UNIVERSIDAD DE LA LAGUNA

An educated introduction to Quantum Measurement Theory and superconducting quantum bits

by

Javier Rivera Deán

Director

Dr. Daniel Alonso Ramírez

Facultad de Física

Departamento de Física

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"- How's your quantum computer prototype coming along?

- Great! The project exists in a simultaneous state of being both totally successful and not even started.

- Can I observe it?

- That's a tricky question."

Dilbert. Scott Adams.

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Abstract

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This end of degree project is an educated review of the texts *Lecture notes for Ph219/CS219* done by John Preskill and *Lecture notes on superconducting quantum circuits* done by Juan José García Ripoll and Borja Peropadre. The aim of it is to do a first approach to some basic concepts of Quantum Information Theory and to see how can information encrypted in a certain quantum system be extracted by performing measurements in a prepared ensemble which is coupled to it. With this purpose, the specific kind of systems that will be object of study are known as superconducting qubits so an introduction to them is given within these pages, focusing mainly in the so-called charge qubits. Thereby, with all these tools, a correlation between a controlled ensemble, i.e., one which shall be completely characterized, and another one that is unknown, will be performed. For so, a particular sort of quantum logic gates known as CNOT gate is used as it shows the main differences between Classical and Quantum Information Theory.

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Este trabajo de fin de grado es una revisión de los textos *Lecture notes for Ph219/CS219* de John Preskill y *Lecture notes on superconducting quantum circuits* de Juan José García Ripoll y Borja Peropadre. Su objetivo es el de llevar a cabo una primera toma de contacto con algunos conceptos de la Teoría Cuántica de la Información y ver cómo información encriptada en un determinado sistema cuántico puede ser extraída realizando medidas en otro que está acoplado a él. Con este proposito, un tipo específico de sistemas conocidos como qubits superconductores serán empleados por lo que una introducción a ellos se hará en estas páginas, centrando la atención básicamente en los conocidos como *charge qubits*. Por lo tanto, con todas estas herramientas, una correlación entre un sistema controlado, i.e., uno que está completamente caracterizado, y otro que es desconocido, será llevada a cabo. Con este fin, se hará uso de un tipo concreto de puerta lógica cuántica denominada como *CNOT gate* pues representa uno de los grandes contrastes existentes entre la Teoría de la Información Clásica y Cuántica.

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Preface

This project is essentially an educated review of the texts *Lecture notes for Ph219/CS219* done by John Preskill and *Lecture notes on superconducting quantum circuits* done by Juan José García Ripoll and Borja Peropadre as well as from other books which are all cited in the bibliography. Therefore, this document does not intend to be original as its main purpose is to do an educated introduction to Quantum Measurement Theory and superconducting quantum bits. Also, most of the pictures we are about to use are extracted from those documents.

Chapter 1

General introduction

"In this chapter, we will introduce some topics which we are going to discuss throughout this dissertation. First of all, we will start by giving the definition of what is a qubit according to the set of lecture notes by John Preskill ([Pre15]) and after we will remember the main characteristics of closed quantum systems. In that way, we can establish the differences with open quantum systems and incorporate the definition of master equations, according to [Car99], which will be useful for our purposes when describing the influence of the environment on our system. Finally, we will give a brief explanation of what is a superconductor as they will be employed later in our systems."

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"En este capítulo, introduciremos algunos temas que serán tratados a lo largo del trabajo. En primer lugar, comenzaremos dando la definición de lo que es un qubit de acuerdo con las notas de John Preskill ([Pre15]) y posteriormente recordaremos las principales caracterísitcas de los sistemas cuánticos aislados. En este sentido, se podrán establecer las diferencias existentes con los sistemas cuánticos abiertos e incorporar la definición de las ecuaciones maestras, de acuerdo con [Car99], las cuales serán de especial importancia pues nos permitirán describir la influencia del entorno en nuestro sistema. Finalmente, explicaremos brevemente qué es un superconductor pues serán empleados en posteriores configuraciones."

1.1 The Qubit

In Classical Information theory, we have that the indivisible unit of information is the bit which takes one of two possible values $\{0, 1\}$. In Quantum Information theory, that role is played by the *quantum bit* or commonly known as *qubit* which describes a state in the simplest possible quantum system and it can be represented by a two-dimensional Hilbert space so in general we can write those states as

$$a\,|0\rangle + b\,|1\rangle \tag{1.1}$$

A geometrical interpretation of these qubits can be given in terms of the spin state of an object with $s = \frac{1}{2}$, so the kets $\{|0\rangle, |1\rangle\}$ now are written in terms of the states $\{|\uparrow\rangle, |\downarrow\rangle\}$. Another interpretation is provided by a photon thanks to their two independent polarizations but they differ from the spin $\frac{1}{2}$ objects in two important ways: photons are massless and they are spin 1 objects.

Basically what we can see is that, for classical bits, there are only two possible outcomes: 0 or 1. For the qubits we also have two possible outcomes but between the input and the outcome there are infinite states whose general form is given by eq. (1.1). This last statement has a very important consequence on the operations we can do with them which are wide as it can be seen for instance in quantum circuit theory where they are known as quantum logic gates. However, we have to first recall the main differences between closed and open quantum systems as the set-up we will study is affected by a greater system: the environment.

1.1.1 Closed quantum systems

As we have seen on previous courses of quantum mechanics, there are five postulates used to develop a description of closed quantum systems. We define that kind of setups as the ones which are perfectly isolated and, because of that, they will not exchange energy and information with the environment. The valid axioms, that we quoted literally from [Pre15], are the following:

1.- A state is a complete description of a physical system. In quantum mechanics, a state is a ray in a Hilbert space. As a ray we understand an equivalent class of vectors that differ by multiplication by a nonzero scalar.

2.- An observable is a property of a physical system that in principle can be measured. In quantum mechanics, an observable is represented by a self-adjoint operator.

3.- A measurement is a process in which information about the state of a physical system is acquired by an observer. In quantum mechanics, we apply an observable $\hat{A}$[I] to a quantum state $|\psi\rangle$ and the outcome of the computation is an eigenvalue a_n of $\hat{A}$. The probability of that outcome is given by

$$\text{Prob}(a_n) = \sum_j^{g_j} \left\| \left\langle u_n^{(j)} \middle| \psi \right\rangle \right\|^2 \tag{1.2}$$

where $u_n^{(j)}$ are the eigenstates that has a_n as an eigenvalue and where j indicates the possible degeneration. After that measurement, the resulting state is given by

$$|\psi'\rangle = \frac{\hat{M}_n |u_i\rangle}{\langle\psi| \hat{M}_n |\psi\rangle} \tag{1.3}$$

where we have defined $\hat{M}_n = \sum_i^{g_i} \left|u_n^{(i)}\right\rangle \left\langle u_n^{(i)}\right|$.

[I] Throughout this text, operators will be denoted by a hat.

4.- Dynamics describes how a state evolves over time. In quantum mechanics, the time evolution is described by an unitary operator.

$$\psi(t') = U(t', t)\psi(t), \quad \text{such that } U(t, t) = \mathbb{1}. \tag{1.4}$$

5.- Let us suppose that we have two systems A and B which are described, respectively, by the Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$. The Hilbert space of the composite systems AB is given by the tensor product $\mathcal{H}_A \otimes \mathcal{H}_B$. Then, if $|i\rangle_A$ and $|\mu\rangle_B$ form a basis for the systems A and B respectively, then a basis for the composite system AB can be written as $\{|i_A\rangle \otimes |\mu_B\rangle\}$ such that any state $|\psi_{AB}\rangle$ is given by

$$|\psi_{AB}\rangle = \sum_{i,\mu} \psi_{i\mu} |i_A\rangle \otimes |\mu_B\rangle . \tag{1.5}$$

1.1.2 Open quantum systems

Open quantum systems interchange energy and information with the environment. So basically, the main differences that open quantum systems have with respect to closed ones are ([Pre15]):

- States are not rays.
- Measurements are not orthogonal projections.
- Evolution is not unitary.

From now on, in the following subsections, we shall analyze each of the points stated in order to justify them. To do that, we will follow the prescriptions given by John Preskill in [Pre15] with the exception of subsection 1.1.2.3 where we will follow [Car99].

1.1.2.1 Density matrices

As a first step, and with the purpose of understanding these systems, we can consider a world which is composed by two systems where we only observe one of them called S. The other one, which we denote by E, is prohibited to us, namely, is *locked* in a vault. Note that for the whole system SE we can apply the axioms analyzed on section 1.1.1. If we use $\{|i_S\rangle\}$ and $\{|\mu_E\rangle\}$ to denote orthonormal basis for parts S and E respectively, then a general state is given by eq. (1.5). Suppose now that we want to measure the mean value of an observable M_S which is only defined in S. This is described by the following operation

$$\langle M_S \rangle = \langle \psi_{SE}| M_S \otimes \mathbb{1}_E |\psi_{SE}\rangle = \mathrm{tr}(M_S \rho_S) \tag{1.6}$$

where ρ_S is the density operator of subsystem S and which can be written like

$$\rho_S = \mathrm{tr}_E(|\psi_{SE}\rangle \langle \psi_{SE}|) = \sum_i p_i |i_S\rangle \langle i_S| \tag{1.7}$$

so we may say that the density operator ρ_S is obtained by performing a partial trace over subsystem E of the density operator for the combined system. From this we make out that if we are looking at a subsystem of a larger quantum system whose state is defined by a ray, the state associated to the subsystem does not need to be a ray, in general is represented by a density operator. In the case that we have a ray, then we will say that it is pure; otherwise we say that is mixed.

1.1.2.2 Measurements

According to what we have discussed for closed quantum systems, when we perform measurements over an isolated system we employ for that a projective operator which involves all the eigenvectors of the observable we are measuring. However, as we are working now with composite systems we can think of performing measurements in one of them in such a way that this measure affects the other ensemble, which we may call S, so the resulting operation in such system does not need to be an orthogonal projection. We can think of S to be a microscopic system that is inaccessible to us and the other one as a macroscopic one on which we can do measurements. This last kind of systems are called typically *ancilla* and we will denote it by letter A, take not confused with the environment which we may call E. To fix ideas we shall consider this system to be a massive particle that in its evolution suffers a wavepacket spread

$$\Delta x \geq \sqrt{\frac{\hbar t}{m}}. \tag{1.8}$$

Hence, the Hamiltonian that describes this whole bipartite system can be written as

$$\mathcal{H} = \mathcal{H}_S + \frac{\hat{P}^2}{2m} + \lambda(t)\hat{O} \otimes \hat{P}, \tag{1.9}$$

where the first term is the unperturbed Hamiltonian for S, $\hat{P}$ is the linear momentum operator so the second term represents the Hamiltonian for our free particle A and the third term is the interaction between both systems, being $\lambda(t)$ a constant that switches on and off the coupling. $\hat{O}$ denotes the operator we want to measure over system S. If we assume that the measurement is done in a very quick way so we can neglect the free evolution of the system, in such case eq. (1.9) is reduced to

$$H \approx \lambda(t)\hat{O} \otimes \hat{P}. \tag{1.10}$$

Considering now that the coupling constant remains switched on during time T in which the measurement takes place, then the time evolution operator will be given by

$$U(T) = e^{-i\lambda T\hat{O}\otimes\hat{P}}. \tag{1.11}$$

Let us see another way in which this operator can be written, by using the spectral representation of $\hat{O}$ as

$$\begin{aligned}
U(T) = e^{-i\lambda T\hat{O}\otimes\hat{P}} = \sum_n \frac{(-i\lambda T\hat{O}\otimes\hat{P})^n}{n!} = \mathbb{1} + (-i\lambda T\hat{O}\otimes\hat{P}) + \frac{1}{2}(-i\lambda T\hat{O}\otimes\hat{P})^2 + ... = \\
= \sum_a |a\rangle\langle a| + \Big(-i\lambda T\sum_a O_a |a\rangle\langle a|\otimes\hat{P}\Big) + \frac{1}{2}\Big(-i\lambda T\sum_a O_a |a\rangle\langle a|\otimes\hat{P}\Big)^2 + ... = \\
= \sum_a |a\rangle\langle a| + \Big(-i\lambda T\sum_a O_a |a\rangle\langle a|\otimes\hat{P}\Big) + \frac{1}{2}\Big(-i\lambda T\otimes\hat{P}\Big)^2\sum_a O_a^2 |a\rangle\langle a| + ... = \\
= \sum_a |a\rangle\langle a|\Big[\mathbb{1} + (-i\lambda TO_a\hat{P}) + \frac{1}{2}(-i\lambda TO_a\hat{P})^2 + ...\Big] = \sum_a |a\rangle\langle a| e^{-i\lambda TO_a\hat{P}}.
\end{aligned}$$

The exponential term located inside the sum has the general form $e^{-ix_o\hat{P}}$ which generates a translation of the initial wavepacket. However there is an important result in this movement that the particle does and is the correlation with the eigenvalues of $\hat{O}$, that is, our massive particle will be displaced a quantity λTO_a so if the initial state of the whole composite system is

$$|\psi_a(x)\rangle = \sum_a \alpha_a |a\rangle\otimes|x\rangle$$

thus after the evolution the final state is

$$\begin{aligned}
U(T)|\psi_a(x)\rangle = \Big[\sum_a |a\rangle\langle a| e^{-i\lambda TO_a\hat{P}}\Big]\sum_a \alpha_a |a\rangle\otimes|x\rangle = \\
= \sum_a \alpha_a |a\rangle\otimes|x-\lambda TO_a\rangle.
\end{aligned}$$

Here we see explicitly that the position of the ancilla has been correlated with the value of the observable, so performing a measurement on the position of the ancilla will lead us to a given preparation of our quantum system S. To perform this measurement it must be satisfied that the width of the wavepacket (eq. (1.8)) has to be very small compared to $\lambda T\Delta O_a$, that is, the minimal gap between the eigenvalues of $\hat{O}$, so we are able to distinguish them.

