Introduction to modern research in Quantum Physics.

Educational review on *"A Classical Channel Model for Gravitational Decoherence"* [9]

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July 2018

Abstract

We present an educational review of an article written by Kafri et al. [9], where they obtain a decoherence model for the system of two resonators which is equivalent to a previous model proposed by Diosi in [1]. In these references it is explored the effects that emerge from treating gravity as a merely classically mediated channel for the case of two gravitationally coupled resonators.

Firstly we introduce the basis of quantum measurement theory that shall be applied to describe the gravitational interaction between the two masses, being of special relevance the Wiseman-Milburn feedback frame [8]. Subsequently, we give both a classical and a quantum description of the two resonators system where some key parameters are introduced, such as their normal mode split frequency which -we will prove- is proportional to the decoherence rate. Finally we study the system replacing gravity by a feedback interaction which will effectively play its role. A master equation is found from which we extract the decoherence and heating rates. After some estimations we will argue that these effects are difficult to observe due to their weak intensity, though this inability to detect them lays on technology and -in principle- not in the model proposed. Not detecting them, provided the necessary technology, would point out that gravity must not be treated as a classical interaction [9].

Abstract (Español)

Presentamos una 'revisión educativa' del artículo escrito por Kafri et al. en [9], en el que obtienen un modelo de decoherencia para el sistema de dos resonadores que es equivalente a un modelo anterior propuesto por Diosi en [1]. En estas referencias se exploran los efectos que emergen de tratar a la gravedad como un canal de comunicación mediado clásicamente para el caso de dos resonadores acoplados gravitatoriamente.

Primero introducimos las bases de la teoría cuántica de la medida, que será aplicada para describir la interacción gravitatoria entre las dos masas, siendo de especial relevancia la teoría de *feedback* de Wiseman-Milburn [8]. A continuación, damos la descripción clásica y cuántica del sistema de los dos resonadores en la cual introducimos algunos parámetros clave, como el desfase en los modos normales, que se relacionará de forma directa con la tasa de decoherencia. Finalmente, estudiamos el sistema remplazando la gravedad por una interacción tipo *feedback* que de forma efectiva actúa como la gravedad. Encontramos la ecuación maestra de la que extraemos las tasas de decoherencia y calentamiento. Tras algunas estimaciones, argumentaremos que estos efectos son difíciles de observar debido a su baja intensidad. Sin embargo, esta incapacidad viene dada por la tecnología y no por el modelo propuesto. No detectar estos fenómenos, dada la tecnología necesaria, indicaría que la gravedad no debe ser tratada como una interacción puramente clásica [9].

Contents

1 Introduction **4**

2 Quantum theory of measurement **5**
2.1 Ideal measurements . . . 5
2.2 Generalized measurements . . . 6
2.3 Simple example: Qubit dynamics . . . 9
2.4 Wiseman-Milburn Feedback . . . 12

3 Quantum system gravitationally coupled **16**
3.1 Classical analysis . . . 16
3.2 Quantum description . . . 19

4 Gravity as a feedback interaction **21**
4.1 System evolution . . . 24
4.2 Decoherence rate . . . 28
4.3 Momentum diffusion rate . . . 29
4.4 Brownian motion analogy and effective temperature . . . 30

5 Conclusions and discussions **32**

6 Síntesis en Español **33**
6.1 Introducción . . . 33
6.2 Teoría cuántica de la medida . . . 33
6.3 Sistema cuántico acoplado gravitatoriamente . . . 33
6.4 Gravedad como una interacción retroalimentada . . . 34
6.5 Conclusiones y discusiones . . . 34

1 Introduction

We know that the theory of general relativity is the one describing gravity, having been tested through several experiments successfully. Nonetheless, we fail when developing a more universal theory that involves gravity and quantum mechanics. One could however wonder why should we find a formalism including both fields. As discussed in [2], in 1936 Bronstein was the first to explore the consequences of applying the uncertainty principle to the general theory of relativity, proving that quantum mechanics prevents the gravitational field to be determined with infinite accuracy in an arbitrarily small region. Many speculative attempts to gather both theories together suggest that time-space should be quantized, being the Planck's time and length the corresponding quanta. Therefore, at times and lengths much larger than those of Planck scale we would not expect to see any quantum gravity effect, and relativity would suffice -for tabletop experiments even a Newtonian description is enough-. As explained in [9], such expectation have recently been called into question by Diósi and Penrose, whose proposals point out that quantum gravitationally induced effects, specifically gravitational decoherence, can become important at larger scales given enough control over the quantum degrees of freedom.

In this report I make an educational review of the article [9] written by Kafri et al. There, a system of two masses gravitationally coupled is studied aiming to obtain a proper decoherence model which may be tested experimentally as far as some technical requirements are achieved. This theoretical setup is motivated by the lately acquired skill to optically cool macroscopic mechanical oscillators close to their ground state [9], which allows us to have a system where both quantum and gravity effects are to be taken into account. Along this review, the procedure followed in [9] is reproduced and explained in detail, for which first I introduce the mathematical and physical frame needed, that is to say, the quantum theory of measurement.

Gravity is treated here as a classical communication channel, though still compatible with quantum mechanics. Effectively, the gravitational field is replaced by a feedback-like interaction: the position of each mass is continuously measured and the information used on the other mass to modify its future dynamics and vice versa. We study the change on the master equation due to both the measurement and the feedback processes, implementing these changes into our system. A decoherence rate shall be obtained along with a heating rate that, after estimation for some fixed parameters, will reveal the strength of the effect and therefore the expectations to detect it.

Preface

Regarding the notation, we have decided not to denote with *hat* ˆ quantum observables. Nonetheless, it should be clear from the context when we refer either to an observable or to a physical quantity.

2 Quantum theory of measurement

In order to introduce gravity effects within the formalism presented in [1] and [9], we shall provide some basic knowledge concerning measurements in the quantum frame.

2.1 Ideal measurements

This section have been written following closely the notes [7], although any book on quantum mechanics shall cover the topics here discussed. The same reference is used in the next section 2.2.

Any measurable physical quantity A can be described by a self-adjoint operator A which is called observable. This quantum operators have a spectral decomposition, such that $A = \sum_n |a_n\rangle \, a_n \, \langle a_n|$. The quantities a_n constitute the spectra of A, and they represent the expected values after measuring the observable A on a given system.

In this frame of ideal measurements, the probability of obtaining a certain value a_n is given by

$$p(a_n) = \langle a_n| \, \rho \, |a_n\rangle \,, \tag{2.1}$$

where ρ is the density operator that represents the state of the system. After the measurement the state changes, according to the von Neumann - L*ü*ders postulate, as

$$\rho \to \rho'_n = p(a_n)^{-1} \Pi(a_n) \rho \Pi(a_n). \tag{2.2}$$

Being $\Pi(a_n) = |a_n\rangle \langle a_n|$ the projector over a_n. We can interpret this result by considering that the original ensemble has been divided into smaller ensembles, each one corresponding to a different value of a_n. By performing the measurement we are restricting the system to one of this sub-ensembles. In this sense, we say we have performed a selective measurement of the system. However it might happen that we have not access to the outcome of the measurement or we simply do not want to use such information. If this is the case we need to describe the system as a superposition of those sub-ensembles, such that

$$\rho \to \rho' = \sum_n p(a_n)^{-1}\Pi(a_n)\rho\Pi(a_n). \tag{2.3}$$

Consequently the system is not restricted to one of the ensembles and we talk about non-selective measurements. It is easy to check that the mean value of A is not affected by the measurement, that is to say $\langle A\rangle = Tr\rho A = Tr\rho' A$. This is the reason why we talk about an ideal measurement.

2.2 Generalized measurements

Previously we have implicitly assumed that the detector has infinite accuracy, so that the profile of the detector related to the measurement is described by a Dirac delta function centered in the output of the measurement. However, in the real case, one would not expect to obtain a perfect knowledge of the spectra of a measured observable, this is due to the fact that the detector has low resolution. Consequently, we must associate a wider profile to the measurement process. Let us now consider this distribution to be gaussian

$$\Omega_f(A) = (2\pi\sigma^2)^{-\frac{1}{4}} \cdot e^{-\frac{(f-A)^2}{4\sigma^2}}, \tag{2.4}$$

where $f \in (-\infty, \infty)$ are the eigenvalues of the observable A. We know that after a measurement, the system's evolution is perturbed. If no measurement is performed the system would evolve according to

$$\partial_t\rho = -\frac{i}{\hbar}\left[H, \rho\right]. \tag{2.5}$$

We shall now find a modified effective master equation that describes the measurement process under certain circumstances. If we measure the observable A, the state immediately after turns out to be, according to the Von-Neumann-Lüders postulate,

$$\rho \to P^{-1}(f) \cdot \Omega_f\rho\Omega_f, \tag{2.6}$$

where $P(f) = Tr(\Omega_f\rho\Omega_f)$ is a normalization factor which corresponds to the probability of obtaining the outcome f. The distribution 2.4 proposed above is already normalized as

$$\int df\Omega_f\Omega_f^\dagger = 1. \tag{2.7}$$

Continuous measurements

We are particularly interested in the case in which the measurement is performed weakly and continuously. In order to model this situation we can consider a large amount of measurements separated in time by an arbitrary quantity τ and afterwards we take the limit when τ goes to zero. We shall also consider the diffusive regime, in which

$$g = \lim_{\tau \to 0} \frac{1}{\tau \sigma^2}. \tag{2.8}$$

This limit would appear naturally later on. As the time τ between the measurement goes to zero, the probability function gets broadened and we obtain less information about the system. For this reason we cannot use the information gained in the measurements, and so we have to restrict to the non-selective measurement regime, in which the state after the measurement is a superposition of all the possible states

$$\rho \to \int df \Omega_f \rho \Omega_f. \tag{2.9}$$

Taking the limit 2.8

$$\Omega_f(A) = \left(\frac{g\tau}{2\pi}\right)^{\frac{1}{4}} \cdot e^{-\frac{g\tau(f-A)^2}{4}} = \left(\frac{g\tau}{2\pi}\right)^{\frac{1}{4}} \cdot e^{-\frac{g(\sqrt{\tau}f-\sqrt{\tau}A)^2}{4}}, \tag{2.10}$$

hence

$$\begin{aligned}\rho &\to \int df \left(\frac{g\tau}{2\pi}\right)^{\frac{1}{2}} \cdot e^{-\frac{g(\sqrt{\tau}f-\sqrt{\tau}A)^2}{4}} \rho \cdot e^{-\frac{g(\sqrt{\tau}f-\sqrt{\tau}A)^2}{4}} \\ &= \left(\frac{1}{2\pi}\right)^{\frac{1}{2}} \int d\phi \cdot e^{-\frac{(\phi-\sqrt{\tau g}A)^2}{4}} \rho \cdot e^{-\frac{(\phi-\sqrt{\tau g}A)^2}{4}},\end{aligned} \tag{2.11}$$

where we have replaced $f \to \dfrac{\phi}{\sqrt{\tau g}}$. To proceed further we expand the exponential function in Taylor series around $\sqrt{\tau} = 0$ as

$$e^{-\frac{(\phi-\sqrt{\tau g}A)^2}{4}} = \left(1 + \phi A \frac{\sqrt{g}}{2}\sqrt{\tau} + \frac{A^2 g}{8}(\phi^2 - 2)\tau\right) \cdot e^{-\frac{\phi^2}{4}} + \mathcal{O}(\tau^{3/2}). \tag{2.12}$$

