic field theory.

2.2 Multimode Squeezed States

The contents of this subsection have been mostly compiled from the book "Quantum Optics" by D.F. Walls and G.J. Milburn [6].

A coherent state is a minimum uncertainty state that the uncertainty of the canonical conjugates (position and momentum) stays constant and contribute equally to he relation.They have dynamics most closely resembling the oscillatory behavior of a classical harmonic oscillator. These states are generated by the abstract unitary displacement operator

$$D(\alpha) \equiv exp\left(\alpha a^{+} - \alpha^{*} a\right)$$

with α an arbitrary complex number, and a^{+}, a the creation and annihilation operators.

A coherent state is an normalized Eigenstate of annihilation operator and, in terms of the ground Fock-state, is given by

$$|\alpha\rangle = D(\alpha)\,|0\rangle$$

Squeezed states are general minimum-uncertainty states. The most general wave function that satisfies the identity above is the squeezed coherent state (in units with $\hbar = 1$)

$$\psi(x) = C \exp\left(-\frac{(x - x_o)^2}{2w_0^2} + ip_o x\right)$$

where C, x_o, w_0, p_0 are constant (a normalization constant, the center of the wavepacket, its width, and the expectation value of its momentum).The new feature relative to a coherent state is the free value of the width w_0, which is the reason why the state is called "squeezed".

In analogy to the coherent states, the squeeze operator

$$S(\epsilon) \equiv exp\left[\frac{1}{2}\epsilon^{*} a^2 - \frac{1}{2}\epsilon (a^{+})^2\right]$$

where $\epsilon = re^{2i\phi}$

The squeeze operator is unitary and obeys the relation

$$S^{+}(\epsilon) = S^{-1}(\epsilon) = S(-\epsilon)$$

Therefore, the definition of a squeezed state is given by

$$|\alpha, \epsilon\rangle = D(\alpha) S(\epsilon)\,|0\rangle$$

To generate a single mode squeezed state, it is necessary a unitary evolution of $S(\epsilon)$, which acts on a vacuum state. This can be obtained,e.g by an Interaction Hamiltonian of the form

$$H_I = \frac{\hbar}{2}\left[\chi (a^{+})^2 + \chi^{*} a^2\right]$$

which describes simultaneous two-photon generation or absorption processes, as ca be realized by second order nonlinear processes where a photon of energy $2\hbar w$ generates two photons, each of energy $\hbar w$.

To generate a multi mode squeezed state, the previous concept shall be generalized to a case of two photons created simultaneously with different energy. Consider a non degenerate parametric amplifier,e.g the generation of two photons of frequencies $w = w_1 + w_2$. The interaction Hamiltonian reads

$$H_I = i\hbar\left[\chi (a_1^{+})^2 (a_2^{+})^2 + \chi^{*} a_1^2 a_2^2\right]$$

The corresponding multi mode squeezed state is given by

$$|\alpha,\epsilon\rangle = D_1(\alpha)D_2(\alpha)S(G)\,|0\rangle$$

with the generalized displacement operator $D_i(\alpha) \equiv exp\left(\alpha a_i^+ - \alpha^* a_i\right)$ and the two mode squeezing operator

$$S(G) \equiv exp\left[G^* a^2 - G(a^+)^2\right] \quad with \quad G = re^{i\Theta}$$

with the properties

$$S^+(G) = S^{-1}(G) = S(-G)$$

$$S^+(G)a_{1,2}S(G) = a_{1,2}cosh(r) - a_{1,2}^+ sinh(r)e^{i\Theta}$$

For $\Theta = 0$, that is G purely real, we find for the expansion of the two mode squeezed vacuum state into Fock-states

$$|\varphi\rangle = S(G)\,|0\rangle = \frac{1}{cosh(r)}\sum_{n\geq 0} tanh^n(r)\,|n,n\rangle$$