Another property of $U(t)$ is that preserves the norm, i.e., is unitary.

$$UU^\dagger = \Big[\sum_a |a\rangle\langle a| e^{-i\lambda TO_a\hat{P}}\Big]\Big[\sum_b |b\rangle\langle b| e^{i\lambda TO_b\hat{P}}\Big] = \sum_{a,b} |a\rangle\langle b|\delta_{ba}e^{(-i\lambda TO_a\hat{P}+i\lambda TO_b\hat{P})} = \mathbb{1}.$$

The sort of measurements we have described until now corresponds to the Von Neumann's model of orthogonal measurement, according to [Pre15]. We see that performing measurements in the fiducial basis of system A let us obtain orthogonal projections of system S so after all it can be said that we are performing *indirect* orthogonal measurements over our quantum system. Nevertheless we have commented at the beginning of this section that measurements on composite quantum system do not need to be orthogonal. To see this let us consider another example in which systems S and A are two qubits which might be called as *target* and *control* respectively. As the *control* system

is the one we can give commands to, we will suppose that it is prepared on state $|0\rangle$ and the correlation between both systems is such that after time T it satisfies the following unitary map

$$U : (\alpha\,|0\rangle + \beta\,|1\rangle)_S \otimes |0_A\rangle \longrightarrow \alpha\,|0_S\rangle \otimes |0_A\rangle + \beta\,|1_S\rangle \otimes |1_A\rangle\,. \tag{1.12}$$

Now the fiducial basis of the ancilla is the $\{|0\rangle\,, |1\rangle\}$ so performing projections onto it we will obtain orthogonal states of our quantum system S. Nevertheless, suppose now that instead of doing projections in that basis we find a way in which we can execute measurements in other basis, for instance $\{|+\rangle\,, |-\rangle\}$ where

$$|+\rangle = \frac{1}{\sqrt{2}}\big[\,|0\rangle + |1\rangle\,\big], \quad |-\rangle = \frac{1}{\sqrt{2}}\big[\,|1\rangle - |0\rangle\,\big].$$

Thereby, measurements performed in the ancilla will not give back orthogonal projections of S and indeed we can write the map shown in eq. (1.12) with this new basis set as follows

$$U : (\alpha\,|0\rangle + \beta\,|1\rangle)_S \otimes |0_A\rangle \longrightarrow \hat{M}_+(\alpha\,|0\rangle + \beta\,|1\rangle) \otimes |+\rangle + \hat{M}_-(\alpha\,|0\rangle + \beta\,|1\rangle) \otimes |-\rangle\,. \tag{1.13}$$

where

$$M_+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M_- = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This sort of measurements are called as *generalized measurements*. In fact, this result for a qubit, i.e., a 2 dimensional basis, can be generalized for a n-dimensional one considering the following map

$$U : |\xi_S\rangle \otimes |0_A\rangle \longrightarrow \sum_a M_a\,|\xi_S\rangle \otimes |a_A\rangle\,,$$

so the probability of obtaining the eigenvalue a corresponding to state $|a_A\rangle$ can be written as

$$\text{Prob}(a) = \langle\xi_S|\hat{M}_a^\dagger \hat{M}_a|\xi_S\rangle = \langle\xi_S|\hat{E}_a|\xi_S\rangle$$

where we have defined $\hat{E}_a = \hat{M}_a^\dagger \hat{M}_a$ which satisfies the following properties

- Hermicity: $\hat{E}_a = \hat{E}_a^\dagger$
- Positivity: $\langle\psi|\hat{E}_a|\psi\rangle \geq 0$
- Completeness: $\sum_a \hat{E}_a = \mathbb{1}$

These properties are demonstrated in appendix A and operators that satisfy them are called as *Positive Operator-Valued Measured* (POVM).

1.1.2.3 Evolution of the density matrices. Master equations

As we have commented above, typically our system is composed by two parts: the system itself, for instance a qubit circuit, and the environment. Therefore we can write

the total Hamiltonian of our system as

$$\mathcal{H} = \mathcal{H}_S + \mathcal{H}_E + \mathcal{H}_{SE} \tag{1.14}$$

where $\mathcal{H}_S$ is the Hamiltonian for the system, $\mathcal{H}_E$ for the environment (which is also called as reservoir) and $\mathcal{H}_{SE}$ the one for the interaction between both. In general, the reservoir is considered to be large enough and maintained in thermal equilibrium so, in the interaction with the system, changes in it can be neglected. However, at $t = 0$ we do not have correlations between both ensembles so the density matrix for the whole system may be written as

$$\chi(0) = \rho(0)R_0 \tag{1.15}$$

where ρ is the density matrix for S and R_0 is the one for the reservoir. In interaction representation we denote the operators with a tilde so the evolution for the density matrix is given by ([Car99])

$$i\hbar\frac{\mathrm{d}\tilde{\chi}(t)}{\mathrm{d}t} = [\tilde{\mathcal{H}}_{SE}(t), \tilde{\chi}(t)] \tag{1.16}$$

being $\tilde{\chi}(t)$ the density matrix that represents the whole composite system.

Integrating now the expression shown above with respect to time between 0 and t, we obtain

$$\tilde{\chi}(t) = \tilde{\chi}(0) + \frac{1}{i\hbar}\int_0^t [\tilde{\mathcal{H}}_{SE}(t'), \tilde{\chi}(t')]\mathrm{d}t'. \tag{1.17}$$

Then, substituting $\tilde{\chi}(t)$ inside the commutator that appears in eq. (1.16)

$$\frac{\mathrm{d}\tilde{\chi}(t)}{\mathrm{d}t} = \frac{1}{i\hbar}[\tilde{\mathcal{H}}_{SE}(t), \tilde{\chi}(0)] - \frac{1}{\hbar^2}\int_0^t \mathrm{d}t' \big[\tilde{\mathcal{H}}_{SE}(t), [\tilde{\mathcal{H}}_{SE}(t'), \tilde{\chi}(t')]\big]. \tag{1.18}$$

The resultant equation describes the time evolution of the whole system. Nevertheless, we are interested in what happens with our system, that is, in how ρ evolves. To obtain that, we perform the partial trace over the environment of our equation, i.e., $\rho = \mathrm{tr}_E\{\tilde{\chi}\}$

$$\frac{\mathrm{d}\rho(t)}{\mathrm{d}t} = \frac{1}{i\hbar}\mathrm{tr}_E\big\{[\tilde{\mathcal{H}}_{SE}(t), \tilde{\chi}(0)]\big\} - \frac{1}{\hbar^2}\int_0^t \mathrm{d}t'\mathrm{tr}_E\big\{\big[\tilde{\mathcal{H}}_{SE}(t), [\tilde{\mathcal{H}}_{SE}(t'), \tilde{\chi}(t')]\big]\big\}. \tag{1.19}$$

Let us start by analyzing the first term of the sum, taking into account eq. (1.15) evaluated at $t = 0$. So as to do that, we will introduce the fact than any interaction operator may be written as $\sum_{\alpha,\beta}\hat{A}_\alpha \otimes \hat{B}_\beta$ where $\hat{A}_\alpha$ acts for instance on our system and $\hat{B}_\beta$ on the environment. Then, we can represent $\mathcal{H}_{SE}$ as

$$\mathcal{H}_{SE} = \sum_{\alpha,\beta}\hat{A}^{(S)}_\alpha \otimes \hat{B}^{(E)}_\beta.$$

Therefore, we have

$$\begin{aligned}\mathrm{tr}_E\big\{[\tilde{\mathcal{H}}_{SE}(t), \tilde{\chi}(0)]\big\} &= \mathrm{tr}_E\big\{\tilde{\mathcal{H}}_{SE}(t)\tilde{\rho}_0\tilde{R}_0 - \tilde{\rho}_0\tilde{R}_0\tilde{\mathcal{H}}_{SE}(t)\big\} = \\ &= \mathrm{tr}_E\Big\{\Big(\sum_{\alpha,\beta}\tilde{A}^{(S)}_\alpha \otimes \tilde{B}^{(E)}_\beta\Big)\big(\tilde{\rho}_0 \otimes \tilde{R}_0\big)\Big\} - \tilde{\rho}_0\mathrm{tr}_E\big\{\tilde{R}_0\tilde{\mathcal{H}}_{SE}(t)\big\} =\end{aligned}$$

$$= \sum_{\alpha,\beta} (\tilde{A}_\alpha^{(S)} \tilde{\rho}_0) \mathrm{tr}_E\{\tilde{B}_\beta^{(E)} \tilde{R}_0\} - \tilde{\rho}_0 \mathrm{tr}_E\{\tilde{R}_0 \tilde{\mathcal{H}}_{SE}(t)\} =$$
$$= \sum_{\alpha.\beta} [\tilde{A}_\alpha^{(S)}, \tilde{\rho}_0] \mathrm{tr}_E\{\tilde{B}_\beta^{(E)} R_0\}$$

Indeed, the mean value we have obtained is equivalent to $\mathrm{tr}_E\{\mathcal{H}_{SE}\tilde{R}_0\}$ so taking it as reference, that is, to be zero, then eq. (1.19) might be rewritten like

$$\frac{\mathrm{d}\rho(t)}{\mathrm{d}t} = -\frac{1}{\hbar^2}\int_0^t \mathrm{d}t' \mathrm{tr}_E\{[\tilde{\mathcal{H}}_{SE}(t), [\tilde{\mathcal{H}}_{SE}(t'), \tilde{\chi}(t')]]\}. \tag{1.20}$$

We have assumed that the coupling between both systems is very weak which means that the reservoir is a large system so its state should be virtually unaffected by the coupling to S. Thereby $\chi(t)$ should only show deviations of order $\mathcal{H}_{SE}$ so at time t we can write

$$\tilde{\chi}(t) = \tilde{\rho}(t) R_0 + \mathcal{O}(\mathcal{H}_{SE}). \tag{1.21}$$

Under this assumptions, we will make what is called as *Born approximation* neglecting terms higher than second order in $\mathcal{H}_{SE}$ in eq. (1.20). Therefore, we obtain

$$\frac{\mathrm{d}\rho(t)}{\mathrm{d}t} = -\frac{1}{\hbar^2}\int_0^t \mathrm{d}t' \mathrm{tr}_E\{[\tilde{\mathcal{H}}_{SE}(t), [\tilde{\mathcal{H}}_{SE}(t'), \tilde{\rho}(t') R_0]]\}. \tag{1.22}$$

A second approximation is to consider that the evolution is Markovian, i.e., the future behaviour of our system depends only on its present state. In eq. (1.22) we see that the evolution depends on all the past history of the system as we are integrating from $t = 0$ to actual values of that variable. Hence, the *Markov approximation* replaces $\rho(t')$ by $\rho(t)$ so for our configuration only matters what is happening in the present. Finally we obtain a master equation in the Born-Markov approximation

$$\frac{\mathrm{d}\rho(t)}{\mathrm{d}t} = -\frac{1}{\hbar^2}\int_0^t \mathrm{d}t' \mathrm{tr}_E\{[\tilde{\mathcal{H}}_{SE}(t), [\tilde{\mathcal{H}}_{SE}(t'), \tilde{\rho}(t) R_0]]\}. \tag{1.23}$$

1.2 Superconductivity

The first registered observation related to the phenomenon of superconductivity was done by Heike Kamerlingh Onnes (1853-1926) in 1911 when he studied the resistance of mercury at very low temperatures. He saw that, below a certain temperature that was measured to be 4.2 K in a range of 0.01 K, the resistance dropped down in a very sharply way achieving non-measurable values. Since then, superconductivity has been found in 25 elements of the periodic table and all of them verifies the following properties, that were stated by N.W. Ashcroft and N.D. Mermin in [AM76] and which are reproduced in the following points:

- A superconductor behaves as if it had no measurable DC electrical resistivity.

- A superconductor can behave as a perfect diamagnetic, that is, when we apply a magnetic field below a certain critical value, then inside the material the magnetic field is zero. In fact, this field decays exponentially when it gets into the superconductor.
- A superconductor behaves as if there were a gap energy centered around Fermi energy.

1.2.1 BCS theory and Cooper pairs

The purpose of this section is to introduce some basic concepts relative to BCS theory and Cooper pairs that will be used in later developments.

Since 1911, a lot of theories arose trying to explain this response of the material but none of them gave a complete characterization. In 1957, Bardeen, Cooper and Schrieffer proposed the known as BCS theory ([BCS57]) which describes this phenomenon as a microscopic effect caused by so-called Cooper pairs. Basically, a Cooper pair is constituted by two electrons whose energy is really close to Fermi energy, which interacts attractively one with the other by the exchange of a phonon. Then, we can say that this new configuration for the electrons conforms the load carriers of superconductors.

One of the most important results that are obtained employing this theory is

$$\vec{J}(\vec{r}) = \frac{\rho q}{m}\left(\hbar\nabla\theta - \frac{q}{c}\vec{A}\right) \tag{1.24}$$

where q and m represents the charge and the mass of a Cooper pair, $\vec{A}$ is the vector potential correspondent to an external field and θ is the phase of the wavefunction that characterize the density of Cooper pairs in the superconductor. Instead of focusing on the derivation of this expression, something that can be seen in [GP00], we are going to perform now the rotational of this relationship taking into account the property $\nabla \times \left(\nabla A\right) = 0$

$$\nabla \times \vec{J}(\vec{r}) = -\frac{\rho q}{m}\left(\nabla \times \vec{A}\right) \tag{1.25}$$

which is commonly known as London equation.