Substituting in the integral it follows

$$\begin{aligned}\rho \to &\left(\frac{1}{2\pi}\right)^{\frac{1}{2}} \int d\phi \cdot e^{-\frac{\phi^2}{2}} \left(1 + \phi A \frac{\sqrt{g}}{2}\sqrt{\tau} + \frac{A^2 g}{8}(\phi^2 - 2)\tau\right) \rho \\ &\cdot \left(1 + \phi A \frac{\sqrt{g}}{2}\sqrt{\tau} + \frac{A^2 g}{8}(\phi^2 - 2)\tau\right) \\ &= \left(\frac{1}{2\pi}\right)^{\frac{1}{2}} \int d\phi \cdot e^{-\frac{\phi^2}{2}} \Big[\rho + \phi \frac{\sqrt{g}}{2}(A\rho + \rho A)\sqrt{\tau} + \phi^2 \frac{g}{4} A\rho A\tau \\ &+ \frac{g}{8}(\phi^2 - 2) \cdot (A^2\rho + \rho A^2)\tau\Big] \\ &= \rho + \frac{g}{8}\left(2A\rho A - A^2\rho - \rho A^2\right)\tau = \rho - \frac{g}{8}[A, [A, \rho]]\,\tau.\end{aligned} \tag{2.13}$$

Thus we obtain

$$\partial_t \rho|_{measure} = \lim_{\tau \to 0} \frac{\rho(t+\tau) - \rho(t)}{\tau} = -\frac{g}{8}\left[A, [A, \rho]\right] \tag{2.14}$$

and the evolution of the system can be described as

$$\partial_t \rho = -\frac{i}{\hbar}\left[H, \rho\right] - \frac{g}{8}\left[A, [A, \rho]\right]. \tag{2.15}$$

There are some subtleties in this deduction which are worth noting. Regarding equation 2.15, it can be solved, for instance, by a Monte Carlo integration, in which case we must have a stochastic equation for the dynamics. In order to obtain the stochastic equation let us observe that the gaussian integration cancels out the term $\phi \frac{\sqrt{g}}{2} \sqrt{\tau}\left(A\rho + \rho A\right)$, while one would expect this term to be relevant as τ goes to zero. Now note that a Wiener process $W(t)$ is associated to a variable which changes continuously in time, and its change ΔW over an interval τ goes as $\Delta W = \phi\sqrt{\tau}$, being ϕ a random variable normally distributed with zero mean and unit variance. Keeping this in mind we see that the term above mentioned shall be proportional to a Wiener increment in the continuous limit satisfying the following conditions

$$\begin{aligned} \langle dW \rangle &= 0 \\ dW^2 &= \tau, \end{aligned} \tag{2.16}$$

hence, we should introduce a term of the form $\frac{\sqrt{g}}{2}\left(A\rho + \rho A\right) dW$. However this would lead to a dynamics in which trace is not preserved, and neither would probability. It is immediate to check that if we subtract the change in the trace, we get a trace-preserving dynamics. Consequently, the dynamics of a system which is being continually measured can be described by the following stochastic master equation (SME)

$$d\rho = -\frac{i}{\hbar}\left[H, \rho\right] dt - \frac{g}{8}\left[A, [A, \rho]\right] dt + \frac{\sqrt{g}}{2}\mathcal{H}[A]\rho dW, \tag{2.17}$$

where we have defined the super-operator $\mathcal{H}$ such that $\mathcal{H}[A]\rho = A\rho + \rho A - 2\langle A \rangle \rho$.

The last thing to note is related with the diffusive limit mentioned previously. In that limit we considered that $\sigma \propto \frac{1}{\tau^{1/2}}$. There is no other dependence which would lead to a desirable description of a system being weakly and continually measured. If the exponent of τ were any smaller, there would be no effect on the dynamics of the system and it would evolve according to its internal degrees of freedom. It corresponds to the case when the measurement is extremely weak. On the other hand, if the exponent were any larger the system would not evolve at all. The measurement would be so strong that the system would be projected into an eigenstate of the observable measured and would remain there, this corresponds to the Quantum Zeno regime.

2.3 Simple example: Qubit dynamics

A better understanding of the consequences that carries out the process of measurement is easily achieved with an illustrative example. Let us consider a quantum system that has only two eigenstates, such system is called qubit. In particular we will study the evolution of a spin momentum that obeys the following Hamiltonian

$$H = \alpha\sigma_x + \beta\sigma_z, \tag{2.18}$$

being σ_x and σ_z the well-known Pauli matrices. We choose the measured observable to be σ_z itself. Hence, the master equation of the system is

$$\partial_t \rho = -i\left[H, \rho\right] - \frac{\gamma}{8}\left[\sigma_z\left[\sigma_z, \rho\right]\right]. \tag{2.19}$$

We have fixed $\hbar = 1$ and we have not included the stochastic term. In the eigenbasis of σ_z it follows

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.20}$$

After some trivial manipulations we obtain a system of four lineal coupled differential equations, one for each density matrix element

$$\begin{aligned}
\partial_t \rho_{11} &= -i\alpha(\rho_{21} - \rho_{12}) \\
\partial_t \rho_{22} &= -\partial_t \rho_{11} \\
\partial_t \rho_{12} &= -i\left(\alpha(\rho_{22} - \rho_{11}) + 2\beta\rho_{12}\right) - \frac{\gamma}{2}\rho_{12} \\
\partial_t \rho_{21} &= i\left(\alpha(\rho_{22} - \rho_{11}) + 2\beta\rho_{21}\right) - \frac{\gamma}{2}\rho_{21}.
\end{aligned} \tag{2.21}$$

Note that the second equation shows explicitly that the trace is preserved, and so are the probabilities. Also note that the two last equations will lead to decoherence, with γ as a weighing factor, induced by the measurement process. In the following we integrate these expressions for some particular cases, all of which have non-zero coherences in the initial state. The simulations have been carried out with the Python framework developed in [5].

We first consider the easiest case, where the observable measured is the energy. This corresponds to $(\alpha, \beta) = (0, 1)$, see figure 1. It is immediate to see that the probabilities do not evolve, the measurement projects the

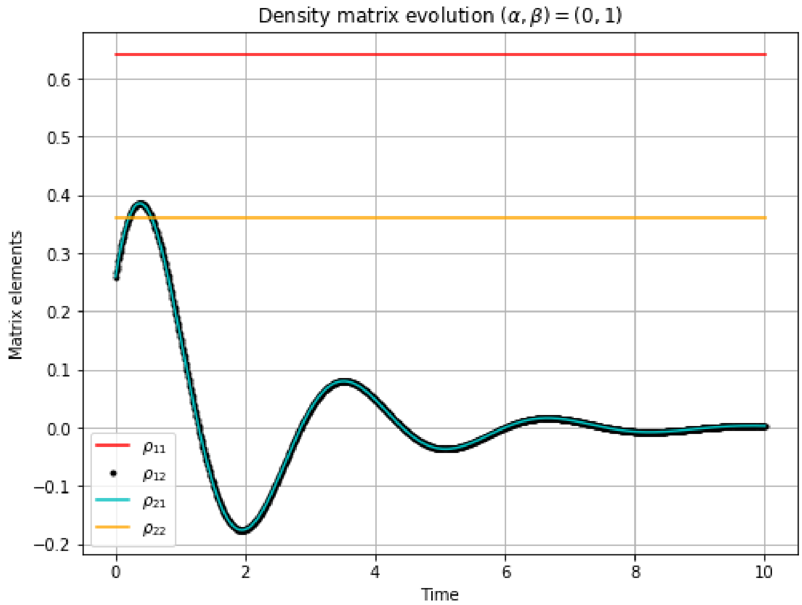

Figure 1: Qubit system obeying the Hamiltonian $H = \sigma_z$. The parameters employed in the simulations are $\alpha = 0$, $\beta = 1$, $\gamma = 1$. The initial state of the system has been chosen as $\rho_{11} = 0.64$, $\rho_{12} = 0.48 \cdot e^{i}$, $\rho_{21} = 0.48 \cdot e^{-i}$, $\rho_{22} = 0.36$. Only the real part is plotted. Time is given in arbitrary units.

system into an eigenstate of σ_z. Furthermore, decoherence appears as a consequence of the measurement.

Afterwards we studied the case where $(\alpha, \beta) = (1, 0)$, see figure 2. Once again decoherence appears as a direct consequence of the measurement. The populations now do not remain constant, instead they keep oscillating until they get stabilized to a certain value. This situation can be understood as follows, between measurements the system evolves freely and it tends to be projected into an eigenstate of the Hamiltonian, nevertheless, when the measurement takes place, it tends to be projected into an eigenstate of the observable measured. The 'competition' between these two processes produces such oscillation.

We finally analyzed the case $(\alpha, \beta) = (1, 1)$, see figure 3. This is intuitively more difficult to comprehend, although it can be regarded as some sort of superposition of the two cases showed above. Decoherence is again induced by the measurements.