Now the concepts have been introduced and we proceed to determinate the wave function of the normal mode ground states, $|0\rangle_+ \bigotimes |0\rangle_-$, which in the coordinate basis of the local center-of-mass coordinates, is a Gaussian two-mode squeezed state [7],

$$|0\rangle_+ \bigotimes |0\rangle_- = \int\int dx_1 dx_2 \psi(x_1,x_2)\,|x_1\rangle \bigotimes |x_2\rangle$$

where the wave function is

$$\psi(x_1,x_2) = Nexp\left(-H/\hbar\right)$$

where N is a normalization constant. To determinate L, consider the two normal modes Hamiltonian

$$H = \frac{1}{2}w_+q_+^2 + \frac{1}{2}w_-q_-^2 = \frac{1}{4}mw_+(x_1+x_2)^2 + \frac{1}{4}mw_-(x_1-x_2)^2 =$$

$$= \frac{1}{4}mw_+\left(x_1^2+x_2^2\right) + \frac{1}{4}mw_-\left(x_1^2+x_2^2\right) + \frac{1}{2}mw_+x_1x_2 - \frac{1}{2}w_-x_1x_2 =$$

$$= x_1^2\left(\frac{1}{4}mw_+ + \frac{1}{4}mw_-\right) + x_2^2\left(\frac{1}{4}mw_+ + \frac{1}{4}mw_-\right) + x_1x_2\left(\frac{1}{2}mw_+ - \frac{1}{2}mw_-\right)$$

Considering the expressions of w_+, w_-, and $\beta = 2K/mw^2$, we express as a matrix equation

$$H = \frac{mw}{4}(x_1,x_2)\begin{bmatrix}1+\sqrt{1-\beta} & 1-\sqrt{1-\beta}\\ 1-\sqrt{1-\beta} & 1+\sqrt{1-\beta}\end{bmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

Therefore,

$$\psi(x_1,x_2) = Nexp\left[-\vec{x}^T L\vec{x}\right]$$

where

$$L = \frac{mw}{4\hbar}\begin{bmatrix}1+\sqrt{1-\beta} & 1-\sqrt{1-\beta}\\ 1-\sqrt{1-\beta} & 1+\sqrt{1-\beta}\end{bmatrix} \quad and \quad \vec{x}^T = (x_1,x_2)$$

As we can appreciate, the Hamiltonian form correspond to the general form for squeezed states.

2.3 A Brief Introduction to Quantum Measurement Theory

The contents of this subsection have been mostly compiled from the books "Mathematical foundations of quantum mechanics" [8], "The theory of open quantum systems" [9],"Quantum computation and quantum information" [10] and "Quantum measurement and control" [11].

2.3.1 Ideal Quantum Measurements

The fundamental quantum measurement postulate gives a description of a quantum measurement. We start by considering an observable characterized by a self-adjoint operator $\hat{x}$, that we assume with discrete spectrum, with unit multiplicity. If $\Pi(x_n)$ defines the eigen-projector operators associated with the spectral values x_n, then

$$\hat{X}\Pi(x_n) = x_n\Pi(x_n)$$

where $\Pi(x_n)\Pi(x'_n) = \delta_{nn'}\Pi(x_n)$ and $\sum_n \Pi(x_n) = 1$. During the measurement, the system undergoes a non-unitary evolution such that

$$\rho \longmapsto \rho_{n'} = \frac{\Pi(x_n)\rho\Pi(x_n)}{Tr\Pi(x_n)\rho\Pi(x_n)} \equiv \frac{\Pi(x_n)\rho\Pi(x_n)}{Tr\Pi(x_n)\rho\Pi(x_n)}P(x_n)$$

where Tr is the trace operation and $P(x_n)$ the probability of the outcome x_n to occur. The state ρ' describes the sub-ensemble of systems for which x_n have been found. This is the von-Neumann-Lüders postulate. Notice that $\sum_n P(x_n) = 1$.

The original ensemble ρ is divided in several sub-ensembles, each one being conditioned on a concrete measurement outcome x_n. In this sense we speak about a selective measurement. However, it might be the case that the measurement outcomes are not known to us or that the observer does not want to use that informations. In such cases the sub-ensembles are mixed again to give

$$\rho' = \sum_n P(x_n)\rho'_n = \sum_n \Pi(x_n)\rho\Pi(x_n)$$

We speak then of a non-selective measurement. The probability distribution $P(x_n)$ defines an entropy as

$$S = -\sum_n P(x_n)logP(x_n)$$

If the initial state is pure, then S is the entropy produced on the measurement. Notice that the overall entropy produced in the process can be higher, for instance when dissipative processes taking place in the interaction between a quantum system and the many degrees of freedom of a detector [12].