Chapter 2

Qubit system. Charge qubit

"In this chapter, we are going to do a review of the lecture notes done by Juan José García Ripoll and Borja Peropadre about superconducting quantum circuits ([GRP14]). We will start introducing which are the basic features that a qubit system must have according to David DiVincenzo requirements ([DiV95]). Next, we define the phase and number operators which will be especially useful in order to incorporate the charge and flux operators that appear in the Hamiltonian of our qubit system: the charge qubit. Lately we will do some approximations to our system obtaining then a two-level ensemble and finally, we will see some operations that can be done with it and consequences of the environment acting on it."

.....

"En este capítulo, se llevará a cabo un análisis de las notas realizadas por Juan José García Ripoll y Borja Peropadre relativas a circuitos cuánticos superconductores ([GRP14]). Comenzaremos introduciendo cuáles son las características básicas que debe tener un sistema de qubits de acuerdo con los requerimientos propuestos por David DiVincenzo ([DiV95]). Seguidamente, definiremos los operadores *número* y *fase* que serán de especial utilidad para incorporar los operadores de *carga* y *flujo* los cuales aparecerán en el Hamiltoniano de nuestro sistema de qubits: el *charge qubit*. Posteriormente se llevarán a cabo algunas aproximaciones a dicho sistema obteniendo así un conjunto de dos niveles y finalmente, veremos algunas operaciones que se pueden llevar a cabo con éste y las consecuencias que tiene el entorno circundante sobre ellos."

2.1 What do we require to a qubit system

As we have seen on section 1.1, a qubit is the basic unit of quantum information and, therefore, a quantum computer is a device which is capable of implementing arbitrary transformations on a set of them as commented by Juan José García Ripoll and Borja Peropadre in [GRP14]. As we are going to study a specific system used in quantum computation, it seems reasonable to introduce which are the requirements that this kind of apparatus must satisfy ([DiV95]).

- **Perfectly distinguishable quantum bits.** Ideally, this implies a physical system with two states. Usually, as in our example, we use the two lowest energy eigenstates of a physical system, ensuring that transitions to other states are suppressed (typically the energy difference between those states is large enough).
- **A procedure to set the qubits to a given zero state.** In our case this can be done by ordinary cooling as we will see later.
- **An apparatus that measures the state of the qubits in some basis.** Sometimes this is not a real apparatus but some mechanism as the action of a qubit on some phasing laser beam that identifies uniquely the state of the qubit.
- **Arbitrary local operations on each qubit.** That is, applying unitary transformations that let us modify the state of the qubit without altering its properties.
- **Quantum gates.** At least one universal quantum gate on each pair of qubits. An example of this kind of gates is the CNOT gate.
- **Error corrections.** Sufficiently small decoherence or errors during the previous operations so that error correction may be implemented.

In this chapter we focus mainly in the first, second and fourth points commented above and in the following chapter we will introduce the CNOT gate and a way to measure the state of a qubit, which covers the fifth and the third requirement respectively. Finally, the last point is not going to be analyzed deeply but an error correction will be introduced in section 2.4 when we talk about dephasing.

2.2 Phase and number operators

In the following sections we will introduce the Hamiltonian for our qubit system and to do that we will use the known as phase and number operator $\hat{\varphi}$ and $\hat{n}$. In terms of this last one, the charge operator is given by $\hat{q} = -2e\hat{n}$. They are defined from the relations

$$\hat{n}\,|m\rangle = m\,|m\rangle\,,\ m \in \mathbb{Z}$$
$$\hat{\phi}\,|\varphi\rangle = \varphi\,|\varphi\rangle\,,\ \varphi \in [0, 2\pi).$$

The closure relations for both basis are

$$\sum_{m\in\mathbb{Z}} |m\rangle\langle m| = \mathbb{1},$$
$$\int_0^{2\pi} \mathrm{d}\varphi\,|\varphi\rangle\langle\varphi| = \mathbb{1}.$$

Additionally, in the phase representation, states $|m\rangle$ are given by

$$\psi_m(\varphi) = \langle\varphi|m\rangle = \frac{1}{\sqrt{2\pi}} e^{-im\varphi}.$$

Later, we will be interested in how the operator $\cos\left(\hat{\phi}\right)$ acts over a state $|m\rangle$ so in order to obtain the correspondent expression, we start by obtaining the action of $e^{-in\hat{\phi}}$ applied to $|m\rangle$

$$e^{-in\hat{\phi}}|m\rangle = \int_0^{2\pi} \mathrm{d}\theta e^{-in\hat{\phi}}|\theta\rangle\langle\theta|m\rangle = \int_0^{2\pi} \mathrm{d}\theta \frac{1}{\sqrt{2\pi}} e^{-i(n+m)\theta}|\theta\rangle = \int_0^{2\pi} \mathrm{d}\theta\,|\theta\rangle\langle\theta|\,|n+m\rangle$$
$$= |n+m\rangle\,.$$

Analogously we have $e^{in\hat{\varphi}}|m\rangle = |m-n\rangle$ and therefore, adding up both operators and dividing that sum by 2, we obtain

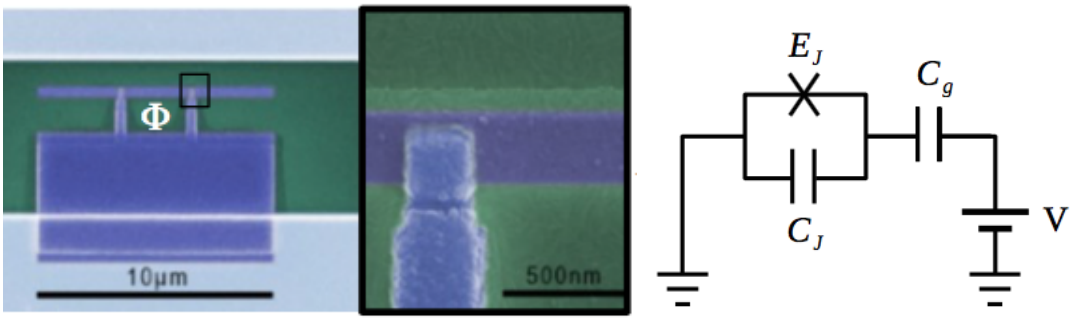
$$\cos\left(\hat{\phi}\right) = \frac{e^{i\hat{\varphi}} + e^{-i\hat{\varphi}}}{2} = \frac{1}{2}\sum_m |m+1\rangle\langle m| + |m\rangle\langle m+1|\,, \tag{2.1}$$

where we have taken into account that using the basis set $\{|m\rangle\}$ we can write the operator $e^{-in\hat{\phi}}$ as

$$e^{-in\hat{\phi}} = \sum_m |m+n\rangle\langle m|\,.$$

2.3 Charge qubit

A charge qubit consists on a simple superconducting island that is connected to a charge reservoir through a tunneling element such as a Josephson junction ([GRP14]). This last element consists of two superconductors coupled by a weak link which can be formed by a short-section of non superconducting-metal according to the definition given by [Con18] (indeed, a representation of this element is shown on figure 2.6 (b)). A picture of a charge qubit is shown in figure 2.1 as well as it equivalent circuit.



(a) Superconducting island coupled to the ground state through two Josephson junctions used in [LFB+07].

(b) Equivalent circuit for the Josephson junction.

FIGURE 2.1: *Josephson junction. Images extracted from [GRP14].*

2.3.1 Quantization of the charge qubit

One of the conditions of DiVincenzo's list mentioned in section 2.1 is to have a physical system with two states. Then, in order to justify the use of charge qubits in quantum

computation, we should quantize this system and therefore see which are the energy levels.

In our analysis we will set the magnetic flux through the loop which contains the junction and the capacitor to zero as this consideration does not affect significantly our final results. Then, after distinguishing the nodes in our circuit we select a direction in which we will pass through each node (figure 2.2) and hence define a flux associated to each circuit element. For the case of the junction and the capacitor C_J we have $\delta_a = \phi_a - \phi_0$ and for C_g we have consequently $\delta_b = \phi_b - \phi_a$.

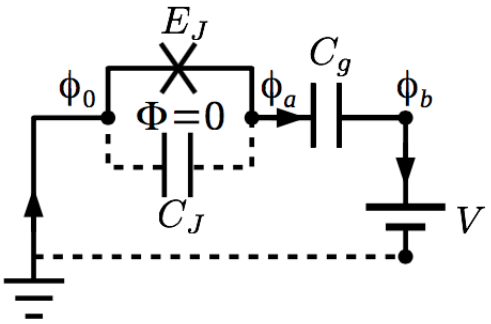


FIGURE 2.2: *Equivalent circuit where the different nodes has been identified. Here E_J represents the Josephson Junction that allow Cooper pairs to get in and out the island being C_J its capacity and, finally, C_g is a capacitor that describes the interaction with the voltage source V. Image extracted from [GRP14] and slightly modified.*

From the charge conservation the following expression must be satisfied

$$I_a = I_b$$

where I_a is the current associated to δ_a and I_b the correspondent to δ_b. We can relate these currents with the fluxes on each branch by the relations for each circuital element

- Capacitors: $I = C\delta\ddot{\phi}$,
- Junctions: $I = -L_J\varphi_0 \sin(\delta\phi/\varphi_0)$, where $E_J = L_J\varphi_0^2$.

Hence, the conservation of charge may be rewritten as it follows

$$C_J\delta\ddot{\phi}_a - \frac{E_J}{\varphi_0}\sin(\delta\phi_a/\varphi_0) = C_g\delta\ddot{\phi}_b. \tag{2.2}$$

From figure 2.2 we can extract some constrains that affect our circuit. In one hand, δ_0 corresponds to the ground plane flux so we can set it to zero. On the other hand, the second restriction comes from the voltage V which imposes that $\dot{\phi}_b = V$. As consequence we obtain

$$C_\Sigma\left(\ddot{\phi}_a - \frac{C_g}{C_\Sigma}\dot{V}\right) = \frac{E_J}{\varphi_0}\sin(\phi_a/\varphi_0) \tag{2.3}$$

where we have defined $C_\Sigma = C_g + C_J$.

The differential equations associated to 2.3 can be derived from an effective Lagrangian $\mathcal{L}$ applying the Euler-Lagrange equations.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}_i}\right) = \frac{\partial \mathcal{L}}{\partial \phi_i}. \tag{2.4}$$

We have written eq. (2.3) in that way on purpose because we can identify the first term of that expression with the one in eq. (2.4) happening the same for the second part of the equality. The main reason for doing this underlies on the fact that we only have zero or second order time derivative terms.

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\phi}_a} = C_\Sigma\left(\ddot{\phi}_a - \frac{C_g}{C_\Sigma}V\right) \Rightarrow \mathcal{L} = C_\Sigma\left(\frac{\dot{\phi}_a^2}{2} - \frac{C_g}{C_\Sigma}V\dot{\phi}_a\right) + \mathcal{O}(\phi_a),$$
$$\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}\phi_a} = \frac{E_J}{\phi_0}\sin(\phi_a/\varphi_0) \Rightarrow \frac{\mathrm{d}\mathcal{O}(\phi_a)}{\mathrm{d}\phi_a} = \frac{E_J}{\phi_0}\sin(\phi_a/\varphi_0) \Rightarrow \mathcal{O}(\phi_a) = -E_J\cos(\phi_a/\varphi_0) + \text{ cte.}$$

The Lagrangian corresponding to these equations, up to a constant term, will be

$$\mathcal{L} = C_\Sigma\left(\frac{\dot{\phi}_a^2}{2} - \frac{C_g}{C_\Sigma}V\dot{\phi}_a\right) + E_J\cos(\phi_a/\varphi_0). \tag{2.5}$$

From here we can obtain the charge variable as the canonically conjugate momenta for the flux ϕ_a

$$q_a\frac{\partial \mathcal{L}}{\partial \dot{\phi}_a} = C_\Sigma\left(\dot{\phi}_a - \frac{C_g}{C_\Sigma V}\right).$$

Performing now the Legendre transformation we can obtain thereby the Hamiltonian for our system

$$\mathcal{H} = q_a\dot{\phi}_a - \mathcal{L} = \frac{1}{2C_\Sigma}(q_a - C_gV)^2 - E_J\cos(\phi/\varphi_0),$$

Defining the quantity $q_g = C_gV$ we finally conclude that the Hamiltonian that characterizes this system is

$$\mathcal{H} = \frac{1}{2C_\Sigma}(q - q_g)^2 - E_J\cos(\phi/\varphi_0), \tag{2.6}$$

where the quantization follows immediately introducing the flux and charge operators in the place of q and ϕ variables.

2.3.2 Approximations. Obtaining a two level system

The Hamiltonian given in eq. (2.6) can be substituted by another one which is much more easier to analyze but, in order to justify this change, let us study each term of eq. (2.6) separately. First of all, for this purpose, we will assume that $E_J = 0$ so it only remains the first contribution. As the charge operator is defined as follows

$$\hat{q}\,|n\rangle = 2en\,|n\rangle\,, \quad n \in \mathbb{Z}$$

where $\{|n\rangle\}$ are defined in terms of the excess or defect of Cooper pairs and $2e$ is the charge of a pair, then its eigenvalues can be written as

$$E_n = 4E_C(n - n_g)^2 \tag{2.7}$$

with $E_C = e^2/2C_\Sigma$ and $n_g = q_g/2e$. Notice that this last term written goes linearly with the external potential V (as $q_g = C_g V$) thus it can be controlled. Indeed, this value will play a fundamental role controlling the fluctuations of the electromagnetic field and also defining a qubit in our system, as we will see below. Plotting in terms of n_g eq. (2.7) we get figure 2.3 where we can see that the different values that n_g takes are almost sufficient to define a qubit. Specifically, at $n_g = 0.5$ we get two states that are degenerated ($n = 0, 1$) and the other one ($n = 2$) is well separated from those. Hence, we have found a system that, under the conditions commented before ($n_g = 0.5$), ensures us having two levels isolated from the rest and that allows us to induce transitions among them. However, we have commented in section 2.1 that we must have a zero state in order to have transitions with another level, namely state one, but in this case both of them have exactly the same energy. It is necessary, therefore, to find an element that can make a split of the degeneracy obtained before. That role is played by the Josephson Junction, which is represented by the second term of the Hamiltonian (eq. 2.6). The effect of introducing this element into our circuit is shown on figure 2.4 and, as we can see there, it achieves the effect we are looking for: splitting the energy of states $|0\rangle$ and $|1\rangle$. In this manner we have found a system that, under certain conditions, let us isolate two levels from the rest.