As a reference we also simulated the case where $\gamma = 0$, which shows the dynamics when no measurement is performed, see figure 4. It is clearly

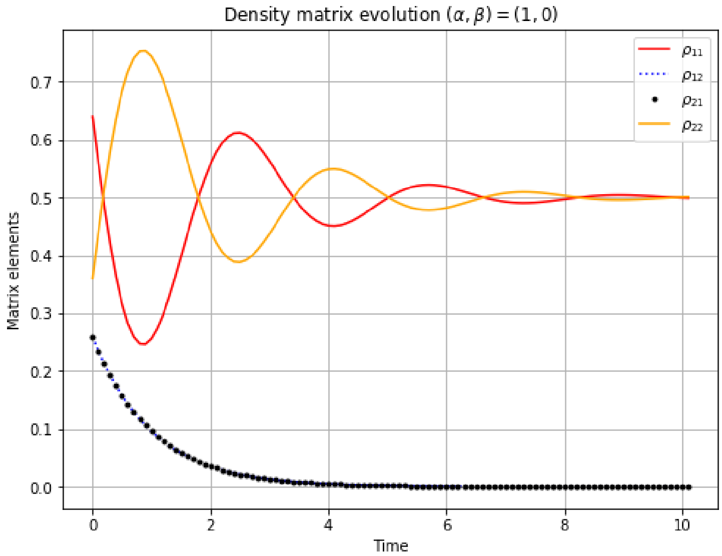

Figure 2: Qubit system obeying the Hamiltonian $H = \sigma_x$. The parameters employed in the simulations are $\alpha = 1$, $\beta = 0$, $\gamma = 2$. The initial state of the system has been chosen as $\rho_{11} = 0.64$, $\rho_{12} = 0.48 \cdot e^{i}$, $\rho_{21} = 0.48 \cdot e^{-i}$, $\rho_{22} = 0.36$. Only the real part is plotted. Time is given in arbitrary units.

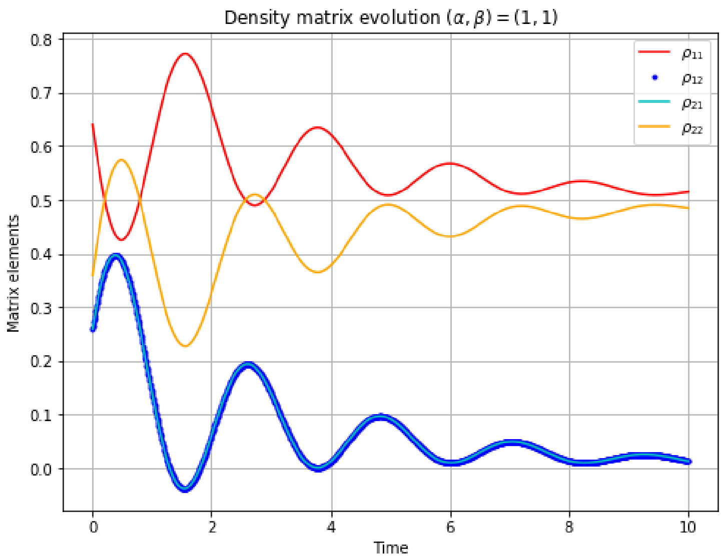

Figure 3: Qubit system obeying the Hamiltonian $H = \sigma_x + \sigma_z$. The parameters employed in the simulations are $\alpha = 1$, $\beta = 1$, $\gamma = 1$. The initial state of the system has been chosen as $\rho_{11} = 0.64$, $\rho_{12} = 0.48 \cdot e^{i}$, $\rho_{21} = 0.48 \cdot e^{-i}$, $\rho_{22} = 0.36$. Only the real part is plotted. Time is given in arbitrary units.

seen than decoherence does not take place now. All this together points out that the only responsible for decoherence is the measurement process, as discussed after 2.21.

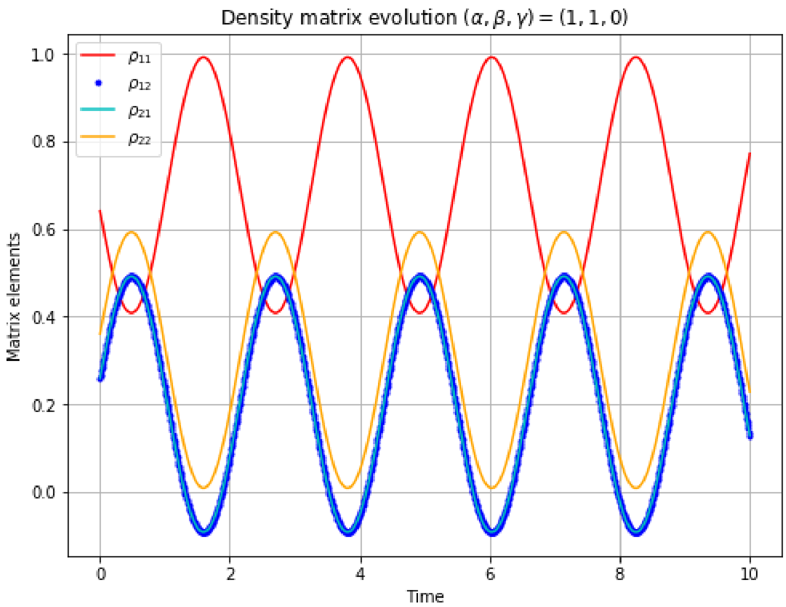

Figure 4: Qubit system evolving freely, i.e. without being measured. The parameters employed in the simulations are $\alpha = 1$, $\beta = 1$, $\gamma = 0$. The initial state of the system has been chosen as $\rho_{11} = 0.64$, $\rho_{12} = 0.48 \cdot e^{i}$, $\rho_{21} = 0.48 \cdot e^{-i}$, $\rho_{22} = 0.36$. Only the real part is plotted. Time is given in arbitrary units.

2.4 Wiseman-Milburn Feedback

In this section we shall introduce some basic concepts on feedback that will be of use in the following sections. Consider you perform a continuous measurement of some observable A and you want the information obtained in the measurement to have a back action on the system so that the dynamics gets conditioned to the outcomes of the measurement. The procedure here described is known as feedback. We aim now to describe how such procedure modifies the master equation. The information included in this section is contained in [8].

If we have a system which evolves according to a certain Hamiltonian H_0, then we will have to include a term associated to the feedback H_{FB}. This extra term takes the form

$$H_{FB} = \frac{dJ}{dt} A. \tag{2.22}$$

Where A is some hermitian operator and J is a classical stochastic measurement record, usually called current, which satisfies the equation

$$\frac{dJ}{dt} = \langle x \rangle + \frac{1}{8k} \xi(t), \tag{2.23}$$

being $\xi(t)$ a noisy contribution. Hence, the Hamiltonian governing the dynamics of the system is

$$H = H_0 + H_{FB}. \tag{2.24}$$

This contribution will lead to a final master equation of markovian nature. This means that the system has no 'memory', in the sense that the following evolution of the system does only depend on its present state.

A real current J has a finite bandwidth (differentiable and continuous), but it is "idealized" as if it were white noise. We then have to apply Itô calculus. The equation 2.23 contains a noise that is not suitable as independent increments dW. It is an equation in Stratonovich sense. Nevertheless, it is always possible to switch from one frame to another by following a mathematical recipe, and construct an Itô equation whose solutions match those of 2.23 [10]. The steps to follow are:

- Replace in 2.23 the real signal by its white-noise limit $\frac{dW}{dt}$.
- Rewrite the resulting equation with the usual rules of calculus, i.e. $(dW)^2 = 0$ instead of $(dW)^2 = dt$.
- Transform the Stratonovich equation into an Itô equation (something well-known in Stochastic Calculus).

Master equation

In order to simplify the following computation we define two super-operators, which will allow to express the results in a compact way. Given an operator A and the density matrix ρ, the super-operators $\mathcal{H}$ and $\mathcal{D}$ are such that

$$\mathcal{H}(A)\rho = A\rho + \rho A^\dagger - \langle A + A^\dagger \rangle \rho \tag{2.25}$$

$$\mathcal{D}(A)\rho = -\frac{1}{2}(A^\dagger A\rho + \rho A^\dagger A - 2A\rho A^\dagger). \tag{2.26}$$

It is easily checked that in the case $A = A^\dagger$, then

$$\mathcal{H}(iA)\rho = i\,[A,\rho] \qquad\qquad \mathcal{D}(A)\rho = \mathcal{D}(iA)\rho = -\frac{1}{2}\,[A,[\;A,\rho]]\,. \tag{2.27}$$

In order to obtain the modified master equation we start looking at the effect of the feedback term in the dynamics. The evolution of the state due to the feedback Hamiltonian is

$$\frac{d\rho_{FB}}{dt} = -\frac{i}{\hbar}\,[H_{FB},\rho] = -\frac{i}{\hbar}\left(\frac{dJ}{dt}\right)[A,\rho] = \frac{1}{\hbar}\left(\langle x\rangle + \frac{1}{\sqrt{8k}}\frac{dW}{dt}\right)\mathcal{H}(-iA)\rho, \tag{2.28}$$

where in the last equality we have directly substituted $\frac{\xi}{8k}$ by its white noise limit. Equivalently, in terms of the increments we get

$$d\rho_{FB} = \frac{1}{\hbar}\left(\langle x\rangle\,dt + \frac{1}{\sqrt{8k}}dW\right)\mathcal{H}(-iA)\rho. \tag{2.29}$$

This equation is still in Stratonovich sense. To obtain the equivalent Itô equation we have to add an extra deterministic term which equals half the square of the stochastic one, i.e.

$$\begin{aligned}\frac{1}{2}\left(\frac{dW}{\hbar\sqrt{8k}}\mathcal{H}(-iA)\right)^2\rho &= \frac{1}{2}\frac{dt}{8k\hbar^2}\mathcal{H}(-iA)\mathcal{H}(-iA)\rho = \frac{1}{2}\frac{dt}{8k\hbar^2}\mathcal{H}(-iA)\left(-i\,[A,\rho]\right)\\ &= \frac{dt}{8k\hbar^2}\left(-\frac{1}{2}\,[A,[A,\rho]]\right) = \frac{1}{8k\hbar^2}\mathcal{D}(iA)\rho dt.\end{aligned} \tag{2.30}$$

Thus, the evolution due to the feedback Hamiltonian written in Itô sense is

$$d\rho_{FB}(\rho) = \frac{1}{\hbar}\left(\langle x\rangle\,dt + \frac{1}{\sqrt{8k}}dW\right)\mathcal{H}(-iA)\rho + \frac{1}{8k\hbar^2}\mathcal{D}(iA)\rho dt. \tag{2.31}$$

The modified master equation can be obtained noticing that the sequence is such that the feedback acts after the measurement. Hence, in a time dt the state evolves by applying the measurement and, later on, the feedback. The change of the state due to the measurement has already been obtained previously, and it can be written in terms of $\mathcal{H}$ and $\mathcal{D}$ as

$$\rho_M = \rho + d\rho_M = \rho + \left(2k\mathcal{D}(x)dt + \sqrt{2k}\mathcal{H}(x)dW\right)\rho. \tag{2.32}$$

Consequently, the total change of the state is

$$\begin{aligned}\rho(t+dt) &= \rho_M + d\rho_{FB}(\rho_M) = \rho + d\rho_M + d\rho_{FB}(\rho + d\rho_M)\\ &= \rho + d\rho_M + d\rho_{FB}(\rho) + d\rho_{FB}(d\rho_M).\end{aligned} \tag{2.33}$$