2.3.2 Continuous Measurements

It is possible to considered continuous sets in non-ideal measurements. In addition, non-sharp measurements can be studied introducing smooth operator effects as it is the case of Gaussian measurements with

$$\Omega_f(A) = (2\pi\sigma^2)^{1/4}e^{-\frac{(f-A)^2}{4\sigma^2}} \quad with \quad f \in (-\infty, \infty)$$

where f is a real number representing the measurement outcomes of the observable A with error σ. The conditional post-measurement state is in this case, following the discrete case

$$\rho \longmapsto P^{-1}(f)\Omega_f\rho\Omega_f$$

The probability $P(f)$ is given by

$$P(f)) = Tr\Omega_f\rho\Omega_f$$

with is normalized accordingly

$$\int df P(f) = Tr\Omega_f \rho \Omega_f = 1$$

At this point one can imagine measuring A continuously in time on a single quantum system. Mathematically this can be conceived as performing measurements periodically in time each τ units of time. The limit $\tau \to 0$ can be introduced if very smooth measurements are taken such that the error σ is proportional $\tau^{-1/2}$ in such limit. Therefore, in the continuous limit

$$g = \lim_{\sigma\to\infty,\tau\to 0} \frac{1}{\tau\sigma^2}$$

In this limit it is possible to deduce a dynamical equation for the non-conditioned density operator describing the continuous measurement problem [13]. For a given system whose dynamics is ruled by a Hamiltonian H, the evolution of a state r when no measurement is performed is given by

$$d\rho = -\frac{i}{\hbar}[H,\rho]dt$$

This unitary, continuous and differentiable dynamics changes when a measurement is continuously made. Hence, some correction is needed to describe the non-unitary evolution associated with the measurement.

The measurement operator Ω_f in the continuous limit has the form

$$\Omega_f(A) = \left(\frac{g\tau}{2\pi}\right)^{1/4} e^{-\frac{g}{4}\left(\sqrt{\tau}f-\sqrt{\tau}A\right)^2}$$

and the post-measurement state in the non-selective case is

$$\rho \longmapsto \int df\Omega_f\rho\Omega_f = \left(\frac{g\tau}{2\pi}\right)^{1/2}\int df e^{-\frac{g}{4}\left(\sqrt{\tau}f-\sqrt{\tau}A\right)^2}\rho e^{-\frac{g}{4}\left(\sqrt{\tau}f-\sqrt{\tau}A\right)^2}$$

Replacing $\sqrt{\tau}f \to \phi$

$$\int df\Omega_f\rho\Omega_f = \left(\frac{g}{2\pi}\right)^{1/2}\int df e^{-\frac{g}{4}\left(\phi-\sqrt{\tau}A\right)^2}\rho e^{-\frac{g}{4}\left(\phi-\sqrt{\tau}A\right)^2}$$

To proceed further the exponentials in the post-state is expanded around $\sqrt{\tau} = 0$

$$e^{-\frac{g}{4}\left(\phi-\sqrt{\tau}A\right)^2} = e^{-g\phi^2/4}e^{-g\sqrt{\tau}A/4} =$$
$$= e^{-g\phi^2/4}\left(1 + \frac{g\sqrt{\tau}\phi A}{2} + \frac{g^2\tau\phi^2A^2}{8} - \frac{g\tau A^2}{4} + O(\tau^{3/2})\right)$$

Therefore

$$\int df\Omega_f\rho\Omega_f =$$
$$= \left(\frac{g}{2\pi}\right)^{1/2}\int d\phi e^{-\frac{g\phi^2}{2}}\left[\rho + \frac{g}{2}\sqrt{\tau}\phi(A\rho+\rho A) + \left(\frac{g}{2}\sqrt{\tau}\phi\right)^2 A\rho A + g\tau\left(\frac{g\phi^2}{8} - \frac{1}{4}\right)(A^2\rho+\rho A^2) + O(\tau^{3/2})\right] =$$
$$= \rho + \frac{g}{4}A\rho A - \frac{g}{8}(A^2\rho + \rho A^2)$$

Then, it happens that the contribution of the measurement to the dynamics of the state is

$$d\rho\mid_{Measurement} = -\frac{g}{8}[A[A,\rho]]dt$$

that gives the dynamical equation of a quantum system evolving with Hamiltonian H and continuously monitored by measurements of an observable A, which is

$$d\rho = -\frac{i}{\hbar}[H,\rho]dt - \frac{g}{8}[A[A,\rho]]dt$$

As it is expected, the measurement induced a non-unitary term in the dynamics [13]. The new term defines a complete positive, trace preserving contracting quantum map.