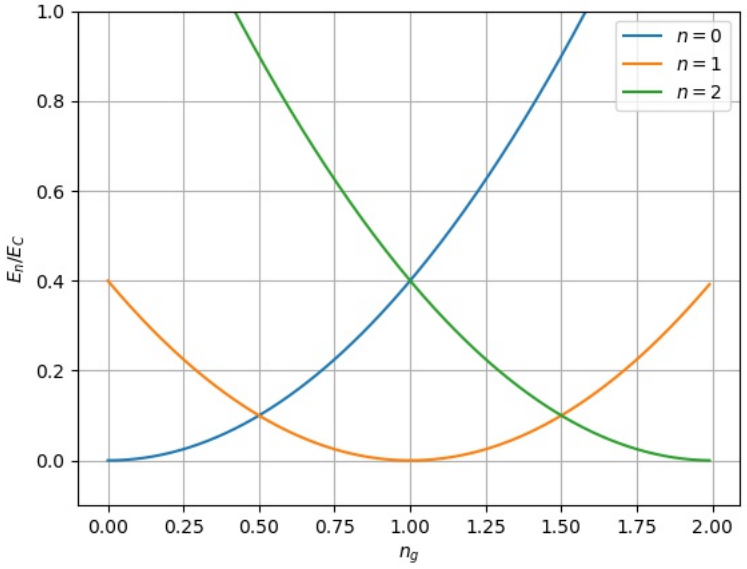


FIGURE 2.3: *First two energy levels of the charge qubit without Josephson Junction ($E_J = 0$) and with tunnelling amplitude $E_J = 0.1E_C$.*

Reached this point, we will try to rewrite our Hamiltonian so we can only work with this two levels. For so, we will take into account relations (2.1) and (2.7) thus we can rewrite eq. (2.6), in the number basis, as

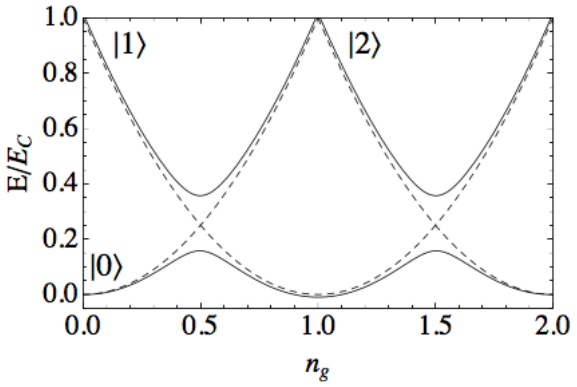


FIGURE 2.4: *Energy levels of the charge qubit with Josephson junction and without it (dashed line). Image extracted from [GRP14].*

$$\mathcal{H} = \sum_n 4E_C(n - n_g)^2 \left|n\right\rangle \left\langle n\right| + \frac{E_J}{2} \sum_n \left|n+1\right\rangle \left\langle n\right| + \left|n\right\rangle \left\langle n+1\right|. \tag{2.8}$$

Restricting ourselves to the lowest levels from figure 2.4 we can truncate the summatories of eq. (2.8) so it only appear states with $n = 0, 1$. This can be done because we are not going to consider transitions to level $|2\rangle$ (note that this state does not correspond to $n = 2$ but for a lineal combination where this number state will appear). Under this assumptions, the effective Hamiltonian that describes our two-level system is, doing a change of our basis set,

$$\mathcal{H}_{\text{eff}} = \frac{\Delta}{2}\sigma_z + \frac{\epsilon}{2}\sigma_x \tag{2.9}$$

where $\sigma_i,\ i = x, y, z$ are the Pauli matrices and parameters Δ and ϵ are defined by

$$\epsilon = 8E_c\left(n_g - \frac{1}{2}\right) \quad ; \quad \Delta = E_J(\phi).$$

The matricial representation of Hamiltonian (2.9) is then

$$\mathcal{H}_{\text{eff}} = \frac{1}{2}\begin{pmatrix} \Delta & \epsilon_0 \\ \epsilon_0 & \Delta \end{pmatrix}.$$

Diagonalizing this matrix we obtain the following eigenvalues

$$E_1 = \frac{1}{2}\sqrt{\epsilon^2 + \Delta^2} \quad ; \quad E_0 = -\frac{1}{2}\sqrt{\epsilon^2 + \Delta^2}. \tag{2.10}$$

In figure 2.5 those energies are plotted where we see that at $\epsilon = 0$ the gap difference is minimal and this is what we call *degeneracy point* because $n_g = 0.5$ and for that point we saw in fig. 2.1 that we had degeneracy of the levels considered. This one is a really important value because fluctuations of the electromagnetic field that affect our system enter at second order as we see by considering the Taylor expansion of the energy gap $\Delta E = E_1 - E_0$

$$\Delta E(\epsilon_{\text{noise}}) = \Delta E(\epsilon = 0) + \frac{1}{2\Delta}\epsilon_{\text{noise}}^2 + \mathcal{O}(\epsilon_{\text{noise}}^4).$$

As it is stated by Juan José García Ripoll and Borja Peropadre in [GRP14], this realization allowed experiments with charge qubits to improve significantly their operation time keeping the qubit as close to the degeneracy point as possible while doing nothing with it.

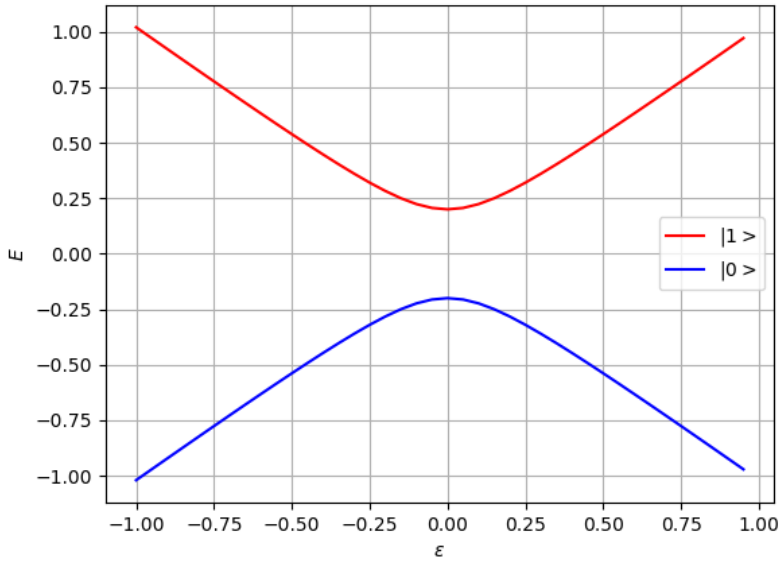
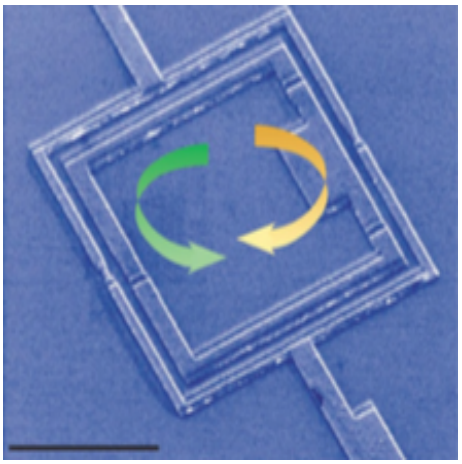


FIGURE 2.5: *Representation of the eigenvalues for the effective Hamiltonian 2.9 with* $\Delta = 0.2$.

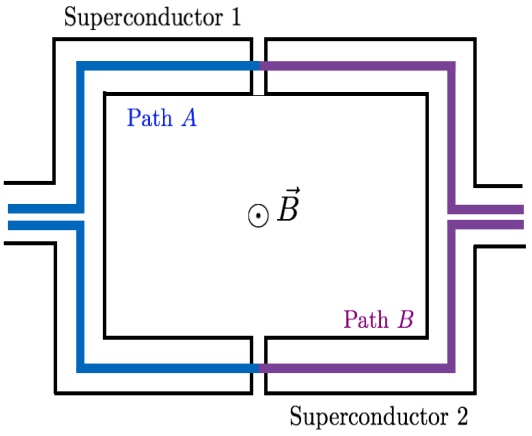
2.3.3 Another example of a qubit system. Flux qubit

As we have commented, we are only going to employ in our realization a unique qubit system which is the charge qubit. However, there are a lot of qubit systems realizations and, particularly, we will talk a little bit about the flux qubit. The main difference between both devices is that the former, as we have seen, depends on the number of Cooper pairs while the later depends on the flux. In this second case states $|0\rangle$ and $|1\rangle$ corresponds to two superconducting currents rotating without dissipation around a certain loop. A typical configuration used as a flux qubit contains three junctions coupled one to the other as it is shown on figure 2.6 (a) in which currents are tuned by using a magnetic field. According to what we have commented in section 1.2 external magnetic fields does not reach the inside part of a superconductor, considering that its width is big enough, so it may seem odd that this magnetic field can induce effects through the internal current.

The main reason why this behavior is obtained has its explanation in the presence of a superconductor junction in our system. However, to analyze this more deeply we are not going to consider the case of a three junction as shown in figure 2.6 (a) but a two junction one as it can be seen in image 2.6 (b). First of all we will start by assuming that if we have a current inside our system then, it should verify that $I = I_A + I_B$ where I_A is



(a) Three junction flux qubit. Image extracted from [BLP+07].



(b) Junction between two superconductors.

FIGURE 2.6: *Flux qubit.*

the supercurrent through path A and I_B the corresponding to path B. According to the expression we write for the current through a junction, then we have that $I_A = I_0 \sin(\delta_A)$ and $I_B = I_0 \sin(\delta_B)$ where we included all the constants in I_0 and δ_A and δ_B represents the phase difference between going through one or another route.

If we do not apply an external flux through the superconductor, we will expect that $\delta_A = \delta_B = \delta_0$ as there is no another circuital element that may generate a deviation for those values. Suppose now that we switch on that magnetic field so, from what we have said about superconductors, the field inside them is zero. As we are not applying any kind of electric field, then from Ampere's law we have that

$$\nabla \times \vec{B} = \mu_o \vec{J} \xrightarrow{\vec{B}=\vec{0}} \vec{J} = \vec{0}$$

which means that inside the superconductor the density current is null and therefore, from eq. (1.24) we get

$$\nabla\theta = \frac{q}{c\hbar}\vec{A}.$$

Integrating this last expression over the two different paths defined in figure 2.6 (b) we get for each of them

$$\delta_A = \frac{q}{c\hbar}\int_A \vec{A}\cdot \mathrm{d}\vec{l}_A, \quad \delta_B = \frac{q}{c\hbar}\int_B \vec{A}\cdot \mathrm{d}\vec{l}_B$$

where $\mathrm{d}\vec{l}_A$ and $\mathrm{d}\vec{l}_B$ are both differential elements for the path and which are defined in such a way that they go from the lower junction to the upper one. It is suitable to combine these two integrals so we can get the flux through our superconductor and hence, we will take into account that $\mathrm{d}\vec{l}_B = -\mathrm{d}\vec{l}_A = \mathrm{d}\vec{l}$ so we can write

$$\delta_B - \delta_A = \frac{q}{c\hbar}\oint \vec{A}\cdot \mathrm{d}\vec{l} = \frac{q}{c\hbar}\int_S \nabla\times\vec{A}\cdot \mathrm{d}\vec{S} = \frac{q}{c\hbar}\int_S \vec{B}\cdot \mathrm{d}\vec{S} = \frac{q\Phi}{c\hbar}$$

where S represents the surface of our junction and ϕ is the flux through that surface. In short, we have found a relationship between the phase difference and the magnetic flux through our superconductor.

As we have mentioned before, if we do not apply any magnetic field we will not expect a difference between paths. Thereby, according to this, when we have the magnetic flux turned on we can write each phase as

$$\begin{cases} \delta_A = \delta_0 - q\Phi/2c\hbar \\ \delta_B = \delta_0 + q\Phi/2c\hbar. \end{cases}$$

So we conclude that the total current $I = I_A + I_B$ through our superconductor will be given by [GP00]

$$I = I_0\Big[\sin(\delta_A) + \sin(\delta_B)\Big] = I_0\Big[\sin(\delta_0 - q\Phi/2c\hbar) + \sin(\delta_0 + q\Phi/2c\hbar)\Big] = \\ = 2I_0 \sin(\delta_0)\cos(q\Phi/2c\hbar).$$

To put in briefly, we see that there is a way in which we can control the current inside our superconductor employing the flux which goes through our two junction system. Therefore, this magnitude is the one which plays the role of the voltage source we have discussed in the charge qubit.