Where for the last equality we have taken into account that $d\rho_{FB}(\rho)$ is linear in ρ. A last computation yields

$$\begin{aligned} d\rho =& \left(2k\mathcal{D}(x)dt + \sqrt{2k}\mathcal{H}(x)dW\right)\rho + \frac{1}{\hbar}\left(\langle x\rangle\, dt + \frac{1}{\sqrt{8k}}dW\right)\mathcal{H}(-iA)\rho \\ &+ \frac{1}{8k\hbar^2}\mathcal{D}(iA)dt\rho + \frac{1}{\hbar}\left(\langle x\rangle\, dt + \frac{1}{\sqrt{8k}}dW\right)\mathcal{H}(-iA)\left(2k\mathcal{D}(x)dt + \sqrt{2k}\mathcal{H}(x)dW\right)\rho \\ &+ \frac{1}{8k\hbar^2}\mathcal{D}(iA)dt\left(2k\mathcal{D}(x)dt + \sqrt{2k}\mathcal{H}(x)dW\right)\rho \\ =& \left\{\sqrt{2k}\mathcal{H}(x)\rho + \frac{1}{\hbar\sqrt{8k}}\mathcal{H}(-iA)\rho\right\}dW + \left\{2k\mathcal{D}(x)\rho + \frac{\langle x\rangle}{\hbar}\mathcal{H}(-iA)\rho\right. \\ &\left. + \frac{1}{8k\hbar^2}\mathcal{D}(iA)\rho + \frac{1}{2\hbar}\mathcal{H}(-iA)\mathcal{H}(x)\rho\right\}dt. \end{aligned} \tag{2.34}$$

Where we have only conserved the terms up to order dt, and we have used $dW^2 = dt$. Before carrying on the computation, we see which form takes the term $\mathcal{H}(-iA)\mathcal{H}(x)\rho$.

$$\begin{aligned} \mathcal{H}(-iA)\mathcal{H}(x)\rho =& \mathcal{H}(-iA)\left(x\rho + \rho x - 2\langle x\rangle\rho\right) \\ =& -i\left[A, x\rho + \rho x\right] - 2\langle x\rangle\mathcal{H}(-iA)\rho. \end{aligned} \tag{2.35}$$

All this together allows us to write

$$\begin{aligned} d\rho =& \sqrt{2k}\mathcal{H}\left(x - \frac{iA}{4\hbar k}\right)dW + 2k\left\{\mathcal{D}(x)\rho + \mathcal{D}\left(-\frac{iA}{4\hbar k}\right)\rho + \frac{\langle x\rangle}{2k\hbar}\mathcal{H}(-iA)\rho\right. \\ &\left. - \frac{i}{8\hbar k}\left[A, x\rho + \rho x\right] - \frac{\langle x\rangle}{2k\hbar}\mathcal{H}(-iA)\rho\right\}dt \\ =& \sqrt{2k}\mathcal{H}\left(x - \frac{iA}{4\hbar k}\right)dW + 2k\left\{\mathcal{D}(x)\rho + \mathcal{D}\left(-\frac{iA}{4\hbar k}\right)\rho - \frac{i}{8\hbar k}\left[A, x\rho + \rho x\right]\right\}dt. \end{aligned} \tag{2.36}$$

Finally, including the free-dynamics term

$$\begin{aligned} d\rho =& -\frac{i}{\hbar}\left[H_0, \rho\right]dt + \sqrt{2k}\mathcal{H}\left(x - \frac{iA}{4\hbar k}\right)dW \\ &+ 2k\left\{\mathcal{D}(x)\rho + \mathcal{D}\left(-\frac{iA}{4\hbar k}\right)\rho - \frac{i}{8\hbar k}\left[A, x\rho + \rho x\right]\right\}dt. \end{aligned} \tag{2.37}$$

This result will be used in section 4 to find the master equation of the resonators when we replace gravity for a feedback interaction as in [9].

3 Quantum system gravitationally coupled

In our attempt to understand the nature of gravity at small scales, as explained in [9], we may grow some intuition by studying simple cases. If one thinks about the simplest problem that concerns gravity, for sure will end up with the simple pendulum. Now, as we want some interaction to take place, we need to introduce another pendulum. From now on we will study the interaction between these two pendulum under several circumstances, as it has been studied in [9].

3.1 Classical analysis

Let us suppose we have a system of two simple pendulums gravitationally coupled with masses m_1 and m_2, separated a distance d and with frequencies ω_1 and ω_2. This system has two degrees of freedom, which will be defined as the distance from the vertical to the centre of mass of each pendulum. Under such conditions, the hamiltonian of the system may be written as follows

$$H = \sum_{i=1}^{2} \frac{p_i^2}{2m_i} + \frac{1}{2} m_i \omega_i^2 x_i^2 + V_{int}, \tag{3.1}$$

where V_{int} is the gravitational interaction energy, classically determined by

$$V_{int} = -\frac{G m_1 m_2}{r}. \tag{3.2}$$

Being r the distance between the two masses: $r = d - (x_1 - x_2)$. As the gravitational interaction is weak, we expect the displacement x_1 and x_2 to be small quantities. With this motivation we expand the potential energy around the point $r = d$, or, equivalently, around $x_1 - x_2 = 0$

$$V_{int}(x_1, x_2) = -\frac{G m_1 m_2}{d} - \frac{G m_1 m_2}{d^2}(x_1 - x_2) - \frac{G m_1 m_2}{d^3}(x_1 - x_2)^2 + \ldots \tag{3.3}$$

The first term is a constant and can be neglected simply redefining the origin of energies. On the other hand, the second term represents a constant force which modifies the equilibrium position of the masses. We now determine these equilibrium positions, that satisfy the condition

$$\left.\frac{dV}{dx_i}\right|_{\bar{x}} = 0. \tag{3.4}$$

Where we define $\bar{x} = (\bar{x_1}, \bar{x_2})$ the equilibrium position seeked, and where V is the total potential energy. Performing the derivatives we obtain

$$\left.\frac{dV}{dx_1}\right|_{\bar{x}} = 0 = m_1 \omega_1^2 \bar{x}_1 - \frac{G m_1 m_2}{d^2} - \frac{2 G m_1 m_2}{d^3}(\bar{x}_1 - \bar{x_2})$$

$$\left.\frac{dV}{dx_2}\right|_{\bar{x}} = 0 = m_2\omega_2^2\bar{x}_2 + \frac{Gm_1m_2}{d^2} + \frac{2Gm_1m_2}{d^3}(\bar{x}_1 - \bar{x_2}). \tag{3.5}$$

Whence, assuming the two pendulums satisfy $m_1\omega_1^2 = \kappa = m_2\omega_2^2$, we get

$$\begin{aligned} \bar{x}_1 &= \frac{Gm_2}{(\omega_1 d)^2}, \quad \text{and} \\ \bar{x}_2 &= -\frac{Gm_1}{(\omega_2 d)^2}. \end{aligned} \tag{3.6}$$

Now we may change our degrees of freedom defining $x_i' = x_i - \bar{x}_i$. Henceforth we refer to this degree of freedom simply as x_i for the sake of simplicity in the notation. With this change the linear term of V_{int} disappears, and the total potential energy -including the harmonic potential- can be written as

$$\begin{aligned} V(x_1, x_2) &= \sum_{i=1}^{2}\left(\frac{1}{2}m_i\omega_i^2x_i^2 - \frac{Gm_1m_2}{d^3}x_i^2\right) + \frac{2Gm_1m_2}{d^3}x_1x_2 \\ &= \sum_{i=1}^{2}\left(\frac{1}{2}m_i\Omega_i^2x_i^2\right) + Kx_1x_2, \end{aligned} \tag{3.7}$$

where we have defined

$$\begin{aligned} K &= \frac{2Gm_1m_2}{d^3}, \\ \Omega_i^2 &= \omega_i^2 - \frac{K}{m_i} \end{aligned} \tag{3.8}$$

and so we can write the Hamiltonian of the system as

$$H = \sum_{i=1}^{2}\left(\frac{p_i^2}{2m_i} + \frac{1}{2}m_i\Omega_i^2x_i^2\right) + Kx_1x_2. \tag{3.9}$$

As we can see, by taking advantage of the low intensity of the gravity interaction, we have reduced the system effectively to two pendulum with a proper frequency slightly lower than the original one. The two degrees of freedom are coupled due to the last term of the Hamiltonian, which is quadratic. This will allow us to find an exact solution for the system. With this aim we determine now its normal modes. In order to do so we solve the secular equation

$$det(V_{ij} - \omega^2T_{ij}) = 0. \tag{3.10}$$

Whence, according to 3.9 we can identify

$$V_{ij} \rightarrow \left(\begin{array}{cc} m_1\Omega_1^2 & K \\ K & m_2\Omega_2^2 \end{array} \right), \quad \text{and} \tag{3.11}$$

$$T_{ij} \rightarrow \left(\begin{array}{cc} m_1 & 0 \\ 0 & m_2 \end{array} \right). \tag{3.12}$$

Then, solving 3.10, we obtain the following frequencies for the normal modes

$$\omega_\pm^2 = \frac{\Omega_1^2 + \Omega_2^2}{2} \pm \frac{1}{2}\left[\left(\Omega_1^2 - \Omega_2^2\right)^2 + 4\frac{K^2}{m_1 m_2}\right]^{\frac{1}{2}}. \tag{3.13}$$

As we said, we are interested in studying the simplest possible case, for this reason we will restrict to the symmetric case, that is $m_1 = m_2 = m$, which leads to $\omega_1 = \omega_2 = \omega$ and $\Omega_1^2 = \Omega_2^2 = \Omega^2$. Substituting in 3.13 we find that the normal modes frequencies are

$$\omega_+ = \omega \qquad \textit{and} \qquad \omega_- = \omega\left[1 - \frac{2K}{m\omega^2}\right]^{\frac{1}{2}}. \tag{3.14}$$

Note that when the gravitational interaction is weak, the two frequencies differ from one another in

$$\Delta = \omega_+ - \omega_- \simeq \frac{K}{m\omega}. \tag{3.15}$$

This quantity is indeed pretty small and consequently it is probably hard to observe in a laboratory. In the macroscopic case we could have $m \sim 1Kg$, $d \sim 1m$ and $\omega \sim 1s^{-1}$, under such conditions we get $\Delta \sim 10^{-10}s^{-1}$. In the microscopic scale the situation is quite similar. If we have two atoms playing the role of the pendulums, then $m \sim 10^{-26}Kg$. As we want the effect to be as high as possible we should choose a small separation distance, for example an easily achievable value would be $d \sim 10^{-9}m$. Also it would be suitable to have a small frequency, although this is not so easy to get. We will assume a frequency around $\omega \sim 1s^{-1}$. All of this would lead to a value of $\Delta \sim 10^{-9}s^{-1}$. The only thing left is to calculate the normal modes themselves.