We can analyze further the resulting equation of the expansion. As we know by the resolution, the term proportional to $\sqrt{\tau}$ is neglected, due to the Gaussian integration. One can consider a Monte Carlo integration as strategy to solve the dynamics, that is, we can consider the integration over ϕ as an average over all possible measurement outcomes which are weighted with the Gaussian around $\phi = 0$, in the limit case of innately many measurements. The outcome of a single measurement however is an inherently stochastic event, and we can introduce a stochastic Weiner process $W(t)$.

A Wiener process, or standard Brownian motion process, is a continuous-time stochastic process which is characterized by Gaussian independent increments, continuous paths and $W(0) = 0$ almost surely. Given an observable $\hat{A}$, the signal $A(t)$ from the continuous measurement through a classical channel has the form of a fluctuation around the quantum-mechanical average due to the stochastic nature of the measurements on the quantum system

$$A(t) = \langle \hat{A} \rangle(t) + \delta A(t)$$

where we can relate the Wiener increment to the time-dependent signal writing heuristically

$$\frac{dW}{dt} = \delta A(t)$$

which gives a correlation for $\delta A(t)$ as

$$\langle \delta A(t) \delta A(\tau) \rangle = \frac{1}{g} \delta(t - \tau)$$

An example of this is the stochastic process defined by

$$X(t) = \mu t + \sigma W(t)$$

which is called a Wiener process with drift μ and infinitesimal variance σ^2.

For the term $\frac{g\sqrt{g}}{2}\sqrt{\tau}\phi(A\rho + \rho A)$, consider the change of the Weiner process over a time interval, the Wiener increment

$$\Delta W = \sqrt{\tau}\phi = \sqrt{\frac{\tau}{g}}\bar{w}$$

where $\bar{w} \in N(0, 1)$ is a random number normally distributed with zero mean and unit variance. In the continuous limit we can write the following conditions

$$\langle dW \rangle = 0$$
$$dW^2 = \frac{d\tau}{g}$$

In the case of a huge, but finite number of measurements, the integral over the measurement outcomes in the expansion becomes a sum over each measurement outcome represented by the stochastic variable $W(t)$. The term proportional to $\sqrt{\tau}$ now is not neglected by temporal average, but replaces by the stochastic term

$$\frac{g}{2}(A\rho + \rho A)dW$$

which has zero mean as expected. The dynamic equation, now caused by measurement process, is described by

$$d\rho = -\frac{i}{\hbar}[H, \rho]dt - \frac{g}{8}[A, [A, \rho]]dt + \frac{g}{2}(A\rho + \rho A)dW$$

Moreover the dynamics must be trace preserving, and this new equation does not preserve trace. One has to subtract the change in trace to keep the state normalized and leads to the stochastic master equation that describes the selective measurement process

$$d\rho = -\frac{i}{\hbar}[H, \rho]dt - \frac{g}{8}[A, [A, \rho]]dt + \frac{g}{2}(A\rho + \rho A - 2\langle A \rangle \rho)dW$$

which is trace preserving.

2.4 Gravity as a Classical Measurement Channel

There are proposals, as the one by Kafri and Taylor's [2], to test the type of channel, quantum or classical, that mediates in long range coherent interactions between two particles. In the case of the named proposal, they define a quantum channel by introducing an ancillary degree of freedom, a harmonic oscillator, which leads, under appropriate circumstances, to an effective direct non-local interaction between the two local systems. A classical channel can then be defined if we allow a continuous measurement.

In the original paper, they take a different approach to defining a classical channel, using methods from quantum stochastic control theory. Rather than a direct quantum interaction of the form $\hat{x_1}\hat{x_2}$, the gravitational center of mass co-ordinate, $\hat{x_i}$, of each particle is continuously measured in a classical stochastic measurement record, $J_k(t)$, carrying this information acts reciprocally as a classical control force on the other mass. The effect on the dynamics of the systems is to produce a Hamiltonian term of the form,

$$H_{grav} = \chi_1 \frac{dJ_1(t)}{dt}\hat{x_2} + \chi_2 \frac{dJ_2(t)}{dt}\hat{x_1}$$

where χ_k has the same units of K. As we are considering a continuous weak measurement of $\hat{x_k}$, the measurement record obeys a stochastic differential equation of the form [11]

$$dJ_k(t) = \langle \hat{x_k} \rangle_c dt + \sqrt{\frac{\hbar}{2\Gamma_k}} dW_k(t)$$

where Γ_k is a constant that determines the rate at which information is gained by the measurement, $dW_{1,2}$ are independent real valued Wiener increments and the average $\langle \hat{x_k} \rangle_c$ is a conditional quantum mechanical average conditioned on the entire history of measurement records up to time t. Note that the unit of $\Gamma_k/\hbar$ are $m^{-2}s^{-1}$.