2.4 Operations with qubits

When one analyze different kind of qubits systems, as it is done in [GRP14], it can be seen that, under the appropriate constrains that lead us to a two level system, all the Hamiltonians that represents each circuit reduces to the one given in eq. (2.9) where the parameters that appear there depends on each configuration. So, reached this point we will see some operations that can be done with these systems.

2.4.1 Single-qubit operations

Let us consider that we are at the degeneracy point, i.e., $\epsilon = 0$ so time-evolution is only governed by σ_z. From now on we will work with the eigenstates of σ_z that we will call $\{|0\rangle, |1\rangle\}$. Under this representation we are able to write operators σ_z and σ_x as

$$\sigma_z = |1\rangle\langle 1| - |0\rangle\langle 0| \quad , \quad \sigma_x = |1\rangle\langle 0| + |0\rangle\langle 1|.$$

The time-evolution of any state of the system is governed by the following operator

$$U(t; \Delta, \epsilon = 0) = e^{-it\Delta/2\hbar}|0\rangle\langle 0| + e^{it\Delta/2\hbar}|1\rangle\langle 1|. \tag{2.11}$$

Assume now that we have prepared our system so the density matrix which characterizes it is given at time $t = 0$ by

$$\rho(0) = \begin{pmatrix} \rho_{11} & 0 \\ 0 & \rho_{00} \end{pmatrix}.$$

The correspondent evolution of our density matrix is such that $\rho(t) = \rho(0), \forall t$. Let us consider now that we apply a very intense constant voltage $\epsilon = \epsilon_0$ so that $\epsilon_0 \gg \Delta$. Under this assumption the gap can be neglected and the operator which governs the time-evolution is

$$U(t; 0, \epsilon_0) = e^{-i\epsilon_0\sigma_x t/2\hbar} = \begin{pmatrix} \cos(\epsilon_0 t/2\hbar) & -i\sin(\epsilon_0 t/2\hbar) \\ -i\sin(\epsilon_0 t/2\hbar) & \cos(\epsilon_0 t/2\hbar) \end{pmatrix}. \tag{2.12}$$

If we now evolve our initial ensemble with this time operator, at time $t = \pi\hbar/\epsilon_0$ we have

$$\rho(t) = U(\pi\hbar/\epsilon_0; 0, \epsilon_0)\rho(0)U^\dagger(\pi\hbar/\epsilon_0; 0, \epsilon_0) = \begin{pmatrix} \rho_{00} & 0 \\ 0 & \rho_{11} \end{pmatrix}.$$

Notice that we have found a method which allows us to exchange population between the $|0\rangle$ and $|1\rangle$ states coherently by performing a rotation of the state of the qubit. Nevertheless, there is a disadvantage about this technique which is that we have to apply intense voltages. In general we do not need to use this kind of values to achieve that effects as there is another procedure which consists on applying a time-dependent field of the form

$$\epsilon(t) = \Omega\cos(\omega_0 t + \varphi) = \frac{\Omega}{2}\big(e^{i\omega_0 t + i\varphi} + e^{-i\omega_0 t - i\varphi}\big).$$

To analyze this procedure we rotate our Hamiltonian on a frame that moves at frequency ω_0 with a unitary operator $U = e^{-i\omega_0 t\sigma_z/2}$. The original state can be written as $|\psi(t)\rangle = U(t)\,|\chi(t)\rangle$ where $|\chi\rangle$ evolves with an effective Hamiltonian of the form

$$\mathcal{H}'_{\text{eff}} = U^\dagger\mathcal{H}_{\text{eff}}U - i\hbar U^\dagger\dot{U}.$$

Let us analyze each term of the last equation separately identifying $\mathcal{H}^{(1)}_{\text{eff}}$ as $(\Delta/2)\sigma_z$ and $\mathcal{H}^{(2)}_{\text{eff}}$ as $(\epsilon_0/2)\sigma_x$

$$\begin{aligned} U^\dagger\mathcal{H}^{(1)}_{\text{eff}}U &= e^{i\omega_0 t\sigma_z/2}\frac{\Delta}{2}\big[\,|1\rangle\langle 1| - |0\rangle\langle 0|\,\big]e^{-i\omega_0 t\sigma_z/2} = \\ &= \frac{\Delta}{2}\big[e^{i\omega_0 t/2}\,|1\rangle\langle 1|\,e^{-i\omega_0 t/2} - e^{-i\omega_0 t/2}\,|0\rangle\langle 0|\,e^{i\omega_0 t/2}\big] = \\ &= \frac{\Delta}{2}\sigma_z. \end{aligned}$$

For the other term we have

$$\begin{aligned} U^\dagger\mathcal{H}^{(2)}_{\text{eff}}U &= \frac{\hbar\epsilon(t)}{2}e^{i\omega_0 t\sigma_z/2}\big[\,|1\rangle\langle 0| + |0\rangle\langle 1|\,\big]e^{-i\omega_0 t\sigma_z/2} = \\ &= \frac{\hbar\epsilon(t)}{2}\big[e^{i\omega_0 t/2}\,|1\rangle\langle 0|\,e^{i\omega_0 t/2} + e^{-i\omega_0 t/2}\,|0\rangle\langle 1|\,e^{-i\omega_0 t/2}\big] = \end{aligned}$$

$$
\begin{aligned}
&= \frac{\hbar\epsilon(t)}{2}\left[e^{i\omega_0 t}\left|1\right\rangle\left\langle 0\right| + e^{-i\omega_0 t}\left|0\right\rangle\left\langle 1\right|\right] = \\
&= \frac{\hbar\Omega}{4}\left[e^{i\omega_0 t + i\varphi} + e^{-i\omega_0 t - i\varphi}\right]\left[e^{i\omega_0 t}\left|1\right\rangle\left\langle 0\right| + e^{-i\omega_0 t}\left|0\right\rangle\left\langle 1\right|\right] = \\
&= \frac{\hbar\Omega}{4}\left[e^{i2\omega_0 t + i\varphi}\left|1\right\rangle\left\langle 0\right| + e^{i\varphi}\left|0\right\rangle\left\langle 1\right| + e^{-i\varphi}\left|1\right\rangle\left\langle 0\right| + e^{-i2\omega_0 t - i\varphi}\left|0\right\rangle\left\langle 1\right|\right) = \\
&= \frac{\hbar\Omega}{4}\left[e^{i(2\omega_0 t+\varphi)}\sigma^- + e^{i\varphi}\sigma^+ + \text{h.c.}\right].
\end{aligned}
$$

where "h.c." stands for the Hermitian conjugate. Finally, the last term give us

$$
U^{-1}\dot{U} = -\sigma_z\frac{i\omega_0}{2}e^{-i\omega_0 t\sigma_z/2}e^{i\omega_0 t\sigma_z/2} = -\sigma_z\frac{i\omega_0}{2}
$$

so the transformed effective Hamiltonian is given by

$$
\mathcal{H}'_{\text{eff}} = \frac{\Delta - \hbar\omega_0}{2}\sigma_z + \frac{\Omega}{4}\left[e^{i(2\omega_0 t+\varphi)}\sigma^- + e^{i\varphi}\sigma^+ + \text{h.c.}\right].
$$

By considering that ω_0 is big enough so that we can neglect the terms $2\omega_0$ because they rotate really fast and tuning ω_0 so it is resonant with the two level transition ($\omega_0 = \Delta/\hbar$) then we obtain for our Hamiltonian

$$
\mathcal{H}'_{\text{eff}} = \frac{\Omega}{4}\left[e^{i\varphi}\sigma^- + e^{i\varphi}\sigma^+ + \text{h.c.}\right] = \frac{\Omega}{2}\cos(\varphi)\sigma_x.
$$

Therefore, the time evolution operator for the state $|\psi\rangle$ is

$$
\begin{aligned}
\tilde{U} = U(t;\omega_0)e^{-iH'_{\text{eff}}t/\hbar}U^\dagger(t;\omega_0) &= \begin{pmatrix} \cos\left(\frac{\Omega\Delta t}{2}\cos\varphi\right) & -ie^{i\omega t}\sin\left(\frac{\Omega\Delta t}{2}\cos\varphi\right) \\ -ie^{i\omega t}\sin\left(\frac{\Omega\Delta t}{2}\cos\varphi\right) & \cos\left(\frac{\Omega\Delta t}{2}\cos\varphi\right) \end{pmatrix} \\
&\approx U(t;0,\Omega/2).
\end{aligned} \tag{2.13}
$$

Basically we rotate the propagator correspondent to $\mathcal{H}'_{\text{eff}}$ to the frame of reference of the laboratory, that is, that one which sees the Hamiltonian rotating and where the state $|\psi\rangle$ is defined.

2.4.2 Dephasing

As it is said in [GRP14], dephasing is a phenomenon caused by perturbations which affect the qubit Hamiltonian and that randomly changes the energies of the states $|1\rangle$ and $|0\rangle$, i.e., $\mathcal{H} \to \mathcal{H} + \delta_{\text{ext}}\sigma_z$. When these fluctuations accumulate in time the main effect that they produce on our density matrix is that its coherences decay exponentially in a typical time T_2 known as dephasing time.

In order to characterize these fluctuations we must take into account that this is a random process so the physical state of our system might be represented as a statistical average over different realizations of that noise. This is mathematically represented by an operation known as quantum channel or positive map ([GRP14])

$$
\varepsilon(\rho;t) = \int_{-\pi}^{\pi} e^{i\varphi\sigma_z}\rho e^{-i\varphi\sigma_z}p(\varphi;t)\mathrm{d}\varphi. \tag{2.14}
$$

Here we can see that the initial state ρ evolves according to a random phase φ whose distribution $p(\varphi;t)$ changes in time. On the other hand, under some conditions ([GRP14]), dephasing can be treated using the following master equation

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \gamma_2(\sigma_z\rho\sigma_z - \rho) \tag{2.15}$$

where $\gamma_2 = 1/T_2$. In matrix representation, eq. (2.15), we get a system of four differential equations that give us

$$\rho(t) = \begin{pmatrix} \rho_{11} & \rho_{10}e^{-t/T_2} \\ \rho_{01}e^{-t/T_2} & \rho_{00} \end{pmatrix}.$$

Nonetheless, dephasing is a type of noise for which exists methods to suppress it. One of them is called as spin-echo in which the state of the qubit is reversed at the middle of the experiment employing one of the operations shown on section 2.1, in particular

$$\rho(\delta t + \delta t) = e^{-i\varphi\delta t\sigma_z}\sigma_x e^{-i\varphi\delta t\sigma_z}\rho(0)e^{i\varphi\delta t\sigma_z}\sigma_x e^{i\varphi\delta t\sigma_z} = \rho(0).$$

Basically, what we are doing is firstly evolving our system according to the corresponding Hamiltonian (that now has the dephase term), then at the middle of the experiment we reverse it and finally let it evolve again with time. Even though these procedure is never perfect because another sources of decoherence, it allow us to extend the lifetime of our qubit state beyond the original one.

Specifically this operation does not give us exactly ρ_0 but a density matrix where the elements are interchanged. We can obtain ρ_0 by applying another rotation around the x axis but the reason why we are writing the expression in that way is because the information that this density matrix contains has not been perturbed, that is, the populations, which are now the coherences, has the same values that they had at the beginning of the experiment.

2.4.3 Relaxation and heating

The dephasing shown above introduces terms to the coherences of our density matrix so at large times they will disappear. On the other hand relaxation and heating are the worst kind of decoherences because, unlike the other case studied above, entail the lost of the qubit itself: the populations and their coherences are destroyed. For example, if we only have heating in our system, at the end of the day our density matrix would be a pure state $|1\rangle\langle 1|$ and if we only have cooling, then we would have a pure state $|0\rangle\langle 0|$. In general, this two processes can be treated uniformly through a single master equation ([GRP14])

$$\begin{aligned}\frac{\mathrm{d}\rho}{\mathrm{d}t} =& [n(\Delta)+1]\frac{\gamma}{2}(2\sigma^-\rho\sigma^+ - \sigma^+\sigma^-\rho - \rho\sigma^+\sigma^-) + \\ &+ n(\Delta)\frac{\gamma}{2}(2\sigma^+\rho\sigma^- - \sigma^-\sigma^+\rho - \rho\sigma^-\sigma^+),\end{aligned} \tag{2.16}$$

where the factor $n(\Delta)$ represents the boson occupation number. In general, this equation lead us to the following system of equations

$$\begin{cases} \dot{\rho}_{11} = \gamma[-(n(\Delta)+1)\rho_{11} + n(\Delta)\rho_{00}] \\ \dot{\rho}_{00} = \gamma[(n(\Delta)+1)\rho_{11} - n(\Delta)\rho_{00}] \\ \dot{\rho}_{10} = -\gamma\left(n+\frac{1}{2}\right)\rho_{10} \\ \dot{\rho}_{01} = -\gamma\left(n+\frac{1}{2}\right)\rho_{01} \end{cases} . \tag{2.17}$$

As the circuits we are analyzing are composed by superconducting elements, they work at very low temperatures so $n(\Delta) \approx 0$ constitutes a very good approximation. Under this conditions the system 2.17 lead us to the following density matrix

$$\rho(t) = \begin{pmatrix} \rho_{11}e^{-t/T_1} & \rho_{10}e^{-t/2T_1} \\ \rho_{01}e^{-t/2T_1} & \rho_{00} + (1-e^{-t/T_1})\rho_{11} \end{pmatrix}.$$

Notice that, as we would expect, when $t \to \infty$ the system cools down and hence all the population which is at the excited level pass to the zero level energy.