The normal modes coordinates

$$q_+ = \frac{1}{\sqrt{2}}(x_1 + x_2) \qquad\qquad q_- = \frac{1}{\sqrt{2}}(x_1 - x_2). \tag{3.16}$$

are associated to the frequencies $\omega_\pm$. From 3.16 we can obtain the momentum associated to each normal mode by performing the temporal derivative, consequently

$$p_+ = \frac{1}{\sqrt{2}}(p_1 + p_2) \qquad\qquad p_- = \frac{1}{\sqrt{2}}(p_1 - p_2). \tag{3.17}$$

Hence, we may write the Hamiltonian of the system in terms of the normal modes coordinates as

$$H(q_\pm, p_\pm) = \frac{p_+^2}{2m} + \frac{1}{2}m\omega_+^2 q_+^2 + \frac{p_-^2}{2m} + \frac{1}{2}m\omega_-^2 q_-^2 = H_+ + H_-. \tag{3.18}$$

As we are in the frame of classical mechanics, we can happily interpret this Hamiltonian as two independent particles and solve the dynamics according to that. However, if we move to the quantum frame we must first check that the canonical commutation relation holds. If we perform the quantization in the original system, this relation can be written as

$$[q_i, p_j] = i\hbar\delta_{ij} \qquad\qquad i, j = \{1, 2\}. \tag{3.19}$$

It is immediate to prove that the same canonical commutation relations are true for the normal modes coordinates. Hence we can also interpret the normal coordinates as the positions and momenta of two independent pendulums.

3.2 Quantum description

In order to make the description within the quantum frame, we are interested in writing the fundamental state of the system $|0\rangle_+ \otimes |0\rangle_- \equiv |0_+, 0_-\rangle$ in terms of the centre of mass coordinates of each particle x_1 and x_2, as these are quantities that can be easily measured. Using the closure relation for $|x_1, x_2\rangle$ we get

$$|0_+, 0_-\rangle = \int dx_1 dx_2 \, |x_1, x_2\rangle \langle x_1, x_2 | 0_+, 0_-\rangle = \int dx_1 dx_2 \psi(x_1, x_2) \, |x_1, x_2\rangle \, . \tag{3.20}$$

Where we must find the form of $\psi(x_1, x_2)$, the wavefunction of the fundamental sate in the centre of mass coordinates representation. This function contains the necessary information for describing the system. As a previous step it may be useful to determine the wavefunction in the representation of the normal modes previously obtained q_+ and q_-. In order to do so we must solve the eigenvalue equation of the system given by

$$H \, |\psi\rangle = E_0 \, |\psi\rangle \, . \tag{3.21}$$

Where E_0 is the energy of the ground state. Luckily for us, and as we commented after 3.19, we can interpret the system as two independent harmonic oscillators with $q_\pm$ as positions and $p_\pm$ as momenta coordinates.

In this representation the Hamiltonian is separable and so we are able to solve them independently from one another. The wavefunction of the total system can be built by multiplying the two wavefunctions of the independent Hamiltonians. Consequently, we focus on solving one of the oscillators, which has the following Hamiltonian

$$h_k = \frac{p_k^2}{2m} + \frac{1}{2}m\omega_k^2 q_k^2. \tag{3.22}$$

Hence the ground state energy for one particle is $E_0^{(k)} = \frac{1}{2}\hbar\omega_k$. Let us recall that the quantum observables act in the spatial coordinates representation such that

$$f(q) \to f(q) \qquad\qquad p \to -i\hbar\partial_q. \tag{3.23}$$

Then we can write 3.21 for one particle as follows

$$\langle q_k| \, h_k \, |0\rangle = E_0^{(k)} \, \langle q_k|0\rangle$$

$$\Rightarrow \left(-\frac{\hbar^2}{2m}\frac{d^2}{dq_k^2} + \frac{1}{2}m\omega_k^2 q_k^2\right)\psi(q_k) = \frac{1}{2}\hbar\omega_k \, \psi(q_k). \tag{3.24}$$

Whence the wavefunction turns out to be a gaussian of the form

$$\psi(q_k) = A \cdot exp\left(-\frac{m\omega_k}{2\hbar}q_k^2\right), \tag{3.25}$$

where A is the normalization constant. As a result, the wavefunction of the total system is given by

$$\Psi(q_+, q_-) = N \cdot exp\left(-\frac{m\omega_+}{2\hbar}q_+^2\right) \cdot exp\left(-\frac{m\omega_-}{2\hbar}q_-^2\right), \tag{3.26}$$

where N is the normalization constant. Substituting 3.14 and 3.16 we get

$$\Psi(x_1, x_2) = N \cdot exp\left(-\frac{m\omega}{2\hbar}\left[\frac{1}{2}(x_1 + x_2)^2 + \frac{1}{2}\sqrt{1-\beta}(x_1 - x_2)^2\right]\right)$$

$$= N \cdot exp\left(-\frac{m\omega}{4\hbar}\left[(1 + \sqrt{1-\beta})x_1^2 + (1 + \sqrt{1-\beta})x_2^2 + 2(1 - \sqrt{1-\beta})x_1 x_2\right]\right). \tag{3.27}$$

In the last equation we have defined $\beta = \dfrac{2K}{m\omega^2}$. This result can be written in a compact form by defining the matrices

$$\vec{x} = (x_1, x_2) \qquad\qquad L = \frac{m\omega}{4\hbar}\begin{pmatrix} 1 + \sqrt{1-\beta} & 1 - \sqrt{1-\beta} \\ 1 - \sqrt{1-\beta} & 1 + \sqrt{1-\beta} \end{pmatrix}, \tag{3.28}$$

thus

$$\Psi(x_1, x_2) = N \exp\left(-\vec{x}^T \cdot L \cdot \vec{x}\right). \tag{3.29}$$

This state has the minimum uncertainty permitted by quantum mechanics. In this sense we can think about it as the closest situation to classical mechanics. This sort of states are known as squeezed states [4], they are indeed gaussian functions which have been rotated and deformed but still satisfy the minimum uncertainty constrain. The off-diagonal elements in 3.28 are responsible for the entanglement of the state, as it is clearly seen in 3.27. If these elements vanish, then the wavefunction would be separable, and so would be the Hamiltonian, implying that the two pendulums are independent and there would be no entanglement. Furthermore, as discussed in [9], the effect of decoherence will be to eliminate the off-diagonal elements, thus reducing entanglement.

4 Gravity as a feedback interaction

In this chapter we closely follow Kafri et al. [9]. In the previous sections we have treated gravity as a direct interaction which Hamiltonian term takes the form $x_1 x_2$. However, Kafri et al. have proposed in [9] to treat gravity as a classical measurement channel, for which we must abandon the sort of interaction term used previously. The idea of the classical channel consists in collecting the information regarding the classical position of one pendulum and, at every instant, using such information to modify the second pendulum's position. Obviously, the process must be reciprocal. Mathematically this can be achieved by continuously measuring the gravitational centre of mass co-ordinate x_j of each particle. Then we define a classical stochastic measurement record, $J_j(t)$, that contains the information of the position. This record will effectively act as a classical control force on the other mass. This describes a feedback-like process. As we have seen in section 2.4, such process can be modelled by introducing a Hamiltonian term of the form

$$H_{grav} = \chi_1 \frac{dJ_1(t)}{dt} x_2 + \chi_2 \frac{dJ_2(t)}{dt} x_1. \tag{4.1}$$

In the continuous weak measurement regime, the currents obey a stochastic differential equation of the form

$$dJ_j(t) = \langle x_j \rangle_c \, dt + \sqrt{\frac{1}{8k_j}} dW_j(t), \qquad \text{where} \qquad k_j = \frac{\Gamma_j}{2\hbar}, \tag{4.2}$$

being Γ_j a constant that determines the rate at which information is gained by the measurement, and $dW_{1,2}$ are two independent Wiener increments.

Before introducing explicitly the feedback effect, we can write the evolution of the conditional density of states ρ_c of the system

$$d\rho_c = -\frac{i}{\hbar}\left[H,\rho_c\right]dt + \sum_{j=1}^{2}\left(\frac{\Gamma_j}{\hbar}\mathcal{D}(x_j)\rho_c dt + \sqrt{\frac{\Gamma_j}{\hbar}}\mathcal{H}(x_j)\rho_c dW_j(t)\right). \quad (4.3)$$

This master equation only takes explicitly into account the process of continuous measurement. We now introduce the effect of the feedback, for which we follow the steps given in the Wiseman-Milburn feedback [8].

The change in the state produced by the feedback term of the Hamiltonian reads

$$\begin{aligned} d\rho_{FB} &= -\frac{i}{\hbar}\left[H_{grav},\rho\right]dt = -\frac{i}{\hbar}\left(\chi_1 dJ_1\left[x_2,\rho\right] + \chi_2 dJ_2\left[x_1,\rho\right]\right) \\ &= -\frac{i}{\hbar}\Bigg(\chi_1\left\langle x_1\right\rangle\left[x_2,\rho\right]dt + \chi_1\sqrt{\frac{1}{8k_1}}\left[x_2,\rho\right]dW_1 \\ &\qquad + \chi_2\left\langle x_2\right\rangle\left[x_1,\rho\right]dt + \chi_2\sqrt{\frac{1}{8k_2}}\left[x_1,\rho\right]dW_2\Bigg). \end{aligned} \quad (4.4)$$

This equation is in Stratonovich sense. We turn it into an Itô sense following the prescriptions previously discussed (see section 2.4).

$$\begin{aligned} d\rho_{FB} = -\frac{i}{\hbar}\Bigg(&\chi_1\left\langle x_1\right\rangle\left[x_2,\rho\right]dt + \chi_1\sqrt{\frac{1}{8k_1}}\left[x_2,\rho\right]dW_1 + \chi_2\left\langle x_2\right\rangle\left[x_1,\rho\right]dt \\ &+ \chi_2\sqrt{\frac{1}{8k_2}}\left[x_1,\rho\right]dW_2\Bigg) - \frac{\chi_1^2}{16k_1\hbar^2}\left[x_2\left[x_2,\rho\right]\right]dt \\ &- \frac{\chi_2^2}{16k_2\hbar^2}\left[x_1\left[x_1,\rho\right]\right]dt. \end{aligned} \quad (4.5)$$