Consider the classical control Hamiltonian

$$H_c = H_0 + H_{grav}$$

As we introduce in the previous subsection, the conditional quantum dynamics of the coupled oscillator system is given by the stochastic master equation

$$d\rho_c = -\frac{i}{\hbar}[H_c, \rho_c]dt - \sum_{k=1}^{2}\left(\frac{\Gamma_k}{2\hbar}[\hat{x_k}, [\hat{x_k}, \rho_c]]dt + \sqrt{\frac{\Gamma_k}{\hbar}}(\hat{x_k}\rho_c + \rho_c\hat{x_k} - 2\langle \hat{x_k} \rangle)dW_k\right)$$

where we consider the observable A as $\hat{x_k}$. Grouping the Weiner term in a single super-operator

$$\hat{x_k}\rho_c + \rho_c\hat{x_k} - 2\langle \hat{x_k} \rangle = \hat{x_k}\rho_c + \rho_c\hat{x_k} - 2Tr(\rho_c\hat{x_k}) = \hat{x_k}\rho_c + \rho_c\hat{x_k} - Tr(\hat{x_k}\rho_c + \rho_c\hat{x_k}) \equiv \mathcal{H}[\hat{x_k}]\rho_c$$

The condensed version of the dynamical stochastic master equation is given by

$$d\rho_c = -\frac{i}{\hbar}[H_c, \rho_c]dt - \sum_{k=1}^{2}\left(\frac{\Gamma_k}{2\hbar}[\hat{x_k}, [\hat{x_k}, \rho_c]]dt + \sqrt{\frac{\Gamma_k}{\hbar}}\mathcal{H}[\hat{x_k}]\rho_c dW_k\right)$$

where the form of equation defines a direct feedback model.

The steps to go from this dynamical stochastic master equation to the unconditional dynamic equation, describing the dynamics averaged over all measurement records, are explained in the Annex (see below, Section 5). Therefore, the corresponding equation

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H_0, \rho] - \frac{i}{2\hbar}\left(\chi_2[\hat{x_1}, \hat{x_2}\rho + \rho\hat{x_2}] + \chi_1[\hat{x_2}, \hat{x_1}\rho + \rho\hat{x_1}]\right) - \sum_{k=1}^{2}\frac{\Gamma_k}{2\hbar}[\hat{x_k}, [\hat{x_k}, \rho]] -$$

$$-\frac{\chi_1^2}{8\hbar\Gamma_1}[\hat{x_2}, [\hat{x_2}, \rho]] - \frac{\chi_2^2}{8\hbar\Gamma_2}[\hat{x_1}, [\hat{x_1}, \rho]]$$

The second terms is the systematic effect of the control protocol. The final two terms represent the effect of feeding back the white noise on the measurement signals to control the dynamics of the other mass.

In the case of highly asymmetric masses, for example the mass of the earth and the mass of a neutron in the experiments on neutron interferometry [14], $m_1 \gg m_2$, so that the relative contribution to the noise in the channel from the major mass to the minor mass is much smaller than the other channel, which is expected, because for major mass the position of the center of mass should be almost classical. Then we expect $\Gamma_1 \gg \Gamma_2$.

Then the carried information rate from the major mass to the minor mass is much greater than the rate from the minor mass to the major mass, that is, the decoherence rate of the minor mass is much less than the decoherence rate of the major mass, with the same idea as before, the major mass acts more classical.

Let us now consider the symmetric case: $m_1 = m_2$ and $\chi_1 = \chi_2 = K$. In this case, we expect $\Gamma_1 = \Gamma_2 = \Gamma$. The noise added by measurement and feedback is a minimum at $\Gamma = \chi/2$ linking the decoherence rate due to the continuous measurement to the scale of the gravitational interaction.

So the resulting unconditional dynamic equation is

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H_0, \rho] - \frac{iK}{\hbar}[\hat{x_1}\hat{x_2}, \rho] - \frac{K}{2\hbar}\sum_{k=1}^{2}[\hat{x_k}, [\hat{x_k}, \rho]]$$

Using a recent result of Kafri and Taylor [2] we can show that, in the case when the two systems are Gaussian, the resulting unconditional dynamic equation can never entangle them. Further, the gravitational decoherence in the dynamics is minimal in the sense that, if it were any smaller, evolution under this equation would immediately entangle the ground state of the uncoupled Hamiltonian H_0.