2.4.3.1 Thermal equilibrium

We said before that equation (2.17) give us the evolution of our system under processes of cooling and heating. Let us now see the case corresponding to thermal equilibrium, i.e., $\dot{\rho}_{ij} = 0$. According to that expression it follows immediately, considering for instance the first equation of the system, that

$$\frac{\rho_{11}}{\rho_{00}} = \frac{n(\Delta)+1}{n(\Delta)} = e^{-\beta\Delta}. \tag{2.18}$$

On the other hand, as we are studying a system at thermal equilibrium, then its density matrix is characterized by the canonical ensemble. Thus, we can write as

$$\rho = \frac{e^{-\beta\mathcal{H}}}{\mathrm{tr}\{e^{-\beta\mathcal{H}}\}}. \tag{2.19}$$

In our case, as we are at the degeneracy point so $\epsilon = 0$, the energies for levels $|1\rangle$ and $|0\rangle$ are respectively $\Delta/2$ and $-\Delta/2$. Applying this last expression to determine the populations the result obtained is the Boltzmann factor shown in eq. (2.18) as expected due to the conditions which we are are working with. Therefore, we see that master equation (2.16) is well suited to describe thermal equilibrium situations.

Chapter 3

Measurements in a charge qubit. CNOT gate

"In this chapter, we are going to employ the features we have seen for the charge qubit in order to design a CNOT gate that allows us to generate a coupling between two qubits so, preparing a certain state in one of them, we can obtain information from the other one. In this way, we see an example of how measurements can be done in a quantum system. For this purpose, we shall introduce what quantum gates are, and then, we will start by analyzing the so-called Pauli's gates. Once defined a CNOT gate, we will present a design of this kind of gate done by charge qubits and proceed to study how to prepare states and how to do measurements employing for this goal superconducting transmission lines."

. . . . . .

"En este capítulo, vamos a emplear las características vistas para un charge qubit con el fin de diseñar una puerta CNOT que nos permitirá generar un acoplamiento entre dos qubits de manera que, configurando uno de ellos, se pueda obtener información del otro. De esta manera, vemos un ejemplo de cómo realizar medidas sobre un sistema cuántico. Con este propósito, introduciremos lo que son las puertas cuánticas y analizaremos para ello las conocidas como puertas de Pauli. Una vez definida la puerta CNOT, presentaremos un diseño de ésta elaborado con charge qubits y se procederá a estudiar cómo preparar estados y cómo realizar medidas empleando para este fin líneas de transmisión superconductoras."

3.1 Quantum gates

In brief, a quantum gate is the equivalent, in quantum circuits, to the logic gates defined in classical circuits. However, there is a very important difference between classical and quantum gates: the latter ones are reversible unlikely most of the first ones. So basically, they allow us to perform unitary operations in a multiple qubit system that can be composed for instance by two charge qubits. In fact, we have seen before these kinds of operations in section 2.4 when we analyzed single qubit operations. By waiting

a suitable interval of time, expression (2.12) (or also eq. (2.13)) can give us the so-called Pauli-X gate (which is the equivalent to the classical NOT gate).

Those quantum gates we have mentioned execute operations on a single qubit, but we can also find other gates that can handle with multiple qubits as the CNOT gate which deals with two qubits.

3.1.1 Pauli's gates

The Pauli-X,Y,Z gates commented above, are quantum gates that operates only with single qubits and that can be characterized by the following matrices

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Hence, if these gates acts over a general configuration $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$ the result will be for each of them

$$X\,|\psi\rangle = \beta\,|0\rangle + \alpha\,|1\rangle\,, \quad Y\,|\psi\rangle = i\beta\,|0\rangle - i\alpha\,|1\rangle\,, \quad Z\,|\psi\rangle = \alpha\,|0\rangle - \beta\,|1\rangle\,.$$

In our case, the first and third operations can be done with our charge qubit as the Hamiltonian of our system contains the Pauli matrices σ_x and σ_z so we can properly omit one of those operators choosing accurately the voltage V. However this give us one limitation: we cannot alter the phase of our initial state applying only charge qubits as there is always an operator which we cannot introduce in our discussion, being in the case considered σ_y.

3.1.2 CNOT gate

Controlled-NOT gate, commonly known as CNOT gate, are a different kind of logic gates as it takes as input two qubits which we may call *target* and *control* qubits. The operation that this logic quantum gate performs is shown on table 3.1 where we see that if the target qubit is set to zero we do not do anything with it while if it is set to be one then the control qubit is flipped. In this chapter we are going to generate this kind

Input		**Outcome**	
Target	**Control**	**Target**	**Control**
$\|0\rangle$	$\|0\rangle$	$\|0\rangle$	$\|0\rangle$
$\|0\rangle$	$\|1\rangle$	$\|0\rangle$	$\|1\rangle$
$\|1\rangle$	$\|0\rangle$	$\|1\rangle$	$\|1\rangle$
$\|1\rangle$	$\|1\rangle$	$\|1\rangle$	$\|0\rangle$

TABLE 3.1: *Truth table for the CNOT gate.*

of quantum gates applying only charge qubits, superconducting transmission lines and classical computers. It is important to remark that this setup has to be understood as a

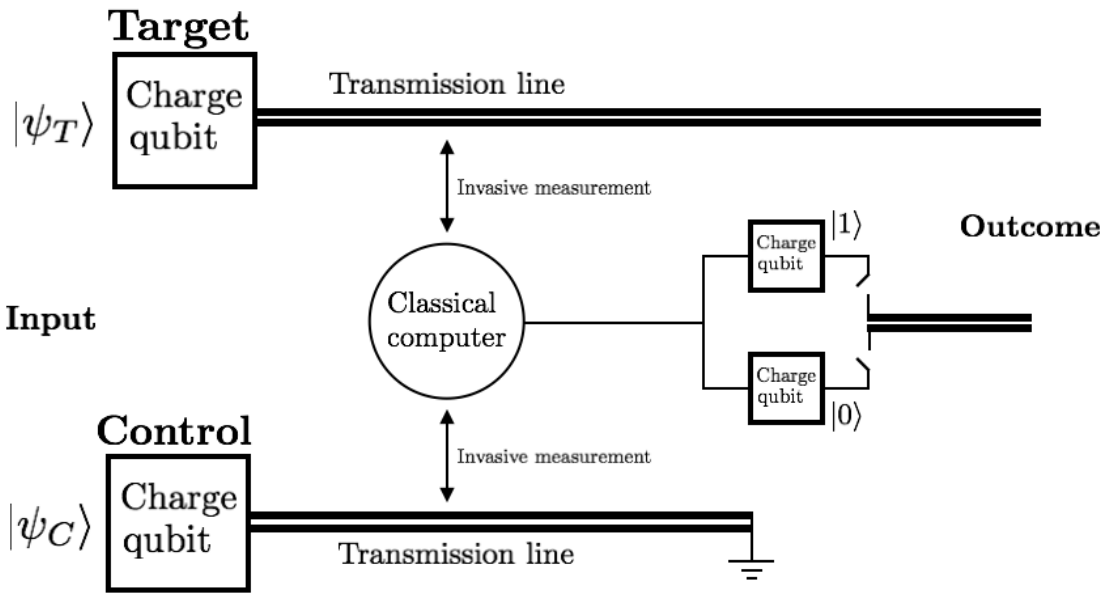


FIGURE 3.1: *CNOT gate desgined with charge qubits and transmission lines.*

mental exercise where we want to apply all the knowledge we have acquired about charge qubits and also some new concepts about transmission lines, not as a real assembly that can be done in a laboratory. The device designed, which is shown on figure 3.1, may be improved if we find another gadget that can act as Pauli-Y gate so we can generate any initial state, but this is something that will be commented within the conclusions.

Relating this to what we have said in section 1.1.2.2, here we have two charge qubits but for us one of them is the relevant one and is the control qubit. In fact, the target can be substituted by a previous connection to other parts of the circuit and we will consider that it is, somehow, inaccessible to us. In consequence, we can understand the control to play the role of the ancilla A and the target to play the role of system S. Hence, to sum it all up, the main goal of this circuit is to generate a coupling between target and control whose mapping is essentially described by table 3.1 so by performing measurements on it we can alter and, if we have enough copies, gain information about the quantum state of the target.

With respect to how this ensemble works, first of all we have two given states on target and control which we might call as $|\psi_T\rangle$ and $|\psi_C\rangle$. Here we will consider that both of them come from two different qubits but as the transmission lines are usually used to transport qubits, as we commented above, we can substitute the charge qubit of the target by the transmission line that follows it. Regardless, we have that the elements that define a state in a charge qubit are Cooper pairs so, for example, if we are in state $|1\rangle$ we will have an excess of Cooper pairs compared with the ground state. Thereby, this fact lets us differ between these two states by measuring the current something that can be done, as we will see below, employing a superconducting transmission line. These measurements are invasive because they are projective measurements that our classical computer shall interpret and, depending on the value obtained, it has to switch on the

connection to the new transmission line of one of both charge qubits located at the right side of figure 3.1, which are prepared in states $|1\rangle$ and $|0\rangle$.

3.2 Preparing the initial state

The preparation of our initial state can be done, as we have mentioned above, with the Pauli gates which in a charge qubit consists basically on finding the proper time interval that we have to wait so the system evolves to a specific final state, employing for that suitable values of the field ϵ applied. Notice that, by doing this, not all the states are accessible because we need the Pauli-Y gate which cannot be performed with our Hamiltonian (eq. (2.9)).

First of all, we will start by generating the simplest possible state, i.e., $|0\rangle$. To do that, we will locate our system at the degeneracy point $n_g = 1/2$ something that can be done searching for an adequate value of the voltage which in general shall be given by $V = e/C_g$. In this way, we can also guarantee that the fluctuations of the electromagnetic field will be minimal. Under that conditions, we leave our system at ideal superconducting conditions in such a way that we can ensure the preparation remains in that zero state.

From what we have said up to now, the reader may ask what happens if we have done previous measurements with our system so there is a remaining state on it. By leaving the charge qubit at superconducting conditions, that is, very low temperatures and for long enough time ($t \to \infty$ ideally), phenomenons of relaxation (see section 2.4.3) will occur. Hence, if ρ denotes our initial state

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{10} \\ \rho_{01} & \rho_{00} \end{pmatrix} \xrightarrow{\text{Relaxation}} \rho(t) = \begin{pmatrix} \rho_{11}e^{-t/T_1} & \rho_{10}e^{-t/2T_1} \\ \rho_{01}e^{-t/2T_1} & \rho_{00} + (1 - e^{-t/T_1})\rho_{11} \end{pmatrix} \xrightarrow{t\to\infty}$$
$$\longrightarrow \rho' = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

at the end of the process all the population will be located at the lowest energy configuration.

Having reached this point, we now can apply Pauli's gates to obtain a particular state. For example, if we want to obtain the excited level $|1\rangle$ with probability one we can apply an intense or a sinusoidal field. If we take the first option, then the evolution of our initial state may be given by

$$\rho(t) = U^\dagger(t;0,\epsilon_0)\,|0\rangle\,\langle 0|\,U(t;0,\epsilon_0) =$$
$$= \begin{pmatrix} \sin^2(\epsilon_0 t/2\hbar) & i\sin(\epsilon_0 t/2\hbar)\cos(\epsilon_0 t/2\hbar) \\ -i\sin(\epsilon_0 t/2\hbar)\cos(\epsilon_0 t/2\hbar) & \cos^2(\epsilon_0 t/2\hbar) \end{pmatrix}.$$

Therefore, proceeding like this, by waiting a time interval given by $\Delta t = \hbar\pi/\epsilon_0$ we obtain in our initial charge qubit the excited stated. For instance, if we wait a time $\Delta t = \hbar\pi/2\epsilon_0$ the resultant state will correspond to $|\psi\rangle = 1/\sqrt{2}(|1\rangle + i\,|0\rangle)$. So we can

conclude that this is an appropriate method that allows us to generate a certain initial state.

3.3 Measuring the state

Once obtained the initial state of our system, the next step as we see in figure 3.1 is to make measurements on it. To do that, we will employ a transmission superconductor line since we can do measurements of the current on it and also isolates very well the system from the environment because of its structure. This device can be composed for instance by a coplanar waveguide (figure 3.2) which consists on a thin superconducting line surrounded by larger planes that are connected to ground or some fixed potential so now the electromagnetic field tunes outside and inside the substrate. This last fact, together with the big size of the ground planes, seems to isolate better the propagating waves from external disturbances as it is stated in [GRP14].

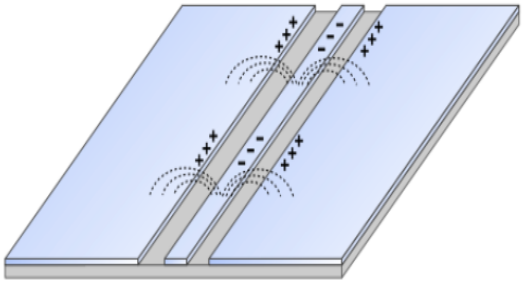


FIGURE 3.2: *Coplanar waveguide. Image extracted from [GRP14].*

3.3.1 Transmission line's Lagrangian

As we did for the charge qubit, we are going to quantize this kind of circuits and for so we will follow the same steps introduced before. First of all, we have to seek for an equivalent circuit which can describe the behaviour of a transmission line and this can be done introducing an array of coupled LC resonators, as shown on figure 3.3, where the capacitors C_0 represent the electric energy stored between the cable and the ground plane which surrounds it and the inductors L_0 represent the natural impedance of the cable against changes in the current.

In order to obtain the correspondent Hamiltonian to that circuit, we restrict ourselves to a specific part of the circuit and once that fraction is well described, we will generalize the result obtained to the whole equivalent circuit. This portion is shown on figure 3.4 where we have taken as well a direction in which flux flows through our system.