To proceed further we recall that the change of the state owing to the measurement is given by

$$d\rho_{M_j} = 2k_j\mathcal{D}(x_j)\rho dt + \sqrt{2k_j}\mathcal{H}(x_j)\rho dW_j. \quad (4.6)$$

Thus, we can write the total change of the state as

$$\rho(t+dt) = \rho(t) + d\rho_M + d\rho_{FB}(\rho) + d\rho_{FB}(d\rho_M). \quad (4.7)$$

Either substituting the contributions above obtained or using the result of the section 2.4, one gets the following master equation

$$
\begin{aligned}
\frac{d\rho}{dt} = &-\frac{i}{\hbar}\left[H_0,\rho\right] - \frac{i}{2\hbar}\left(\chi_1\left[x_2, x_1\rho + \rho x_1\right] + \chi_2\left[x_1, x_2\rho + \rho x_2\right]\right) \\
&-\sum_{j=1}^{2} k_j\left[x_j,\left[x_j,\rho\right]\right] - \frac{\chi_1^2}{16k_1\hbar^2}\left[x_2\left[x_2,\rho\right]\right] - \frac{\chi_2^2}{16k_2\hbar^2}\left[x_1\left[x_1,\rho\right]\right],
\end{aligned}
\tag{4.8}
$$

where we have already averaged over the noise, so that the terms linear in the Wiener increments have vanished. This equation describes the unconditional dynamics. Taking a look at 2.14, it is clear that the third term represents the effect of the continuous weak measurement of each position. The two last terms have the same kind of commutator, which denotes that the observable is being measured, however, the constant of proportionality depends on variables concerning to the other mass. Consequently, it seems reasonable to identify these two terms as the effect of the feedback to control the dynamics of the masses. Finally, the second term is just the direct effect of the control protocol. It is the analogous to the term Kx_1x_2 treated in previous sections. We shall prove this result later on. As a last step we substitute the value of k_j that we defined in 4.2 and collect some terms to obtain

$$
\begin{aligned}
\frac{d\rho}{dt} = &-\frac{i}{\hbar}\left[H_0,\rho\right] - \frac{i}{2\hbar}\left(\chi_1\left[x_2, x_1\rho + \rho x_1\right] + \chi_2\left[x_1, x_2\rho + \rho x_2\right]\right) \\
&-\left(\frac{\Gamma_2}{2\hbar} + \frac{\chi_1^2}{8\hbar\Gamma_1}\right)\left[x_2\left[x_2,\rho\right]\right] - \left(\frac{\Gamma_1}{2\hbar} + \frac{\chi_2^2}{8\hbar\Gamma_2}\right)\left[x_1\left[x_1,\rho\right]\right].
\end{aligned}
\tag{4.9}
$$

Note that if we fix $\chi_1 = \chi_2 = K$ then the second term turns out to be

$$
\chi_1\left[x_2, x_1\rho + \rho x_1\right] + \chi_2\left[x_1, x_2\rho + \rho x_2\right] = 2K\left[x_1x_2,\rho\right]. \tag{4.10}
$$

Whence we can see that we recover the Hamiltonian interaction term $H_{int} = Kx_1x_2$ considered previously in section 3.1. This fact provides some background support to the model studied in this section, as it matches the known interaction as a particular case. In the following we will restrict to this situation.

Let us take a look at the case of highly asymmetric masses $m_1 \gg m_2$. According to the definition of Γ_j we also expect that $\Gamma_1 \gg \Gamma_2$. Therefore, the noise contribution, which is proportional to Γ_k^{-1}, from the larger mass to the smaller one is much smaller than reverse situation. Now remember from section 2.3 that the measurement term unavoidably leads to decoherence, consequently, the bigger the proportional factor of the measurement term, the bigger the decoherence. Combining these arguments we can conclude that the highly asymmetric case would lead to a greater decoherence rate for the larger mass in comparison to the smaller one.

We now focus on the symmetric case studied in the previous sections, where $m_1 = m_2 = m$. Here we expect the coefficients Γ_j to have the same value. We choose this value to be the one which minimizes the contribution of the measurement and the feedback to the noise, pursuing the dynamics where the decoherence rate is minimum. So we need to minimize

$$\frac{\Gamma}{2\hbar} + \frac{\chi^2}{8\hbar\Gamma}, \tag{4.11}$$

whence we get $\Gamma = \frac{\chi}{2} = \frac{K}{2}$. Substituting this result into the master equation leads to

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}\left[H_0, \rho\right] - \frac{i}{\hbar}K\left[x_1 x_2, \rho\right] - \frac{K}{2\hbar}\sum_{j=1}^{2}\left[x_j\left[x_j, \rho\right]\right]. \tag{4.12}$$

Rewriting in terms of dimensionless operators $\tilde{x} = \left(\frac{m\omega}{\hbar}\right)^{1/2} x$ we obtain

$$\begin{aligned}\frac{d\rho}{dt} &= -\frac{i}{\hbar}\left[H_0, \rho\right] - \frac{iK}{m\omega}\left[\tilde{x}_1\tilde{x}_2, \rho\right] - \frac{K}{2m\omega}\sum_{j=1}^{2}\left[\tilde{x}_j\left[\tilde{x}_j, \rho\right]\right] \\ &= -\frac{i}{\hbar}\left[H_0, \rho\right] - ig\left[\tilde{x}_1\tilde{x}_2, \rho\right] - \frac{1}{4}\sum_{i,j=1}^{2} Y_{ij}\left[\tilde{x}_j\left[\tilde{x}_j, \rho\right]\right],\end{aligned} \tag{4.13}$$

where we have defined $g = \frac{K}{m\omega}$ and the matrix $Y_{ij} = 2g\delta_{ij}$

4.1 System evolution

In order to understand the dynamics of the system we will work on equation 4.12. Solving the equation directly might be considerably complicated. For this reason, instead of finding the temporal evolution of the density operator, we shall focus on the evolution of the mean values of the positions and momentums for each particle, extracting afterwards the convenient information.

The mean value of a certain operator A can be written as

$$\langle A \rangle = \mathrm{Tr}\left(\rho A\right) = \mathrm{Tr}\left(A\rho\right). \tag{4.14}$$

Hence, we can perform the time evolution of the mean value as

$$\partial_t \mathrm{Tr}\left(A\rho\right) = \mathrm{Tr}\left(A\partial_t\rho\right). \tag{4.15}$$

where we have considered that, in Schrödinger picture, the observables do not evolve with time. Now we shall substitute 4.12 in 4.15, obtaining

$$\partial_t \langle A \rangle = -\frac{i}{\hbar}\mathrm{Tr}\left(A\left[H_0,\rho\right]\right) - \frac{iK}{\hbar}\mathrm{Tr}\left(A\left[x_1x_2,\rho\right]\right) - \frac{K}{2\hbar}\sum_{k=1}^{2}\mathrm{Tr}\left(A\left[x_k,\left[x_k,\rho\right]\right]\right), \tag{4.16}$$

and therefore

$$\partial_t \langle A \rangle = -\frac{i}{\hbar}\langle [A,H_0]\rangle - \frac{iK}{\hbar}\langle [A,x_1x_2]\rangle - \frac{K}{2\hbar}\sum_{k=1}^{2}\langle [[A,x_k],x_k]\rangle . \tag{4.17}$$

Now we apply this equation to the momentum and position observables, as well as to the quadratic moments. Essentially, the computation can be done easily just taking into account the canonical commutation relations given in 3.19 and some properties of the commutators, such as

$$[AB,C] = A\left[B,C\right] + \left[A,C\right]B \;\;\wedge\;\; [A,B] = -\left[B,A\right]. \tag{4.18}$$

The time evolution equations for the mean values and the quadratic moments are

$$\begin{aligned}
&\partial_t \langle x_k\rangle = \frac{\langle p_k\rangle}{m}\\
&\partial_t \langle p_k\rangle = -\left(m\omega_k^2\langle x_k\rangle + K\langle x_i\rangle\right)\\
&\partial_t \left\langle x_k^2\right\rangle = 2\frac{\langle p_kp_k\rangle}{m}\\
&\partial_t \langle x_kp_k\rangle = \frac{1}{m}\left\langle p_k^2\right\rangle - m\omega_k^2\left\langle x_k^2\right\rangle - K\langle x_1x_2\rangle\\
&\partial_t \langle x_1x_2\rangle = \frac{1}{m}\left(\langle p_1x_2\rangle + \langle x_1p_2\rangle\right)\\
&\partial_t \langle x_ip_k\rangle = \frac{1}{m}\langle p_1p_2\rangle - m\omega_k^2\langle x_1x_2\rangle - K\left\langle x_i^2\right\rangle\\
&\partial_t \langle p_1p_2\rangle = -m\omega_k^2\left(\langle p_1x_2\rangle + \langle x_1p_2\rangle\right) - K\left(\langle p_1x_1\rangle + \langle x_2p_2\rangle\right)\\
&\partial_t \left\langle p_k^2\right\rangle = -2m\omega_k^2\langle p_kx_k\rangle - 2K\langle x_ip_k\rangle + \hbar K.
\end{aligned} \tag{4.19}$$

Note that the evolution of the system does only depend on moments up to order 2. This means that if the system is originally in a Gaussian state (and we have seen that this is the case for the ground state in section 3.2), the system will remain Gaussian forever owing to the fact that a Gaussian function can be perfectly described only with the first and second moments.