To be able to see this we write the unconditional dynamic equation in terms of the dimensionless operators $\tilde{x_k} = \hat{x_k}(mw/\hbar)^{1/2}$,

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H_0, \rho] - i\tilde{g}[\hat{\tilde{x}}_1\tilde{x_2}, \rho] - \frac{1}{4}\sum_{k=1}^{2} Y_{ij}[\tilde{x_i}, [\tilde{x_j}, \rho]]$$

where $\tilde{g} = K/mw$ measures the strength of the gravitational interaction and $Y_{ij} = \left(\frac{2K}{mw}\delta_{ij}\right)$ is the decoherence matrix. From the Kafri and Taylor's paper, we note that entanglement is never generated if and only if the matrix $Y - 2i\tilde{g}\sigma$ has no negative eigenvalues, where σ is the 2x2 symplectic matrix. Noting that $Y - 2i\tilde{g}\sigma$ has eigenvalues 0 and $4g$ we see that a slightly less noisy matrix $Y - \epsilon\delta_{ij}$ produces entanglement for any positive ϵ.

3 Experimental Test

We now consider the prospects for an experimental observation of the model proposed here. In this test we assume the symmetric case of two mechanical resonators with same mass and frequency. In the unconditional dynamic equation without [2] considerations, the last term is driving a diffusion process in momentum of each oscillators at rate $\hbar K$, which is the gravitational heating rate D_{grav}.

We define this rate in terms of the rate of change of the phonon number, such as, the average mechanical energy divided by $\hbar w$

$$R_{grav} = \frac{K}{2mw}$$

The last term also leads to the decay of off-diagonal coherence in the position basis of each mechanical oscillator

$$\frac{d\langle x_k'|\,\rho\,|x_k\rangle}{dt} = (...) - \frac{K}{\hbar^3}\left(x_k' - x_k\right)^2$$

which shows the rate of decay of coherence quickly increases the greater the separation of the superposed states. We can use the natural length scale proceeded by the zero-point position fluctuations in the ground state of each resonator to rewrite the decoherence rate as

$$\Lambda_{grav} = \frac{K}{\hbar^3}\Delta x_0^2 = \frac{K}{2\hbar^2 mw}$$

In natural units, these rates can be expressed in terms of the normal mode splitting for weak gravitational interaction,

$$R_{grav} = \Lambda_{grav} = \frac{\Delta}{2}$$

As we can see, thee two important parameters responsible for gravitational decoherence are are of the order of the normal mode splitting between the two mechanical resonators due to their gravitational coupling.

In order to test the detection of gravitational decoherence in this model, it is necessary to obtain the normal mode splitting as larger as possible. Writing this in terms of the Newton constant, the splitting has the form

$$\Delta = \frac{Gm}{wd^3}$$

Assuming the case of two spheres of radius r, now the splitting is written in terms of the density of the material,

$$\Delta = \frac{4\pi G\rho}{w}\left(\frac{r}{d}\right)^3$$

As $d < 2r$, this quantity is bounded

$$\Delta \leq \frac{\pi G\rho}{6w}$$

We need to use a material with a large density and a mechanical frequency as small as possible to raise the heating rates. The original article proposes as example depleted uranium spheres and a mechanical frequency of one Hertz, so the splitting becomes $\Delta \sim 10^{-7}s^{-1}$, a value so small that a terrestrial experiment would be challenging.

Considering low frequency mechanical resonators in a realistic experiment, thermal noise and frictional damping will be unavoidable. We can estimate the relative size of these effects using the quantum Brownian motion master equation [15]

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H_R, \rho] - \sum_{j=1}^{2}\left(\frac{i}{\hbar}\gamma_j\left[\hat{x_j}, \{\hat{p_j}, \rho\}\right] + \frac{2\gamma_j m_j k_B T}{\hbar^2}[\hat{x_k}, [\hat{x_k}, \rho]]\right)$$

where γ_k is the dissipation rate for each of the mechanical resonators assumed to be interacting with a common thermal environment at temperature T. If we compare the form of thermal noise in this equation to the form of gravitational decoherence, in the symmetric case we can assign an effective temperature to the gravitational decoherence rate given by

$$T_{grav} = \frac{\hbar^3 K}{2m\gamma k_B}$$

If we write this in terms of the quality factor for mechanical resonators Q,

$$k_B T_{grav} = \hbar Q\Delta$$

Using the previous example with uranium spheres, for the relatively high value of $Q = 10^9$, we find that $T_{grav} \sim 10^{-9}$, then one would need an ambient temperature less than this to clearly distinguish gravitational decoherence from environmental effects. The original article postulates that gravitationally coupled Bose–Einstein condensates of atomic gases possibly could reach this regime.