Denoting I_1 and I_2 the currents that enter to the node and I_3 the one which is exiting (see figure 3.4), from charge's conservation we get

$$I_1 + I_2 = I_3.$$

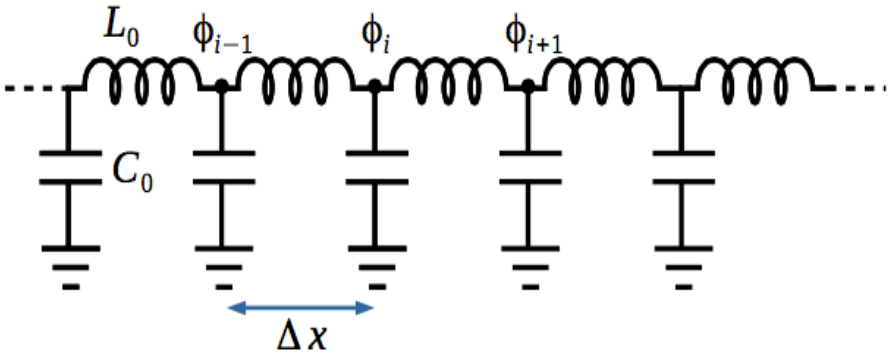


FIGURE 3.3: *Equivalent circuit for our transmission line. Image extracted from [GRP14].*

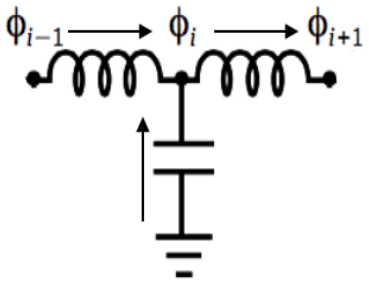


FIGURE 3.4: *Portion considered of the equivalent circuit. Imaged extracted from [GRP14] and slightly modified.*

We saw in section 2.3.1 how is the relation between the current and the flux when we have a capacitor as a circuital element. When we are introducing inductors, the correspondent relationship may be written as follows

$$I = \frac{\delta\phi}{L}$$

being $\delta\phi$ the flux difference between its nodes.

Hence, for the currents written above we have

$$I_1 = \frac{\phi_i - \phi_{i-1}}{L_0}, \quad I_2 = \ddot{\phi}_i C_0, \quad I_3 = \frac{\phi_{i+1} - \phi_i}{L_0}.$$

Substituting these values into the equation for the current we obtain

$$\frac{\phi_i - \phi_{i-1}}{L_0} + \ddot{\phi}_i C_0 = \frac{\phi_{i+1} - \phi_i}{L_0} \Rightarrow \ddot{\phi}_i C_0 = \frac{\phi_{i+1} - \phi_i}{L_0} + \frac{\phi_{i-1} - \phi_i}{L_0}.$$

As we have told in the previous chapter, this last relationship satisfies the Euler-Lagrange equations so we can obtain the Lagrangian of our system as follows

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}_i}\right) = \ddot{\phi}_i C_0 \Rightarrow \mathcal{L} = \frac{1}{2}C\dot{\phi}_i^2 + \mathcal{O}(\phi_i),$$
$$\frac{\partial\mathcal{L}}{\partial\phi_i} = \frac{\phi_{i+1} - \phi_i}{L_0} + \frac{\phi_{i-1} - \phi_i}{L_0} \Rightarrow \frac{\partial\mathcal{O}(\phi_i)}{\partial\phi_i} = \frac{\phi_{i+1} - \phi_i}{L_0} + \frac{\phi_{i-1} - \phi_i}{L_0} \Rightarrow$$

$$\Rightarrow \mathcal{O}(\phi_i) = -\frac{(\phi_{i+1}-\phi_i)^2}{2L_0} - \frac{(\phi_{i-1}-\phi_i)^2}{2L_0}.$$

Hence, we conclude that the Lagrangian which characterizes the portion of the system considered is given by

$$\mathcal{L} = \frac{1}{2}C_0\dot{\phi}_i^2 - \frac{(\phi_{i+1}-\phi_i)^2}{2L_0} - \frac{(\phi_{i-1}-\phi_i)^2}{2L_0}. \tag{3.1}$$

However, as we have told, the equivalent circuit consists on an infinite set of these elements so we must generalize that Lagrangian but this is something that can be done easily considering as *fundamental unit* one capacitor coupled to an inductor being these configuration repeated all over the space, or rather the length of the transmission line considered. Thereby, the generalization of eq. (3.1) is

$$\mathcal{L} = \sum_{i=1}^{N}\left[\frac{1}{2}C_0\dot{\phi_i}^2 - \frac{1}{2L_0}(\phi_{i+1}-\phi_i)^2\right]. \tag{3.2}$$

For our CNOT gate we will employ a $\lambda/2$ transmission line which has a finite length d and that is connected to the remaining elements of the circuit via two capacitors. What is relevant to us in this configuration is that we have to incorporate length scales defining capacitance and inductance densities c_0 and l_0 respectively

$$C_0 = c_0\Delta x, \quad L_0 = l_0\Delta x,$$

where Δx is the typical length between two consecutive nodes in our circuit (figure 3.3). Thus, eq. (3.2) might be rewritten as

$$\mathcal{L} = \sum_i \left[\frac{1}{2}c_0\dot{\phi}_i^2 + \frac{1}{2l_0}\left(\frac{\phi_{i+1}-\phi_i}{\Delta x}\right)^2\right]\Delta x.$$

As we want our analysis to be valid in the $\lambda/2$ case, then $\Delta x \to 0$ and the nodes will be infinitely closed to each other. Thereby, in this continuous limit the Lagrangian shall be written as

$$\mathcal{L} = \int_0^d \mathrm{d}x\left[\frac{1}{2}c_0\dot{\phi}^2 - \frac{1}{2l_0}\left(\frac{\partial\phi}{\partial x}\right)^2\right], \tag{3.3}$$

with Neumann boundary conditions for $\phi(x)$ coming from the fact that we have reflection at the edges.

$$\left.\frac{\partial\phi}{\partial x}\right|_{x=0} = \left.\frac{\partial\phi}{\partial x}\right|_{x=d} = 0.$$

Basically, inside our transmission line we have defined a scalar field $\phi(x,t)$ that represents the flux trapped inside this element. Therefore the main task now is to quantize that field and for doing so we will start by obtaining the equations of motion of that Lagrangian by applying Hamilton's action principle. In consequence, we define the functional $\mathcal{S}$ as

$$\mathcal{S} = \int_{t_1}^{t_2}\mathrm{d}t\int_0^d \mathrm{d}x\left[\frac{1}{2}c_0\dot{\phi}^2 - \frac{1}{2l_0}\left(\frac{\partial\phi}{\partial x}\right)^2\right]. \tag{3.4}$$

Introducing a small perturbation η that verifies $\eta(t_1) = \eta(t_2) = 0$ to ϕ we get that, neglecting second order terms in η, $\delta\mathcal{S}$ is given by

$$\delta\mathcal{S} = \int_{t_1}^{t_2} \mathrm{d}t \int_0^d \mathrm{d}x \left[c_0 \frac{\partial\phi}{\partial t}\frac{\partial\eta}{\partial t} - \frac{1}{l_0}\frac{\partial\phi}{\partial x}\frac{\partial\eta}{\partial x}\right]. \tag{3.5}$$

Taking into account that for a general variable x_i we have

$$\frac{\partial}{\partial x_i}\left[\frac{\partial\phi}{\partial x_i}\eta\right] = \frac{\partial^2\phi}{\partial x_i^2}\eta + \frac{\partial\eta}{\partial x_i}\frac{\partial\phi}{\partial t} \Rightarrow \frac{\partial\eta}{\partial x_i}\frac{\partial\phi}{\partial t} = \frac{\partial}{\partial x_i}\left(\frac{\partial\phi}{\partial x_i}\eta\right) - \frac{\partial^2\phi}{\partial x_i^2}\eta,$$

we can write eq. 3.5 as follows

$$\delta\mathcal{S} = \int_{t_1}^{t_2} \mathrm{d}t \int_0^d \mathrm{d}x \Bigg\{ \left[c_0 \frac{\partial}{\partial t}\left(\frac{\partial\phi}{\partial t}\eta\right) - \frac{1}{l_0}\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\eta\right)\right] - \left[c_0\frac{\partial^2\phi}{\partial t^2} - \frac{1}{l_0}\frac{\partial^2\phi}{\partial t^2}\right]\eta \Bigg\}.$$

The first part of the integral, the one which includes a derivation over η, turns out to be zero. A way in which we can see that is because for the second term of that part, if we integrate over x we have to evaluate the derivative of ϕ with respect to x in 0 and d which is zero because of the boundary conditions of our problem. On the other hand, for the first term we can interchange the order of the derivations, that is, start by integrating over time obtaining as consequence the evaluation of η in t_2 and t_1 which is also zero. Therefore, we get

$$\delta\mathcal{S} = -\int_{t_1}^{t_2} \mathrm{d}t \int_0^d \mathrm{d}x \left[c_0\frac{\partial^2\phi}{\partial t^2} - \frac{1}{l_0}\frac{\partial^2\phi}{\partial t^2}\right]\eta. \tag{3.6}$$

From Hamilton's action principle we have that $\delta\mathcal{S} = 0$ and, since η is a general perturbation, the term inside the integral should be zero

$$c_0\frac{\partial^2\phi}{\partial t^2} - \frac{1}{l_0}\frac{\partial^2\phi}{\partial x^2} = 0. \tag{3.7}$$

This last differential equation can be solved applying separation of variables method assuming a solution of the form $\phi(x,t) = \sum_\mu \psi_\mu(t)\chi_\mu(x)$. Introducing this into eq. (3.7)

$$\frac{1}{c_0 l_0}\frac{1}{\chi}\frac{\partial^2\chi_\mu}{\partial x^2} = \frac{1}{\psi_\mu}\frac{\partial^2\psi_\mu}{\partial t^2} = -\omega_\mu^2,$$

where we have chosen the constant to be ω_μ^2 because of the similarity of our Lagrangian to the harmonic oscillator. For the spacial part of the expression given above we get the known as Sturn-Lioville problem that, under the constrain of having $l_0, c_0 > 0$ which is true in our case as both constants are the inductance and capacitive density respectively, its eigenfunctions form an orthonormal basis which satisfies

$$\frac{1}{\Omega}\int_0^d \chi_n(x)\chi_m(x)w(x)\mathrm{d}x = \delta_{nm} \tag{3.8}$$

where Ω is a normalization constant and $w(x) > 0$ is a weight function that in our case coincide with the capacitive density c_0. Indeed, if we continue developing the spacial part we get that the normalized oscillating modes are given by

$$\chi_\mu(x) = \sqrt{\frac{2}{d}} \cos\Big(\frac{\mu\pi}{d}x\Big), \quad \omega_\mu \sqrt{c_0 l_0} = \frac{\mu\pi}{d} \tag{3.9}$$

with $\mu \in \mathbb{N}$. Introducing this into eq. (3.9) it follows immediately that $\Omega = c_0$.

Reached this point, we bring our solution $\phi(x,t)$ to the Lagrangian of eq. (3.3) having then

$$\begin{aligned}\mathcal{L} &= \int_0^d \mathrm{d}x \left[\frac{c_0}{2}\left(\frac{\partial}{\partial t}\sum_\mu \psi_\mu(t)\chi_\mu(x)\right)^2 - \frac{1}{2l_0}\left(\frac{\partial}{\partial x}\sum_\mu \psi_\mu(t)\chi_\mu(x)\right)^2\right] = \\ &= \sum_\mu \frac{c_0}{2}\frac{\mathrm{d}\phi_\mu^2}{\mathrm{d}t} - \sum_{\mu\nu}\psi_\mu\psi_\nu \frac{1}{2l_0}\int_0^d \mathrm{d}x \frac{\mathrm{d}\chi_\mu}{\mathrm{d}x}\frac{\mathrm{d}\chi_\nu}{\mathrm{d}x}\end{aligned}$$

where we have used in eq. (3.8) to solve the first part of the integral. On the other hand, for the second one we integrate by parts obtaining finally

$$\mathcal{L} = \frac{c_0}{2}\sum_\mu \Big[\dot{\psi}_\mu^2 - \omega_\mu^2 \psi_\mu^2\Big]. \tag{3.10}$$

3.3.2 Transmission line's Hamiltonian. Quantization

From eq. (3.10) we see that the conjugated momenta φ_μ is given by

$$\varphi_\mu = \frac{\partial \mathcal{L}}{\partial \dot{\psi}_\mu} = c_0 \dot{\psi}_\mu \Rightarrow \dot{\psi}_\mu = \frac{\varphi_\mu}{c_0}$$

so performing the Legendre transformation of our Lagrangian we conclude that the Hamiltonian that characterizes our transmission line is given by

$$\mathcal{H} = \sum_\mu \left[\frac{\varphi_\mu^2}{2c_0} + \frac{1}{2}c_0\omega_\mu^2\psi_\mu^2\right]. \tag{3.11}$$

This Hamiltonian is the one of a set of harmonic oscillators whose quantization can be done substituting the variables φ_μ and ψ_μ by the corresponding operators $\hat{\varphi}_\mu$ and $\hat{\psi}_\mu$.

$$\mathcal{H} = \sum_\mu \left[\frac{\hat{\varphi}_\mu^2}{2c_0} + \frac{1}{2}c_0\omega_\mu^2\hat{\psi}_\mu^2\right]. \tag{3.12}$$

Equation (3.12) can be rewritten in terms of the creation and annihilation operators using the following definitions and assuming $\hbar = 1$ as

$$\hat{\varphi}_\mu = \frac{1}{\sqrt{2c_0\omega_\mu}}(a_\mu + a_\mu^\dagger), \quad \hat{\psi}_\mu = i\sqrt{\frac{c_0\omega_\mu}{2}}(a_\mu^\dagger - a_\mu).$$