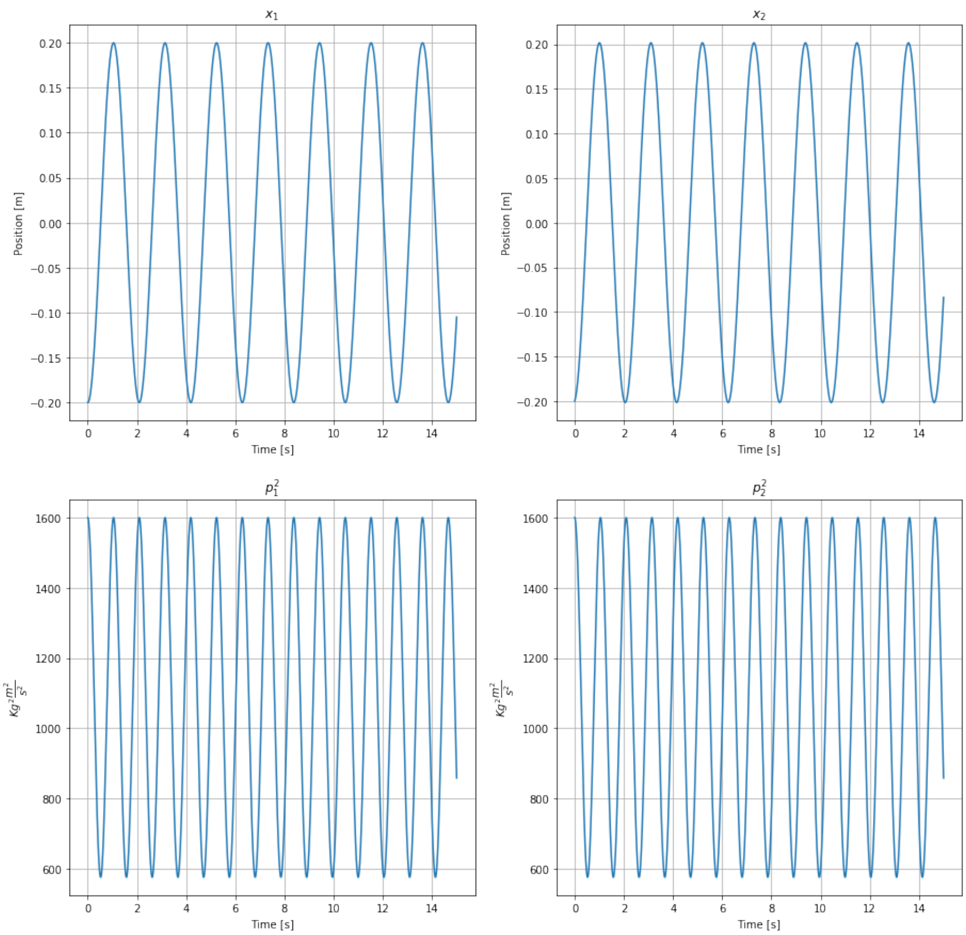

Figure 5: Simulation of equation 4.19 for the pendulum of LIGO: $m = 40\,kg$, $L = 1\,m$, $d = 1\,m$, $\omega = 3\,s^{-1}$. As the initial configuration we separate both pendulum from their equilibrium position and then let the system evolve. Moment diffusion is not observed.

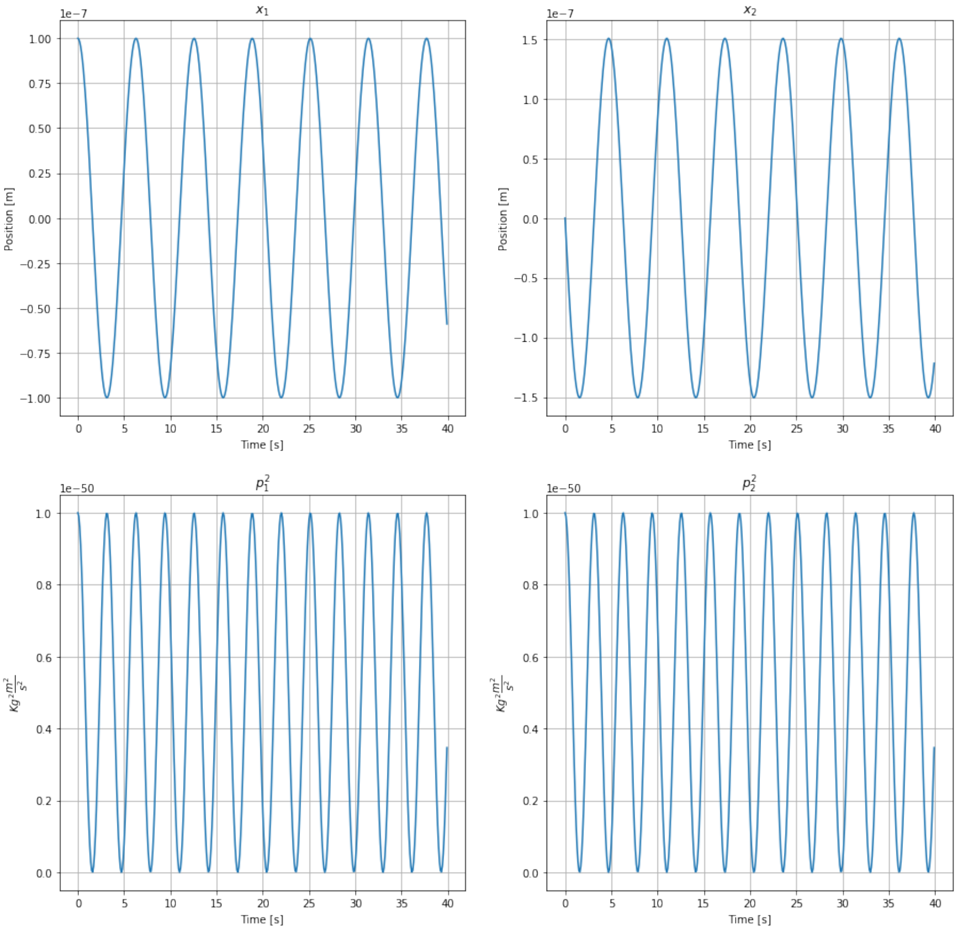

Figure 6: Simulation of equation 4.19 for two trapped Ca^+ ions optically cooled: $m = 40\,uma$, $d = 1\,\mu m$, $\omega = 1\,s^{-1}$. The initial configuration consists in the two masses displaced from the equilibrium position and with a small initial velocity. Moment diffusion is not observed.

Also note that the last term of the last equation leads to moment diffusion, i.e. a continuous increase of the momentum. This effect arises from the last term of 4.12, which is also the responsible of decoherence. Thus these two effects are directly related and we can obtain information on decoherence by taking a look at the evolution of the momentum. We shall now study in more detail both decoherence and diffusion processes.

The figures 5 and 6 show the dynamics predicted by the equation 4.19 in the case of a macroscopic and microscopic system respectively. The mean values and quadratic moments that are not plotted exhibit the same oscillating behaviour without change in the amplitude. In none of both figures we can observe the effect of decoherence through moment diffusion. As we will prove later, for observing such phenomena we should wait an immense amount of time. The gravitational interaction between the masses is not seen either due to its low intensity.

4.2 Decoherence rate

The master equation 4.12 describes the dynamics of the system considered along this paper. Once again, from section 2.3 we know that the only responsible for decoherence is the last term of 4.12. We now project the last term of the equation into the position space of one of the masses in order to obtain the decoherence rate at which it is submitted.

$$
\begin{aligned}
\frac{d\langle x'_k|\,\rho\,|x_k\rangle}{dt} =&(\dots) - \frac{K}{2\hbar}\langle x'_k|\,[x_k\,[x_k,\rho]]\,|x_k\rangle \\
=&(\dots) - \frac{K}{2\hbar}\langle x'_k|\left(x_k^2\rho - 2x_k\rho x_k + \rho x_k^2\right)|x_k\rangle \\
=&(\dots) - \frac{K}{2\hbar}(x_k^2 - 2x_k x'_k + x_k'^2)\langle x'_k|\,\rho\,|x_k\rangle \\
=&(\dots) - \frac{K}{2\hbar}(x_k - x'_k)^2\langle x'_k|\,\rho\,|x_k\rangle\,.
\end{aligned}
\tag{4.20}
$$

Whence we identify the decoherence rate as

$$
\Lambda_{grav} = \frac{K}{2\hbar}(x_k - x'_k)^2 = \frac{K}{2\hbar}\Delta x_0^2, \tag{4.21}
$$

where for the last equality we have considered we are in the ground state. Let us now compute the deviation $\left(\Delta x^2\right)_{ground} = \left\langle x^2\right\rangle_{ground} - \langle x\rangle_{ground}^2$. The position operator can be written in terms of the raising and lowering operators as

$$
x = \sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a). \tag{4.22}
$$

Consequently, the mean value x for the ground state is null. Regarding the mean value of x^2 we have

$$x^2 = \frac{\hbar}{2m\omega}(a^{\dagger 2} + a^2 + 1 + 2N), \tag{4.23}$$

being N the number operator and 1 the identity operator. Therefore, the mean value of x^2 turns out to be $\frac{\hbar}{2m\omega}$. Collecting these results and recalling the definition 3.15 we can rewrite the decoherence rate as

$$\Lambda_{grav} = \frac{K}{4m\omega} = \frac{\Delta}{4}. \tag{4.24}$$

So the key parameter to take into account when studying decoherence is Δ, the normal mode split for weak interaction. This parameter can be rewritten using the definition 3.8, obtaining

$$\Delta = \frac{2Gm}{\omega d^3}. \tag{4.25}$$

Now if we have spherical masses, we can rewrite this expression in terms of the density of the material such that

$$\Delta = \frac{8\pi G\rho}{3\omega}\left(\frac{r}{d}\right)^3 = \frac{\pi G\rho}{3\omega}\left(\frac{2r}{d}\right)^3. \tag{4.26}$$

Obviously, the relation $d > 2r$ must hold always, the quantity is bounded, satisfying the inequality

$$\Delta \leq \frac{\pi G\rho}{3\omega} \quad \Rightarrow \quad \Lambda_{grav} \leq \frac{\pi G\rho}{12\omega}. \tag{4.27}$$

After this computation we are in conditions of making some estimation similar to the one made after equation 3.15. In order to detect the decoherence rate we need a large value for the density and a rather small value for the frequency. An optimistic estimation given by Kafri [9] provides $\Lambda_{grav} \sim 10^{-7}$, which it is argued to be out of range for observing with the current technology.

4.3 Momentum diffusion rate

It is worth to keep working on the last term of 4.12. As we have seen, this term not only predicts the decoherence of the system but also a momentum diffusion, which has a special interest for our discussion as it is a classical measurable effect. We have already seen explicitly how this effect emerges naturally from the master equation when written in terms of the mean values of the observables 4.19.

An alternative way of obtaining such result is by noticing that the last term in 4.12 corresponds to the averaging of a random force with correlation function [6]

$$\langle F(t)F(t+\tau)\rangle = K\hbar\delta(\tau). \tag{4.28}$$

where the factor multiplying the delta function is the diffusion rate.

Let us now consider two different systems in order to evaluate the momentum diffusion rate. First of all we shall explore a microscopic situation, for example two trapped ions. Provided that they have a mass around $m \sim 238\ uma$ -corresponding to uranium-, a small frequency $\omega \sim 1s^{-1}$ and a separation of $d \sim 10^{-6}m$, then we obtain $K \sim 10^{-41}$. Hence the diffusion rate takes $D \sim 10^{-75}(kg\ \ m/s)^2/s$.

This means that if we want to observe a change on the momentum δp due to the diffusion, we should wait a time $t = (\delta p)^2/D$. For example, if we want to detect a velocity increase of mm/s we would measure $\delta p \sim 10^{-28}$, so we should wait around $t \sim 10^{19}s$. Obviously this can never be observed, as the age of the universe is around $t \sim 10^{16}s$.