4 Conclusions

The original article have presented a two resonators model with gravitational interaction mediated by a purely classical channel with continuous weak measurements, which is equivalent to a gravitational decoherence model first proposed by Diosi. It is also a concrete example of a general theory of a classically mediated non entangling force law considered by Kafri and Taylor [2], which using a result from this paper we find that this kind of channel based on continuous weak measurement can never entangle Gaussian systems.

In a experimental test of two similar gravitationally coupled oscillators, this result is shown as a direct scaling between the normal mode splitting induced by the gravitational force and the gravitational decoherence rate. However an experimental test using two gravitationally coupled opto-mechanical resonators would be difficult nowadays, because of the ladder of the results. Aside the proposed coupled Bose–Einstein condensates of atomic gases system, the original article also proposes laser cooling techniques to prepare harmonically trapped particles of large mass in the ground state, which affects both effective temperature and quality factor, so their application has to be discussed more. The advantage on these technologies will allow to clearly observe gravitational decoherence from environmental effects, possibly rule out treating the interaction as purely classical.

5 Annex: Obtaining the Unconditional Dynamic Equation

The following section describes a method to obtain the corresponding unconditional dynamic equation (that is, the dynamic equation averaged in time over all measurements records). The idea is to apply the evolution operator feedback of measurement records to the post-measurement state $\rho + d\rho_M$. The demonstration could not be as rigorous as expected, such as terms diverging in factor, however it servers the purpose of this review.

That is

$$\rho + d\rho \longrightarrow V(dt)\rho + d\rho_M V^{+}(dt)$$

where $V(dt)$ is the evolution operator for the feedback of measurement records through the classical channel, evolving infinitesimally.

Before that, we introduce the Wiseman-Hilburn feedback concept. Consider the continuous measurement of $\hat{x}$. Then the channel evolves

$$\frac{dJ}{dt} = \langle \hat{x_k} \rangle + \frac{1}{8K}\xi(t)$$

where $\xi(t)$ is noise. The feedback appears in the Hamiltonian as

$$H = H_0 + \left(\frac{dJ}{dt}\right) A$$

where A is some hermitian operator.

A real J has finite width (differentiable and continuous), but it is idealized as white noise approach in the Ito Calculus, that is, expanding until second order and then neglecting all terms in the expansion for t, aside the zero and first order. We also consider for infinitesimal Weiner increments the conditions in continuous limit, $dW_k^2 = dt/g_k$. Also, there are no crossing terms between two different infinitesimal Weiner increments, $dW_1 dW_2 = 0$.

In general, our Hamiltonian contains noise that is not suitable an independent Weiner Increment, that is, the equation is given in Stratonovich sense. However it is possible to construct an Ito equation whose solutions match those of our Stratonovich Hamiltonian.

The process to do so is: replace the the signal by its white-noise limit dW/dt, then applies Ito Calculus considerations, and finally transforms to Ito equation adding an additional term.

Consider the evolution of the state due to the feedback Hamiltonian

$$d\rho_F B = -\frac{i}{\hbar}\left(\frac{dJ}{dt}\right)[A,\rho] = \frac{1}{\hbar}\left(\langle\hat{x_k}\rangle + \frac{1}{8K}dW\right)\mathcal{H}(-iA)\rho$$

To pass into Ito Calculus, we have to add an extra term which is equals half the square of the stochastic one.

$$d\rho_F B = -\frac{i}{\hbar}\left(\frac{dJ}{dt}\right)[A,\rho] = \frac{1}{\hbar}\left(\langle\hat{x_k}\rangle + \frac{1}{8K}dW\right)\mathcal{H}(-iA)\rho + \frac{1}{16K\hbar^2}[A,[A,\rho]]$$

Therefore, we proceed to obtain the form of the evolution operator. Consider the initial state ρ which evolves infinitesimally to $\rho + d\rho_F B$, strictly for the feedback. The evolution operator for the feedback of the corresponding post-measurement system has the following proposed form

$$V(dt) = exp\left(-\frac{i}{\hbar}Adt - \frac{i}{\hbar}B_1 dW_1 - \frac{i}{\hbar}B_2 dW_2\right)$$

where A, B_1, B_2 are operators to determinate.