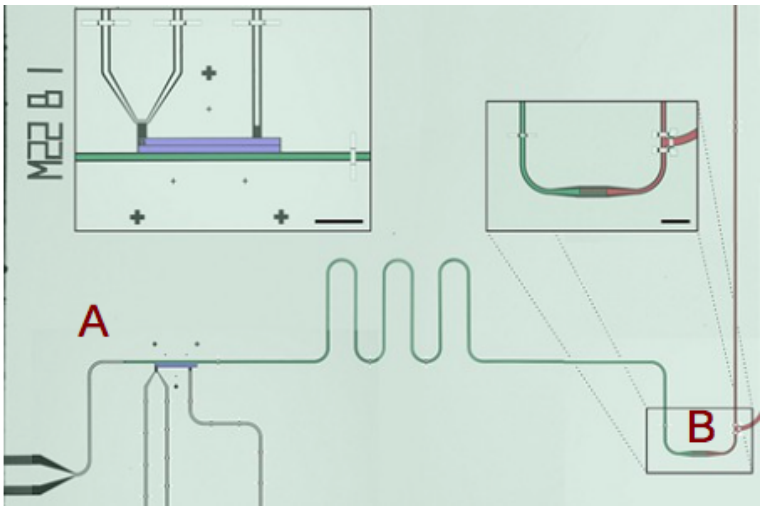


FIGURE 3.5: *Configuration of a circuit which includes a $\lambda/2$ transmission line. Image extracted from [GRP14].*

Then, according to the solution we have considered for our problem, we obtain that the flux in our transmission line is given by

$$\hat{\phi}(x,t) = \sum_{\mu} \hat{\psi}_{\mu}(t)\chi_{\mu}(x) = \sum_{\mu} \frac{1}{\sqrt{2c_0\omega_{\mu}}}\chi_{\mu}(x)(a_{\mu}^{\dagger} + a_{\mu}) \tag{3.13}$$

where the functions $\chi_{\mu}(x)$ are given by eq. (3.9). Deriving this expression with respect to time, we obtain that the conjugated momenta, that is, the charge density is

$$\hat{\rho}(x,t) = i\sum_{\mu} \sqrt{\frac{c_0\omega_{\mu}}{2}}\chi_{\mu}(x)(a_{\mu}^{\dagger} - a_{\mu}). \tag{3.14}$$

This last result is really important for us because that means we can measure the density charge, and for so the intensity, in our transmission line. Thereby, the implementation we have considered in our design of the CNOT gate (figure 3.1) seems plausible.

On figure 3.5 we show a configuration of a $\lambda/2$ resonator which is coupled to one qubit (point A in the image) and to another part of the circuit that is not shown here (point B in the image). In this case, the connections are done through two capacitors located in both points reaching in that way the $\lambda/2$ configuration that we have been talking about along this section. This is a very important figure because shows us how the system we have to quantize can be performed experimentally and establish a curious parallelism between our idealization (figure 3.3) and what we face to in real life.

3.3.3 States for the transmission line

Typically, superconducting transmission lines are elements which allow the guided transport of photons so we can select from a wide variety of states that can be transported through our waveguide. Concretely, we will focus in the known as thermal states which are states defined in a certain cavity that is in thermal equilibrium. The reason why

we want to employ this kind of states is because the temperature of the thermal cavity let us distinguish between states $|0\rangle$ and $|1\rangle$. For example, we can let our current run through a certain resistor element, having in consequence a certain amount of heat dissipated. As the main difference between the states $|0\rangle$ and $|1\rangle$ is in the number of Cooper pairs that we have, the temperature of the cavity that encloses that resistor will differ depending on if we have one state or the another.

So as to have a starting point, we will take into consideration the fact that the Hamiltonian which characterizes the electromagnetic field can be quantized as a set of harmonic oscillators of the form[I]

$$\mathcal{H}_{\text{em}} = \sum_{\mu} \omega_{\mu} a_{\mu}^{\dagger} a_{\mu}. \tag{3.15}$$

Concretely, we will suppose that we are applying a perfect monochromatic laser field. In such a way, we will select only one possible mode being the expression written above reduced to

$$\mathcal{H}_{\text{em}} = \omega a^{\dagger} a. \tag{3.16}$$

As we have basically photons enclosed in a thermal cavity, we can employ to characterize the density of states for these kind of systems a Bose-Einstein distribution whose partition function is given by $\mathcal{Z} = (1 - e^{-\beta\omega})^{-1}$. Thereby, the density matrix that describes our set of photons is

$$\rho = \sum_{n} e^{-\beta n\omega}(1 - e^{-\beta\omega}) \left|n\right\rangle \left\langle n\right|. \tag{3.17}$$

Assuming that this frequency we have selected can *habit* inside our transmission line, that is, it coincides with one of the possible modes given by eq. (3.9), then the charge density operator of eq. (3.14) might be written in this case as[II]

$$\hat{\rho}(x,t) = i\sqrt{\frac{c_0\omega}{2}}\chi(x)(a^{\dagger} - a). \tag{3.18}$$

However, we can see immediately that if we try to perform the mean value of this operator over the configuration given by eq. (3.17) the result will be zero so we are not detecting anything. Nevertheless, as we are working with creation and annihilation operators we can seek for another option which is employing a certain kind of coherent states called as thermal coherent states, which can be obtained, according to [OVMR91], displacing the thermal mixed state of eq. (3.17), having then

$$\rho(\alpha) = \frac{1}{\mathcal{Z}} D(\alpha) e^{-\beta\omega a^{\dagger} a} D^{\dagger}(\alpha) = \sum_{n} e^{-\beta n\omega}(1 - e^{-\beta\omega}) D(\alpha) \left|n\right\rangle \left\langle n\right| D^{\dagger}(\alpha), \tag{3.19}$$

with $D(\alpha) = e^{\alpha a^{\dagger} - \alpha^* a}$ the so called displacement operator. Let us evaluate now the mean value of the charge density operator under this kind of states

$$\left\langle \hat{\rho} \right\rangle = \text{tr}\left\{ k(x)(a^{\dagger} - a) \sum_{n} e^{-\beta n\omega} \mathcal{Z}^{-1} D(\alpha) \left|n\right\rangle \left\langle n\right| D^{\dagger}(\alpha) \right\} =$$

[I] Remember that we have considered $\hbar = 1$.

[II] Note that we denote by ρ the density matrix operator and by $\hat{\rho}$ the charge density operator.

$$= \sum_n \kappa(x,n,T)\,\mathrm{tr}\Big\{(a^\dagger - a)D(\alpha)\,|n\rangle\,\langle n|\,D^\dagger(\alpha)\Big\}$$

where $k(x)$ is a complex function of the position is $\kappa(x,n,T)$ is another one which contains $k(x)$ and whose main characteristic is the dependence it has with T and which is given by

$$\kappa(x,n,T) = i\sqrt{\frac{c_0\omega}{2}}\chi(x)e^{-\beta n\omega}\mathcal{Z}^{-1}.$$

Taking out this term out of the trace, we can focus mainly on the other part of the expression obtaining

$$\begin{aligned}
&\mathrm{tr}\Big\{(a^\dagger - a)D(\alpha)\,|n\rangle\,\langle n|\,D^\dagger(\alpha)\Big\} = \sum_m \langle m|\,(a^\dagger - a)D(\alpha)\,|n\rangle\,\langle n|\,D^\dagger(\alpha)\,|m\rangle = \\
&= \langle n|\,(a^\dagger - a)D(\alpha)\,|n\rangle\,\underbrace{\langle n|\,D^\dagger(\alpha)\,|n\rangle}_{1} = \langle n|\,(a^\dagger - a)\sum_{l=0}\frac{(\alpha a^\dagger - \alpha^* a)^l}{l!}\,|n\rangle = \\
&= \langle n|\,(a^\dagger - a)(|n\rangle + \alpha\sqrt{n+1}\,|n+1\rangle - \alpha^*\sqrt{n}\,|n-1\rangle + ...) = \\
&= -[\alpha^* n + \alpha(n+1)].
\end{aligned}$$

So we conclude that the mean value of the charge density is

$$\langle\hat{\rho}\rangle = -\kappa(x,n,T)[\alpha^* n + \alpha(n+1)]. \tag{3.20}$$

This is result indicates a way in which we can measure the current through our transmission line, taking into account that the charge density is related to that magnitude, that let us distinguish between ground and excited states as we have a clear dependence with the temperature of the thermal cavity. Therefore, we have found an ideal superconducting circuit arrangement which, in principle, can play the role of a CNOT gate and act as a coupling between target and control.

Chapter 4

Conclusions

Summing it all up, we have started this project studying the main difference between closed quantum systems and open quantum systems to see how can we perform measurements on an inaccessible system by performing a coupling of another system that we can control and to see how can we introduce the effects of greater systems like the environment on it. Once characterized a qubit system like the charge qubit, we have been allowed to see how the surroundings affect on it and to employ that disturbance as an advantage to prepare certain states in our setup. With all this knowledge, we have generated employing charge qubits, transmission lines and classical computers a certain kind of coupling between two quantum systems (control and target) characterized by a CNOT gate.

Nevertheless, there are some open fronts that we have not discussed. Taking advantage of a controlled environment and employing charge qubits, we have seen a way in which we can prepare certain configurations in it, but as we saw there is a limitation due to the fact that the Hamiltonian presented in eq. (2.9) does not incorporate σ_y. However, as we are working with microwave radiation in certain parts of our circuit, we can introduce an interferometer in this setup that allows us to generate a phase. Typically we can control only one qubit, namely the known as *control*, and from our mapping (table 3.1) we only care about which is the state not on its phase but in order to do measurements on the transmission line, the temperature of the waveguide has to be the same of the resistor which *converts* Cooper pairs states into thermal-photon states. In this conversion maybe we can lose information about the phase of the target but as we have commented, the target does not need to be a qubit, it might come from other parts of the circuit through a transmission line.

All this discussion lead us now in how can we infer the state of the target from the measurements of the qubit. Taking into account that the initial state of the target shall be written as

$$|\psi\rangle_{\text{target}} = \alpha\,|0\rangle + \beta\,|1\rangle$$

where α and β are complex numbers, the module of those values can be obtained performing a suitable number of measurements over our state so we can see how many times a certain value is repeated, obtaining then the probability of having a certain state. On

the other hand, suppose that $|0\rangle$ states correspond to a horizontally polarized wave and $|1\rangle$ states to a vertical one. If at the beginning of our experiment we prepare independently from the rest two charge qubits in those states and perform always for them some optical operations so for every experiment those two states are prepared equally, we can associate to each of them a given phase. Now, once we obtain states $|0\rangle$ and $|1\rangle$ coming from the *target* and we have managed to isolate them, they can be compared with the other ones mentioned above so a phase difference between them can be measured. Then, if the state under study has been generated by another person who is trying to send a message according to a previously established *alphabet* formed by phase differences and probabilities ($\|\alpha\|^2$, $\|\beta\|^2$), we can decode it by this method. Notice that we have to agree previously how those initial states must be prepared, in order to have the same phase differences.

In conclusion, we consider that our main objective of studying how quantum measurements can be done and how it affects our system has been achieved.

Appendix A

Demonstration of POVM properties

Recalling the properties shown on section 1.1.2.2 for POVM operators, in this appendix we are going now to prove them taking into account that these sort of operators are generally given by $\hat{E}_a = \hat{M}_a^\dagger \hat{M}_a$.

- Hermicity. $\hat{E}_a = \hat{E}_a^\dagger$

  This property can be easily proved taking considering the following properties for operators:

  $$(AB)^\dagger = B^\dagger A^\dagger, \quad (A^\dagger)^\dagger = A.$$

  So applying this to our case we obtain:

  $$\hat{E}_a^\dagger = (\hat{M}_a^\dagger \hat{M}_a)^\dagger = \hat{M}_a^\dagger (\hat{M}_a^\dagger)^\dagger = \hat{M}_a^\dagger \hat{M}_a = \hat{E}_a \Rightarrow \boxed{\hat{E}_a = \hat{E}_a^\dagger}.$$

- Positivity. $\langle\psi|\hat{E}_a|\psi\rangle \geq 0$

  To prove this we will focus on the meaning of $\langle\psi|\hat{E}_a|\psi\rangle$. As we commented in 1.1.2.2, the probability of obtaining eigenvalue a for system B is given by

  $$\text{Prob}(a) = \langle\xi|\hat{E}_a|\xi\rangle \tag{A.1}$$

  where $|phi\rangle_S$ was the initial state for our quantum system. Therefore the quantity $\langle\psi|\hat{E}_a|\psi\rangle$ defines a probability and for so it has to be greater or equal to zero, so we conclude:

  $$\boxed{\langle\psi|\hat{E}_a|\psi\rangle \geq 0}.$$

- Completeness. $\sum_a \hat{E}_a = \mathbb{1}$

  To prove this last property we will take into account that the time evolution operator U is unitary so it preserves the norm. That means that if the norm of our initial state is 1 when the measure is finished, the norm of the final state has

also to be one having then

$$1 = \left\| \sum_a \hat{M}_a \left|\xi\right\rangle \otimes \left|a\right\rangle_B \right\| = \sum_{a,b} \langle\xi|\hat{M}_a^\dagger \hat{M}_a|\xi\rangle \, \delta_{ab} = \sum_a \langle\xi|\hat{M}_a^\dagger \hat{M}_a|\xi\rangle = \sum_a \langle\xi|\hat{E}_a|\xi\rangle \, .$$

As this is valid for any initial state $|\xi\rangle$ we conclude

$$\boxed{\sum_a \hat{E}_a = \mathbb{1}} .$$

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