We move now to the macroscopic setup. For the computation we consider $m \sim 40\,kg$, $\omega \sim 1\,s^{-1}$ and $d = 1\,m$, thus leading to $K \sim 10^{-7}$. Therefore the diffusion rate is around $D \sim 10^{-41}$.

Once again we would like to detect a change of the velocity of the order of mm/s, for which we must have $\delta p \sim 10^{-2}$. Consequently we should wait $10^{37}\,s$, so the situation gets even more complicated for the observations.

4.4 Brownian motion analogy and effective temperature

The alternative reasoning in the previous section 4.3 for determining the diffusion rate through the comparison with the random force averaging strongly suggest an analogy with the Brownian motion.

Essentially, the Brownian motion is a stochastic process where the system is affected by both a diffusive and friction process. Furthermore it is a process of Markovian nature, that is to say that the immediate future dynamics of the system only depends on its present state. This Markovian nature is mathematically described by the correlation function 4.28.

Although the Brownian motion can be satisfactorily described with the classical model proposed by Langevin, we will use here the quantum description for comparison purposes. There is a particularly interesting term in the quantum master equation of Brownian motion [9]

$$\partial_t \rho = (\dots) - \sum_{i=1}^{2} 2\gamma k_B T m \left[x_j \left[x_j, \rho\right]\right]. \tag{4.29}$$

It has the same dependence that the term in 4.12. The factor γ represents the dissipation rate of the resonators in contact with a common thermal bath of temperature T. Now it seems convenient to compare both factors representing the dissipation rate. Thus we get an effective temperature for the gravitational decoherence

$$2\gamma k_B T m = K\hbar \quad \Rightarrow \quad T = \frac{K\hbar}{2\gamma k_B m}. \tag{4.30}$$

This result can be rewritten in terms of the quality factor, defined as $Q = \omega/2\gamma$, thus

$$T = \frac{\hbar Q \Delta}{k_B}. \tag{4.31}$$

From [6] we see that a high and achievable value of the quality factor is $Q \sim 10^6$. Recalling the numerical estimation of 4.2 we also have $\Delta \sim \Lambda_{grav} \sim 10^{-7}$. Using these values, the effective temperature is $T \sim 10^{-12}$. If we were to observe the decoherence of the system we should have a lower environment temperature than the one obtained. However this temperature is extremely low and could hardly be achieved in the desired conditions. In [9] they propose a more optimistic value of the quality factor $Q \sim 10^9$. Even with this value the temperature is of the order of nano-Kelvin, still very low.

Currently the system of interest that is closer to the desired conditions is a chain of trapped ions. Doppler cooling is the theoretical background to construct such systems. Basically the ions are submitted to the action of two lasers in each axis slightly detuned to the red of the resonance frequency of the atom [3]. A classical Doppler effect acts together with radiative forces giving as a final result the slowing down, and therefore cooling, of the ions. Furthermore the ions are submerged in a common harmonic trap to compensate the Coulomb interaction. This setup allows us to have relatively heavy/dense atoms separated a distance of the order of micrometers, thus increasing the value of Δ and our expectations to detect gravitational decoherence.

5 Conclusions and discussions

This work has been a detailed educational review of the paper [9], where gravity is introduced as a purely classical communication channel. This has been achieved by considering that each of the masses is being continuously and weakly measured. In this context the quantum theory of measurement applies, providing a mathematical frame for describing such monitoring. In this model, the result of measurement is somehow stored and feedback to the other mass, inducing the right gravitational force and reproducing thus the dynamics.

On our way to find the master equation some assumptions were made. First of all we accounted for the symmetrical case, as both pendulums have the same mass then their reciprocal effect must be equal, and so must the weighing factor χ_i. Therefore, the master equation simplifies and we naturally recover the classical interaction Hamiltonian Kx_1x_2 deduced in section 3.1. This fact alone does not prove anything, but is a reason to believe that the model proposed indeed accurately pictures gravity. The second key imposition is to choose a value for the rate at which information is gained Γ such that the noise contribution from the measurement to the system is minimum, being thus proportional to the gradient of the gravitational field K. Its motivation comes from the fact that we want to detect decoherence, which seems to be considerably weak, so having a big amount of noise in the system would complicate our task.

We extract information from the evolution of the mean values and quadratic moments instead of solving the master equation for the density operator. Specifically the most relevant parameter for our discussion is the decoherence rate, which happens to be proportional to the normal modes split of the resonator. Its value turns out to be considerably small. Furthermore, two classical effects associated with decoherence are studied. On one hand we have the momentum diffusion, if decoherence is taking place then we should expect the resonators to get heated. After estimations we see that for detection of such phenomenon we should wait a large amount of time. On the other hand, we find an effective temperature below which decoherence could not be distinguished from environmental effects [9]. A simple computation shows that this temperature is of the order of nano-Kelvin. All this together pose some technological requirements that must be fulfilled before trying to detect decoherence.

6 Síntesis en Español

6.1 Introducción

A día de hoy no hemos sido capaces de desarrollar una teoría que aúne gravedad y cuántica dentro del mismo marco. Aparentemente a escalas mucho mayores que las de Planck, los fenómenos que emergen de una ”gravedad cuántica” no deberían ser observados. Sin embargo, recientemente Diósi y Penrose han propuesto un marco en el que dichos efectos podrían observarse a escala de laboratorio. El trabajo aquí presentado es una revisión con fines esencialmente educativos del artículo [9] publicado por Kafri et al. Estudiaremos un sistema acoplado gravitatoriamente en el marco de la mecánica cuántica. La idea básica sobre la que se sustenta nuestro estudio es el tratamiento de la gravedad como un proceso de realimentación: la posición de cada masa es medida, y dicha información se transmite a la segunda masa de forma que se induzca la fuerza gravitatoria. Obtendremos una tasa de decoherencia que revelará su intensidad.

6.2 Teoría cuántica de la medida

Comenzamos por introducir lo que entendemos como medida ideal. Básicamente se trata de una medida que no altera el valor medio de los observables, aunque sí que cambia el estado cuántico. Este proceso puede ser generalizado al caso en el que el detector tiene resolución finita, eliminando el carácter ideal de la mediada. En concreto estudiamos el caso de una medida que se realiza de forma continuada. En este contexto, introducimos el límite difusivo a fin de encontrar el cambio en la ecuación maestra debido al proceso de medida. Finalmente incorporamos dos términos, uno de carácter determinista y otro de carácter estocástico. A continuación ilustramos el papel que juega el proceso de medida en la evolución de un sistema, para ello simulamos la evolución de un qubit. Por último, incorporamos un segundo proceso en el sistema, la retroalimentación o *feedback*. De nuevo analizamos el cambio que se produce en la ecuación maestra, demostrando que debemos añadir varios términos de carácter tanto determinista como estocástico.

6.3 Sistema cuántico acoplado gravitatoriamente

En este capítulo describimos un sistema formado por dos péndulos acoplados gravitatoriamente. En primer lugar damos un análisis desde una perspectiva clásica, asumiendo que el efecto gravitatorio es débil, de forma que podemos escribir el Hamiltoniano del sistema en forma cuadrática. Posteriormente hallamos los modos normales y la frecuencia que los separa. En segundo lugar abordamos el sistema a través del formalismo cuántico, hallando la autofunción del estado fundamental, que resulta ser un estado *squeezed*. Dichos estados tienen mínima incertidumbre. Escrita en una forma compacta surge una matriz en la que se espera que los elementos no diagonales desaparezcan como consecuencia de la decoherencia.

6.4 Gravedad como una interacción retroalimentada

Remplazamos en el Hamiltoniano el potencial gravitatorio por un Hamiltoniano que describe el proceso de *feedback*. Dicho proceso consiste en que la posición de cada masa es medida y esa información almacenada en una función y empleada para modificar la dinámica futura de su compañera de igual forma que lo haría la gravedad. Determinamos la ecuación maestra 4.8 en base a lo discutido en el capítulo sobre teoría de la medida. Particularizamos para el caso simétrico, en el que las masas son iguales, recuperando el Hamiltoniano de interacción ya conocido de la mecánica clásica. Imponemos en la ecuación una contribución mínima al ruido, obteniendo una tasa de transmisión de información entre las masas que es proporcional al gradiente del campo gravitatorio. El siguiente paso es resolver la ecuación maestra. En lugar de resolverla directamente, optamos por encontrar la evolución de los valores medios y de los momentos cuadráticos que aportarán de forma alternativa la información necesaria. A partir de la ecuación maestra determinamos la tasa de decoherencia, que resulta proporcional al desfase entre los modos normales de los osciladores. Asimismo, estudiamos dos efectos clásicos que surgen del proceso de decoherencia. Por una parte, encontramos la difusión del momento, cuya tasa obtenemos a partir de la evolución de los momentos cuadráticos. En segundo lugar, en analogía con el movimiento Browniano, definimos una temperatura efectiva por encima de la cual no podremos detectar la decoherencia, pues quedará oculta por la agitación térmica. Estimaciones de los tres parámetros mecionados revelan bajo que condiciones debe estar el sistema a fin de detectar la decoherencia. En líneas generales podemos decir que aún no disponemos de la tecnología necesesaria para satisfacer tales condiciones, pero los avances en "opto-mecánica" son prometedores.

6.5 Conclusiones y discusiones

Hemos descrito la interacción gravitatoria mediante un modelo de *feedback* estudiado por Kafri et al. en [9]. Para la obtención de la ecuación maestra nos hemos apoyado en la teoría cuántica de la medida. Una vez obtenida, realizamos una serie de simplificaciones, en primer lugar consideramos el caso simétrico en el que las masas son iguales, recuperando el término de interacción de la mecánica clásica. Si bien esto no prueba nada, es una razón para creer que el modelo propuesto describe de forma adecuada a la gravedad. La segunda simplificación es minimizar la señal de ruido que proviene del proceso de medida, a fin de poder distinguir más claramente el proceso de decoherencia. Determinamos la tasa de decoherencia, cuyo valor es considerablemente pequeño y por tanto difícil de detectar en el laboratorio. Asimismo, encontramos la tasa de difusión del momento, un efecto clásico que tiene como origen la decoherencia. Dicho valor revela que el efecto se hace notable tras esperar un tiempo del orden de la edad del universo, y es por tanto indetectable e irrelevante. Por último, determinamos una temperatura por debajo de la cual debe estar el sistema si no queremos que el proceso de decoherencia nos pase desapercibido. Concluimos entonces que, visto el estado del arte de la tecnología, la detección de este fenómeno se muestra complicada por el momento.

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