We assume two considerations: first, the exponential as a Taylor Expansion; second, in any expansion we consider Ito Calculus. Note that these considerations are using across the section.

Given these considerations,

$$V(dt) = 1 - \frac{i}{\hbar}Adt - \frac{i}{\hbar}B_1 dW_1 - \frac{i}{\hbar}B_2 dW_2 + \frac{1}{2\hbar^2}B_1 dt + \frac{1}{2\hbar^2}B_2 dt$$

Therefore, the post-measurement

$$\rho + d\rho \longrightarrow V(dt)\rho V^{+}(t) =$$

$$= \left[\rho - \frac{i}{\hbar}A\rho dt - \frac{i}{\hbar}B_1\rho dW_1 - \frac{i}{\hbar}B_2\rho dW_2 - \frac{g_1^{-1}}{2\hbar^2}B_1^2\rho dt - \frac{g_2^{-1}}{2\hbar^2}B_2^2\rho dt\right]\cdot$$

$$\cdot\left[1 + \frac{i}{\hbar}A^{+}dt + \frac{i}{\hbar}B_1^{+}dW_1 + \frac{i}{\hbar}B_2^{+}dW_2 - \frac{g_1^{-1}}{2\hbar^2}B_1^{+2}dt - \frac{g_2^{-1}}{2\hbar^2}B_2^{+2}dt\right] =$$

$$= \rho + dt\left[-\frac{i}{\hbar}\left(A\rho - \rho A^{+}\right) - \frac{1}{2\hbar^2}g_1^{-1}\left(B_1^2\rho + \rho B_1^{+2} - 2B_1\rho B_1^{+}\right) - \frac{1}{2\hbar^2}g_2^{-1}\left(B_2^2\rho + \rho B_2^{+2} - 2B_2\rho B_2^{+}\right)\right] -$$

$$-\frac{i}{\hbar}\left(B_1\rho - \rho B_1^{+}\right)dW_1 - \frac{i}{\hbar}\left(B_2\rho - \rho B_2^{+}\right)dW_2$$

where we recognize $A = \hat{x_1} + \hat{x_2}$, $B_1 = \chi_1\hat{x_2}/\sqrt{8\Gamma_1}$ and $B_2\chi_2\hat{x_1}/\sqrt{8\Gamma_2}$.Then the actual form of the evolution operator for the dynamics is given by

$$V(dt) = exp\left(-\frac{i}{\hbar}(\hat{x_1} + \hat{x_2})dt - \frac{i}{\hbar}\frac{\chi_1}{\sqrt{8\Gamma_1}}\hat{x_2}dW_1 - \frac{i}{\hbar}\frac{\chi_2}{\sqrt{8\Gamma_2}}\hat{x_1}dW_2\right)$$

Given these considerations

$$V(dt) =$$

$$= 1 - \frac{i}{\hbar}(\hat{x_1} + \hat{x_2}) - \frac{i}{\hbar}\frac{\chi_1}{\sqrt{8\Gamma_1}}\hat{x_2}dW_1 - \frac{i}{\hbar}\frac{\chi_2}{\sqrt{8\Gamma_2}}\hat{x_1}dW_2 +$$

$$+ \frac{\chi_1^2\hat{x_2}^2}{16\hbar\Gamma_1}dt + \frac{\chi_2^2\hat{x_1}^2}{16\hbar\Gamma_2}dt$$

Given $d\rho_M$ as

$$d\rho_M = -\frac{i}{\hbar}[H_c, \rho_c]dt - \sum_{k=1}^{2}\left(\frac{\Gamma_k}{2\hbar}[\hat{x_k}, [\hat{x_k}, \rho_c]]dt + \sqrt{\frac{\Gamma_k}{\hbar}}(\hat{x_k}\rho_c + \rho_c\hat{x_k} - 2\langle\hat{x_k}\rangle)dW_k\right)$$

Then the unconditional master equation is calculated by obtaining the post-measurement state

$$\rho + d\rho \longrightarrow V(dt)(\rho + \rho_M)V^{+}(dt)$$

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